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One-point connectifications of subspaces of the Euclidean line

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RIASSUNTO: *Uno spazio connesso di Hausdorff Y è detto connettificazione con un punto di uno spazio X se X è immerso in Y e $Y \setminus X$ ha esattamente un punto. In questo lavoro si caratterizzano i sottospazi della retta euclidea che hanno una connettificazione con un punto. Inoltre vengono dati alcuni esempi per dimostrare che tale caratterizzazione non è più valida nel caso del piano euclideo.*

ABSTRACT: *A connected Hausdorff space Y is called one-point connectification of a space X if X is embedded in Y and $Y \setminus X$ has exactly one point. In this paper we characterize the subspaces of the Euclidean line which have a one-point connectification. Several examples are given to show how different is the situation in the Euclidean plane.*

A space X is called connectifiable if it can be densely embedded in a connected Hausdorff space Y, in such a case Y is called a connectification of X (see [6], [5], [1]). Obviously every one-point connectification of a space X is a connectification of X.

Recently the authors have introduced the related concept of pathwise connectifiable space [3] : a space X is called pathwise connectifiable if it can be densely embedded in a pathwise connected Hausdorff space Y

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(Y will be called a pathwise connectification of X). Similarly we say that a pathwise connected Hausdorff space Y is a one-point pathwise connectification of a space X if X is embedded in Y and $|Y \setminus X| = 1$.

Since a subspace of the Euclidean line R is connected if and only if it is pathwise connected, it is natural to ask if a subspace of R is connectifiable if and only if it is pathwise connectifiable. Although the answer to this question is negative in general (the subspace $\{0\} \cup \bigcup\{(\frac{1}{2n+2}, \frac{1}{2n+1}] : n \in \mathbb{N} \cup \{0\}\}$ of R is connectifiable but it is not pathwise connectifiable, see Example 2.4 in [3]), in this paper we show that the situation changes radically if we consider only one-point connectifications. More precisely we will show that a subspace X of R has a one-point connectification if and only if it has a one-point pathwise connectification, and it will be also shown that the above conditions are equivalent to the fact that every component of X is open and non compact.

We will conclude this paper with some examples showing the different situation occurring in the Euclidean plane.

We refer the reader to [2] for notations and terminology not explicitly given.

THEOREM. *Let X be a subspace of the Euclidean line R. Then the following conditions are equivalent:*

i) *X has a one-point connectification;*
ii) *X has a one-point pathwise connectification;*
iii) *every component of X is open and non compact;*
iv) *X is locally connected and every component of X is not compact.*

PROOF. ii) $\Rightarrow$ i) is obvious.

iii) $\Rightarrow$ ii) Let us suppose that every component C_α is open and non compact. Therefore we may assume that every C_α has the form $[a_\alpha, b_\alpha)$ or (a_α, b_α), with $a_\alpha, b_\alpha \in \tilde{R} = R \cup \{-\infty, +\infty\}$. Let $S = \{b_\alpha\}_\alpha$ and, for every α, set $D_\alpha = C_\alpha \cup \{b_\alpha\}$. Clearly the members of $\{D_\alpha\}_\alpha$ are pairwise disjoint. Let Z be the quotient of the sum $\oplus_\alpha D_\alpha$ (every D_α has the subspace topology of the extended Euclidean line $\tilde{R}$) obtained identifying S to a point. Obviously X is embedded in Z and $|Z \setminus X| = 1$. Moreover Z is a T_2-space (S is a closed subset of the T_3-space $\oplus_\alpha D_\alpha$). It remains to show that Z is pathwise connected. Let $p : \oplus_\alpha D_\alpha \to Z$ be the natural mapping. Now $\{p(D_\alpha)\}_\alpha$ is a family of pathwise connected subspaces of Z with non empty intersection whose union Z. Therefore Z is pathwise connected.

i) $\Rightarrow$ iii) Since X has a one-point connectification, it contains no non empty compact open subsets. Therefore it is enough to show that every component of X is open. Let us suppose that C is a component of X which is not open, and let us pick some $a \in C \setminus \text{int}_X(C)$. Let $Y = X \cup \{p\}$ be a one-point connectification of X and let us take two disjoint open subsets U and V of Y such that $a \in U$ and $p \in V$. We may take $U = (a - \epsilon, a + \epsilon) \cap X$ for some $\epsilon > 0$. Observe that there is some $\alpha \in (a - \epsilon, a + \epsilon) \setminus X$ (otherwise $(a - \epsilon, a + \epsilon)$ would be a connected subset of X containing a, so $(a - \epsilon, a + \epsilon) \subset C$ and $a \in \text{int}_X(C)$, a contradiction). We may assume, without loss of generality, that $\alpha < a$. Now let $b \in (X \setminus C) \cap (\alpha, a + \epsilon)$ (if $(X \setminus C) \cap (\alpha, a + \epsilon) = \emptyset$ then $(\alpha, a + \epsilon) \cap X$ is an open neighbourhood of a in X which is contained in C, a contradiction). Since a and b are in different components, there is some $\beta \in R \setminus X$ between them. Now $(\alpha, \beta) \cap X$ is a proper non empty clopen subset of Y, a contradiction.

iii) $\Leftrightarrow$ iv) It is enough to observe that a subspace X of R is locally connected if and only if every component of X is open.

Remark 1. A Hausdorff space is called H-closed if it is closed in every Hausdorff space in which it can be embedded. It is worth noting that if X has a one-point pathwise connectification then every path component of X is not H -closed. In fact let $Z = X \cup \{p\}$ be a one-point pathwise connectification of X and let $\{C_\alpha\}_\alpha$ be the family of path components of X. We claim that $p \in \text{cl}_Z(C_\alpha)$ for every α (and therefore every C_α is not H-closed). Let $x \in C_\alpha$ and let $f : I \to Z$ be an embedding such that $f(0) = x$ and $f(1) = p$. Since $f([0, 1))$ is a pathwise connected subset of X containing x, it follows that $f([0, 1)) \subset C_\alpha$. By the continuity of f it follows that $p \in \text{cl}_Z(C_\alpha)$.

The following examples will show that the above theorem is no more valid for subspaces of the Euclidean plane.

Example 1. Let F be the Knaster-Kuratowski fan (see [2], 6.3.23) and let $X = F \setminus \{(\frac{1}{2}, \frac{1}{2})\}$. F is a one-point connectification of X (in the terminology of [4] X is called pulverized), but X has no one-point pathwise connectifications. In fact X is hereditarily disconnected (i.e., it does not contain connected subsets of cardinality larger than one) and therefore every path component of X is H-closed (= compact).

However X is pathwise connectifiable, in fact it is a dense subspace of the cone over the Cantor set with vertex in $(\frac{1}{2}, \frac{1}{2})$.

Remark 2. Regarding example 1, observe that it is also possible to find a one-point connectifiable subspace of the Euclidean plane which is not pathwise connectifiable at all. In fact, let $X = A \cup \{(0,0)\}$ where $A = \{(x, \sin\frac{\pi}{x}) : 0 < x \leq 1\}$. X is one-point connectifiable (if $p \in \{(0,y) : -1 \leq y \leq 1, y \neq 0\}$, then $X \cup \{p\}$ is a one-point connectification of X), but X is not pathwise connectifiable. Assume the contrary and consider a pathwise connected Hausdorff space Z in which X is densely embedded.

First let us show that for every $x \in (0,1)$ the set $G(x) = \{(y, \sin\frac{\pi}{y}) : y \in (x, 1]\}$ is open in Z. Since $G(x)$ is open in Z, there is an open set W of Z such that $W \cap X = G(x)$. We claim that $G(x) = Z$. If not, take a $z \in W \setminus G(x)$, then $z \notin X$ and so $z \in \mathrm{cl}_Z(G(x)) = \{(y, \sin\frac{\pi}{y}) : y \in [x, 1]\}$. Since $\mathrm{cl}_Z(G(x))$ is compact, there are two disjoint open subsets U and V of Z such that $z \in U$ and $\mathrm{cl}_Z(G(x)) \subset V$. Set $H = U \cap W$, then $H \cap \mathrm{cl}_Z(G(x)) = \emptyset$ and $H \cap X \subset W \cap X = G(x)$. So $H \cap X = \emptyset$, a contradiction (X is dense in Z). Therefore $W = G(x)$ and $G(x)$ is open in Z.

Now let $f : I \to Z$ be an embedding such that $f(0) = (1, 0)$ and $f(1) = (0,0)$. We claim that $A \subset f(I)$. If not, take $(x, \sin\frac{\pi}{x}) \in A \setminus f(I)$, then $G(x) \cap f(I) = \mathrm{cl}_Z(G(x)) \cap f(I)$ is a proper non empty clopen subset of $f(I)$, a contradiction. So $A \subset f(I)$ and $Z = \mathrm{cl}_Z(A) = f(I)$.

Now take three distinct points $z_1, z_2, z_3 \in Z \setminus A$ (since $\{(0,0)\}$ is a path component of X, it follows by remark 1 that $|Z \setminus X| \geq 2$) and let $t_1, t_2, t_3 \in I$ such that $f(t_i) = z_i$, $i = 1, 2, 3$. Since f is a homeomorphism of I onto Z, it follows that $\tilde{Z} = Z \setminus \{z_1, z_2, z_3\}$ and $\tilde{I} = I \setminus \{t_1, t_2, t_3\}$ are homeomorphic, a contradiction ($\tilde{Z}$ is connected while $\tilde{I}$ is not).

Therefore X is not pathwise connectifiable.

Remark 3. Other examples of one-point connectifiable spaces which are not pathwise connectifiable can be obtained in the following way. Let X be a continuum which is not pathwise connected. If there is a point $p \in X$ such that $Y = X \setminus \{p\}$ is not pathwise connected, then Y is not pathwise connectifiable (although Y is obviously one-point connectifiable). In fact, let us suppose that Z is a pathwise connected Hausdorff space in which Y is densely embedded, and let $H = Z \setminus Y$. We claim that the map $f : Z \to X$, defined by $f(z) = z$ if $z \in Y$ and $f(z) = p$ if $z \in H$, is continuous. Let C be a closed subset of X. If $p \notin C$ then $f^{-1}(C) = C$ is closed in Z (observe that C is compact). If $p \in C$ then $f^{-1}(C) = (C \setminus \{p\}) \cup H$. H is closed in Z (Y is a locally compact dense subspace of the Hausdorff space Z, therefore Y is open in Z), moreover $C \setminus \{p\} = C \cap Y = \mathrm{cl}_Y(C \cap Y) = \mathrm{cl}_Z(C \cap Y) \cap Y$, therefore $\mathrm{cl}_Z(C \setminus \{p\}) = \mathrm{cl}_Z(C \cap Y) = (\mathrm{cl}_Z(C \cap Y) \cap Y) \cup (\mathrm{cl}_Z(C \cap Y) \cap H) \setminus (C \setminus \{p\}) \cup H$. Hence $\mathrm{cl}_Z(f^{-1}(C)) = \mathrm{cl}_Z(C \setminus \{p\}) \cup \mathrm{cl}_z(H) \subset (C \setminus \{p\}) \cup H = f^{-1}(C)$ and $f^{-1}(C)$ is closed in Z. By the continuity of f it follows that X is pathwise connected, a contradiction. Therefore Y is not pathwise connectifiable.

Observe that the compactness condition on X cannot be omitted. Let T be the extended long line and let X be the quotient of the sum $(T \setminus \{0\}) \oplus [-2, -1]$ obtained identifying $\{\omega_1, -2\}$ to a point. X is a (non compact) connected Hausdorff space which is not pathwise connected. Nevertheless $Y = X \setminus \{-1\}$ is a pathwise connectifiable space which is not pathwise connected.

Example 2. For every $n \in \mathbb{N}$ let L_n be the segment joining $(0, 0)$ with $(1, \frac{1}{n})$ and set $X = \bigcup\{L_n : n \in \mathbb{N}\}$. The only (path) component of X (which is X itself) is open and non compact. Nonetheless X has no one-point pathwise connectifications. In fact let us suppose that $Z = X \cup \{p\}$ ($p \notin X$) is a pathwise connected Hausdorff space. Observe that $p \notin \mathrm{cl}_Z(L_n) = L_n$ for every $n \in \mathbb{N}$. Since Z is T_2 there are two disjoint open sets U and V such that $0 \in U$ and $p \in V$. Take an embedding $f : I \to Z$ such that $f(0) = p$ and $f(1) = 0$. Let ϵ be a positive number such that $f([0, \epsilon)) \subset V$ and set $G = f([0, \epsilon))$. Take a natural number n such that $G \cap L_n \neq \emptyset$. Since $L_n \setminus \{0\}$ is open in Z, it is easy to see that $G \cap L_n = G \cap (L_n \setminus \{0\})$ is a non empty proper clopen subset of G. Since G is connected, we have a contradiction.

Example 3. Let X be as in example 2 and let $Y = X \cup (L \setminus (\frac{1}{2}, 0))$ where L is the segment joining $(0, 0)$ and $(1, 0)$. Now $Y \cup \{(\frac{1}{2}, 0)\}$ is a one-point pathwise connectification of Y but the path component $\{(x, 0) \in Y : x \in (\frac{1}{2}, 1]\}$ of Y is not open in Y.

Example 4. Let X and L be as in example 3 and let $Y = (X \cup L) \setminus \{(0, 0)\}$. Clearly $X \cup L$ is a one-point pathwise connectification of Y, but the component $L \setminus \{(0, 0)\}$ of Y is not open.

Motivated by the above examples we conclude this paper with the following

PROBLEM. Characterize those subspaces of the Euclidean plane which have a one-point (pathwise) connectification.

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