Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 615-622

A new solution of the biconfluent Heun equation

H. EXTON

RIASSUNTO: *Si determina una soluzione esplicita della equazione biconfluente di Heun (BHE) con il metodo della trasformata di Laplace. La soluzione trovata non è menzionata nella letteratura precedente ed è presentata nella forma di una serie convergente tripla di tipo ipergeometrico.*

ABSTRACT: *The Laplace transform and inverse are applied to a soluble system of differential equations to deduce an explicit solution of the biconfluent Heun equation wich has not previously been mentioned in the literature. This is presented in the form of a triple convergent series of hypergeometric type. The computer algebra package MAPLE V is used deal with an ancillary system of non-linear algebraic equation.*

1 – Introduction

The purpose of this study is to deduce a new explicit representation of a solution of the biconfluent Heun equation (BHE) in the form of a convergent triple series of hypergeometric type. The BHE is a linear ordinary differential equation of the second order with two singularities, one regular at the origin and the other irregular of the fourth type at infinity. The canonical form of the BHE used here is

$$XY''+(1+\alpha-\beta X-2X^2)Y'+[(\gamma-\alpha-2)X-\{\delta+(1+\alpha)\beta\}/2]Y=0 \tag{1.1}$$

with the exponents zero and $-\alpha$.

KEY WORDS AND PHRASES: *Biconfluent – Heun – hypergeometric.*
A.M.S. CLASSIFICATION: 34A05 – 33C50 – 33E20 – 81Q05

For a general overview of Heun's equation and its confluent forms, the reader is referred to Ronveaux [3], in particular Part D. This paper is motivated, in part, by the fact that the BHE is associated with a number of quantum interactions, see Ronveaux [3], 199.

$$\text{If} \qquad U = X^{1/2+\alpha/2}\exp(-\beta X/2 - X^2/2)Y(X), \tag{1.2}$$

the normal form of (1.1) is seen to be

$$U'' + [(1-\alpha^2)X^2/4 - \delta X/2 + \gamma - \beta^2/4 - \beta X - X^2]U = 0\,, \tag{1.3}$$

wich will be used below.

The method of tackling (1.3) and, in turn, (1.1) used here consists of applying the Laplace transform and its inverse to a soluble system of differential equations.

A subsidiary system of non-linear algebraic equations is shown to be consistent in a manner readily capable of numerical implementation by means of the computer algebra package MAPLE V. Compare Exton [2] where this method has been used to obtain solutions of Heun's equation.

The Pochlammer symbol $(a,n) = \Gamma(a+n)/\Gamma(a)$ is used during the course of the working out and it is taken that all indices of summation run over all of the non-negative integers. Any values of parameters leading to results which do not make sense are tacitly excluded and any constant multipliers which have no essential bearing on the nature of the main result are also left out. Term-by-term integration is justified in each case on account of the uniform convergence of the series concerned. If

$$Y(\alpha,\beta,\gamma,\delta;X) \tag{1.4}$$

is a solution of (1.1), then so also is

$$X^{-\alpha}Y(-\alpha,\beta,\gamma,\delta;X) \tag{1.5}$$

See Ronveaux [3], 195.

2 – A soluble differential system

Consider the differential equations

$$Ktv''' + av'' + bv' + cv = 0 \tag{2.1}$$

and

$$tu''/k + fu' + gu = v\,, \tag{2.2}$$

which are equivalent to the equation of the fifth order

$$\begin{aligned} &t^2u^v + (fk + 3 + a/k)tu^{1v} + [(gk + b/k)t + af + 2a/k]u'''+\\ &\quad + [ct/k + ag + bf + b/k]u'' + (bg + cf)u' + cgu = 0. \end{aligned} \tag{2.3}$$

The inverse Laplace transform is written as

$$u(t) = \int \exp(-xt)y(x)dx, \tag{2.4}$$

where we have put

$$x = 1/X = \tag{2.5}$$

where the contour of integration consist of a simple path closed on the Riemann surface of the integrand such that this integrand remains unchanged after the completion of one circuit.

Is is found that the function $y(x)$ is a solution of the differential equation

$$\begin{aligned} &x^5y'' + [(5 - fk - a/k)x^4 + (gk + b/k)x^3 - cx^2/k]y'+\\ &\quad + [(af + 8 - 4fk - 2a/k)x^3 + (3gk + 2b/k - ag - bf)x\ {-}^2 +\\ &\quad + -(bg + cf - 2c/k)x - cg]y = 0. \end{aligned} \tag{2.6}$$

If

$$U'' + IU = 0 \tag{2.7}$$

and

$$U = \exp[-(gk - b/k)/(2x) + c/(4kx^2)]x^{5/2 - fk/2 - a/(2k)}y \tag{2.8}$$

then,

$$\begin{aligned} I = {} & (17k^2 - a^2 + 2k^2af - f^2k^4 - 8k^3f)/(4k^2x^2) + \\ & + (2ab + 6k^3g + 2fk^4g - 2k^2bf + 2bk - 2k^2ag)/(4k^2x^3) + \\ & + (2k^2bg - 4ck - g^2k^4 - b^2 - 2ac + 2k^2cf)/(4k^2x^4) + \\ & + (bc - cgk^2)/(2k^2x^5) - c^2/(4k^2x^6). \end{aligned} \tag{2.9}$$

If we put

$$\begin{aligned} (17k^2 - a^2 + 2k^{2af - f^2}k^4 - 8k^3f)/(4k^4) &= A, \\ (2ab + 6k^3g + 2fk^4g - 2k^2bf + 2bk - 2k^2ag)/(4k^2) &= B, \\ (2k^2bg - 4ck - g^2k^4 - b^2 - 2ac + 2k^2cf)/(4k^2) &= C, \\ (bc - cgk^2)/(2k^2) &= D \\ \text{and} \quad c^2/(4k^2) &= E, \end{aligned} \tag{2.10}$$

where A, B, C, D and E are to be determined, then the system (2.10) can be shown to be consistent and manipulated numerically as required by means of the computer algebra package MAPLE V. The explicit algebraic results are too lengthy to be recorded here and this would be unnecessary in any case in view of the convenience of numerical manipulation when required. One of the quantities a, b, c, f, g and k remains arbitrary and this fact will be employed in Section 4 of this paper.

From (2.5), we recall that $x = 1/X$, and with this new variable, (2.7) becomes

$$U'' - 2U'/X + (AX^{-2} + BX^{-1} + C + DX + EX^2)U = 0. \tag{2.11}$$

Let

$$V = XU, \tag{2.12}$$

when we have

$$V'' + (EX^2 + DX + C + BX^{-1} + AX^{-2})V = 0. \tag{2.12}$$

On putting $E = (1-\alpha^2)/4$, $D = -\delta/2$, $C = \gamma - \beta^2/4$, $B = -\beta$ and $A = -1$, (2.12) is seen to be identified with (1.3), the normal form of the BHE, a solution to the canonical form of wich (1.1) can be obtained by means the inverse of (2.4) from the solution of (2.1) and (2.2).

3 – The solution of the equations (2.1) and (2.2)

Since (2.1) is a Laplace linear equation, a solution can be written as

$$v = \int_{-\infty}^{(0+)} \exp(st)\eta(s)ds, \tag{3.1}$$

in which

$$\eta'/\eta = [(a/k-3)s^2 = bs/k + c/k]s^{-3}\,, \tag{3.2}$$

when

$$\begin{aligned} v &= \int_{-\infty}^{0+} \exp[st - b/ks^{-1} - c/(2ks^2)]s^{a/k-3}ds \\ &= t^{2a/k} \sum_{m,n} [(-bt/k)^m(-ct^2/(2k))^n]/[(3-a/k, m+2n)m!n!] \end{aligned} \tag{3.3}$$

By the theory of the solution of inhomogeneous linear differential equations, the equation (2.2), which is written as

$$\begin{aligned} &tu'' + kfu' + kgu = \\ &= t^{2-a/k} \sum_{m,n} [(-bt/k)^m(-ct^2/(2k))^n]/[(3-a/k, m+2n)m!n!], \end{aligned} \tag{3.4}$$

has the solution

$$\begin{aligned} u = {} & [(b/(2k^2)^m(-c/2g^2k^3)^n]/[(2-a/k, m+2n)m!n!]\times \\ & \times {}_0f_{1,\,3-a/k+m+2n}[-, kf; -kgt], \end{aligned} \tag{3.5}$$

when the inhomogeneous hypergeometric function ${}_0f_{1,\,3-a/k+m+2n}[-kf; -kgt]$ is given by

$$\begin{aligned} &(-kgt)^{3-a/k+m+2n}/[(3-a/k+m+2n)(2-a/k+kf+m+2n)]\times \\ &\times {}_1F_2[1; 4-a/k+m+2n, 3-a/k+kf+m+2n; -kgt]. \end{aligned} \tag{3.6}$$

See Babister [1], 277. Hence, u is proportional to

$$t^{3'-a/k}\sum_{m,n,p}{}' [(2-a/k+kf,m+2n)(-bt/k)^m(-ct^2/(2k))^n(-kgt)^p]/ \\ /[(4-a/k,m+2n+p)(3-a/k+kf,m+2n+p)m!n!]. \tag{3.7}$$

4 – A solution of the BHE

From (2.4) by inversion,

$$y(x)=\int \exp(xt)u(t)dt, \tag{4.1}$$

so that, if the contour of integration is taken to be a simple loop beginning and ending at ∞ if $\mathrm{Re}(x)<0$ at $-\infty$ if $\mathrm{Re}(x)>0$ and encircling the origin, then

$$y(x)=x^{a/k-4}\sum_{m,n,p}[(2-a/k+kf,m+2n)(-b/(kx))^m\times \\ \times(-c/(2kx^2))^n(-kg/x)^p]/ \\ /[(3-a/k+kf,m+2n+p)m!n!]. \tag{4.2}$$

From (1.2), (1.3), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12),

$$Y(X)= \\ =X^{2-a/(2k)+fk/2-\alpha/2}\exp[(\beta-gk-b/k)X/2+(c/4+1)X^2/2]\times \\ \times\sum_{m,n,p}[(2-a/k+kf,m+2n)(-bX/k)^m(-cX^2/(2k))^n(-kgX)^p]/ \\ /[(3-a/k+kf,m+2n+p)m!n!]. \tag{4.3}$$

Put

$$2-a/(2k)+fk/2-\alpha/2=0 \tag{4.4}$$

and (4.4) is found by MAPLE V to be consistent with the equations of the system (2.10). The expansion (4.3) is then a convergent triple series representation of the solution with zero exponent of the BHE relative to the regular singularity at the origin. Bearing in mind (1.4) and (1.5), a second independent solution relative to the same singular point can be deduced.

Triple hypergeometric series of this type have already been encountered in the literature and have been studied by Srivastava [5]. If the parameters are not too large, the series solution (4.3) can be implemented numerically for $|x| <= 50$.

5 – A representation of a solution relative to the irregular singularity at infinity

Bearing in mind from (4.4) that the exponent of (4.3) has been put equal to zero, write this expression in the form

$$Y(X) = \exp[(\beta - gk - b/k)X/2 + (c/4+1)X^2/2]S \tag{5.1}$$

where the series S can be expressed as a double series of confluent hypergeometric functions, namely

$$\begin{aligned} S = \sum_{m,n} & [(2 - a/f + kf, m+2n)(-bX/k)^m(-cX^2/(2k))^n]/ \\ & /[(3 - a/k + bf, m+2n)m!n!]\times \\ & \times {}_1F_1[1; 3 - a/k + kf + m + 2n; -kgX]. \end{aligned} \tag{5.2}$$

If

$$\begin{aligned} & kgX \to -\infty, \qquad {}_1F_1[1; 3 - a/k + kf + m + 2n; -kgX] \\ & \sim \Gamma(3 - a/k + kf + m + 2n)\exp(-kgX)X^{a/k-kf-2-m-2n}{}_2F_0/ \\ & /[1, a/k - 1 - kf - m - 2n; -; 1/(kgX)]. \end{aligned} \tag{5.3}$$

(Slater, [4], 60).

The required formal solution relative to the irregular singularity of the BHE at infinity can be deduced if (5.2) and (5.3) are inserted into (5.1).

REFERENCES

[1] A. W. Babister: *Transcendental functions satisfying nonhomogeneous linear differential equations*, Macmillan, London, 1970.

[2] H. Exton: *New solutions of Heun's equation*, S.E. Asian Bull. Math., 1998 to appear.

[3] A. Ronveaux: *Heun's differential equations*, Oxford University Press, 1995.

[4] L. J. Slater: *Confluent hypergeometric functions*, Cambridge University Press, 1960.

[5] H.M. Srivastava: *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc. **63** (1967), 425-429.

Lavoro pervenuto alla redazione il 9 ottobre 1997
modificato il 12 aprile 1998
ed accettato per la pubblicazione il 19 maggio 1998.
Bozze licenziate il 2 ottobre 1998

Indirizzo dell'Autore:

H. Exton -"Nyuggel"- Lunabister - Dunrossness - Shetland ZE2 9JH - United Kingdom