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Timelike shock waves in a generic matter-energy field in general relativity

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RIASSUNTO: *Si studiano le onde d'urto del genere tempo per una generica distribuzione di materia-energia sotto diverse condizioni sulla struttura dello spazio-tempo, basate sull'esistenza di Ricci collineation e di vettori di Killing. Infine si studia la conservazione del carattere temporale di un'onda.*

ABSTRACT: *Timelike shock waves for a generic matter-energy field are studied under different structural conditions for the space time, based upon the existence of Ricci collineations and Killing vectors. The conservation of the timelike kind of a wave is also studied.*

1 – Introduction

A wave in the sense of Hadamard is a field which is regularly discontinuous across a moving surface, which is called the wave-front. However, for the sake of simplicity, the name "wave" is commonly given to the wave-front, rather than to the solution.

Waves in the sense of Hadamard in any case correspond to the propagation of discontinuities of physical quantities describing either fields (in mathematical physics essentially electromagnetic field and gravitational

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field) or the motion of a fluid, or together, like in magnetohydrodynamics, the changes in time of a field and of a fluid (see e.g. [4], [5], [25]).

In this framework, an ordinary gravitational wave is a discontinuity hypersurface for the Riemann curvature tensor: $[R_{\alpha\beta\rho\sigma}] \not\equiv 0$ (for complete details, see Section 2.1, Definition 1). To avoid any confusion we have to remark that, with different terminology, this kind of waves are elsewhere called gravitational shock waves (see e.g. [31]); however, following Cattaneo and Lichnerowicz, we prefer to adopt the latter denomination only in the particular case of discontinuity of the first derivatives of the gravitational potentials, which we are not going to study in the present paper.

In general relativity the properties of matter and fields are summarized by the stress-energy tensor, which, by Einstein equations, plays the role of source of the gravitational field. Thus the Einstein equations rule the interactions between gravitation and the evolution of matter and energy. Consequently, in a matter-energy field, a gravitational wave may also carry a discontinuity for the stress-energy tensor, so as to define, in the terminology of this paper, a shock wave for the matter energy (Section 2.2, Definition 2). If a shock wave for the matter energy is timelike (i.e. the normal vector to the wave-front is spacelike), for the sake of brevity we call it a material wave (Section 3.1).

There is a vast literature on ordinary and shock waves for matter and energy in relativity (expecially on magnetohydrodynamical waves) and also, separately, on gravitational waves, but the problem of their interaction in general relativity, due to Einstein equations, does not seem to have aroused the main interest of the scientists.

In particular, during the 60s and the 70s, Lichnerowicz published a series of papers with the derivation of a complete theory of relativistic magnetohydrodynamical waves (for a synthesis of that theory see e.g. [25]). About the same years, the theory of gravitational discontinuity waves was developed by a number of valuable scientists, including Lichnerowicz, Pirani, Papapetrou, Treder and Trautman (for a survey of the theory of gravitational waves and radiation see e.g. [36]). There seems to be only one paper, that published by Dautcourt in 1963 [7], where generic shock waves for the stress-energy tensor are approached by means of their coupled gravitational discontinuity.

The aim of this paper is to study material waves in terms of the discontinuity of metric, Riemann and Ricci tensors, rather than more familiar terms as the discontinuity of density, pressure and other dynamical variables. With this method, it is possible to characterize their propagation in a generic matter-energy field under very general hypothesis on the structure of the space time. In particular, a special class of waves, characterized by a continuous curvature scalar, seems worth to be studied. Special material waves are a generalization of Alfven shock waves of magneto-hydrodynamics; under suitable conditions, they are the only possible material waves.

The study of this kind of link between material and gravitational waves is interesting, since it can display the differences between special relativistic and general relativistic dynamics. Moreover, in a time when several experiments for the detection of general relativity are in preparation, it also suggests the possibility, in principle, of indirect revelation of general relativistic properties of gravity by means of observations on the motion of fluids.

However in this paper we are going to study the problem under a mathematical point of view only, leaving possible experimental applications as open problems still to be investigated.

The formalism used here is that of waves in the sense of Hadamard (see e.g. [4], [5], [10]), both for what concernes gravitational and material waves. This permits to recognize their links and to obtain results which are very general, in the sense that they do not depend on the choice of a particular solution of the Einstein equations. Thus the very difficult problem of solving that equations for some model allowing the presence of both matter and gravitational radiation is avoided at this level.

In Section 2 general definitions of gravitational wave (Definition 1) and matter-energy shock wave (Definition 2) are given in terms of the jump of the Riemann tensor and of the energy tensor.

In Section 3 matter-energy shock waves are studied: here their classification in terms of propagation speed is given (Section 3.1). The following concepts are then introduced: material waves (Section 3.1), essential (i.e. gauge invariant) metric discontinuity (Section 3.2), third order tensor potentials formalism (Section 3.3), special material waves (Section 3.4).

In Section 4 matter-energy shock waves are studied in the presence of Ricci collineations: general theorems are given (Section 4.1); a simple model of wave transmission from empty space to matter is introduced (Section 4.2); the case of magnetohydrodynamics is considered; a mistake contained in an earlier paper by the writer is corrected (Section 4.3).

In Section 5 material waves are studied in the presence of Killing vectors: a property of tangence is proved (Section 5.1); the conservation in time of the timelike kind of a wave is studied (Section 5.2).

In Section 6 the examples of a perfect charged fluid, of a radiating fluid and of a plane simmetric neutral fluid are considered.

2 – Definitions

Let V_4 be the space time of general relativity, i.e. (see for ex. [17], [25]) an oriented differentiable manifold of Dimension 4, of class (C^2, piecewise C^4), provided with a metric tensor $g_{\alpha\beta}$ of strictly hyperbolic type, signature $- + + +$ and class (C^1, piecewise C^2).

Greek indices run from 0 to 3, latin indices from 1 to 3, except where otherwise stated.
Units are chosen in order to have the speed of light in empty space $c = 1$.

Let $\Omega \subset V_4$ be an open domain with compact closure and $\Sigma \subset \Omega$ be a hypersurface of equation $f(\mathbf{x}) = 0$, where $f \in C^2(\Omega)$, $\ell_\alpha \stackrel{\text{def}}{=} \partial_\alpha f$, $\ell_0 \neq 0$. Let $g \in C^1(\Omega) \cap C^2(\Omega \backslash \Sigma)$ and let its second derivatives be regularly discontinuous (eventually with null discontinuity) across Σ. Let us denote by $[\varphi]$ the jump across Σ of a generic regularly discontinuous function φ.

2.1 – Gravitational waves

Definition 1. If $[R_{\alpha\beta\rho\sigma}] \neq 0$, then Σ is called an (ordinary discontinuity) **gravitational wave** (see for ex. [33], [18], [19], [36], [25], [11]).

The metric discontinuity is a well defined field on Σ, denoted by $\partial^2 g_{\alpha\beta}$, such that the following Hadamard compatibility conditions hold (see for ex. [4], [10]):

$$(1) \qquad [\partial_\alpha \partial_\beta g_{\rho\sigma}] = \ell_\alpha \ell_\beta \partial^2 g_{\rho\sigma} \,.$$

In terms of $\partial^2 g_{\alpha\beta}$, from the usual definition of the curvature tensor (see for ex. [30], [8]) and from (1) we have:

$$(2) \quad [R_{\alpha\beta\rho\sigma}] = (1/2)(\ell_\beta \ell_\rho \partial^2 g_{\alpha\sigma} - \ell_\beta \ell_\sigma \partial^2 g_{\alpha\rho} - \ell_\alpha \ell_\rho \partial^2 g_{\sigma\beta} + \ell_\alpha \ell_\sigma \partial^2 g_{\beta\rho}) \,.$$

The metric discontinuity is a tensor with respect to C^3 coordinate transformations. Any piecewise C^3 transformation of the so called "tangent to identity" kind instead produces a "gauge change" like the following:

$$\partial^2 g_{\alpha\beta} \ \leftrightarrow \ \partial^2 g_{\alpha\beta} + \ell_\alpha q_\beta + \ell_\beta q_\alpha \,, \tag{3}$$

(see for ex. [19], [32] p. 174, [1] p. 98, [25] chap IV), where the vector $\mathbf{q}$, defined on Σ, is determined by the discontinuity of the third derivatives of the transformation. Therefore there is a gauge freedom for $\partial^2 g_{\alpha\beta}$. In particular $\partial^2 g_{\alpha\beta} \neq 0$ is not an invariant condition (it is invariant only for C^3 coordinate transformations).

$[R_{\alpha\beta\rho\sigma}]$ is instead invariant for gauge changes (3). In particular $[R_{\alpha\beta\rho\sigma}] \neq 0$ is an invariant condition. This is the reason why it is correct to define gravitational waves by requiring $[R_{\alpha\beta\rho\sigma}] \neq 0$ rather than $\partial^2 g_{\alpha\beta} \neq 0$. For example, a metric discontinuity of the kind $\partial^2 g_{\alpha\beta} = \ell_\alpha q_\beta + \ell_\beta q_\alpha$, is "inessential", since it is due to the gauge choice (i.e. it can be eliminated by a suitable gauge change) and produces no discontinuity for the Riemann tensor: $[R_{\alpha\beta\rho\sigma}] = 0$ (see (2)).

From (2) we have the following "Bianchi like" formulae:

$$\ell_{[\alpha}[R_{\beta\rho]\sigma\nu}] = 0 \,, \tag{4}$$

and the following expressions for the jump of the Ricci tensor $R_{\beta\rho} \stackrel{\text{def}}{=} R_{\alpha\beta\rho}{}^{\alpha}$ and that of the scalar $R \stackrel{\text{def}}{=} R_\alpha{}^\alpha$:

$$\begin{aligned} [R_{\beta\rho}] &= (1/2)\{\ell_\beta \ell_\rho (g^{\sigma\nu}\partial^2 g_{\sigma\nu}) - 2\ell^\nu \ell_{(\rho}\partial^2 g_{\beta)\nu} + (\boldsymbol{\ell}\cdot\boldsymbol{\ell})\partial^2 g_{\beta\rho}\} \,, \\ [R] &= (\boldsymbol{\ell}\cdot\boldsymbol{\ell})(g^{\sigma\nu}\partial^2 g_{\sigma\nu}) - \ell^\sigma \ell^\nu \partial^2 g_{\sigma\nu} \,. \end{aligned} \tag{2}$$

From (5) we also have the relation:

$$[R_{\beta\rho}]\ell^\rho = (1/2)[R]\ell_\beta \,. \tag{6}$$

It is not difficult to prove the following theorem, which is well known, and has a number of different equivalent proofs in the literature (see for ex. [34], [18], [35], [19], [31], [11]):

Theorem 1. $[R_{\alpha\beta\rho\sigma}] \not\equiv 0$, $[R_{\beta\rho}] = 0 \Rightarrow (\boldsymbol{\ell}\cdot\boldsymbol{\ell}) = 0$.

Gravitational waves have been studied by many authors in the case of a continuous Ricci tensor: $[R_{\alpha\beta}] = 0$ (see for ex. [34], [19], [7], [11]); for example, this is the case of empty space-time. For the sake of brevity in the following we will call *pure wave* a wave such that $[R_{\alpha\beta}] = 0$.

2.2 – Matter-energy shock waves

Let $T_{\alpha\beta} \in C^0(\Omega \backslash \Sigma)$ denote the energy tensor of the matter-energy distribution in the domain Ω. We suppose that $T_{\alpha\beta}$ is regularly discontinuous across Σ (eventually with null discontinuity) and satisfies the conservation equations

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{7}$$

in the ordinary sense in $\Omega \backslash \Sigma$, and in the weak sense globally in Ω. This implies that if $[T_{\alpha\beta}]$ is not null then the Rankine-Hugoniot jump conditions must be satisfied:

$$\ell_\beta [T^{\alpha\beta}] = 0 \tag{8}$$

(see for ex. [22], [23], [25]).

Definition 2. If $[T_{\alpha\beta}] \neq 0$ then Σ is called a **matter-energy shock wave**.

3 – Gravitational waves in a matter-energy field

Let us now consider the jump of the Einstein equations in the presence of a gravitational wave (Definition 1):

$$R_{\beta\rho} - (1/2) R g_{\beta\rho} = -\chi T_{\beta\rho} \Rightarrow [R_{\beta\rho}] - (1/2)[R] g_{\beta\rho} = -\chi [T_{\beta\rho}]\,. \tag{9}$$

Since $[R] = \chi[T]$, ($T \stackrel{\text{def}}{=} T_\alpha{}^\alpha$), we have $[R_{\beta\rho}] = 0 \Leftrightarrow [T_{\beta\rho}] = 0$.

The case $[R_{\beta\rho}] \neq 0$, $[T_{\beta\rho}] \neq 0$ therefore necessarily corresponds to the presence of matter (or energy) in $\Omega \subset V_4$; in particular it defines a matter-energy shock wave. This is probably usually taken for granted, since $[T_{\beta\rho}] \neq 0$, but if we adopt Definition 2 for matter-energy shock waves we have to verify that the Rankine-Hugoniot conditions are also satisfied; this is not difficult to prove, so let us formalize this result by the following theorem.

THEOREM 2. *Let Σ be a gravitational wave and let $[R_{\alpha\beta}] \neq 0$. Then Σ is a matter-energy shock wave. Conversely, if Σ is a matter-energy shock wave it is also a gravitational wave.*

PROOF. Definition 1 is compatible with Definition 2 in the case $[R_{\alpha\beta}] \neq 0$, $[T_{\alpha\beta}] \neq 0$, since, due to the Einstein equations (9), (6) is equivalent to the Rankine-Hugoniot conditions (8). This proves that, if Σ is a gravitational wave and $[R_{\alpha\beta}] \neq 0$, then it is also a matter-energy shock wave. On the other hand, if $[T_{\alpha\beta}] \neq 0$, then $[R_{\alpha\beta}] \neq 0$ and consequently $[R_{\alpha\beta\rho\sigma}] \neq 0$. This proves the converse. □

3.1 – Material waves and limit waves

LICHNEROWICZ ([21], [22], [23], [25]) showed that, under reasonable compressibility hypothesis, magneto-hidrodynamic shock waves are timelike (i.e. $(\boldsymbol{\ell}{\cdot}\boldsymbol{\ell}) > 0$). Moreover condition $(\boldsymbol{\ell}{\cdot}\boldsymbol{\ell}) > 0$ assures the admissibility of the propagation speed of the wave with respect to the generic observer. Actually, if $\mathbf{u}$ is a timelike unit vector field representing a reference frame, we have:

$$(10) \qquad (v_\Sigma(\mathbf{u}))^2 = (\mathbf{u}\cdot\boldsymbol{\ell})^2/\{(\mathbf{u}\cdot\boldsymbol{\ell})^2 + (\boldsymbol{\ell}\cdot\boldsymbol{\ell})\}\,,$$

and therefore, if $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) > 0$ then $|v_\Sigma(\mathbf{u})| < 1$, $\forall\ \mathbf{u}$.

Therefore, it is natural to study matter-energy shock waves ($[T_{\alpha\beta}]{\neq}0$) in the following cases:

a) $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) > 0$; for the sake of brevity in the following we will simply call this kind of waves **material waves**.

b) $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) = 0$; this is the limiting case of a), for the sake of brevity in the following we will simply call them **limit waves**.
As said before, case b) cannot occur in compressible magneto-hydro-dy-namics. Limit waves can instead appear in pure electro-magnetism (see for ex. [19], [25]) and in the relativistic version of incompressible hydro-dynamics (see [17], [21], [24]).

Of course we are not going to consider the case:

c) $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) < 0$;
since it violates the causality principle.

The above classification is local, since it in principle is not invariant from event to event of the domain Ω (nor from domain to domain of the space-time manifold). However we are going to see (Section 5) that, if a Killing vector field is present, the classification gains some global significance.

3.2 – Essential metric discontinuity for material waves

One may wonder if it is possible to neglect the inessential part of the metric discontinuity, due to the gauge choice, and work with gauge-invariant quantities.

This is certainly possible for material waves: since $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) \neq 0$, we can define:

$$\mathcal{G}_{\alpha\beta} \overset{\text{def}}{=} (\delta_\alpha{}^\mu - (\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}\ell_\alpha\ell^\mu)(\delta_\beta{}^\nu - (\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}\ell_\beta\ell^\nu)\partial^2 g_{\mu\nu}\,. \tag{11}$$

Since $\mathcal{G}_{\alpha\beta}$ is clearly invariant for gauge changes (3), we can name it the *essential metric discontinuity*. It is always possible to choose the gauge such that $\partial^2 g_{\alpha\beta} = \mathcal{G}_{\alpha\beta}$: it suffices to choose:

$$q_\alpha = -(\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}\ell^\nu\partial^2 g_{\alpha\nu} + (1/2)(\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-2}\ell^\nu\ell^\mu\partial^2 g_{\nu\mu}\ell_\alpha \tag{12}$$

in the gauge change (3). We call this gauge the *natural gauge* for the material wave. The natural gauge is unique, in the sense that it is always determined by a unique gauge change (the one with q_α defined by (12)).

Therefore $\mathcal{G}_{\alpha\beta}$ is at the same time:

- the completely orthogonal component of $\partial^2 g_{\alpha\beta}$ with respect to $\boldsymbol{\ell}$;
- the essential part, i.e. gauge-invariant, of $\partial^2 g_{\alpha\beta}$;
- the representation of $\partial^2 g_{\alpha\beta}$ in the natural gauge.

In particular $\mathcal{G}_{\alpha\beta}$ has a precise *geometrical meaning*, besides one in terms of gauge-invariance. This means that results expressed in terms of $\mathcal{G}_{\alpha\beta}$, different that in terms of $\partial^2 g_{\alpha\beta}$, are invariant and have no degrees of freedom.

In terms of $\mathcal{G}_{\alpha\beta}$, we have:

$$
\begin{aligned}
[R_{\alpha\beta\rho\sigma}] &= (1/2)(\ell_\beta \ell_\rho \mathcal{G}_{\alpha\sigma} - \ell_\beta \ell_\sigma \mathcal{G}_{\alpha\rho} - \ell_\alpha \ell_\rho \mathcal{G}_{\sigma\beta} + \ell_\alpha \ell_\sigma \mathcal{G}_{\beta\rho}) \\
[R_{\beta\rho}] &= (1/2)\{\ell_\beta \ell_\rho \mathcal{G}_\nu{}^\nu + (\boldsymbol{\ell}\cdot\boldsymbol{\ell})\mathcal{G}_{\beta\rho}\} \\
[R] &= (\boldsymbol{\ell}\cdot\boldsymbol{\ell})\mathcal{G}_\nu{}^\nu \,.
\end{aligned} \tag{13}
$$

The essential metric discontinuity was first introduced in [9], without mention of its properties of gauge-invariance, as a very useful tool for splitting the jump of the Einstein equations; material waves were first studied in terms of metric discontinuity in [7], in a tethrad formalism.

3.3 – Third order potentials of the Riemann tensor

It is useful to introduce a method for the study of gravitational wavefronts in terms of third order potentials.

Lanczos ([16], [2]), showed that the Weyl tensor has a third order tensor potential. In case the Riemann tensor has the same property one has interesting consequences ([3]), but this occurs only in a few particular situations ([2], [29]).

In the case of gravitational waves, however, in our continuity hypothesis for the metric, the jump of the Riemann tensor is equal to that of the complex of the second derivatives of the metric: $[R_{\alpha\beta\rho\sigma}] = [R'{}_{\alpha\beta\rho\sigma}]$, where:

$$
R'{}_{\alpha\beta\rho\sigma} \stackrel{\text{def}}{=} (1/2)(\partial_\beta\partial_\rho g_{\alpha\sigma} - \partial_\beta\partial_\sigma g_{\alpha\rho} - \partial_\alpha\partial_\rho g_{\sigma\beta} + \partial_\alpha\partial_\sigma g_{\beta\rho}) \,. \tag{14}
$$

$R'{}_{\alpha\beta\rho\sigma}$, which is not a tensor, is generated by the following third order "potential":

$$
H_{\alpha\beta\rho} \stackrel{\text{def}}{=} (1/2)\partial_{[\alpha} g_{\beta]\rho} \,, \tag{15}
$$

according to the formula:

$$
R'{}_{\alpha\beta\rho\sigma} = \partial_\sigma H_{\alpha\beta\rho} - \partial_\rho H_{\alpha\beta\sigma} + \partial_\beta H_{\rho\sigma\alpha} - \partial_\alpha H_{\rho\sigma\beta} \,. \tag{16}
$$

We have $H_{\alpha\beta\rho} \equiv (1/2)\Gamma_{\rho[\alpha\beta]}$ and $H_{\alpha\beta\rho} \in C^0(\Omega) \cap C^1(\Omega\backslash\Sigma)$. In our continuity hypothesis $[H_{\alpha\beta\rho}] = 0$, while $\partial H_{\alpha\beta\rho}$ is a tensor field (with support on Σ), with respect to C^3 local coordinate transformations, which is not null unless $g_{\alpha\beta} \in C^2(\Omega)$.

Consequently in this case we can follow [3] just adding square brackets to the formulae quoted there. Thus, or also directly from (16), we have:

$$
\begin{aligned}
[R_{\alpha\beta\rho\sigma}] &= \ell_\sigma \partial H_{\alpha\beta\rho} - \ell_\rho \partial H_{\alpha\beta\sigma} + \ell_\beta \partial H_{\rho\sigma\alpha} - \ell_\alpha \partial H_{\rho\sigma\beta} \\
[R_{\alpha\beta}] &= 2\ell_{(\alpha}\partial V_{\beta)} + g_{\alpha\beta}\ell^\sigma \partial V_\sigma - \ell^\sigma(\partial K_{\alpha\sigma\beta} + \partial K_{\beta\sigma\alpha}) \\
[R] &= 4\ell^\sigma \partial V_\sigma \,,
\end{aligned}
\tag{17}
$$

with:

$$
V_\alpha \stackrel{\text{def}}{=} g^{\beta\rho} H_{\alpha\beta\rho}, \quad K_{\alpha\beta\rho} \stackrel{\text{def}}{=} (2/3)V_{[\beta}g_{\alpha]\rho} - H_{\alpha\beta\rho} \Rightarrow g^{\beta\rho}K_{\alpha\beta\rho} = 0\,. \tag{18}
$$

We will use $(17)_3$ in the following section for defining the class of special material waves.

3.4 – Special material waves

Let us consider the complex: $V_\alpha \in C^0(\Omega) \cap C^1(\Omega\backslash\Sigma)$; its derivatives are regularly discontinuous across Σ, such that the following Hadamard compatibility conditions hold:

$$
[V_\alpha] = 0, \quad \partial_\beta[V_\alpha] = \ell_\beta \partial V_\alpha \,, \tag{19}
$$

where: $\partial V_\alpha = (1/2)g^{\beta\rho}\ell_{[\alpha}\partial^2 g_{\beta]\rho}$. The derivative $\partial_\beta[V_\alpha]$ can be uniquely defined on Σ with the help of arbitrary regular prolongations; we have: $\partial_\beta[V_\alpha] = [\partial_\beta V_\alpha]$ (see for ex. [4], [10]).

From a classic decomposition theorem (see for ex. [3], [28] p. 49), any vector field is the sum of a gradient plus a solenoidal part. Let us make use of this property for $[V_\alpha]$:

$$
[V_\alpha] = \partial_\alpha \Phi + \eta_\alpha{}^\mu{}_{\rho\sigma} \nabla_\mu \Psi^{\rho\sigma} \,, \tag{20}
$$

where Φ is a suitable scalar field, $\Psi^{\rho\sigma}$ a suitable antisymmetric 2-tensor with support on Σ and η is the antysimmetric Ricci tensor. For the compatibility conditions (19) to be satisfied, there must exist a suitable scalar field ϕ and a suitable antisymmetric 2-tensor $\psi^{\rho\sigma}$, of class $\in C^1(\Omega) \cap C^2(\Omega\backslash\Sigma)$, with regularly discontinuous second derivatives, such that $\Phi = [\phi]$, $\Psi^{\alpha\beta} = [\psi^{\alpha\beta}]$, $\partial_\alpha\partial_\beta[\phi] = \ell_\alpha\ell_\beta\partial^2\phi$ and $\partial_\alpha\partial_\beta[\psi^{\rho\sigma}] = \ell_\alpha\ell_\beta\partial^2\psi^{\rho\sigma}$; therefore:

$$\partial V_\alpha = \ell_\alpha\partial^2\phi + \eta_\alpha{}^\beta{}_{\rho\sigma}\ell_\beta\partial^2\psi^{\rho\sigma} \,. \tag{21}$$

From $(17)_3$ and (21) we directly have: $[R] = 4(\boldsymbol{\ell}\cdot\boldsymbol{\ell})\partial^2\phi$. Let $\varphi \stackrel{\text{def}}{=} 4\phi$; we have just proved the following theorem.

Theorem 3. *Let Σ be a gravitational wave. Then there is a function $\varphi \in C^1(\Omega)$ such that*

$$[R] = (\boldsymbol{\ell}\cdot\boldsymbol{\ell})\partial^2\varphi\,; \tag{22}$$

φ is a suitable function of the metric $g_{\alpha\beta}$, but in the general case it is unknown.

Corollary 3.1. *$[R] = 0 \Rightarrow (\boldsymbol{\ell}\cdot\boldsymbol{\ell}) = 0$, or $\partial^2\varphi = 0$.*

This corollary generalizes, in a sense, Theorem 1, since, if we neglect the special case when $\varphi \in C^2(\Omega)$ even if $g_{\alpha\beta} \in C^1(\Omega) \cap C^2(\Omega\backslash\Sigma)$, then we have:

$$[R_{\alpha\beta\rho\sigma}] \not\equiv 0, [R] = 0 \Rightarrow (\boldsymbol{\ell}\cdot\boldsymbol{\ell}) = 0\,,$$

while Theorem 1 needs the stronger hypothesis: $[R_{\alpha\beta}] = 0$.

We are therefore led to define **special material waves** those such that $[R] = 0$, $(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) > 0$, $\partial^2\varphi = 0$. In the following we will see that this kind of waves has notable properties related to the propagation of gravitational waves in matter.

An example of special material wave according to our definition is given by Alfven magneto-hydrodynamical shock waves (see for ex. [20], [21], [25]), which are such that the thermodynamical variables ρ and p are continuous across the wave-front, and consequently $[T] = 0$ and $[R] = 0$.

We moreover have the following interesting property of limit waves:

Corollary 3.2. *For a limit wave one necessarily has: $[R] = 0$.*

Therefore, Corollary 3.1 can be equivalently expressed as follows:

Corollary 3.3. *If* $[R] = 0$, *a gravitational wave can be a pure wave, a limit wave or a special material wave; the case of a general material wave is forbidden.*

4 – Material waves and Ricci collineations

In this section we wish to study the propagation of matter-energy shock waves (in a generic matter-energy field) under additional hypothesis about the structure of the space-time based on the existence of Ricci collineations.

4.1 – Ricci collineations

Let us recall a useful conservation property about Ricci collineations:

Theorem 4. *Let* $\mathcal{R}_{\alpha\beta}$ *be a symmetric tensor field such that* $\nabla_\beta(\mathcal{R}_\alpha{}^\beta - (1/2)\mathcal{R}_\nu{}^\nu\delta_\alpha{}^\beta) = 0$. *If there is a vector field* $\mathbf{Y}$ *such that* $\mathcal{L}_Y\mathcal{R}_{\alpha\beta} = 0$, *then the following conservation law holds:*

$$\nabla_\alpha(R^\alpha{}_\beta Y^\beta) = 0\,. \tag{23}$$

Proof. Written explicitely, equation $\mathcal{L}_Y\mathcal{R}_{\alpha\beta} = 0$ is:

$$Y^\mu\nabla_\mu\mathcal{R}_{\alpha\beta} + \nabla_\alpha Y^\mu\mathcal{R}_{\mu\beta} + \nabla_\beta Y^\mu\mathcal{R}_{\alpha\mu} = 0\,;$$

by contraction on α and β we have:

$$\nabla_\alpha(\mathcal{R}_\mu{}^\alpha Y^\mu) + Y^\mu\nabla_\alpha\big((1/2)\mathcal{R}_\nu{}^\nu\delta_\mu{}^\alpha - \mathcal{R}_\mu{}^\alpha\big) = 0\,,$$

which leads to the thesis. ∎

Theorem 4 actually is a slight generalization of the analogous result for Ricci collineations ($\mathcal{R}_{\alpha\beta} = R_{\alpha\beta}$, see [6]; for the case $R = 0$ see [13]). We have introduced this generalization in view of application to the case $\mathcal{R}_{\alpha\beta} = [R_{\alpha\beta}]$ (see Theorem 6).

The existence of Ricci collineations are general properties of many space-times (see for ex. [14], [12]).

The following theorem holds.

THEOREM 5. *If there is* $\mathbf{Y} \in C^0(\Omega)$ *such that:*

$$\mathcal{L}_Y R_{\alpha\beta} = 0\,, \tag{24}$$

then a necessary condition for Σ *to be a material wave is:*

$$(\boldsymbol{\ell} \cdot \mathbf{Y})\partial^2\varphi = 0\,. \tag{25}$$

PROOF. From (23) we have that $\mathcal{L}_Y R_{\alpha\beta} = 0$ implies the conservation law: $\nabla_\alpha(R^\alpha{}_\beta Y^\beta) = 0$. From the shock condition (8) and the continuity of $\mathbf{Y}$, we then have on Σ: $\ell^\alpha Y^\beta [R_{\alpha\beta}] = 0$; in the case of a material wave, from $(13)_2$:

$$(\boldsymbol{\ell} \cdot \boldsymbol{\ell})(\boldsymbol{\ell} \cdot \mathbf{Y})\mathcal{G}_\nu{}^\nu = 0\,.$$

Therefore we must have $(\boldsymbol{\ell} \cdot \mathbf{Y}) = 0$, or $\mathcal{G}_\nu{}^\nu = 0$, which, from $(13)_3$, implies $[R] = 0$. In this case from (22) we have $(\boldsymbol{\ell} \cdot \boldsymbol{\ell}) = 0$ (no material wave) or $\partial^2\varphi = 0$. □

COROLLARY 5.1. *If the hypothesis of Theorem* 5 *hold in* Ω *and moreover* $(\boldsymbol{\ell} \cdot \mathbf{Y}) \neq 0$, *then in* Ω *gravitational waves which propagate in matter are necessarily pure waves, limit waves or special material waves; the case of general material waves is forbidden. Therefore in this case the only possible material waves are special waves.*

4.2 – A toy model of wave transmission in the matter

Let us now consider a domain of space-time $\Omega \subset V_4$ where the following elements are present:

- a regular world tube $\mathcal{V} \subset \Omega$, such that $T_{\alpha\beta} \not\equiv 0$ in $\mathcal{V}$;

- an external empty space $\Omega\backslash\mathcal{V}$, such that $T_{\alpha\beta} \equiv 0$ in $\Omega\backslash\mathcal{V}$;
- a gravitational wave $\Sigma \subset \Omega$ such that $\Sigma \cap \partial\mathcal{V} \neq \emptyset$.

In this situation Ω is divided into four parts by Σ and $\partial\mathcal{V}$. $\mathcal{V}$ may be the world tube of a star and $\Omega\backslash\mathcal{V}$ the outer space, so as to represent the transmission of an ingoing wave in the matter. Since $\Omega\backslash\mathcal{V}$ is empty, Σ is necessarily a pure wave in $\Omega\backslash\mathcal{V}$, while in $\mathcal{V}$ it may or not become a material wave or a limit wave. It is rather reasonable to make the continuity hypothesis: $[R_{\alpha\beta}] = 0$ on $\Sigma \cap \partial\mathcal{V}$. We can find conditions which, in this model, are sufficient to determine the kind of the wave in $\mathcal{V}$.

THEOREM 6. *If the following structural condition holds:*

$$\mathcal{L}_\ell[R_{\alpha\beta}] = 0\,, \tag{26}$$

then the wave can be transmitted in $\mathcal{V}$ as a pure wave, a limit wave or a special material wave; the case of a general material wave is forbidden.

PROOF. The jump of the Riemann tensor on a wave-front Σ satisfies the Bianchi identities $\nabla_{[\alpha}[R_{\beta\rho]\sigma\nu}] = [\nabla_{[\alpha}R_{\beta\rho]\sigma\nu}] = 0$, where covariant derivatives of the jump are well defined by the use of the regular prolongation method (see [4], [5], [10]). Consequently the jump of the Ricci tensor also satisfies the Bianchi contracted identities: $\nabla_\beta\big([R_\alpha{}^\beta] - (1/2)[R_\nu{}^\nu]\delta_\alpha{}^\beta\big) = 0$, and we can make use of Theorem 4. From (23) we thus have: $\nabla_\alpha([R^\alpha{}_\beta]\ell^\beta) = 0$. From (6) we then have the conservation law:

$$\nabla_\alpha([R]\ell^\alpha) = 0\,. \tag{27}$$

Therefore in this case our model must have $[R] = 0$ in $\mathcal{V}$. Consequently general material waves are forbidden. □

COROLLARY 6.1. *If there is a vector field $\boldsymbol{\ell} \in C^1(\Omega)$ which coincides, on the hypersurface Σ, with the normal vector of Σ, and which is such that*

$$\mathcal{L}_\ell R_{\alpha\beta} \in C^0(\Omega) \tag{28}$$

then again the wave Σ can be transmitted in $\mathcal{V}$ as a pure wave, a limit wave or a special material wave; the case of a general material wave is forbidden.

We have found several simple structural conditions under which no general material wave can be transmitted in a matter-energy field. The most interesting surviving case is that of special material waves, since it is the only possible case where $(\boldsymbol{\ell} \cdot \boldsymbol{\ell}) > 0$, i.e. the propagation speed of the wave is less than the speed of light.

COROLLARY 6.2. *If, in our transmission model, the structural condition* (24) *holds with* $(\mathbf{Y} \cdot \boldsymbol{\ell}) \neq 0$*, or if* (26) *holds, or if* (28) *holds, then a necessary condition for Σ to be a material wave is:* $\partial^2\varphi = 0$.

Consequently, under these hypothesis, the only possible material waves are the special ones.

4.3 – Gravitational waves and Alfven shock waves

In perfect magneto-hydrodynamics an example of special waves are Alfven shock waves (see for ex. [26], [25]); moreover, if certain hypothesis are satisfied, Alfven shock waves are the only magneto-hydrodynamical special material waves:

THEOREM 7. *Given a special material wave Σ in a perfect charged fluid, if* $[R_{\alpha\beta}]D^\alpha = 0$*, where* $\mathbf{D}$ *is the current density vector, then Σ is an Alfven shock wave.*

PROOF. The current density vector is a particular vector which is orthogonal to both the 4-velocity of the fluid and the magnetic field (for its definition see [26] and [25] p. 211). A theorem due to LUKACEVIC ([26]), gives the following necessary and sufficient conditions for a material wave to be an Alfven shock wave:

$$[R_{\alpha\beta}]\ell^\beta = 0, \quad [R_{\alpha\beta}]D^\beta = 0\,. \tag{29}$$

Condition $(29)_1$ is verified, for special material waves, as a consequence of (6). Condition $(29)_2$ is contained in the hypothesis of the theorem. □

As a corollary, we have a uniqueness property of Alfven shock waves:

Corollary 7.1. *Let us consider a perfect charged fluid in the hypothesis of Corollary* 5.1 *or of Theorem* 6 *or of Corollary* 6.1*. If moreover condition* $[R_{\alpha\beta}]D^\alpha = 0$ *holds, then any material wave necessarily is an Alfven shock wave.*

The connection between Alfven shock waves and gravitational waves in the matter was already studied in some particular cases (see [26], [27]). In an earlier paper ([9]) the writer studied the compatibility conditions which relate the weak discontinuity of the metric to the shocks of the dynamical variables, when a gravitational wave is coupled with a material wave. In Section 9 of that work it is concluded that Alfven shock waves are the only possible material waves of magneto-hydrodynamics. We have just seen that this is true if some suitable structural conditions hold, but not in the general case, since this would contrast Theorem 2 (every shock wave is a gravitational wave). Therefore that conclusion must be wrong. We realize now of an error contained in [9], which caused the mistake: an incorrect "-1" appears in equation $(9.4)_2$ instead of the correct "h^2". Actually, equation (9.9) and (9.13) are identities; which invalidates the results of Section 9, where they are instead considered as the relations identifying Alfven shocks and matchable (with gravitational waves) hydrodynamical shocks, respectively.

5 – Material waves and Killing vectors

In this section we wish to study the propagation of matter-energy shock waves (in a generic matter-energy field) under additional hypothesis about the structure of the space-time based on the existence of Killing vectors.

5.1 – Killing vectors

Let us suppose there exists a vector field $\boldsymbol{\xi} \in C^1(\Omega) \cap C^2(\Omega \backslash \Sigma)$ which satisfies the Killing equation:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \,. \tag{30}$$

We therefore have:

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = 2\Gamma_{\alpha\beta}{}^{\nu}\xi_\nu\,, \tag{31}$$

and, from the definition of curvature tensor:

$$\partial_\rho\partial_{(\alpha}\xi_{\beta)} - \partial_\alpha\partial_{(\rho}\xi_{\beta)} = (R_{\alpha\rho\beta}{}^{\nu} - 2\Gamma_{\mu(\rho}{}^{\nu}\Gamma_{\alpha)\beta}{}^{\mu})\xi_\nu + 2\Gamma_{\beta[\alpha}{}^{\nu}\partial_{\rho]}\xi_\nu\,. \tag{32}$$

Theorem 8. *Let $\boldsymbol{\xi}$ have regularly discontinuous second derivatives across a hypersurface Σ. Then, a necessary condition for Σ to be a material wave is $(\boldsymbol{\ell}\cdot\boldsymbol{\xi}) = 0$.*

Proof. Let us consider the weak disontinuity of order 2 of ξ_α and denote it by $\partial^2\xi_\alpha$; this is a well defined field on Σ, such that Hadamard compatibility conditions hold: $[\partial_\alpha\partial_\beta\xi_\rho] = \ell_\alpha\ell_\beta\partial^2\xi_\rho$. From our continuity hypothesys and from (32) we have:

$$\ell_\beta(\ell_\rho\partial^2\xi_\alpha - \ell_\alpha\partial^2\xi_\rho) = 2[R_{\alpha\rho\beta}{}^{\nu}]\xi_\nu\,. \tag{33}$$

From $(13)_1$ we consequently have:

$$\ell_\beta\ell_{[\rho}(\mathcal{G}_{\alpha]}{}^{\nu}\xi_\nu - \partial^2\xi_{\alpha]}) + (\boldsymbol{\ell}\cdot\boldsymbol{\xi})\ell_{[\alpha}\mathcal{G}_{\rho]\beta} = 0\,. \tag{34}$$

A splitting of the equation above along $\boldsymbol{\ell}$ and its orthogonal subspace gives the following equivalent system:

$$\begin{aligned} \partial^2\xi_\alpha - (\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}(\ell^\nu\partial^2\xi_\nu)\ell_\alpha - \mathcal{G}_\alpha{}^{\nu}\xi_\nu &= 0\,,\\ (\boldsymbol{\ell}\cdot\boldsymbol{\xi})\mathcal{G}_{\alpha\beta} &= 0\,. \end{aligned} \tag{35}$$

If a material wave is present ($\mathcal{G}_{\alpha\beta}\neq 0$) we must therefore have $(\boldsymbol{\ell}\cdot\boldsymbol{\xi}) = 0$. □

Condition $(\boldsymbol{\ell}\cdot\boldsymbol{\xi}) = 0$ gives $\ell^\nu\partial^2\xi_\nu = \partial^2(\boldsymbol{\ell}\cdot\boldsymbol{\xi}) - \ell^\nu\partial^2\ell_\nu = \partial^2(\boldsymbol{\ell}\cdot\boldsymbol{\xi}) - (\boldsymbol{\ell}\cdot\boldsymbol{\xi})\partial^3 f = 0$. Therefore the following corolary holds:

Corollary 8.1. *Let Σ be a material wave and $\boldsymbol{\xi}$ a Killing vector in the hypothesys of Theorem 8. Then we have:*

$$\partial^2\xi_\alpha = \mathcal{G}_\alpha{}^{\nu}\xi_\nu, \quad (\boldsymbol{\ell}\cdot\boldsymbol{\xi}) = 0\,. \tag{36}$$

5.2 – Conservation of the timelike kind of a wave

An interesting property, which extends the concept of material wave from a single event $E \in \Sigma$ to some portion of Σ, is expressed by the following theorem.

THEOREM 9. *Let Σ be a material wave and $\boldsymbol{\xi}$ a Killing vector in the hypothesis of Theorem 8. Then the scalar $(\boldsymbol{\ell} \cdot \boldsymbol{\ell})$ is constant along the integral lines of $\boldsymbol{\xi}$.*

PROOF. We wish to prove that $\xi^\alpha \nabla_\alpha (\boldsymbol{\ell} \cdot \boldsymbol{\ell}) = 0$. Making use of the Killing property $\nabla_{(\alpha} \xi_{\beta)} = 0$ for $\boldsymbol{\xi}$ and of the gradient property $\nabla_{[\alpha} \ell_{\beta]} = 0$ for $\boldsymbol{\ell}$, we have:

$$\xi^\alpha \nabla_\alpha (\boldsymbol{\ell} \cdot \boldsymbol{\ell}) = 2\ell^\beta \xi^\alpha \nabla_\beta \ell_\alpha = 2\ell^\beta \nabla_\beta (\boldsymbol{\xi} \cdot \boldsymbol{\ell}) \, .$$

Therefore $\xi^\alpha \nabla_\alpha (\boldsymbol{\ell} \cdot \boldsymbol{\ell})$ vanishes since, by Theorem 8, we have $(\boldsymbol{\ell} \cdot \boldsymbol{\xi}) = 0$. $\square$

As a consequence, condition $(\boldsymbol{\ell} \cdot \boldsymbol{\ell}) > 0$ can be transported from a given event $E \in \Sigma$ to a line laying on Σ, and eventually, from a 2-dymensional section of Σ, to the whole $\Sigma \cap \Omega$. Let us introduce a reference frame in Ω, determined by a 1-parameter family of spacelike leafs. We may consider sections of Σ with different leafs as the (2-dymensional) wave-front at different times. We have just proved the following corollary:

COROLLARY 9.1. *In the hypothesys of Theorem 8, if Σ is a material wave at a given time, it is a material wave always (in Ω). Conversely, if it is a limit wave at a given time, then it is a limit wave always.*

6 – Examples

It is interesting to see what essential metric discontinuity actually are in a known example. Let us consider a perfect charged fluid in a world tube $\mathcal{T} \subset \Omega$. The energy tensor is (see [20], [21], [22], [23], [25]):

$$T_{\alpha\beta} \stackrel{\text{def}}{=} (\rho + p + \mu h^2) U_\alpha U_\beta + (p + (1/2)\mu h^2) g_{\alpha\beta} - \mu h_\alpha h_\beta \, , \tag{37}$$

where μ is a positive constant (magnetic permeability), $\mathbf{h}$ is the magnetic field and $h^2 \stackrel{\text{def}}{=} (\mathbf{h} \cdot \mathbf{h}) > 0$. If a material wave Σ is present, we have

$[T_{\alpha\beta}]\ell^\beta = 0$, or:

$$[(\rho + p + \mu h^2)U_\alpha(\mathbf{U}\cdot\boldsymbol{\ell}) + (p + (1/2)\mu h^2)\ell_\alpha - \mu h_\alpha(\mathbf{h}\cdot\boldsymbol{\ell})] = 0\,.$$

From Theorem 2, a gravitational ordinary wave corresponds to Σ; from equations (9) and (13) we have:

$$\mathcal{G}_\nu{}^\nu = \chi(\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}[3p - \rho] = \partial^2\varphi$$

$$\mathcal{G}_{\alpha\beta} = -2\chi(\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}[(\rho + p + \mu h^2)U_\alpha U_\beta - \mu h_\alpha h_\beta] + \tag{38}$$

$$+ (1/2)[(\boldsymbol{\ell}\cdot\boldsymbol{\ell})^{-1}(3p - \rho)\ell_\alpha\ell_\beta + (\rho + p + \mu h^2)g_{\alpha\beta}]\,.$$

The latter is the expressions of the essential metric discontinuity in terms of shocks of the dynamical variables. For a magneto-hydrodynamical shock wave to be a special wave, we must have $\mathcal{G}_\nu{}^\nu = \partial^2\varphi = 0$; therefore the special wave condition, in the considered example, is: $[\rho] = 3[p]$. Consequently, in the particular case where the equation of state is $\rho = 3p$ any shock wave is a special one. This equation of state defines a situation called "incoherent radiation" (see e.g. [15] p. 75). In the general case, instead, an example of special wave is given by Alfven shock waves, which are such that $[\rho] = [p] = [h^2] = 0$ (see e.g. [21], [23], [25]).

These considerations have been made without specifying the metric. Let us now consider a perfect fluid exact solution, as given by Taub (see [15] p. 161). The line element:

$$ds^2 = -e^{2\nu(z)}dt^2 + z^2(dx^2 + dy^2) + z/F(z)dz^2 \tag{39}$$

corresponds to a neutral perfect fluid of the kind:

$$T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p g_{\alpha\beta}, \qquad \rho = \rho(p) \tag{40}$$

if the following relations hold between the variables:

$$2zp'(\rho + p)^{-1} = 1 - \chi p z^3 F^{-1}\,, \qquad F' + \chi\rho z^2 = 0, \qquad (\rho + p)\nu' = p'\,.$$

Let us consider a (C^1, piecewise C^2) match of two solutions of the kind (39) across the hypersurface Σ of equation $z = z_0$. This corresponds to a gravitational ordinary discontinuity wave and, from Theorem 2, to a matter-energy shock wave. We have $\ell_\alpha = \delta_\alpha{}^z$ and $(\boldsymbol{\ell} \cdot \boldsymbol{\ell}) = F/z$. Since $g = -e^{2\nu} z^3/F < 0$, we must have $F/z > 0$ and therefore Σ is timelike and the wave is a material wave. The metric discontinuity is:

$$\partial^2 g_{\alpha\beta} = -2e^{2\nu_0}[\nu'']\delta_\alpha{}^t \delta_\beta{}^t - z_0/F_0^2[F'']\delta_\alpha{}^z \delta_\beta{}^z\,;$$

and its essential component is:

$$\mathcal{G}_{\alpha\beta} = -2e^{2\nu_0}[\nu'']\delta_\alpha{}^t \delta_\beta{}^t, \qquad \mathcal{G}_\nu{}^\nu = 2[\nu'']\,. \tag{42}$$

We also have:

$$[R_{\alpha\beta}] = \{-(F_0/z_0)e^{2\nu_0}\delta_\alpha{}^t \delta_\beta{}^t + \delta_\alpha{}^z \delta_\beta{}^z\}[\nu''], \qquad [R] = 2(F_0/z_0)[\nu'']\,. \tag{43}$$

The essential metric discontinuity $\mathcal{G}_{\alpha\beta}$ is perpendicular to $\boldsymbol{\ell}$; the Rankine-Hugoniot shock condition $[T_{\alpha\beta}]\ell^\beta = 0$ is therefore automatically satisfied.

Finally, the conservation of the timelike character of the wave is, in this example, expressed by the indipendence of $(\boldsymbol{\ell} \cdot \boldsymbol{\ell}) = F/z$ on the variable t.

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