Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 649-653

The points on curves whose coordinates are U-numbers

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RIASSUNTO: *Si dimostra che su ogni curva, che ammetta una rappresentazione parametrica mediante funzioni razionali con coefficienti razionali, ci sono infiniti punti con coordinate che sono numeri di tipo* U_m.

ABSTRACT: *In this paper it is shown that if a curve* C *has a parametrization by non constant rational functions having rational coefficients, then there are infinitely many points* P *on* C *such that the coordinates of* P *are* U_m*-numbers.*

It is known that if $y = f(x)$ is a function satisfying some conditions, then there are infinitely many U_1 numbers (Liouville numbers) γ such that $f(\gamma) \in U_1$ (see, [8], [7], [2]). In 1992 Bertnik and Dombrovskij proved that there are infinitely many points $P(x, y)$ on some certain curves in $\mathbb{R}^2$ such that x, $y \in U_3$. Recently POLLINGTON [5] has shown that for every $m \in \mathbb{Z}^+$, there are infinitely many points on the line $y = x + r$ $(r \in \mathbb{R})$ whose coordinates are U_m-numbers. In this paper we consider the same problem for some algebraic curves in $\mathbb{R}^n$. Theorem 2 and Theorem 3 are the main results of this paper. For this we recall

KEY WORDS AND PHRASES: *Liouville numbers – U-numbers*
A.M.S. CLASSIFICATION: 10F35 – 10K15

DEFINITION[1]. Let $\gamma \in \mathbb{C}$ and $m \geq 1$ an integer. The number γ is called an U_m-number if for every $w > 0$ there is an algebraic number α of degree m with

$$0 <| \gamma - \alpha |< H(\alpha)^{-w}$$

where $H(\alpha)$ is the maximum of the absolute values of coefficients of the minimal polynomials of α, and if there exist constants c, $k > 0$ depending only on γ and m such that the relation $| \gamma - \beta |> cH(\beta)^{-k}$ holds for every algebraic number β of degree $< m$.

LEMMA 1. *Let $\alpha_1, \alpha_2, \dots, \alpha_k$ $(k \geq 1)$ be algebraic numbers which belong to an algebraic number field K of degree g, η be algebraic and $F(y, x_1, \dots, x_k)$ be polynomial with integral coefficients so that its degree with respect to y is at least one. Next assume that $F(\eta, \alpha_1, \dots, \alpha_k) = 0$. Then degree of $\eta \leq dg$ and*

$$h_\eta \leq 3^{2dg+(l_1+\cdots+l_k)g} H^g h_{\alpha_1}^{l_1 g} \dots h_{\alpha_k}^{l_k g},$$

where h_η is the height of η, h_{α_i} $(i = 1, 2, \dots, k)$ is the height of α_i $(i = 1, 2, \dots, k)$, H is the maximum of absolute values of coefficients of F, l_i $(i = 1, 2, \dots, k)$ is the degree of F in x_i $(i = 1, 2, \dots, k)$ and d is the degree of F with respect to y.

THEOREM 1. *Let $\{\alpha_i\}$ be a sequence of algebraic numbers with*

$$\deg \alpha_i = m_i \leq k, \qquad \lim_{i\to\infty} H(\alpha_i) = \infty, \tag{1}$$

$$0 <| \alpha_{i+1} - \alpha_i |= H(\alpha_i)^{-w(i)} \qquad \text{where } \lim_{i\to\infty} w(i) = \infty, \tag{2}$$

$$| \alpha_{i+1} - \alpha_i |< H(\alpha_{i+1})^{-\rho} \qquad \text{for some } \rho > 0. \tag{3}$$

Then $\lim_{i\to\infty} \alpha_i \in U_m$ where $m = \liminf_{i\to\infty} m_i$.

We quote a couple of results which can be found in [4] and [1] respectively. Now we generalize the Theorem 2 in [1] as:

THEOREM 2. *Let $m \in \mathbb{Z}^+$ and $P_i(x)$, $Q_i(x) \in \mathbb{Z}[x]$, where $(P_i(x), Q_i(x)) = 1$, $\min(\deg P_i(x), \deg Q_i(x)) \geq 1$ $(i = 1, \dots, k)$. Then there are infinitely many U_m-numbers γ such that $\frac{P_i(\gamma)}{Q_i(\gamma)} \in U_m$ for $i = 1, 2, \dots, k$.*

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(1) We note that, we have, in fact, defined a Koksma U_m^*-number instead of a Mahler U_m-number. However, it is known that they are the same (see [6], [9]).

PROOF. Let α be a real algebraic number of degree m and let $\alpha^{(1)} = \alpha$, $\alpha^{(2)}, \dots, \alpha^{(m)}$ denote the conjugates of α. We set $K_i(x) = \frac{P_i(x)}{Q_i(x)}$ $(i = 1, 2, \dots, k)$. Let us consider the equation

$$K_i(\alpha^{(r)} + y) = K_i(\alpha^{(s)} + y)\,. \tag{4}$$

For fixed r, s, i, (4) is equivalent to some polynomial equation $c_t y^t + \cdots + c_1 y + c_0 = 0$ where the coefficient c_i are algebraic numbers. Since $\alpha^{(r)} \neq \alpha^{(s)}$ for $r \neq s$, we have $c_t \neq 0$, and so (4) has only finitely many solutions in $y \in \mathbb{R}$. Therefore for large $n \in \mathbb{Z}^+$ we see that $\deg K_i(\alpha + n^{-1}) = m$ and $Q_i(\alpha + n^{-1}) \neq 0$ for $i = 1, 2, \dots, k$. Let $\{w(n)\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} w(n) = \infty$. We define α_i and positive integers n_i as:

$$\deg K_i(\alpha + n_1^{-1}) = m \quad (i = 1, 2, \dots, k), \quad \alpha_1 = \alpha + n_1^{-1} \tag{5}$$

$$\begin{gathered} \deg K_t(\alpha_i + n_{i+1}^{-1}) = m \ (t = 1, 2, \dots, k)\,, \\ H(\alpha_i)^{w(i)} < n_{i+1}, \quad n_i^2 < n_{i+1} \quad (i \geq 1) \end{gathered} \tag{6}$$

$$\alpha_{i+1} = \alpha_i + n_{i+1}^{-1}\,. \tag{7}$$

It follows from (5) and (7) that $\alpha_{i+1} = \alpha + \sum_{j=1}^{i+1} n_j^{-1}$. Applying Lemma 1 we find (using (6))

$$H(\alpha_{i+1}) < n_{i+1}^{2m+2} \qquad (i \text{ large})\,. \tag{8}$$

A combination of (7) and (8) gives us

$$|\ \alpha_{i+1} - \alpha_i\ | \leq H(\alpha_{i+1})^{-1/(2m+2)} \qquad (i \text{ large})\,. \tag{9}$$

On the other hand it follows from (6) and (7) that

$$|\ \alpha_{i+1} - \alpha_i\ | \leq H(\alpha_i)^{-w(i)} \qquad (i \geq 1)\,. \tag{10}$$

Thus $\{\alpha_i\}$ satisfies the conditions (1), (2) and (3) and so we have $\gamma = \lim \alpha_i \in U_m$. Now we show that $K_t(\gamma) \in U_m$ $(t = 1, \dots, k)$. Since

$Q_i(x) = 0$ has only finitely many solutions in $x \in \mathbb{R}$, we may assume that $Q_t(\alpha_i) \neq 0$ for $t = 1, \dots, k$ and $i \in \mathbb{Z}^+$. Put $K_t(\alpha_i) = \beta_i$. Then

$$(11) \qquad | \beta_{i+1} - \beta_i | = | \alpha_{i+1} - \alpha_i || K'_t(\theta) |$$

where $\alpha_i < \theta < \alpha_{i+1}$ and $K'_t(x)$ is the derivative of $K_t(x)$. Since γ is not algebraic, we have $Q_t(\gamma) \neq 0$ $(t = 1, \dots, k)$ and so there is a constant $c(t) > 0$ depending only on $K_t(x)$ and γ such that $| K_t(\theta) | < c(t)$. Set $c_1 = \max_{t=1}^{k}\{c(t)\}$. An application of Lemma 1 gives us

$$(12) \qquad H(\beta_i) \leq H(\alpha_i)^{qm+1} \qquad (i \text{ large})$$

where q is a positive constant depending only $\deg P_i$, $\deg Q_i$ $(i=1, \dots, k)$. Using (12) and (10) in (11)

$$| \beta_{i+1} - \beta_i | \leq H(\beta_i)^{-w(i)+1/(qm+1)} \qquad (i \text{ large}),$$

and combining (9) and (12) we find

$$| \beta_{i+1} - \beta_i | \leq c_1 | \alpha_{i+1} - \alpha_i | < | \alpha_{i+1} - \alpha_i |^{1/2} \leq H(\beta_{i+1})^{-\rho} \qquad (i \text{ large})$$

where $\rho = (4m+4)^{-1}(qm+1)^{-1}$. Thus by Theorem 1 we obtain

$$\lim \beta_i = K_t(\lim \alpha_i) = K_t(\gamma) \in U_m \quad (t = 1, \dots, k).$$

We note that using the arguments in Theorem 3 in [1], one can prove that

THEOREM 3. *Let $m \in \mathbb{Z}^+$ and $\{\frac{p_i(x)}{Q_i(x)}\}$ be a sequence of rational functions with $P_i(x)$, $Q_i(x) \in \mathbb{Z}[x]$, $(P_i(x), Q_i(x)) = 1$, $\min(\deg P_i, \deg Q_i) \geq 1$. Then there are infinitely many U_m-numbers γ such that $\frac{P_i(\gamma)}{Q_i(\gamma)} \in U_m$ $(i = 1, 2, \dots)$.*

The following is a consequence of Theorem 2.

COROLLARY 1. *Let C be a curve in $\mathbb{R}^n$ such that C has a parametrization by non constant rational functions having rational coefficients. Then there are infinitely many points $P(x_1, x_2, \dots, x_n)$ on C where $x_i \in U_m$ $(i = 1, 2, \dots, n)$.*

For $n = 2$ we can give the circle $x^2 + y^2 = a^2$ $(a \in \mathbb{Q})$, the lemniscate curve and the curves $y^m = x^n$ $(m, n \in \mathbb{Z})$ as examples for such curve C.

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Lavoro pervenuto alla redazione il 9 dicembre 1997
ed accettato per la pubblicazione il 20 maggio 1998.
Bozze licenziate il 17 novembre 1998

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