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Natural Lagrangians for quantum structures over 4-dimensional space-time

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RIASSUNTO: *Si assegna una classificazione delle funzioni lagrangiane della teoria einsteiniana generale della meccanica quantistica relativistica. Si utilizzano metodi di fibrati di tipo gauge-naturali ed operatori naturali. Si riconosce che tutte le lagrangiane quantistiche naturali possono essere descritte mediante una base funzionale costituita da quattro elementi. Si fornisce anche una descrizione invariante di queste lagrangiane.*

ABSTRACT: *The natural quantum Lagrangians which appear in Einstein general relativistic quantum mechanics are classified by using methods of gauge-natural bundles and natural operators. It is proved that all natural quantum Lagrangians for scalar particles have a functional base formed by four Lagrangians. The invariant description of these Lagrangians is given.*

– Introduction

In [1], [3], [4] the authors have proposed a new geometric formulation of quantum mechanics of a classical charged particle, with given gravitational and electromagnetic classical fields, in the framework of a general

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relativistic Galilei spacetime. An important role in this theory is played by the distinguished quantum Lagrangian constructed naturally from the metric tensor, a potential of the electromagnetic field and a section of the quantum bundle. In [5] all natural quantum Lagrangians in the Galilei approach are classified by using the theory of natural operators, [8], [9].

Recent papers [6], [7] have proposed the Einstein analogue to some results of [1], [3], [4]. The aim of this paper is to introduce the Einstein analogue of the distinguished quantum Lagrangian in the Galilei approach and to classify all natural quantum Lagrangians for scalar particles in the Einstein approach. We prove that the set of natural quantum Lagrangians has a functional base formed by four Lagrangians and the only Lagrangian in the base involving both gravitational and electromagnetic structure of the spacetime is the distinguished quantum Lagrangian which is constructed by using a quantum connection. For the natural basic Lagrangians we give the corresponding Euler-Lagrange operators and generalized Euler-Lagrange equations.

We assume the following fundamental unit spaces [1]:

(1) the oriented 1-dimensional vector space $\mathbb{T}$ over $\mathbb{R}$ of *time intervals*,

(2) the positive 1-dimensional semi-vector space $\mathbb{L}$ over $\mathbb{R}^+$ of *lengths*,

(3) the positive 1-dimensional semi-vector space $\mathbb{M}$ over $\mathbb{R}^+$ of *masses*.

Moreover, we refer to the *light velocity* and the *Planck's constant*

$$c \in \mathbb{T}^{+^*} \otimes \mathbb{L}\,, \qquad \hbar \in \mathbb{T}^{+^*} \otimes \mathbb{L}^2 \otimes \mathbb{M}\,,$$

and consider a classical particle with *mass* and *charge*

$$m \in \mathbb{M}\,, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}\,.$$

1 – Geometry of phase space

In this section we recall basic geometric properties of the phase space of a classical particle in the Einstein general relativistic framework. For further details and proofs see [6].

1.1 – Phase space

We start with the basic assumptions and definitions.

We assume *spacetime* to be a 4-dimensional oriented and time-oriented manifold M equipped with a scaled Lorentzian metric of signature $(+---)$

$$g : M \to \mathbb{L}^2 \otimes T^*M \underset{M}{\otimes} T^*M \,.$$

We denote by $\bar{g} : M \to \mathbb{L}^{*2} \otimes TM \otimes_M TM$ the inverse metric.

Local coordinate charts on M will be denoted by (x^λ), $\lambda = 0, 1, 2, 3$.

The coordinate expression of g and $\bar{g}$ are then

$$g = g_{\lambda\mu} d^\lambda \otimes d^\mu \,, \quad g_{\lambda\mu} : M \to \mathbb{L}^2 \otimes \mathbb{R} \,,$$
$$\bar{g} = g^{\lambda\mu} \partial_\lambda \otimes \partial_\mu \,, \quad g^{\lambda\mu} : M \to \mathbb{L}^{*2} \otimes \mathbb{R} \,.$$

In what follows we shall use local coordinate charts such that the vector ∂_0 is time-like and time oriented and $\partial_1, \partial_2, \partial_3$ are space-like; hence $g_{00} > 0$, g_{11}, g_{22}, $g_{33} < 0$.

Latin indices $i, j, p, \ldots$ will span space-like coordinates, while Greek indices λ, μ, ϕ, $\ldots$ will span space-time coordinates.

A 1-jet of 1-dimensional submanifolds of M at $x \in M$ is defined to be an equivalence class of 1-dimensional submanifolds touching each other at x with a contact of order 1, [11]. The k-jet of a 1-dimensional submanifold $l \subset M$ at $x \in l$ is denoted by $j_1 l(x)$. The set of all 1-jets of 1-dimensional submanifolds of M can be equipped, in a natural way, with a smooth structure. The corresponding manifold is denoted by $J(M, 1)$ and its projection on M is denoted by $\pi^1_0 : J(M, 1) \to M$.

We have the canonical fibred isomorphism over M of $J(M, 1)$ with the Grassmannian bundle of dimension 1

$$J(M, 1) \to \mathrm{Grass}(M, 1) : \phi \mapsto L_\phi \,,$$

where $\phi \in J(M, 1)$ and $L_\phi \subset T_{\underline{\phi}}M$ is the tangent space at $\underline{\phi} = \pi^1_0(\phi)$ of 1-dimensional submanifolds generating ϕ.

A local chart on M is said to be *divided* if the set of its coordinate functions is divided into two subsets of 1 and $(\dim M - 1)$ elements. Our typical notation for a divided chart will be

$$(x^0, x^i) \,, \qquad 1 \le i \le \dim M - 1 \,.$$

A divided chart and a 1-dimensional submanifold $l \subset M$ are said to be *related* if the submanifold l can be expressed locally by formulas of the type

$$x^i = l^i(x^0)\,;$$

i.e., more precisely $x^i|l = l^i \circ x^0|l$, with $l^i : \mathbb{R} \to \mathbb{R}$.

Every divided chart on M determines canonically a local fibred chart

$$(x^0, x^i; x^i_0)$$

on $J(M,1)$. We shall always refer to such charts.

A *motion* in M is defined to be a 1-dimensional time-like submanifold $l \subset M$.

We define the *phase space*

$$UM \subset J(M,1)$$

to be the subspace of all 1-jets of motions. Hence, $\phi = j_1 l(\underline{\phi}) \in J(M,1)$ belongs to UM if and only if $L_\phi = T_{\underline{\phi}} l$ lies inside the light cone.

1.2 – Contact structure

The geometric structure of the phase space and the Lorentz metric allow us to recover the contact structure.

We have the following *contact maps*

$$D : UM \to \mathbb{T}^* \otimes TM\,, \qquad \tau^\natural := \frac{g^\flat}{c^2} \circ D : UM \to \mathbb{T} \otimes T^*M\,,$$

with coordinate expressions

$$(1.1)\quad D = c\,\alpha\, D_0 = c\,\alpha\,(\partial_0 + x^i_0 \partial_i)\,, \qquad \tau^\natural \equiv \tau^\natural_\lambda\, d^\lambda = \frac{\alpha}{c}\,(g_{0\lambda} + g_{i\lambda} x^i_0) d^\lambda\,,$$

where

$$\alpha = 1/\|D_0\| = 1/\sqrt{g_{00} + 2 g_{0j} x^j_0 + g_{ij} x^i_0 x^j_0} \in \mathbb{L}^*\,.$$

We have

$$D \lrcorner \tau^\natural = 1\,,$$

i.e., in coordinates,

$$c\,\alpha\,(\tau^\natural_0 + \tau^\natural_h x^h_0) = 1\,. \tag{1.2}$$

1.3 – Splitting of the tangent and cotangent spaces of spacetime

The contact structure induces a natural splitting of the tangent and cotangent spaces of spacetime into two orthogonal components over the phase space.

We define the vector bundles over UM

$$T^\parallel M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L_\phi\}\,,$$
$$T^\perp M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L^\perp_\phi\}\,,$$

and

$$T^*_\parallel M := \{(\phi, \omega) \in UM \underset{M}{\times} T^*M \mid \langle \omega, L^\perp_\phi \rangle = 0\}\,,$$
$$T^*_\perp M := \{(\phi, \omega) \in UM \underset{M}{\times} T^*M \mid \langle \omega, L_\phi \rangle = 0\}\,.$$

which yield the splittings over UM

$$UM \underset{M}{\times} TM = T^\parallel M \underset{UM}{\oplus} T^\perp M\,, \qquad UM \underset{M}{\times} T^*M = T^*_\parallel M \underset{UM}{\oplus} T^*_\perp M\,.$$

The following mutually dual local bases of vector fields and forms are adapted to the above splittings

$$D_0 := \partial_0 + x^i_0 \partial_i\,, \qquad b_i := \partial_i - c\,\alpha \tau^\natural_i D_0\,, \tag{1.3}$$
$$\lambda^0 := d^0 + c\,\alpha \tau^\natural_i \theta^i = c\,\alpha \tau^\natural\,, \qquad \theta^i := d^i - x^i_0 d^0\,. \tag{1.4}$$

It is convenient to introduce the matrices, with respect to a natural base,

$$h_{i\mu} := g_{i\mu} - c^2 \tau^\natural_i \tau^\natural_\mu\,, \qquad h^{i\mu} := g^{i\mu} - x^i_0 g^{0\mu}\,,$$

and the mutually inverse matrices, with respect to an adapted base,

$$g^\perp_{ij} := g_{ij} - c^2 \tau^\natural_i \tau^\natural_j\,, \qquad g^{ij}_\perp := g^{ij} - g^{i0} x^j_0 - g^{j0} x^i_0 + g^{00} x^i_0 x^j_0\,. \tag{1.5}$$

Moreover, the parallel and orthogonal projections

$$\lambda : UM \underset{M}{\times} TM \to T^{\|}M\,, \qquad \theta = 1_M - \lambda : UM \underset{M}{\times} TM \to T^{\perp}M\,,$$

and

$$\lambda^* : UM \underset{M}{\times} T^*M \to T^*_{\|}M\,, \qquad \theta^* = 1^*_M - \lambda^* : UM \underset{M}{\times} T^*M \to T^*_{\perp}M\,,$$

have the coordinate expressions

$$\lambda = \lambda^0 \otimes D_0\,, \qquad \theta = \theta^i \otimes b_i\,,$$

and

$$\lambda^* = D_0 \otimes \lambda^0\,, \qquad \theta^* = b_i \otimes \theta^i\,.$$

In what follows we shall use the following

LEMMA 1.1. *We have*

$$g^{ij}_{\perp}\tau_j = \frac{1}{c\,\alpha}(g^{00}x^i_0 - g^{0i})\,,$$
$$g^{ij}_{\perp}\tau_i\tau_j = \frac{1}{c^2\alpha^2}(g^{00} - \alpha^2)\,.$$

PROOF. It follows immediately from (1.1) and (1.5). □

Additionally, we have a natural linear fibred isomorphism over UM

$$v^{\perp} : VUM \to \mathbb{T}^* \otimes T^{\perp}M\,,$$

with the coordinate expression

$$v^{\perp} = c\,\alpha\, d^i_0 \otimes b_i\,, \qquad v^{\perp\,-1} = \frac{1}{c\,\alpha}\theta^i \otimes \partial^0_i\,.$$

1.4 – Connections and 2-forms

A linear connection of spacetime induces naturally on the phase space a connection and a 2-form.

A linear connection K on the vector bundle $\pi_M : TM \to M$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$K : TM \to T^*M \underset{TM}{\otimes} TTM\,, \qquad \nu_K : TM \to T^*TM \underset{TM}{\otimes} TM\,,$$

respectively, with coordinate expressions

$$K = d^\phi \otimes (\partial_\phi + K_\phi{}^\mu{}_\psi\, \dot{x}^\psi\, \dot{\partial}_\mu)\,, \qquad \nu_K = (\dot{d}^\mu - K_\phi{}^\mu{}_\psi\, \dot{x}^\psi\, d^\phi) \otimes \partial_\mu\,,$$

where $K_\phi{}^\mu{}_\psi \in C^\infty(M)$ and $(x^\phi, \dot{x}^\phi)$ is the induced coordinate chart on TM.

We observe that a linear connection ν_K on $TM \to M$ induces a linear connection $\nu'_K : T(\mathbb{T}^* \otimes TM) \to \mathbb{T}^* \otimes TM$ on the vector bundle $\mathbb{T}^* \otimes TM \to M$, with coordinate expression

$$\nu'_K = u^0 \otimes (\dot{d}^\mu_0 - K_\phi{}^\mu{}_\psi\, \dot{x}^\psi_0\, d^\phi) \otimes \partial_\mu\,,$$

where u^0 is a base of $\mathbb{T}^*$ and $(x^\phi, \dot{x}^\phi_0)$ denotes the induced chart on $\mathbb{T}^* \otimes TM$.

A connection Γ on UM can be expressed, equivalently, by a tangent valued form, or by a vector valued form

$$\Gamma : UM \to T^*M \underset{UM}{\otimes} TUM\,, \qquad v^\perp \circ \nu_\Gamma : UM \to T^*UM \underset{UM}{\otimes} (\mathbb{T}^* \otimes T^\perp M)\,,$$

with coordinate expressions

$$\Gamma = d^\phi \otimes (\partial_\phi + \Gamma_\phi{}^i_0\, \partial^0_i)\,, \qquad v^\perp \circ \nu_\Gamma = c\, \alpha (d^i_0 - \Gamma_\phi{}^i_0\, d^\phi) \otimes b_i\,,$$

respectively, where $\Gamma_\phi{}^i_0 \in C^\infty(UM)$.

For any linear connection K on TM the map

$$\nu_\Gamma = v^{\perp\,-1} \circ \theta \circ \nu'_K \circ TD$$

turns out to be a connection on the bundle $UM \to M$ with coordinate expression

$$\Gamma_{\phi 0}^{\,i} = K_{\phi}{}^{i}{}_{j}\, x_0^j + K_{\phi}{}^{i}{}_{0} - x_0^i (K_{\phi}{}^{0}{}_{j}\, x_0^j + K_{\phi}{}^{0}{}_{0})\,.$$

A connection Γ on UM and the metric g yield the scaled 2-form on UM

$$\Omega(g,\Gamma) := (v^{\perp} \circ \nu_{\Gamma}) \bar{\wedge}\, \theta : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* UM\,,$$

where $\bar{\wedge}$ denotes the wedge product and the contraction via the metric g.

We have the coordinate expression

$$\Omega(g,\Gamma) = c\, \alpha\, h_{i\mu}\, (d_0^i - \Gamma_{\phi 0}^{\,i}\, d^{\phi}) \wedge d^{\mu}\,. \tag{1.6}$$

1.5 – Gravitational objects

First of all, we consider the objects introduced in the above Subsection, that come from the Lorentz metric.

The metric g yields the *gravitational* connection on TM and the *gravitational* connection on UM

$$K^{\natural} := \varkappa\,, \qquad \nu_{\Gamma^{\natural}} := v^{\perp -1} \circ \theta \circ \nu'_{K^{\natural}} \circ TD\,,$$

respectively, where $\varkappa$ is the Levi-Civita connection with the Christoffel symbols

$$\varkappa_{\phi\psi}^{\sigma} = -\frac{g^{\sigma\tau}}{2} (\partial_{\phi} g_{\tau\psi} + \partial_{\psi} g_{\tau\phi} - \partial_{\tau} g_{\phi\psi})\,.$$

Moreover, the *gravitational* 2-form

$$\Omega^{\natural} := \Omega(g, \Gamma^{\natural})$$

turns out to be the contact 2-form generated by $c^2 \tau^{\natural}$; namely, we obtain the equality

$$\Omega^{\natural} = c^2 d\tau^{\natural}\,.$$

The metric g admits the canonical scaled volume form and the canonical scaled 4-vector

$$\epsilon : M \to \mathbb{T}\otimes\mathbb{L}^3\otimes\wedge^4 T^*M\,, \quad \bar{\epsilon} : M \to \mathbb{T}^*\otimes\mathbb{L}^{*3}\otimes\wedge^4 TM\,,$$

with coordinate expressions

$$\epsilon = \frac{1}{c}\sqrt{|\det(g)|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \tag{1.7}$$

and

$$\bar{\epsilon} = \frac{c}{\sqrt{|\det(g)|}}\, \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3\,, \tag{1.8}$$

respectively. Clearly $\langle \bar{\epsilon}, \epsilon \rangle = 1$.

1.6 – Total connections and 2-forms

Next, we deform the above geometric structures by a suitable coupling with electromagnetic field.

We assume the *electromagnetic field* to be a closed scaled 2-form on M

$$F : M \to (\mathbb{T}^*\otimes\mathbb{L}^{3/2}\otimes\mathbb{M}^{1/2})\otimes\wedge^2 T^*M\,.$$

We denote a local potential of F by

$$A : M \to (\mathbb{T}^*\otimes\mathbb{L}^{3/2}\otimes\mathbb{M}^{1/2})\otimes T^*M\,.$$

Here, by definition we set $2dA = F$. Let us remark, that A is given locally up to a "gauge". We shall be involved with this fact later in Section 2.

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures of the phase space. Namely, we obtain *total* objects obtained correcting the gravitational objects by an electromagnetic term, in such a way to preserve their original relations.

For this purpose we need a suitable coupling constant. So, we consider a particle with a given mass and charge

$$m \in \mathbb{M}\,, \qquad q \in \mathbb{T}^*\otimes\mathbb{L}^{3/2}\otimes\mathbb{M}^{1/2}\,,$$

and refer to the coupling constant

$$\frac{q}{m} \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{*1/2} .$$

We have the obvious coupling of the electromagnetic field with the gravitational contact 2-form on UM. Accordingly, we define the *total 2-form* to be

$$\Omega := \Omega^\natural + \frac{q}{2mc} F : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* UM .$$

Of course, we obtain a local potential of Ω in the form

$$\tau = c^2 \tau^\natural + \frac{q}{mc} A$$

and

$$d\Omega = 0 .$$

2 – Quantum bundle

In this section we introduce the quantum bundle, which is a line complex bundle over spacetime and we define quantum connections which live on the pullback of the quantum bundle with respect to the projection $UM \to M$.

2.1 – Quantum bundle

We assume the *quantum bundle* to be a Hermitian line bundle over spacetime

$$\pi : Q \to M ,$$

i.e., $\pi : Q \to M$ is a Hermitian complex vector bundle with one-dimensional fibres. Let us denote by $h : Q \times_M Q \to \mathbb{C}$ the Hermitian product. Let $\mathbf{b} : M \to Q$ be a (local) base of Q such that $h(\mathbf{b}, \mathbf{b}) = 1$. Such a local base is said to be *normal* and the fibred coordinate chart (x^λ, z) induced by a normal base of Q is said to be a *normal coordinate chart* on Q. In any fibred normal coordinate chart $h(\Psi, \Phi) = \bar{\psi}\varphi$ for every sections Ψ, $\Phi : M \to Q$, with $\Psi = \psi\mathbf{b}$, $\Phi = \varphi\mathbf{b}$.

It is easy to see that Q admits a bundle atlas constituted by normal fibred coordinate charts; the associated cocycle takes its value in the group $U(1,\mathbb{C})$. Hence Q can be viewed as an associated gauge-natural vector bundle functor defined on the category $\mathcal{PB}_4(U(1,\mathbb{C}))$ of principal $U(1,\mathbb{C})$-bundles over 4-dimensional bases, [5]. Hence any local principal bundle isomorphism (called the *change of gauge*) $\varphi \in \mathrm{Mor}\mathcal{PB}_4(U(1,\mathbb{C}))$, covering an isomorphism f, can be viewed as the linear fiber diffeomorphism $\varphi : Q \to Q$, covering f.

Let $(\underline{x}^\lambda, \underline{z})$ be a new normal coordinate chart on Q, then the corresponding transformation relations are of the form

$$\underline{x}^\lambda = \underline{x}^\lambda(x), \quad \underline{z} = e^{2\pi i \vartheta(x)} z\,. \tag{2.1}$$

The Liouville vector field $I : Q \to VQ = Q \times_M Q$ will be identified with $I = \mathrm{id}_Q : M \to Q^* \otimes_M Q$. In a normal base $I = z \otimes \mathbf{b}$.

2.2 – Quantum connection

Let us consider the pullback bundle parametrized by observers $\pi^\uparrow : Q^\uparrow := UM \times_M Q \to UM$ of the quantum bundle $\pi : Q \to M$ with respect to $UM \to M$.

A connection $C : Q^\uparrow \to T^*UM \otimes_{UM} TQ^\uparrow$ is said to be a *quantum connection* if, [3], [4], [12], [13],

(1) C is Hermitian,

(2) C is a universal connection,

(3) the curvature of C is given by

$$R[C] = i\,\frac{m}{\hbar}\,\Omega \otimes I : Q^\uparrow \to \wedge^2 T^*UM \underset{UM}{\otimes} Q^\uparrow\,.$$

For the definition of universal connection see [4], [10]. Very briefly: if $\{\xi[o]\}$ is a system of Hermitian connections on the bundle $\pi : Q \to M$, parametrized by the family of observers (sections of UM) $\{o : M \to UM\}$, then there exists a unique connection C on the bundle $Q^\uparrow \to UM$ such that, for each observer $o : M \to UM$, the pullback o^*C equals $\xi[o]$. This connection is said to be universal.

A pair (Q, C) is said to be a *quantum structure*.

The coordinate expression of a universal Hermitian connection C on $Q^{\uparrow}$ is

$$C = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + i C_{\lambda} d^{\lambda} \otimes I \,. \tag{2.2}$$

The curvature of C is

$$R[C] : Q^{\uparrow} \to \wedge^2 T^* UM \underset{UM}{\otimes} Q^{\uparrow} \,,$$

with coordinate expression

$$R[C] = i\, d(C_{\lambda}) \wedge d^{\lambda} \otimes I \,.$$

Since a local potential of Ω is of the type $c^2 \tau^{\natural} + \frac{q}{mc} A$, the condition (3) implies that, in a normal coordinate chart, the coordinate expression of a quantum connection is of the kind

$$C = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + i \frac{m}{\hbar} \Big(c^2 \tau_{\lambda}^{\natural} + \frac{q}{mc} A_{\lambda} \Big) d^{\lambda} \otimes I \,, \tag{2.3}$$

where $A = A_{\lambda} d^{\lambda}$ is a local particular electromagnetic potential (determined by C).

2.3 – Quantum electromagnetic connection

A connection $\tilde{C} : Q^{\uparrow} \to T^* UM \underset{UM}{\otimes} TQ^{\uparrow}$ is said to be a *quantum electromagnetic connection* if

(1) $\tilde{C}$ is Hermitian,

(2) $\tilde{C}$ is a universal connection,

(3) the curvature of $\tilde{C}$ is given by

$$R[\tilde{C}] = i \, \frac{q}{2\hbar \, c} \, F \otimes I : Q^{\uparrow} \to \wedge^2 T^* M \underset{UM}{\otimes} Q^{\uparrow} \,.$$

A pair (Q, F) is said to be a *quantum electromagnetic structure.*

The coordinate expression of a quantum electromagnetic connection is of the kind

$$\tilde{C} = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + i \frac{q}{\hbar \, c} A_{\lambda} d^{\lambda} \otimes I \,. \tag{2.4}$$

Remark 2.1. Let us stress that quantum electromagnetic connections live in fact on $\pi : Q \to M$, but from the technical reasons we prefer to define them on $Q^{\uparrow} \to UM$. The difference between a quantum connection and a quantum electromagnetic connection corresponding to the same local electromagnetic potential is

$$C - \tilde{C} = i\frac{mc^2}{\hbar}\pi^{\uparrow}\tau^{\natural}\otimes I\,,$$

where $\pi^{\uparrow}\tau^{\natural}$ is the pullback of $\tau^{\natural}$.

2.4 – Transformation relations for an electromagnetic potential

Let us suppose the change of gauge given by (2.1) and assume an universal Hermitian connection C on $Q^{\uparrow}$ given in a new normal coordinate chart by

$$C = \underline{d}^{\lambda}\otimes\underline{\partial}_{\lambda} + \underline{d}^{i}_{0}\otimes\underline{\partial}^{0}_{i} + i\underline{C}_{\lambda}\underline{z}\,\underline{d}^{\lambda}\otimes\underline{\mathbf{b}}\,.$$

Then from the linearity of C and (2.1) we get the transformation relations

$$\underline{C}_{\lambda} = (C_{\mu} + 2\pi\partial_{\mu}\vartheta)\frac{\partial x^{\mu}}{\partial\underline{x}^{\lambda}}\,,$$

which implies together with (2.3) that coefficients of an electromagnetic potential A are transformed by

$$\underline{A}_{\lambda} = \Big(A_{\mu} + \frac{2\pi\hbar\, c}{q}\partial_{\mu}\vartheta\Big)\frac{\partial x^{\mu}}{\partial\underline{x}^{\lambda}}\,. \tag{2.5}$$

The transformation (2.5) implies that A is a section of a 1st order gauge-natural bundle on the category $\mathcal{PB}_4(U(1,\mathbb{C}))$. We shall call this bundle the *bundle of electromagnetic potentials* and denote it by $\mathcal{P}M$. Let us note that $\mathcal{P}M$ contains the unit spaces $\mathbb{T}, \mathbb{L}, \mathbb{M}$.

Remark 2.2. The flat quantum electromagnetic connection given by a chosen normal coordinate chart corresponds to the electromagnetic potential $A = df$, where f is a $\mathbb{T}^*\otimes\mathbb{L}^{3/2}\otimes\mathbb{M}^{1/2}$-valued real function on M given up to a "gauge". It means that if we change a normal coordinate chart (gauge) by the formula (2.1), we have

$$f(\underline{x}) = f(x) + \frac{2\pi\hbar\, c}{q}\vartheta(x)\,.$$

2.5 – Existence of quantum and quantum electromagnetic structures

In [12], [13] a topological necessary and sufficient condition for the existence of a quantum structure was found.

Theorem 2.1. *The following conditions are equivalent:*

(1) *there exists a quantum structure* (Q, C)*;*

(2) *the closed form* $\frac{m}{\hbar}\Omega$ *determines a cohomology class in the subgroup*

$$\Big[\frac{m}{\hbar}\Omega\Big] = \Big[\frac{q}{\hbar\, c}F\Big] \in i(H^2(M,\mathbb{Z})) \subset H^2(M,\mathbb{R})\,,$$

where $i : (H^2(M,\mathbb{Z})) \to H^2(M,\mathbb{R})$ *is the canonical group morphism.* □

As a consequence of the above theorem the existence of a quantum structure is equivalent to the existence of a quantum electromagnetic structure.

In what follows we shall assume that a quantum structure on M exists.

2.6 – Quantum and quantum electromagnetic covariant differential

For a section $\Psi : M \to Q$ (quantum history) we have the *quantum covariant differential*

$$\nabla\Psi := \nabla[C]\Psi^{\uparrow} : UM \to T^*M \otimes Q$$

with coordinate expression

$$\nabla\Psi = \Big(\partial_\lambda\psi - i\frac{m}{\hbar}\psi(c^2\tau^{\natural}_{\lambda} + \frac{q}{mc}A_\lambda)\Big)d^\lambda \otimes \mathbf{b}\,. \tag{2.6}$$

Similarly the *quantum electromagnetic covariant differential*

$$\widetilde{\nabla}\Psi := \nabla[\tilde{C}]\Psi : M \to T^*M \otimes Q\,,$$

has coordinate expression

$$\widetilde{\nabla}\Psi = \Big(\partial_\lambda \psi - i\frac{q}{\hbar\, c}\psi A_\lambda\Big) d^\lambda \otimes \mathbf{b}\,. \tag{2.7}$$

Clearly

$$(\widetilde{\nabla}\Psi)^\uparrow = \nabla\Psi + i\frac{mc^2}{\hbar}\tau^\natural \underset{UM}{\otimes} \Psi\,.$$

The *parallel component* of the quantum covariant differential is defined by

$$\nabla_\| \Psi := C \lrcorner \nabla\Psi : UM \to \mathbb{T}^* \otimes Q$$

with coordinate expression

$$\nabla_\| \Psi = \Big(c\,\alpha(\partial_0\psi + \partial_p\psi x_0^p) - i\frac{mc^2}{\hbar}\psi - i\frac{q\alpha}{\hbar}\psi(A_0 + A_\lambda x_0^\lambda)\Big)\otimes\mathbf{b}\,,$$

which can be written as

$$\nabla_\| \Psi = \Big(D.\psi - i\frac{mc^2}{\hbar}\psi - i\frac{q}{\hbar\, c}\psi(D\lrcorner A)\Big)\otimes\mathbf{b}\,. \tag{2.8}$$

The *orthogonal component* of the quantum covariant differential is defined by

$$\nabla_\perp \Psi = \theta^* \lrcorner \nabla\Psi : UM \to T^*_\perp M \otimes Q$$

with coordinate expression

$$\nabla_\perp \Psi = \Big(\partial_j\psi - c\,\alpha\tau^\natural_j(\partial_0\psi + \partial_p\psi x_0^p) - i\frac{q}{\hbar\, c}(A_j - c\,\alpha\tau^\natural_j(A_0 + A_p x_0^p))\psi\Big)\theta^j\otimes\mathbf{b}\,,$$

which can be written as

$$\nabla_\perp \Psi = \Big(\partial_j\psi - i\frac{q}{\hbar\, c}A_j\psi + i\frac{q}{\hbar\, c}\psi\tau^\natural_j(D\lrcorner A) - \tau^\natural_j D.\psi\Big)\theta^j\otimes\mathbf{b}\,. \tag{2.9}$$

Lemma 2.1. *We have*

$$\nabla\Psi = \nabla_\perp\Psi + \tau^\natural \otimes \nabla_\| \Psi\,.$$

PROOF. It follows from (2.6), (2.8) and (2.9). □

3 – Quantum Lagrangians

In this section we shall introduce a distinguished quantum Lagrangian which is the Einstein equivalent to the Galilei quantum Lagrangian, [1], [3], [4].

3.1 – Scaled functions

Let us consider a section $\Psi : M \to Q$. We define the following scaled real functions on UM

$$\begin{aligned}
\ell_{\|}(\Psi) &= \frac{1}{2} h(\nabla_{\|}\Psi, \nabla_{\|}\Psi) : UM \to \mathbb{T}^{*2}\otimes\mathbb{R}\,,\\
\ell_{o}(\Psi) &= \frac{1}{2}\big(h(\Psi, i\nabla_{\|}\Psi) + h(i\nabla_{\|}\Psi, \Psi)\big) : UM \to \mathbb{T}^{*}\otimes\mathbb{R}\,,\\
\ell_{\perp}(\Psi) &= \frac{1}{2}(\bar{g}\otimes h)(\nabla_{\perp}\Psi, \nabla_{\perp}\Psi) : UM \to \mathbb{L}^{*2}\otimes\mathbb{R}\,.
\end{aligned}$$

LEMMA 3.1. *We have*

$$\begin{aligned}
\ell_{\|}(\Psi) &= \frac{1}{2} D.\psi D.\bar{\psi} + i\frac{mc^2}{2\hbar}(\bar{\psi}D.\psi) - \psi D.\bar{\psi}) +\\
&\quad + i\frac{q}{2\hbar\, c}(\bar{\psi}D.\psi - \psi D.\bar{\psi})(D\lrcorner A) +\\
&\quad + \frac{m^2c^4}{2\hbar^2}\bar{\psi}\psi + \frac{qmc}{\hbar^2}\bar{\psi}\psi(D\lrcorner A) + \frac{q^2}{2\hbar^2c^2}\bar{\psi}\psi(D\lrcorner A)^2\,,
\end{aligned} \tag{3.1}$$

$$\ell_{o}(\Psi) = \frac{mc^2}{\hbar}\bar{\psi}\psi + \frac{q}{\hbar\, c}\bar{\psi}\psi(D\lrcorner A) + \frac{i}{2}(\bar{\psi}D.\psi) - \psi D.\bar{\psi})\,, \tag{3.2}$$

$$\begin{aligned}
\ell_{\perp}(\Psi) &= \frac{1}{2} g^{\lambda\mu}\partial_\lambda\bar{\psi}\partial_\mu\psi + \frac{q^2}{2\hbar^2c^2} g^{\lambda\mu}A_\lambda A_\mu\bar{\psi}\psi +\\
&\quad + i\frac{q}{2\hbar\, c} g^{\lambda\mu}A_\lambda(\bar{\psi}\partial_\mu\psi - \psi\partial_\mu\bar{\psi}) - \frac{q^2}{2\hbar^2c^4}\bar{\psi}\psi(D\lrcorner A)^2 +\\
&\quad - i\frac{q}{2\hbar\, c^3}(\bar{\psi}D.\psi - \psi D.\bar{\psi})(D\lrcorner A) - \frac{1}{2c^2}D.\psi D.\bar{\psi}\,.
\end{aligned} \tag{3.3}$$

PROOF. (3.1) and (3.2) follow directly from (2.8). To prove (3.3) we have to use (2.9) and Lemma 1.1. ∎

The functions $\ell_{\|}(\Psi)$, $\ell_o(\Psi)$, $\ell_{\perp}(\Psi)$ are defined on UM, i.e. are observer dependent. We would like to find their linear combination projectable on M (observer independent). We have

LEMMA 3.2. *There is a unique (up to a multiplicative factor) linear combination (with coefficients dependent on c, $\hbar$, m) of the functions $\ell_{\|}(\Psi)$, $\ell_o(\Psi)$, $\ell_{\perp}(\Psi)$ projectable on M with values in $\mathbb{R}$, namely*

$$\ell(\Psi) = \frac{\hbar^2}{m^2c^4}\ell_{\|}(\Psi) + \frac{\hbar^2}{m^2c^2}\ell_{\perp}(\Psi) - \frac{\hbar}{mc^2}\ell_o(\Psi) : M \to \mathbb{R} \tag{3.4}$$

with coordinate expression

$$\begin{aligned}\ell(\Psi) = {}& \frac{\hbar^2}{2m^2c^2} g^{\lambda\mu}\partial_\lambda\bar{\psi}\partial_\mu\psi + i\frac{q\hbar}{2m^2c^3} g^{\lambda\mu}A_\lambda(\bar{\psi}\partial_\mu\psi - \psi\partial_\mu\bar{\psi}) + \\ & + \Big(\frac{q^2}{2m^2c^4} g^{\lambda\mu}A_\lambda A_\mu - \frac{1}{2}\Big)\bar{\psi}\psi\,.\end{aligned} \tag{3.5}$$

PROOF. It follows directly from (3.1)–(3.3). ∎

Moreover, we define the real function

$$\tilde{\ell}(\Psi) = \frac{\hbar^2}{2m^2c^2}(\bar{g}\otimes h)(\widetilde{\nabla}\Psi, \widetilde{\nabla}\Psi) : M \to \mathbb{R}\,,$$

with coordinate expression

$$\begin{aligned}\tilde{\ell}(\Psi) = {}& \frac{\hbar^2}{2m^2c^2} g^{\lambda\mu}\partial_\lambda\bar{\psi}\partial_\mu\psi + i\frac{q\hbar}{2m^2c^3} g^{\lambda\mu}A_\lambda(\bar{\psi}\partial_\mu\psi - \psi\partial_\mu\bar{\psi}) + \\ & + \frac{q^2}{2m^2c^4} g^{\lambda\mu}A_\lambda A_\mu\bar{\psi}\psi\,.\end{aligned} \tag{3.6}$$

Clearly we have

$$\ell(\Psi) = \tilde{\ell}(\Psi) - \frac{1}{2}h(\Psi, \Psi)\,. \tag{3.7}$$

3.2 – Distinguished quantum Lagrangians

By using the functions $\ell(\Psi)$ and $\tilde{\ell}(\Psi)$ we can define two distinguished *quantum Lagrangians* by

$$\begin{aligned}\mathcal{L}[\Psi] &:= \ell(\Psi)\cdot\epsilon : M \to \mathbb{T}\otimes\mathbb{L}^3\otimes\wedge^4 T^*M\,,\\ \tilde{\mathcal{L}}[\Psi] &:= \tilde{\ell}(\Psi)\cdot\epsilon : M \to \mathbb{T}\otimes\mathbb{L}^3\otimes\wedge^4 T^*M\,,\end{aligned}$$

where $\mathcal{L}[\Psi] = \tilde{\mathcal{L}}[\Psi] - \frac{1}{2}h(\Psi,\Psi)\cdot\epsilon$.

The coordinate expression of the distinguished quantum Lagrangian $\mathcal{L}[\Psi]$ is of the form

$$\begin{aligned}\mathcal{L}[\Psi] = \Big[&\frac{\hbar^2}{2m^2c^3}g^{\lambda\mu}\partial_\lambda\bar{\psi}\partial_\mu\psi + i\frac{q\hbar}{2m^2c^4}g^{\lambda\mu}A_\lambda(\bar{\psi}\partial_\mu\psi - \psi\partial_\mu\bar{\psi}) + \\ &+\Big(\frac{q^2}{2m^2c^5}g^{\lambda\mu}A_\lambda A_\mu - \frac{1}{2c}\Big)\bar{\psi}\psi\Big]\sqrt{|\det(g)|}\,d^0\wedge d^1\wedge d^2\wedge d^3\,.\end{aligned} \tag{3.8}$$

The distinguished quantum Lagrangian $\mathcal{L}[\Psi]$ looks to be the Einstein equivalent to the distinguished Galilei quantum Lagrangian.

3.3 – Natural quantum Lagrangians

According to the theory of natural operators, [8], [9], the Lagrangian $\mathcal{L}$ can be regarded as a fibred mapping over M

$$\mathcal{L} : T^*M\otimes T^*M \underset{M}{\times} \mathcal{P}M \underset{M}{\times} J_1Q \to \mathbb{T}\otimes\mathbb{L}^3\otimes\wedge^4 T^*M \tag{3.9}$$

by setting $\mathcal{L}\circ j_1\Psi = \mathcal{L}[\Psi]$. Moreover $\mathcal{L}$ is equivariant with respect to changes of gauge and changes of bases in the unit spaces. The coordinate expression can be deduced from (3.8).

We define a *natural quantum Lagrangian* to be a fibred morphism over M

$$\mathcal{F} : T^*M\otimes T^*M \underset{M}{\times} \mathcal{P}M \underset{M}{\times} J_1Q \to \mathbb{T}\otimes\mathbb{L}^3\otimes\wedge^4 T^*M \tag{3.10}$$

equivariant with respect to changes of gauge and changes of bases in the unit spaces.

4 – Classification of natural quantum Lagrangians

From the point of view of natural geometry, [8], [9], the distinguished quantum Lagrangians described in Section 3 are natural operators transforming the metric g, the electromagnetic potential A and the section $\Psi : M \to Q$ into $\mathbb{T}\otimes\mathbb{L}^3$-valued 4-forms on M. Moreover, these operators are of order 1 with respect to ψ (i.e., they depend on the first derivatives of Ψ) and are invariant with respect to changes of bases in the unit spaces $\mathbb{T}, \mathbb{L}, \mathbb{M}$. Recall that naturality expresses the fact, that the operator is invariant with respect to changes of fibred local coordinates on $\pi : Q \to M$ (changes of gauge). In this section we shall classify all natural Lagrangians of the type described above.

4.1 – Bundle J_1Q

In the coordinate expression of the distinguished quantum Lagrangian there are the first order derivatives of a section Ψ. It means that the Lagrangian is defined on 1-jet bundle J_1Q of $\pi : Q \to M$. For a normal coordinate chart on Q we get the induced fibred coordinate chart $(x^\lambda, z; z_\lambda)$ on J_1Q and the transformation relations

$$\underline{z}_\lambda = e^{2\pi i\vartheta}(z_\mu + 2\pi i z \partial_\mu \vartheta)\frac{\partial x^\mu}{\partial \underline{x}^\lambda}\,. \tag{4.1}$$

Moreover, we need the transformation relations for the inverse metric tensor

$$\underline{g}^{\lambda\mu} = g^{\rho\sigma}\frac{\partial \underline{x}^\lambda}{\partial x^\rho}\frac{\partial \underline{x}^\mu}{\partial x^\sigma}\,. \tag{4.2}$$

4.2 – Invariant functions

It is easy to see that any natural quantum Lagrangian is of the type $\mathcal{F} = f\cdot\epsilon$, where f is an invariant real function on $T^*M\otimes T^*M \times_M \mathcal{P}M \times_M J_1Q$. So to classify natural quantum Lagrangians it is sufficient to classify all invariant functions.

LEMMA 4.1. *A functional base of invariant functions on $T^*M \otimes T^*M \times_M \mathcal{P}M \times_M J_1Q$ is constituted by four functions which have, in a fibred normal coordinate chart on Q, the following expressions*

$$f_1 = \frac{q^2}{\hbar\, c}\,, \tag{4.3}$$

$$f_2 = \frac{1}{2}\bar{z}z\,, \tag{4.4}$$

$$f_3 = \frac{\hbar^2}{2m^2c^2} g^{\lambda\mu}(z\bar{z}_\lambda + z_\lambda\bar{z})(z\bar{z}_\mu + z_\mu\bar{z})\,, \tag{4.5}$$

$$\begin{aligned} f_4 = {} & \frac{\hbar^2}{2m^2c^2} g^{\lambda\mu}\bar{z}_\lambda z_\mu + i\frac{q\hbar}{2m^2c^3} g^{\lambda\mu}A_\lambda(\bar{z}z_\mu - z\bar{z}_\mu) + \\ & + \frac{q^2}{2m^2c^4} g^{\lambda\mu}A_\lambda A_\mu \bar{z}z\,, \end{aligned} \tag{4.6}$$

i.e. any invariant function is in the form $f = f(f_1, f_2, f_3, f_4)$, where f is a function of four real variables.

PROOF. According to the general theory of equivariant mappings, [9], we get from (2.1), (2.5), (4.1) and (4.2) that any invariant function f on $T^*M \otimes T^*M \times_M \mathcal{P}M \times_M J_1Q$ has to be a solution of the following system of partial differential equations

$$\begin{aligned} c\frac{\partial f}{\partial c} + \hbar\frac{\partial f}{\partial \hbar} + q\frac{\partial f}{\partial q} + A_\lambda\frac{\partial f}{\partial A_\lambda} &= 0\,, \\ c\frac{\partial f}{\partial c} + 2\hbar\frac{\partial f}{\partial \hbar} + \frac{3}{2}q\frac{\partial f}{\partial q} - 2g^{\lambda\mu}\frac{\partial f}{\partial g^{\lambda\mu}} + \frac{3}{2}A_\lambda\frac{\partial f}{\partial A_\lambda} &= 0\,, \\ \hbar\frac{\partial f}{\partial \hbar} + \frac{1}{2}q\frac{\partial f}{\partial q} + m\frac{\partial f}{\partial m} + \frac{1}{2}A_\lambda\frac{\partial f}{\partial A_\lambda} &= 0\,, \\ 2g^{\lambda\nu}\frac{\partial f}{\partial g^{\mu\nu}} - A_\mu\frac{\partial f}{\partial A_\lambda} - z_\mu\frac{\partial f}{\partial z_\lambda} - \bar{z}_\mu\frac{\partial f}{\partial \bar{z}_\lambda} &= 0\,, \\ \frac{\hbar\, c}{q}\frac{\partial f}{\partial A_\lambda} + iz\frac{\partial f}{\partial z_\lambda} - i\bar{z}\frac{\partial f}{\partial \bar{z}_\lambda} &= 0\,, \\ z\frac{\partial f}{\partial z} - \bar{z}\frac{\partial f}{\partial \bar{z}} + z_\lambda\frac{\partial f}{\partial z_\lambda} - \bar{z}_\lambda\frac{\partial f}{\partial \bar{z}_\lambda} &= 0\,. \end{aligned}$$

Let us remark that the first three equations correspond to changes of bases in the unit spaces $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{M}$, respectively, and the last three equations corresponds to a change of gauge.

This system is complete and has 28 independent variables and 24 equations, so a functional base is formed by four solutions. Clearly f_1, f_2, f_3, f_4 are functionally independent solutions and form such a base. □

REMARK 4.1. The invariant function corresponding to the distinguished quantum Lagrangian $\tilde{\mathcal{L}}$ is $\tilde{\ell} = f_4$ and the function corresponding to the distinguished quantum Lagrangian $\mathcal{L}$ is $\ell = f_4 - f_2$.

4.3 – Natural quantum Lagrangians

Finally we shall classify all natural quantum Lagrangians.

THEOREM 4.1. *Let $\Psi : M \to Q$ be a section. All natural quantum Lagrangians are of the form*

$$\mathcal{F}[\Psi] = f(f_1(\Psi), f_2(\Psi), f_3(\Psi), f_4(\Psi)) \cdot \epsilon \,,$$

where

$$f_1(\Psi) = \frac{q^2}{\hbar\, c}\,, \tag{4.7}$$

$$f_2(\Psi) = \frac{1}{2} h(\Psi, \Psi)\,, \tag{4.8}$$

$$f_3(\Psi) = \frac{\hbar^2}{2m^2c^2}\bar{g}(dh(\Psi, \Psi), dh(\Psi, \Psi))\,, \tag{4.9}$$

$$f_4(\Psi) = \frac{\hbar^2}{2m^2c^2}(\bar{g} \otimes h)(\tilde{\nabla}\Psi, \tilde{\nabla}\Psi)\,, \tag{4.10}$$

and f is a function of four real variables.

PROOF. It is easy to see that, in a normal fibred coordinate chart, the coordinate expression of (4.7) is given by (4.3), (4.8) by (4.4), (4.9) by (4.5) and (4.10) by (4.6). Our theorem is now a direct consequence of the above Lemma 4.1. □

REMARK 4.2. The Lagrangian $\mathcal{F}_1 = f_1 \cdot \epsilon$ is a constant multiple of the canonical volume form. The natural quantum Lagrangians $\mathcal{F}_2 = f_2 \cdot \epsilon$ and $\mathcal{F}_3 = f_3 \cdot \epsilon$ have only gravitational meaning, because they do not depend on the potential of the electromagnetic fields. The unique natural quantum Lagrangian in the base which has both gravitational and electromagnetic meaning is $\mathcal{F}_4 = \tilde{\mathcal{L}}$.

REMARK 4.3. In the construction of the distinguished quantum Lagrangians we have used essentially the properties of the Lorentz metric. Finally we can see from Theorem 4.1 that the resulting natural quantum Lagrangians are independent of the dimension of the underlying manifold M and of the signature of the metric g.

5 – Euler-Lagrange operators

In this section we shall describe the Euler-Lagrange operator and the corresponding field equations for the natural Lagrangians which form the base of quantum Lagrangians.

5.1 – Euler-Lagrange operator

Let us consider a Lagrangian

$$\mathcal{F} : T^*M \otimes T^*M \underset{M}{\times} \mathcal{P}M \underset{M}{\times} J_1 Q \to \mathbb{T} \otimes \mathbb{L}^3 \otimes \wedge^4 T^*M$$

given in coordinates in the form

$$\mathcal{F} = F \, d^0 \wedge d^1 \wedge d^2 \wedge d^3$$

and consider the Euler-Lagrange operator

$$E(\mathcal{F}) : J_1(T^*M \otimes T^*M) \underset{M}{\times} J_1 \mathcal{P}M \underset{M}{\times} J_2 Q \to V^*Q \otimes \wedge^4 T^*M$$

given by

$$E(\mathcal{F}) = \Big(\frac{\partial F}{\partial z} - D_\lambda \frac{\partial F}{\partial z_\lambda} \Big) d^z \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \,,$$

where D_λ is the formal derivative with respect to x^λ.

By using the identification of V^*Q with $Q \times Q^*$, the isomorphism $\Re(h) : Q^* \to Q$ given by the real part of h and the contraction with $\bar{\epsilon}$, we can characterize the Euler-Lagrange operator by the fibred morphism over M

$$\mathbf{E}(\mathcal{F}) := \langle \Re(h)(E(\mathcal{F})), \bar{\epsilon} \rangle : J_1(T^*M \otimes T^*M) \underset{M}{\times} J_1\mathcal{P}M \underset{M}{\times} J_2Q \to Q.$$

$\mathbf{E}(\mathcal{F})$ has the following coordinate expression

$$\mathbf{E}(\mathcal{F}) = \frac{c}{\sqrt{|\det(g)|}} \Big[\frac{\partial F}{\partial z} - \Big(\frac{\partial}{\partial x^\lambda} + z_\lambda \frac{\partial}{\partial z} + z_{\lambda\mu} \frac{\partial}{\partial z_\mu} \Big) \Big(\frac{\partial F}{\partial z_\lambda} \Big) \Big] \mathbf{b} \,. \tag{5.1}$$

For a section $\Psi : M \to Q$ we set $\mathbf{E}(\mathcal{F}[\Psi]) = \mathbf{E}(\mathcal{F}) \circ j_2\Psi$.

5.2 – Euler-Lagrange operators of the distinguished Lagrangians

THEOREM 5.1. *We have*

$$\mathbf{E}(\mathcal{F}_1[\Psi]) = 0 \,, \tag{5.2}$$

$$\mathbf{E}(\mathcal{F}_2[\Psi]) = \Psi \,, \tag{5.3}$$

$$\begin{aligned} \mathbf{E}(\mathcal{F}_3[\Psi]) = \frac{\hbar^2}{m^2c^2} g^{\lambda\mu} (-\psi^2 \partial_{\lambda\mu} \bar{\psi} - \bar{\psi}\psi \partial_{\lambda\mu}\psi + \\ - 2\psi \partial_\lambda \psi \partial_\mu \bar{\psi} - \varkappa^\nu_{\lambda\mu} (\psi \partial_\nu \bar{\psi} + \bar{\psi} \partial_\nu \psi)\psi) \mathbf{b} \,, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \mathbf{E}(\mathcal{F}_4[\Psi]) = \frac{\hbar^2}{m^2c^2} g^{\lambda\mu} \Big(- \partial_{\lambda\mu}\psi + 2i \frac{q}{\hbar\, c} A_\lambda \partial_\mu \psi + i \frac{q}{\hbar\, c} \psi \partial_\lambda A_\mu + \\ + \frac{q^2}{\hbar^2 c^2} \psi A_\lambda A_\mu - \varkappa^\nu_{\lambda\mu} \partial_\nu \psi + i \frac{q}{\hbar\, c} \varkappa^\nu_{\lambda\mu} \psi A_\nu \Big) \mathbf{b} \,. \end{aligned} \tag{5.5}$$

PROOF. (5.2) and (5.3) can be obtained immediately from (4.3) and (4.4).

To calculate the Euler-Lagrange operator for the quantum Lagrangians $\mathcal{F}_3$ and $\mathcal{F}_4 = \tilde{\mathcal{L}}$ we have to use the formula

$$\frac{\partial_\lambda (g^{\lambda\mu} \sqrt{|\det(g)|})}{\sqrt{|\det(g)|}} = g^{\rho\sigma} \varkappa^\mu_{\rho\sigma} \,. \tag{5.6}$$

Now from (4.5) and (4.6) we get (5.4) and (5.5). □

REMARK 5.1. For the distinguished quantum Lagrangian $\mathcal{L} = \mathcal{F}_4 - \mathcal{F}_2$ we have the Euler-Lagrange operator

$$\mathbf{E}(\mathcal{L}[\Psi]) = \mathbf{E}(\mathcal{F}_4[\Psi]) - \mathbf{E}(\mathcal{F}_2[\Psi])\,.$$

5.3 – Covariant differential with respect to the Levi-Civita connection

Let us recall that the metric tensor g admits the Levi-Civita connection $\varkappa$ on TM and its dual connection $\varkappa^*$ on T^*M. For a 1-form $\beta : M \to T^*M$ its covariant differential with respect to $\varkappa^*$ is $\nabla[\varkappa^*]\beta : M \to T^*M \otimes T^*M$ with coordinate expression $\nabla[\varkappa]\beta = (\partial_\lambda \beta_\mu + \varkappa^\nu_{\lambda\mu}\beta_\nu) d^\lambda \otimes d^\mu$.

Applying the covariant differential with respect to $\varkappa^*$ on the 1-form $dh(\Psi, \Psi)$ we have

$$\nabla[\varkappa^*](dh(\Psi, \Psi)) : M \to T^*M \otimes T^*M$$

with coordinate expression

$$\begin{aligned}\nabla[\varkappa^*](dh(\Psi, \Psi)) = \big(&\bar\psi \partial_{\lambda\mu}\psi + \partial_\lambda\psi\partial_\mu\bar\psi + \partial_\mu\psi\partial_\lambda\bar\psi + \psi\partial_{\lambda\mu}\bar\psi + \\ &+ \varkappa^\nu_{\lambda\mu}(\psi\partial_\nu\bar\psi + \bar\psi\partial_\nu\psi)\big) d^\lambda \otimes d^\mu\,.\end{aligned} \tag{5.7}$$

THEOREM 5.2. *We have*

$$\mathbf{E}(\mathcal{F}_3[\Psi]) = -\frac{\hbar^2}{m^2c^2}\langle \bar g, \nabla[\varkappa](dh(\Psi, \Psi))\rangle \Psi\,.$$

PROOF. It follows from (5.4) and (5.7). □

5.4 – Second order quantum electromagnetic covariant differentials

Let us consider the connection $\varkappa^* \underset{UM}{\otimes} \tilde C$ on $T^*M \underset{UM}{\otimes} Q^\uparrow$. For the quantum electromagnetic covariant differential $\widetilde\nabla\Psi : M \to T^*M \otimes_M Q$ we can define the second order covariant differential by

$$\widetilde\nabla\widetilde\nabla\Psi := \nabla[\varkappa^* \underset{UM}{\otimes} \tilde C](\widetilde\nabla\Psi)^\uparrow : M \to T^*M \underset{M}{\otimes} T^*M \underset{M}{\otimes} Q\,.$$

We have

$$\widetilde{\nabla}\widetilde{\nabla}\Psi = \Big(\partial_{\lambda\mu}\psi - i\frac{q}{\hbar\, c}(A_\lambda \partial_\mu \psi + A_\mu \partial_\lambda \psi) - i\frac{q}{\hbar\, c}\psi \partial_\mu A_\lambda + \\ - \frac{q^2}{\hbar^2 c^2}\psi A_\lambda A_\mu + \varkappa^\nu_{\lambda\mu}\partial_\nu \psi - i\frac{q}{\hbar\, c}\varkappa^\nu_{\lambda\mu}\psi A_\nu\Big) d^\mu \otimes d^\lambda \otimes \mathbf{b}\,, \tag{5.8}$$

Theorem 5.3. *We have*

$$\mathbf{E}(\tilde{\mathcal{L}}[\Psi]) = -\frac{\hbar^2}{m^2 c^2}\langle \bar{g}, \widetilde{\nabla}\widetilde{\nabla}\Psi\rangle\,,$$
$$\mathbf{E}(\mathcal{L}[\Psi]) = -\frac{\hbar^2}{m^2 c^2}\langle \bar{g}, \widetilde{\nabla}\widetilde{\nabla}\Psi\rangle - \Psi\,.$$

Proof. It follows immediately from (5.3), (5.5) and (5.8). □

5.5 – Euler-Lagrange equations

Now we can write down the Euler-Lagrange equations $\mathbf{E}(\mathcal{F}[\Psi])=0$ for an unknown section Ψ for all quantum Lagrangians in the base. For the Lagrangians $\mathcal{F}_1$ and $\mathcal{F}_2$ the corresponding equations are obvious. For $\mathcal{F}_3$ we get from Theorem 5.2 the Euler-Lagrange equation in the form

$$\langle \bar{g}, \nabla[\varkappa](dh(\Psi,\Psi))\rangle \Psi = 0\,.$$

For the distinguish quantum Lagrangian $\tilde{\mathcal{L}} = \mathcal{F}_4$ we get from Theorem 5.3 the corresponding equation in the form

$$\langle \bar{g}, \widetilde{\nabla}\widetilde{\nabla}\Psi\rangle = 0\,,$$

and finally, for the distinguish quantum Lagrangian $\mathcal{L} = \mathcal{F}_4 - \mathcal{F}_2$ we get the corresponding equation in the form

$$\langle \bar{g}, \widetilde{\nabla}\widetilde{\nabla}\Psi\rangle + \frac{m^2 c^2}{\hbar^2}\Psi = 0\,,$$

i.e., we get the generalized Klein-Gordon equation.

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