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Curves which are obstructions to the existence of Kähler metrics on threefolds

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RIASSUNTO: *Si discute il seguente problema: "Sia M una varietà analitica complessa liscia compatta di dimensione tre, e C una sua curva liscia. Supponendo che $M-C$ abbia una metrica Kähleriana, sotto quali condizioni M è Kähleriana, oppure è bimeromorfa ad una varietà Kähleriana?"*

ABSTRACT: *The following problem is discussed: "Let M be a compact complex threefold and C a smooth curve on M. If $M-C$ has a Kähler metric, when is M itself Kähler, or bimeromorphic to a Kähler manifold?".*

1 – Introduction

The problem we shall consider here is the following:

(*Q*) *Let M be a compact complex manifold of dimension $n=3$, C a smooth curve of genus g on M, such that $M-C$ has a Kähler metric. Is M itself Kähler, or bimeromorphic to a Kähler manifold (i.e., in the class $\mathcal{C}$ of Fujiki)?*

This kind of problem arises in algebraic geometry, as the search of a non-projective Moishezon manifold M. The first example was given, in

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dimension 3, by Hironaka; in that case M, which is a modification of $\mathbf{P}_3$, is not Kähler (hence not projective) because it contains a smooth rational curve C homologous to zero. Nevertheless, by blowing-up this curve we get a projective variety M_1, so that $M - C$ receives from M_1 a Kähler metric.

Our question (Q) does not concern projective geometry, but only Kähler geometry; moreover, we will not assume that, using suitable modifications, we get from M a projective or a Kähler manifold. Our starting point is only a (closed) Kähler form on $M - C$; in this situation, we study the conditions which guarantees that M itself is Kähler, or in the class $\mathcal{C}$ (in this case, M carries a balanced metric, as shown in [1]).

The case $g = 0$ has been studied in the second part of [2]: here we analyze the "genus g"-case for a threefold M, using heavily some strong results on positive currents, in particular the Main Theorem in [4]. Moreover, we use a sequence of blow-ups:

$$\cdots \longrightarrow M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M_0 := M$$

where α_1 is the blow-up of M with center C and exceptional divisor E_1, C_1^∞ is a section of minimal self-intersection in $E_1(C_1^\infty \cdot C_1^\infty = -e_1)$, α_2 is the blow-up of M_1 with center C_1^∞, and so on. The exceptional divisor of M_n is the surface E_n which contains a curve C_n^∞ of minimal self-intersection: $C_n^\infty \cdot C_n^\infty = -e_n$.

Our main result in the case $g > 0$ (see Theorem 4.5 and its corollaries) is that if for every n, $e_{n+1} \geq 0$ and $E_{n+1} \cdot C_{n+1}^\infty \geq 0$, then the following facts are equivalent:

(i) M is Kähler
(ii) M is bimeromorphic to a Kähler manifold
(iii) C is not homologous to zero in the Aeppli group $V_{\mathbb{R}}^{2,2}(M)$.

The hypothesis can be explained as follows (see also the Appendix): none of the conormal bundles $N^*_{C|M}, N^*_{C_1^\infty|M_1}, \ldots$ is stable, and the intersection number $E_{n+1} \cdot C_{n+1}^\infty$ in M_{n+1}, which corresponds to the degree of the line bundle $(N^*_{E_{n+1}|M_{n+1}})|_{C_{n+1}^\infty}$ on C_{n+1}^∞, is never negative.

The above result holds in particular when $e_{n+1} > 0$ for every n, that is, when on every exceptional divisor of the sequence of blow-ups, there is a curve with negative self-intersection. When $g = 0$, this corresponds to say that in the sequence of blow-ups, the rational ruled surface $\mathbf{P}_1 \times \mathbf{P}_1$ does never appear.

In Section 5 we give some results using only the first blow-up M_1 of M along C and $E_1 \cdot C_1^\infty$. Similar problems are studied in the projective case for instance in [3], [12] and [13]; there, typical algebraic-geometric techniques are added to currents techniques (see at the end of Section 2).

The (standard) machinery on ruled surfaces has been collected in the Appendix, to which we refer for the notation, whether the notation and the non-standard results about positive currents are collected in the first part of Section 2. Some technical results which are very similar to those stated in [2] are not proved here.

2 – Positive currents

The standard back-ground about positive currents can be found in [10] or [15] or [2]; we call *real* (p, p)*-currents* on M the elements of the dual space of the space of (smooth) real $(n-p,\, n-p)$-forms on M, where $n = \dim M$.

As regards closed positive currents, the corner-stone of our techniques is the following result ([4], Main Theorem 1.1):

Theorem 2.1. *Let M be a compact complex manifold, and let u be the Kähler form of a hermitian metric on M such that $c_1(O_{TM}(1), h) + \pi^* u \geq 0$. Let T be a closed positive $(1,1)$-current on M and γ be a continuous real $(1,1)$-form on M such that $T \geq \gamma \geq 0$.*

Then there are a sequence $\{\phi_k\}$ of closed $(1,1)$-forms on M in the same Aeppli class as T in $\Lambda^{1,1}_{\mathbb{R}}(M)$ (that is, $T = \phi_k + i\partial\bar\partial\psi_k$), and a non-increasing sequence $\{\lambda_k\}$ of continuous non-negative functions on M such that $\lim_{k\to\infty} \lambda_k(x) = n(T,x)$ at every point x, and $\phi_k \geq \gamma - \lambda_k u$.

Notice that we can always choose a smooth hermitian metric h on $O_{TM}(1)$ and a strictly positive (1,1)-form u on M such that $c_1(O_{TM}(1), h) + \pi^* u > 0$ (see [9]). Moreover, $n(T,x)$ denotes the Lelong number of T in x.

Nevertheless, the obstruction to the existence of a Kähler metric on a manifold M is given in terms of positive currents which are not closed, in general (see the characterization of compact Kähler manifolds, [8] Theorem 14); so let us recall some definitions.

DEFINITION 2.2. A real (p,p)-current T is said *pluriharmonic* if $i\partial\bar\partial T = 0$, and *plurisubharmonic* if $i\partial\bar\partial T \geq 0$; T is said to be *the* $(n-p,$ $n-p)$*-component of a boundary* if there exists a current R such that $\partial\overline{R} + \bar\partial R = T$, that is, T is the component of a d-exact current, which means that the class $\langle T \rangle$ in the Aeppli group $V_{\mathbb{R}}^{p,p}(M)$ vanishes.

The following results about plurisubharmonic and pluriharmonic currents are proved in [2] for a compact complex manifold of dimension n:

PROPOSITION 2.3. *If T is a positive plurisubharmonic $(p,p,)$-current on M, and Y is a submanifold of M of codimension p, then there exists a constant $c \geq 0$ such that $\chi_Y T = c[Y]$* ([2], 2.1 and 2.2) (We shall denote the current $[Y]$ also by Y).

PROPOSITION 2.4. *Let Y be a submanifold of M with $\operatorname{codim} Y \geq 2$, and let $\alpha : M' \to M$ be the blow-up with center Y and exceptional divisor $E := \alpha^{-1}(Y)$. If T is a closed positive $(1,1)$-current on M, there exists a unique closed positive $(1,1)$-current on M', denoted by $\alpha^* T$, such that*

$$\text{(i)} \qquad \alpha_* \alpha^* T = T$$
$$\text{(ii)} \qquad \alpha^* T \in \alpha^* \langle T \rangle \,.$$

(*Notice that the pull-back of smooth forms induces a map on classes $\alpha^* : \Lambda_{\mathbb{R}}^{1,1}(M) \to \Lambda_{\mathbb{R}}^{1,1}(M')$; we denote by $\langle T \rangle$ the class of T in the Aeppli group $\Lambda_{\mathbb{R}}^{1,1}(M)$*).

Moreover, if $n(T,Y) := \inf\{n(T,x), x \in Y\}$ (and the same for $n(\alpha^*T, E)$), it holds $n(T,Y) = n(\alpha^*T, E)$ ([2], 3.3 and 3.4).

Now let us recall what we got about question (Q) in the first part of [2]. There, we considered the case where M has dimension $n \geq 3$ and C is an irreducible curve on M (not smooth, in general). In this case, we have: ([2], Theorem 5.5).

THEOREM 2.5. *If $M - C$ has a Kähler metric, one and only one of the following cases may occur:*

(i) *M is Kähler,*

(ii) *C is the $(1,1)$-component of a boundary,*

(iii) C *is part of the* $(1,1)$*-component of a boundary, that is, there exists a positive* $(n-1,\ n-1)$*-current* $S \neq 0$ *on* M *such that* $S + C$ *is the* $(1,1)$*-component of a boundary and* $\chi_C S = 0$.

We give here the proof of this result to show how the machinery of currents works.

Proof. If (i) does not hold, by ([8], Theorem 14) there exists a positive current $T \neq 0$ which is the (1,1)-component of a boundary. Hence, by Proposition 2.3, there is a constant $c \geq 0$ such that $\chi_C T = c[C]$ so that

$$T = S + c[C]\,,$$

where $S := T - \chi_C T \geq 0, \partial\bar{\partial} S = 0$ and $\chi_C S = 0$.

Let ω be a closed Kähler form on $M - C$, which extends to a closed positive current on M (see [7]), also called ω, to which we apply Theorem 2.1.

Therefore

$$0 = \omega \cdot T = \omega \cdot S + \omega \cdot cC \tag{2.1}$$

because T is the (1,1)-component of a boundary; but, by Theorem 2.1,

$$\omega \cdot S = \phi_k \cdot S \geq S(\gamma) - S(\lambda_k u)\,.$$

Call μ the positive measure on M given by $\mu(A) := S(\chi_A u)$, where A is a Borel subset of M. Since $0 \leq \lambda_k \leq \lambda_0$, we get:

$$\lim_{k\to\infty} S(\lambda_k u) = \lim_{k\to\infty} \int_M \lambda_k d\mu = \int_M n(\omega, x) d\mu = 0\,,$$

(in fact, ω is smooth on $M - C$, hence $\{x \in M : n(\omega, x) \neq 0\} \subset C$ and $\mu(C) = 0$ because $\chi_C S = 0$). Thus

$$\omega \cdot S \geq S(\gamma) \geq 0\,. \tag{2.2}$$

Moreover, we can take γ strictly positive on every fixed compact subset of $M - C$, so that

$$\omega \cdot S = 0 \Leftrightarrow S = 0\,. \tag{2.3}$$

Now, c cannot vanish, otherwise from (2.1) and (2.3) we would obtain $T = 0$. So we get (ii) if $S = 0$ and (iii) if $S \neq 0$.

Let us look now at the relationships among the cases: if (ii) or (iii), (i) is not allowed. Finally, if (ii) and (iii), then $S + C$ and C are the (1,1)-component of a boundary, so that S is the (1,1)-component of a boundary too, and by (2.3), $S = 0$ because $\omega \cdot S = 0$; this is a contradiction.

If ω denotes the Kähler form of a Kähler metric on $M - C$, in terms of the intersection number we have ([2], 5.7):

Theorem 2.6.

(i) *If there exists a Kähler metric on $M - C$ whose Kähler form ω satisfies $\omega \cdot C > 0$, then M is Kähler.*

(ii) *If there exists a Kähler metric on $M - C$ whose Kähler form ω satisfies $\omega \cdot C = 0$, then M is Kähler or C is the $(1,1)$-component of a boundary.*

(iii) *If there exists a Kähler metric on $M - C$ whose Kähler form ω satisfies $\omega \cdot C < 0$, then M is Kähler or C is part of the $(1,1)$-component of a boundary.*

Theorem 2.5 tells us that C always plays a role among the obstructions to the existence of a Kähler metric on M. Moreover, if C "moves" in M, points where the Lelong numbers of ω does not vanish can be avoided (see the proof of Theorem 2.5), and we get $\omega \cdot C > 0$: hence ([2], 5.8 and 5.9):

Corollary 2.7. *If there exists a curve C' homologous to C, such that C and C' have no common component, then M is Kähler.*

Corollary 2.8. *Let C be a smooth curve of genus g in M; if $N_{C|M} \cdot C > (n-1)(g-1)$, then M is Kähler.*

Another point of view is given in the following theorem.

Theorem 2.9. *Suppose that M is a compact complex manifold and C an irreducible curve on M, and that there exist $p \in C$ and a ball $B(p,r)$ such that $(M - C) \cup B(p,r)$ is Kähler. Then M is Kähler.*

Proof. Denote $C \cap B(p,r)$ by C_r and let ω be a Kähler form on $(M-C)\cup B(p,r)$, extended as a closed positive current across C_r. Since $C - C_r$ has an open Stein neighborhood in M, we can use Lemma 3.4 in [9]:

For every strictly positive (1,1)-form γ on M, for every (1,1)-form $\theta \geq 0$ on M such that $\omega \geq \theta$, and for every $\varepsilon > 0$, there exists a (1,1)-form ω_ε in the same Aeppli class as ω such that $\omega_\varepsilon + \varepsilon\gamma \geq \theta$.

Now let T be a positive (1,1)-component of a boundary in M:

$$0 = T\cdot\omega = T(\omega_\varepsilon) \geq T(\theta) - \varepsilon T(\gamma)\,,$$

that is, $\varepsilon T(\gamma) \geq T(\theta)$ for every $\varepsilon > 0$.

Hence $T(\theta) = 0$, but θ can be choosen strictly positive on every compact subset of $(M-C)\cup B(p,r)$, thus supp $T \subseteq C$, and $T = c[C]$. But T has to vanish in $B(p,r)$, hence also on C_r: this gives $c = 0$ and $T = 0$.

As we said in the introduction, also the case of Moishezon manifolds may be handled using currents, as is done f.i. in [12] and [3]. These authors investigate the obstructions to the algebraicity of a Moishezon manifold X in terms of closed currents which are effective (i.e. given by the integration on effective curves) or weak limits of effective curves (i.e. in $\hat{A}_1^+(X)$). In [3] the following results are proved: "Let X be a Moishezon manifold. X is projective if and only if for every $T \in \hat{A}_1^+(X)$ with $\langle T\rangle = 0$, it holds $T = 0$ (the involved cohomology group is $H^{n-1,n-1}(X,\mathbb{R})$)". "Let X be a 1-projective non projective threefold; then there exist an effective curve C and $T \in \hat{A}_1^+(X)$ such that $\langle C\rangle + \langle T\rangle = 0$."

Notice that, in Kähler geometry, their problem can be translated as follows: "Let $M \in \mathcal{C}$; when is M Kähler?", which is much more restrictive than question (Q), in dimension three (see f.i. the proof of Corollary 4.6). Also the tools are different: for instance, due to the projective situation, they may consider closed currents (and the usual cohomology), instead of pluriharmonic currents.

3 – The sequence of blow-ups

Let M be a compact complex threefold, and C a smooth curve of genus g on M such that $M - C$ is Kähler. A first answer to question (Q)

can be given using Corollary 2.8: If deg $N_{C|M} > 2g-2$, that is, if $K_M \cdot C < 0$, then M is Kähler, because C moves in M. To handle the other cases, we shall associate to (M, C) a sequence of blow-ups (this is a well-known technique: see f.i. [11] in the case of an exceptional curve).

Definition and notation 3.1. (See also the Appendix) Let $M_1 \xrightarrow{\alpha_1} M_0 := M$ be the blow-up of M with center C and exceptional divisor E_1; if $N_0^* := N^*_{C|M}$ is unstable, there exists a unique section C_1^∞ on E_1 such that $C_1^\infty \cdot C_1^\infty = -e_1 < 0$; in this case, since C_1^∞ cannot move in E_1, we blow-up M_1 with center C_1^∞.

If N_0^* is semistable, then $e_1 \leq 0$ and in E_1 there are no curves with negative self-intersection; in this case we blow-up a section C_1^∞ with minimal self-intersection in E_1 (hence $C_1^\infty \cdot C_1^\infty = -e_1 \geq 0$). Recall that by a theorem of Nagata, it holds $e_1 \geq -g$, so that in the case of rational ruled surfaces, N_0^* is semistable if and only if $e_1 = 0$.

In this manner we get the following diagram:

$$(3.1) \qquad \cdots \longrightarrow M_{n+1} \xrightarrow{\alpha_{n+1}} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\alpha_2} M_1 \xrightarrow{\alpha_1} M_0 := M$$

The exceptional divisor of M_n is the surface E_n which contains a curve C_n^∞ of minimal self-intersection: $C_n^\infty \cdot C_n^\infty = -e_n$; the generic fibre is called F_n and the conormal bundle $N^*_{C_n^\infty|M_n}$ is denoted by N_n^*. We call (3.1) *the sequence of blow-ups associated to* (M, C).

Notation 3.2. Let $1 \leq m < n$; we call $E_{m,n}$ the proper transform of the exceptional divisor E_m of M_m by the modification $\alpha_{m+1} \circ \cdots \circ \alpha_n : M_n \to M_m$.

$E_{m,n}$ is birational to E_m, but in general not biholomorphic to it. Let us denote by $C^\infty_{m,n}$ and $F_{m,n}$ the curves that correspond to C_m^∞ and F_m in $E_{m,n}$ via the birational map $E_{m,n} \to E_m$.

In this section we collect some observations on the ties between the degrees of the involved conormal bundles and the e_j's.

Remark 3.3. For every $n \geq 0$

$$(3.2)_{n+1} \qquad 0 \to (N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \to N^*_{n+1} \to N^*_{C^\infty_{n+1}|E_{n+1}} \to 0$$

is an exact sequence, and

$$(3.3)_{n+1} \qquad B_n := \deg(N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} = -E_{n+1} \cdot C^\infty_{n+1},$$

$$(3.4)_{n+1} \qquad \deg N^*_{C^\infty_{n+1}|E_{n+1}} = -C^\infty_{n+1\cdot E_{n+1}} C^\infty_{n+1} = e_{n+1}\,,$$

$$(3.5)_{n+1} \qquad \begin{aligned} A_n :&= \deg N^*_{n+1} = \deg(N^*_{E_{n+1}|M_{n+1}})_{C^\infty_{n+1}} + \deg N^*_{C^\infty_{n+1}|E_{n+1}} = \\ &= B_n + e_{n+1}\,. \end{aligned}$$

In particular, B_n does not depend on the choice of C^∞_{n+1}; moreover,

$$(3.6)_n \qquad \deg N^*_n = A_n + B_n, \text{ so that } A_n = A_{n+1} + B_{n+1}\,.$$

Indeed, it holds $(N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \sim aC^\infty_{n+1} + bF_{n+1}$, with $a = -E_{n+1|E_{n+1}} \cdot F_{n+1} = 1$, and $B_n = -E_{n+1} \cdot C^\infty_{n+1} = -e_{n+1} + b$, so that

$$(N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \sim C^\infty_{n+1} + A_n F_{n+1}\,,$$

and (see [5] p. 610)

$$\begin{aligned} \deg N^*_n = (N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \cdot (N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} = (C^\infty_{n+1} + A_n F_{n+1})\cdot \\ \cdot (C^\infty_{n+1} + A_n F_{n+1}) = 2A_n - e_{n+1} = A_n + B_n\,. \end{aligned}$$

Remark 3.4. Let $n \geq 0$, and suppose that N^*_n is unstable. Then there is a unique exact sequence (see the Appendix)

$$(3.7)_n \qquad 0 \to \mathcal{L}_n \to N^*_n \to \mathcal{M}_n \to 0$$

where $\deg \mathcal{L}_n > \deg \mathcal{M}_n$ (and obviously $\deg N^*_n = \deg \mathcal{L}_n + \deg \mathcal{M}_n$).

We have in (3.1) the blow-up $M_{n+1} \xrightarrow{\alpha_{n+1}} M_n$, and the exact sequence $(3.2)_{n+1}$. Using the definition of C^∞_n, it is easy to prove the following claim:

$$(N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \cong \mathcal{M}_n \text{ and } N^*_{C^\infty_{n+1}|E_{n+1}} \cong \mathcal{L}_n \otimes (\mathcal{M}_n)^{-1}\,.$$

So $(3.2)_{n+1}$ becomes $0 \to \mathcal{M}_n \to N^*_{n+1} \to \mathcal{L}_n \otimes (\mathcal{M}_n)^{-1} \to 0$, and

$$(3.8)_n \qquad \deg \mathcal{M}_n = B_n;\ \deg \mathcal{L}_n = A_n\,.$$

Proposition 3.5. (i) *If $e_{n+1} < B_n$, then $A_{n+1} = B_n$ and $B_{n+1} = A_n - B_n = e_{n+1}$, therefore $e_{n+2} = A_{n+1} - B_{n+1} = B_n - e_{n+1} > 0$.*
(ii) *If $e_{n+1} \geq B_n$, then $B_n \leq B_{n+1}$ and $A_{n+1} \leq A_n - B_n = e_{n+1}$; thus $e_{n+2} \leq e_{n+1} - B_n$.*
(iii) *If $e_{n+1} = B_n$, N^*_{n+1} is semistable.*

PROOF. In the first case, by $(3.2)_{n+1}$, N^*_{n+1} is unstable, hence (by the uniqueness in $(3.7)_{n+1}$)

$$e_{n+1} = \deg \mathcal{M}_{n+1} = B_{n+1}, \qquad B_n = \deg \mathcal{L}_{n+1} = A_{n+1}\,.$$

If $e_{n+1} \geq B_n$ and N^*_{n+1} is unstable, we have the exact sequences

$$(3.2)_{n+1} \qquad 0 \to (N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \xrightarrow{\alpha} N^*_{n+1} \xrightarrow{\beta} N^*_{C^\infty_{n+1}|E_{n+1}} \to 0$$

$$(3.7)_{n+1} \qquad 0 \to \mathcal{L}_{n+1} \xrightarrow{a} N^*_{n+1} \xrightarrow{b} \mathcal{M}_{n+1} \to 0$$

where $\deg \mathcal{L}_{n+1} > \deg \mathcal{M}_{n+1}$.

Recall that if $\gamma : \mathcal{L} \to \mathcal{M}$ is a homomorphism of line bundles on a curve, and $\deg \mathcal{L} > \deg \mathcal{M}$, then $\gamma = 0$, since Hom $(\mathcal{L} \otimes \mathcal{M}) \cong \mathcal{L}^{-1} \otimes \mathcal{M}$ has no non trivial sections.

Suppose $B_n > B_{n+1}$: then by $(3.5)_{n+1}$ and $(3.6)_{n+1}$ we get $A_{n+1} = A_n - B_{n+1} > A_n - B_n$ so that $b \circ \alpha = 0$, $\beta \circ a = 0$, and from this we can prove that

$$0 \to (N^*_{E_{n+1}|M_{n+1}})|_{C^\infty_{n+1}} \xrightarrow{\alpha} N^*_{n+1} \xrightarrow{b} \mathcal{M}_{n+1} \to 0$$

is exact; since $B_n > B_{n+1}$, it is isomorphic to $(3.7)_{n+1}$, hence by $(3.8)_{n+1}$ $A_{n+1} = B_n$, which gives $e_{n+1} = A_n - B_n < A_{n+1} = B_n$, a contradiction.

Moreover, if $e_{n+1} = B_n$, we get $A_n = e_{n+1} + B_n = 2B_n$ so that $2B_n = A_n = A_{n+1} + B_{n+1} \geq A_{n+1} + B_n$. Hence (recall that $e_{n+2} > 0$ because N^*_{n+1} is supposed to be unstable): $B_n \geq A_{n+1} > B_{n+1} \geq B_n$, which is a contradiction. Thus, if $e_{n+1} = B_n$, N^*_{n+1} has to be semistable.

If $e_{n+1} \geq B_n$ and N^*_{n+1} is semistable, it holds $0 \geq e_{n+2} = A_{n+1} - B_{n+1}$. Suppose $B_n > B_{n+1}$: then $A_{n+1} > A_n - B_n = e_{n+1} \geq B_n > B_{n+1}$, which is a contradiction.

REMARK 3.6. If $g = 0$, C is biholomorphic to $\mathbf{P}_1$, hence the conormal bundle $N^*_{C|M}$ is decomposable, and also the conormal bundle of C^∞_n is decomposable:

$$N^*_{C^\infty_n|M_n} \cong O(a_n) \oplus O(b_n), \qquad a_n \geq b_n\,,$$

so $e_{n+1} = a_n - b_n$. Notice that N_n^* is semistable $\Leftrightarrow e_{n+1} = 0$, since the normalized sheaf is $E_n \cong (O(-a_n) \oplus O(-b_n)) \otimes O(b_n)$ whose degree is $b_n - a_n$.

Hence N_n^* is unstable if and only if $a_n > b_n$, and $B_n = -E_{n+1} \cdot C_{n+1}^\infty = b_n$.

PROPOSITION 3.7. *Suppose that* $e_{n+1} \geq 0 \; \forall n \in \mathbb{N}$, *and that*

$$B_n \leq 0 \quad \forall n \quad \text{if} \quad g > 0, \qquad e_{n+1} > 0 \quad \forall n \quad \text{if} \quad g = 0\,.$$

Then in both cases we have:

(i) $B_0 \leq B_1 \leq \ldots \leq B_n \leq \ldots \leq 0$,

(ii) $\forall k \geq 0$, $\forall m \geq 1 : e_{k+m} \leq A_k - mB_k$.

PROOF. If $g = 0$, let us prove that $B_n \leq 0 \; \forall n$.

We may assume, by contradiction, that $B_n > 0$; we need only to check that

$$A_n > A_{n+1} \geq B_{n+1} > 0 \tag{3.9}$$

(this obviously implies that there exists m such that $A_m = B_m$, thus $e_{m+1} = 0$). From Proposition 3.5, if $e_{n+1} < B_n$, we get $A_{n+1} = B_n < A_n$ and $B_{n+1} = e_{n+1} > 0$; if $e_{n+1} \geq B_n$, then $A_{n+1} \leq A_n - B_n < A_n$ and $B_{n+1} \geq B_n > 0$, hence (3.9) is proved.

Thus for every g, $B_n \leq 0 \; \forall n$; this implies that we can never have $e_{n+1} < B_n$, but only $e_{n+1} \geq B_n$, so that by Proposition 3.5 it holds $B_n \leq B_{n+1}$.

It remains to prove: $\forall k \geq 0$, $\forall m \geq 1 : e_{k+m} \leq A_k - mB_k$.

Let us fix $k \geq 0$, and argue by induction on m : $e_{k+m+1} = A_{k+m} - B_{k+m} \leq A_{k+m-1} - 2B_{k+m-1} = e_{k+m} - B_{k+m-1} \leq A_k - mB_k - B_{k+m-1} \leq A_k - (m+1)B_k$.

REMARK. The same proof can be used in the case $g > 0$, $e_{n+1} > 0 \; \forall n$, to give the same results. Hence in particular $e_{n+1} > 0 \; \forall n$ implies that $B_n \leq 0 \; \forall n$.

PROPOSITION 3.8. *For* $n \geq 1$ *it holds:*

(i) $C_{n,n+1}^\infty \sim C_{n+1}^\infty + (e_n - B_n)F_{n+1}$, *with* $e_n - B_n \geq 0$.

(ii) $C_1^\infty - B_0 F_1 \in \alpha_1^* \langle C \rangle$ *in* $V_{\mathbb{R}}^{2,2}(M_1)$, $C_{n+1}^\infty - B_n F_{n+1} \in \alpha_{n+1}^* \langle C_n^\infty \rangle$ *in* $V_{\mathbb{R}}^{2,2}(M_{n+1})$.

PROOF.

(i) By A.2 there exists $b \geq 0$ such that $C^{\infty}_{n,n+1} \sim C^{\infty}_{n+1} + bF_{n+1}$. By definition

$$-B_n = E_{n+1} \cdot C^{\infty}_{n+1} = E_{n+1} \cdot (C^{\infty}_{n,n+1} - bF_{n+1}) = -e_n + b\,,$$

since $E_{n+1 \cdot M_{n+1}} C^{\infty}_{n,n+1} = C^{\infty}_n \cdot C^{\infty}_n = -e_n$; thus $b = e_n - B_n$.

(ii) In fact, $E_1 \cdot (C^{\infty}_1 - B_0 F_1) = 0$, and similarly $E_{n+1} \cdot (C^{\infty}_{n+1} - B_n F_{n+1}) = 0$.

4 – The main result

Let us consider the following hypothesis on (M, C) (compare Proposition 3.7):

$$\text{(H)} \qquad \begin{array}{c} e_{n+1} \geq 0 \;\forall\, n \in \mathbb{N} \;(\textit{this is obvious when } g = 0), \textit{ and} \\ B_n \leq 0 \;\forall\, n \quad \textit{if} \quad g > 0, \;\; e_{n+1} > 0 \;\forall\, n \quad \textit{if} \quad g = 0\,. \end{array}$$

In this case, we prove that the only obstruction to the existence of a Kähler metric on M is the vanishing of the Aeppli class of C (Theorem 4.5), and that M belongs to the class $\mathcal{C}$ if and only if M is Kähler (Corollary 4.6).

Notice that, by Proposition 3.7, condition (H) implies that, if $B_0 = 0$, then $B_n \leq 0 \;\forall\, n$; hence the simplest example where condition (H) is verified is the following:

take a smooth curve C of genus zero in a threefold M, such that $N^*_{C|M} \cong O(0) \oplus O(2)$ (as a matter of fact, if $N^*_{C|M} \cong O(0) \oplus O(k)$ with $k < 2$, then M is Kähler by Corollary 2.8). Recall that, if

$$0 \to O(a) \to O(a') \oplus O(b') \to O(b) \to 0$$

is an exact sequence of bundles on $\mathbf{P}_1$, then $\{a', b'\} = \{a+p, b-p\}$, where $0 \leq p \leq (b-a)/2$. By applying this result to the sequence in $(3.2)_1$, we get that $N^*_{C^{\infty}_1|M_1} \cong O(0) \otimes O(2)$ or $N^*_{C^{\infty}_1|M_1} \cong O(1) \otimes O(1)$. Take the case when $N^*_{C^{\infty}_n|M_n} \cong O(0) \otimes O(2)$ for every n; here, $e_{n+1} = 2 \;\forall\, n$, so that our results apply to (M, C).

About this example, Reid ([14], page 165) notes moreover that, if there is a neighborhood of C which is an open subset of a compact Moishezon manifold, then the curve moves in M; therefore, by Corollary 2.7, M

is Kähler. Notice that, if M itself is Moishezon, we get the same result by our Corollary 4.6, since Moishezon manifolds belong to the class $\mathcal{C}$.

The idea of the proof of Theorem 4.5 is the following: given a closed Kähler form ω on $M - C$, in order to apply Theorem 2.6, we have to calculate $\omega \cdot C$, using Theorem 2.1. We get as usual $\omega \sim \varphi_k \geq \gamma - \lambda_k u$ and thus

$$\omega \cdot C \geq \int_C \gamma - n(\omega, C) \int_C u\,;$$

therefore we must pay attention to the Lelong number $n(\omega, C)$ because the last term is negative if $n(\omega, C) \neq 0$.

To overcome this difficulty, we blow-up C; in M_1 we consider a closed positive form ω_1 (see Definition 4.3) which coincides with $\alpha_1^*\omega$ in $M_1 - E_1$ and has vanishing Lelong numbers almost everywhere in E_1 (but of course ω_1 may have non-vanishing Lelong numbers on some curve of E_1); then if $n(\omega_1, C_1^\infty) \neq 0$ we blow-up again, and so on. Thus we shall estimate $\omega \cdot C$ by means of a suitable sequence $\{\omega_n\}_{n\geq 1}$ of closed positive currents such that their Lelong numbers on the exceptional divisor E_n go to zero as n goes to infinity.

Lemma 4.1. *Suppose* (H) *and consider the line bundle on E_n given by $\mathcal{L} = 2C_n^\infty + (3e_n + 2g + 1)F_n$. Then both $\mathcal{L}$ and $TM_{n|E_n} \otimes \mathcal{L}$ are ample.*

Proof. $\mathcal{L}$ is ample by A.5. Consider now the exact sequence of bundles on E_n:

$$0 \to TE_n \otimes \mathcal{L} \to TM_{n|E_n} \otimes \mathcal{L} \to N_{E_n|M_n} \otimes \mathcal{L} \to 0\,. \tag{4.1}$$

A routine computation gives: $N_{E_n|M_n} \sim -C_n^\infty + (-B_{n-1} - e_n)F_n$, therefore $N_{E_n|M_n} \otimes \mathcal{L} \sim C_n^\infty + (2e_n - B_{n-1} + 2g + 1)F_n$ is ample.

Consider also the exact sequence

$$0 \to \operatorname{Ker} \alpha_{n*} \to TE_n \to \alpha_n^* TC_{n-1}^\infty \to 0 \tag{4.2}$$

where $\alpha_n^* TC_{n-1}^\infty \sim aC_n^\infty + bF_n$, with $a = 0$ and $b = 2 - 2g$, since $a = \alpha_n^* TC_{n-1}^\infty \cdot F_n = 0$, and $b = bF_n \cdot C_n^\infty = \alpha_n^* TC_{n-1}^\infty \cdot C_n^\infty = TC_{n-1}^\infty \cdot C_{n-1}^\infty = -K_{C_{n-1}^\infty} \cdot C_{n-1}^\infty = 2 - 2g$. And also

$$\operatorname{Ker} \alpha_{n*} \sim aC_n^\infty + bF_n, \text{ with } a = 2 \text{ and } b = e_n\,,$$

in fact by (4.2)

$$-K_{E_n} = TE_n \wedge TE_n \cong \operatorname{Ker} \alpha_{n*} \otimes \alpha_n^* TC_{n-1}^\infty \text{ but also } K_{E_n} = \\ = -2C_n^\infty + (-e_n + 2g - 2)F_n \,.$$

This gives that both $\operatorname{Ker} \alpha_{n*} \otimes \mathcal{L} \sim 4C_n^\infty + (4e_n + 2g + 1)F_n$ and $\alpha_n^* TC_{n-1}^\infty \otimes \mathcal{L} \sim 2C_n^\infty + (3e_n + 3)F_n$ are ample by A.5, therefore also $TE_n \otimes \mathcal{L}$ is ample, and we get the thesis from (4.1).

Proposition 4.2. *If* (H), *then for every* $n \in \mathbb{N}$ *there exist a hermitian metric* h_n *on* $O_{TM_n}(1)$ *and a hermitian metric on* M_n *with Kähler form* u_n *such that:*

$$\text{(i)} \qquad c_1(O_{TM_n}(1), h_n) + \pi^* u_n > 0$$

$$\text{(ii)} \qquad \int_{C_n^\infty} u_n = e_n + 2g + 1.$$

The proof of this Proposition is the same as that of Proposition 7.2 of [2] given in the case $g = 0$, the point is only to use $\mathcal{L} = 2C_n^\infty + (3e_n + 2g + 1)F_n$ instead of $2C_n^\infty + (3e_n + 1)F_n$; hence we don't give it here.

Let us consider on each threefold M_n a closed positive current ω_n as follows: take the Kähler form ω of a Kähler metric on $M - C$, extended as a positive current on the whole of M (this can be done because $\dim C < 2 =$ bidimension of ω, see [7]). Let $c_0 := n(\omega, C) = n(\alpha_1^*\omega, E_1)$ (Proposition 2.4) and consider the current $\omega_1 := \alpha_1^*\omega - c_0[E_1]$; ω_1 is positive because $n(\alpha_1^*\omega, E_1) = \chi_{E_1}\alpha_1^*\omega$ ([15], Proposition 12.3), and is smooth on $M_1 - E_1$. Let us now give an inductive definition:

Definition 4.3. Let ω be the Kähler form of a Kähler metric on $M - C$, extended as a positive current on the whole of M; define

$$\omega_1 := \alpha_1^*\omega - c_0[E_1] \qquad \text{where } c_0 := n(\omega, C)\,,$$
$$\omega_{n+1} := \alpha_{n+1}^*\omega_n - c_n[E_{n+1}] \qquad \text{where } c_n := n(\omega_n, C_n^\infty)\,.$$

Notice that ω_n is a closed positive current on M_n, smooth on $M_n - \cup_{k=1}^n E_{k,n}$.

Proposition 4.4.

a) *The sequence* $\{c_n\}$ *given by* $c_n = n(\omega_n, C_n^\infty)$ *is not increasing.*

b) *The sequence* $\{c_n\}$ *satisfies:* $\sum_{n=0}^{\infty} c_n^2 < \infty$.

Proof. a) First of all, $\omega_n \cdot F_n = (\alpha_n^* \omega_{n-1} - c_{n-1}[E_n]) \cdot F_n = -c_{n-1} E_n \cdot F_n = c_{n-1}$. Thus (see 3.2)

$$\omega_n \cdot F_{n-1,n} = (\alpha_n^* \omega_{n-1} - c_{n-1}[E_n]) \cdot F_{n-1,n} = \omega_{n-1} \cdot F_{n-1} - c_{n-1} = \\ = c_{n-2} - c_{n-1} \,.$$

All fibres $F_{n-1,n}$ on $E_{n-1,n} \cong E_{n-1}$ are homologous one to each other, hence we can suppose that ω_n has vanishing Lelong numbers on $F_{n-1,n}$; by Theorem 2.1 applied to ω_n with $\gamma = 0$ we get $\omega_n \cdot F_{n-1,n} = \phi_k \cdot F_{n-1,n} \geq -\lambda_k u \cdot F_{n-1,n} \underset{k\to\infty}{\longrightarrow} 0$, so $\omega_n \cdot F_{n-1,n} \geq 0$.

The proof of b) is involved but very similar to that of Proposition 7.6 of [2], hence we remaind the reader to that paper.

So we can give our main Theorem in this section.

Theorem 4.5. *Let* M *be a compact complex threefold, which is Kähler outside a smooth curve* C *of genus* g. *Suppose* (H); *then* M *is Kähler if and only if* C *is not the* $(1,1)$-*component of a boundary.*

Proof. For every $n \geq 0$, let us choose in $O_{TM_n}(1)$ a hermitian metric h_n, and a hermitian metric on M_n whose Kähler form u_n satisfy Proposition 4.2.

Let us consider the current ω given by the extension of a closed Kähler form on $M - C$, and let ω_n on M_n be defined as in 4.3. Then it holds for every $n \geq 1$

$$\omega \cdot C = -B_0 c_0 - \ldots - B_{n-1} c_{n-1} + \omega_n \cdot C_n^\infty \,. \tag{4.3}$$

In fact, by Proposition 3.8 (ii),

$$\omega \cdot C = \alpha_1^* \omega \cdot \alpha_1^* C = (\omega_1 + c_0[E_1]) \cdot (C_1^\infty - B_0 F_1) = \omega_1 \cdot C_1^\infty - B_0 \omega_1 \cdot F_1 = \\ = -B_0 c_0 + \omega_1 \cdot C_1^\infty$$

since $\omega_1 \cdot F_1 = (\alpha_1^* \omega - c_0[E_1]) \cdot F_1 = c_0$. This proves (4.3) for $n = 1$. Assume (4.3): then

$$\begin{aligned}\omega_n \cdot C_n^\infty &= \alpha_{n+1}^* \omega_n \cdot \alpha_{n+1}^* C_n^\infty = (\omega_{n+1} + c_n[E_{n+1}] \cdot (C_{n+1}^\infty - B_n F_{n+1}) = \\ &= -B_n c_n + \omega_{n+1} \cdot C_{n+1}^\infty\end{aligned}$$

because $\omega_{n+1} \cdot F_{n+1} = (\alpha_{n+1}^* \omega_n - c_n[E_{n+1}]) \cdot F_{n+1} = c_n$.

By using Theorem 2.1 applied to the current ω_n with $\gamma = 0$, we get $\omega_n \cdot C_n^\infty \geq -\int_{C_n^\infty} \lambda_k u_n$, hence, if $k \to \infty$, we obtain $\omega_n \cdot C_n^\infty \geq -n(\omega_n, C_n^\infty) \int_{C_n^\infty} u_n = -c_n(e_n + 2g + 1)$.

By Proposition 3.7, $B_n \leq 0$ and there exists an index k such that $B_k = B_{k+n}$ for every n, therefore: $\omega \cdot C \geq -B_0 c_0 - \ldots - B_{n-1} c_{n-1} - c_n(e_n + 2g + 1) = -B_0 c_0 - \ldots - B_k(c_k + \ldots + c_{k+n-1}) - c_{k+n}(e_{k+n} + 2g + 1)$, but $e_{k+n} \leq A_k - nB_k$, thus

$$\begin{aligned}\omega \cdot C \geq -B_0 c_0 - \ldots - B_{k-1} c_{k-1} - B_k(c_k + \ldots + c_{k+n-1}) + \\ - c_{k+n}(A_k - nB_k + 2g + 1) \geq -B_0 c_0 - \ldots - B_{k-1} c_{k-1} + \\ - B_k n(c_{k+n-1} - c_{k+n}) - c_{k+n}(A_k + 2g + 1)\,.\end{aligned}$$

Since $c_{k+n-1} - c_{k+n} \geq 0$ and the last term goes to zero, as $n \to \infty$ (Proposition 4.4), it follows $\omega \cdot C \geq 0$. We conclude by Theorem 2.6.

COROLLARY 4.6. *If M and C satisfy* (H), *but M is not Kähler, then:*

(i) $K_M \cdot C = 0$
(ii) $\forall\, n \in \mathbb{N}, M_n$ *is not Kähler*
(iii) M *cannot belong to the class $\mathcal{C}$.*

PROOF. (i) By Theorem 4.5, C is the (1,1)-component of a boundary.
(ii) By Proposition 3.8 (ii) we get that

$$C_n^\infty - B_0 F_{1,n} - \ldots - (B_0 + \ldots + B_{n-1}) F_n \in (\alpha_1 \circ \ldots \circ \alpha_n)^* \langle C \rangle\,,$$

but this class vanishes, so that we have on M_n a positive current which it the (1,1)-component of a boundary.
(iii) A compact manifold M is bimeromorphic to a Kähler manifold if and only if there exists a modification $f : M' \to M$ such that M' is

Kähler ([16]). Suppose we are in this case, and call θ a closed Kähler form on M'; take $\omega := f_*\theta$ and define ω_n on M_n as in Definition 4.3. Since we can always find a smooth, positive definite (1,1)-form γ on M such that $\omega \geq \gamma$, we get also $\omega_n \geq (\alpha_1 \circ \ldots \circ \alpha_n)^*\gamma$.

Claim. If $n(\omega, C) \neq 0$, there exist $k \in \mathbb{N}$ such that $n(\omega_k, C_k^\infty) = 0$.

The proof of this claim is based on a local analysis of the modification f, considered as a sequence of blow-ups with smooth centers: for details, see [2], Theorem 7.13.

Now, (4.3) give

$$\omega \cdot C = -B_0 c_0 - \ldots - B_{k-1} c_{k-1} + \omega_k \cdot C_k^\infty, \text{ and}$$
$$\omega_k \cdot C_k^\infty \geq \int_C \gamma - n(\omega_k, C_k^\infty) \int_{C_k^\infty} u_k = \int_C \gamma > 0\,,$$

which gives $\omega \cdot C > 0$, i.e. M is Kähler.

Corollary 4.7. (i) *If M is Kähler outside a smooth* rational *curve, and no E_j is biholomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$, then M is Kähler if and only if C is not the $(1,1)$-component of a boundary.* (See also [2] Theorem 7.10).
(ii) *If M is Kähler outside a smooth elliptic curve, and $A_0 + B_0 < 0$, then M is Kähler by Corollary 2.8. If $A_0 + B_0 \geq 0$, suppose $e_{n+1} \geq 0$ and no B_n is positive; then M is Kähler if and only if C is not the $(1,1)$-component of a boundary. This happens in particular if $e_{n+1} > 0\ \forall\, n$, or if there is k such that $e_{k+1} = e_{k+2} = 0$, as the following lemma shows.*

Lemma 4.8. *Let C be an elliptic curve, and let us suppose that $A_0 + B_0 \geq 0$, $e_{n+1} \geq 0\ \forall\, n$ and that $e_{k+1} = 0$. Then $A_k = B_k$ and $B_{k+1} = 0$. If moreover $e_{k+2} = 0$, then $(A_n, B_n) = (0,0)\ \forall\, n \in \mathbb{N}$.*

Proof. Let us prove by induction that $A_n + B_n \geq 0\ \forall\, n$. Since $A_n \geq B_n$ because $e_{n+1} \geq 0$, we get $2(A_{n+1} + B_{n+1}) = 2A_n \geq A_n + B_n \geq 0$.

Obviously, $A_k = B_k \geq 0$. By Proposition 3.8 (i), $B_{k+1} \leq e_{k+1} = 0$. If $B_{k+1} < 0$, we get a contradiction: if $B_k > 0$, by Proposition 3.5 (i) we would get $B_{k+1} = e_{k+1} = 0$; if $B_k = 0$, by Proposition 3.5 (iii) we would get $e_{k+2} = 0$, hence $A_{k+1} = B_{k+1} < 0$.

If moreover $e_{k+2} = 0$, then $A_{k+1} = B_{k+1} = 0$, hence also $A_k = B_k = 0$, since $A_k = A_{k+1} + B_{k+1}$.

Now $e_{k+2} = 0$ implies $B_{k+2} = 0$, so that $A_{k+2} = A_{k+1} - B_{k+2} = 0$, and $e_{k+3} = 0$, and so on. On the other hand, remark that $B_{k-1} \le e_k$, because $e_{k+1} = 0$ (Proposition 3.5 (i)); hence $B_{k-1} \le B_k = 0$; on the other hand, $A_{k-1} = A_k + B_k = 0$, hence $B_{k-1} = 0$ and $e_k = 0$ and so on.

By Corollary 4.7 (i), it seems interesting to look at this question: What about a rational curve with E_j biholomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$? This case is discussed in [2] chapter 8.

5 – Some results

Let us collect here some results on the existence of a Kähler metric on M in terms of the numbers $K_{M|C} \cdot C$ and $B_0 = -E_1 \cdot C_1^\infty$. Since if $K_M \cdot C < 0$, then M is Kähler, let us assume from now on that $K_M \cdot C \ge 0$.

Proposition 5.1. *If $B_0 \le 0$, then M is Kähler if and only if M_1 is Kähler.*

Proof. Suppose that M_1 is Kähler but M is not Kähler: hence by Theorem 2.5, C is the (1,1)-component of a boundary or C is part of the (1,1)-component of a boundary. In the first case, by Proposition 3.8 (ii) it holds $C_1^\infty - B_0 F_1 \in \alpha_1^* \langle C \rangle = 0$, which is impossible because M_1 is Kähler. If $C + S$ is the (1,1)-component of a boundary, take a closed Kähler form θ on M_1 and call $\omega := \alpha_{1*}\theta$. By Theorem 2.6, $\omega \cdot C < 0$, but

$$\omega \cdot C = \alpha_{1*}\theta \cdot C = \theta \cdot \alpha_1^* C = \theta \cdot (C_1^\infty - B_0 F_1) = \int_{C_1^\infty} \theta - B_0 \int_{F_1} \theta \,,$$

which is positive

Proposition 5.2. a) *If $B_0 > 0$ and $K_M \cdot C = 0$, then $g = 0$, $A_0 = B_0 = 1$, M_1 is Kähler.*
b) *If $B_0 > 0$ and $K_M \cdot C > 0$, then it holds $K_{M_1} \cdot C_1^\infty \ge 0$; if in particular $K_{M_1} \cdot C_1^\infty = 0$, then we have only three possibilities: $g = 0$ and $e_1 \in \{0, 1\}$, or $g = 1$, $e_1 = -1$, $A_0 = 0$, $B_0 = 1$.*

Proof. a) $\deg N_0^* = 2-2g = A_0+B_0 = e_1+2B_0$: hence e_1 is even. If $e_1<0$, we have $e_1 \geq -g$, so that $2-2g \geq -g+2B_0$, that is, $2-g \geq 2B_0 \geq 2$, thus $g=0$, which is impossible. If $e_1 \geq 0, 2-2g \geq 2B_0 \geq 2$, hence $g=0$, which gives $e_1=0$, $A_0=B_0=1$.
b) Notice that M is Kähler if and only if C is not part of the (1,1)-component of a boundary. Moreover, $K_{M_1}\cdot C_1^\infty = 2g-2+B_0+e_1$: in fact, $K_{E_1} \sim -2C_1^\infty + (2g-2-e_1)F$, so that

$$K_{E_1}\cdot C_1^\infty = 2e_1+2g-2-e_1 = e_1+2g-2 = K_{M_1}\cdot C_1^\infty + E_1\cdot C_1^\infty = \\ = K_{M_1}\cdot C_1^\infty - B_0\,.$$

If $K_{M_1}\cdot C_1^\infty < 0$, we get $0<B_0<2-2g-e_1$, hence $2g+e_1 \leq 0$ but if $e_1<0$, this implies $2g \leq -e_1 \leq g$, so that $g=0$, a contradiction; if $e_1 \geq 0$, the only possibility is $g=0$, $e_1=0$, hence $0<B_0<2$, so that $A_0=B_0=1$, but $A_0+B_0>2-2g$, a contradiction.

If in particular $K_{M_1}\cdot C_1^\infty = 0$, it holds $0<B_0=2-2g-e_1$, hence $2g+e_1 \leq 1$. If $e_1 \geq 0$, then $g=0$ and $e_1 \in \{0,1\}$, that is, $B_0=A_0=2$ in the first case, and $B_0=1$, $A_0=2$ in the second case. If $e_1<0$, we get $2g-1 \leq -e_1 \leq g$, so $g \leq 1$; the only possibility is $g=1$, $e_1=-1$, $B_0=1$, $A_0=0$.

Remark 5.3. A natural question is the following: to which extent do the numbers A_0, B_0, $K_M\cdot C$, ... determine the global geometric situation? Let us check it in the simplest non trivial situation, that given in Proposition 5.2 a): $g=0$, $A_0=B_0=1$, where M_1 is Kähler. In this case the normal bundle to C in M is weakly negative and C has an open neighborhood which is biholomorphic to an open neighborhood of the zero section of the normal bundle $N_{C|M}$ ([11] Theorem 3.5).

If this is also the global situation (i.e. M is the total space of $O(-1)\oplus O(-1)$ and C is the zero section) (to have a compact manifold, take the projectivization of the fibers), then M is Kähler. In the example of Hironaka ([6], p. 444) $g=0$, $A_0=B_0=1$ but M is not Kähler, because C is a boundary (hence, a (1,1)-component of a boundary). A third example shows that also the remaining case in Theorem 2.5 (i.e. M is not Kähler because C is part of the (1,1)-component of a boundary) happens.

Let Q be a conic on a projective plane π in $\mathbf{P}_3$ and choose a straight line R which intersects Q in two distinct point: p' and p''. Blow-up $\mathbf{P}_3$

with center R getting a projective threefold X; then blow-up X with center the proper transform of Q, getting again a projective threefold M'. The fibers of p' and p'' in X have as proper transforms in M' two smooth rational curves C' and C''. They are homologous one to each other and $N_{C'|M'} \cong N_{C''|M'} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Now it is easy to check that blowing-up M' with center C', we get a projective threefold $M_1 \xrightarrow{\beta} M'$ and that there exists a contraction $M_1 \xrightarrow{\alpha_1} M$ onto a smooth threefold such that the exceptional divisor E_1 of β is contracted to a smooth rational curve C on M with normal bundle $N_{C|M} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $C \sim -\alpha_1(\beta^{-1}(C''))$.

$$\begin{array}{ccccc} M' & \longrightarrow & X & \longrightarrow & \mathbf{P}_3 \\ {\scriptstyle\beta}\uparrow & & & & \\ M_1 & \xrightarrow{\alpha_1} & M & & \end{array}$$

Thus $M - C$ is Kähler and $\alpha_1(\beta^{-1}(C'') + C$ is a positive current on M which is the (1,1)-component of a boundary.

This proves that in general the global situation cannot be determined simply by looking at numerical information on the curve C, nor at a neighborhood of it.

Proposition 5.4. *Let us suppose that M_1 is Kähler. Then M is Kähler too if one of the following conditions holds* (*and obviously also when $K_M \cdot C < 0$*)*:*

(i) $B_0 \leq 0$

(ii) $e_1 > 0$ *and* $K_M \cdot C < 2g + 1$.

(iii) $\deg N_0^* < 0$

Proof. (i) By Proposition 5.1. (ii) In fact, if $B_0 > 0$, then $A_0 = e_1 + B_0 \geq 2$ and we get a contradiction: $3 \leq A_0 + B_0 = K_M \cdot C + 2 - 2g$. (iii) Since $0 > A_0 + B_0 = e_1 + 2B_0$, there exists $\varepsilon > 0$ such that $B_0 + e_1/2 + \varepsilon < 0$.

Call $D := C_1^\infty + (e_1/2 + \varepsilon)F_1 : D$ is ample by A.5 and $D.D = 2\varepsilon > 0$. Hence nD is effective for $n >> 0$ (see [6] p. 363). Let θ be a Kähler form on M_1: then

$$\begin{aligned}\alpha_{1*}\theta \cdot nC &= \alpha_1^*(\alpha_{1*}\theta) \cdot n\alpha_1^* C = \theta \cdot n(C_1^\infty - B_0 F_1) = \\ &= \theta \cdot n(C_1^\infty + (e_1/2 + \varepsilon)F_1 - (e_1/2 + \varepsilon + B_0)F_1) = \\ &= \theta \cdot nD - \theta \cdot n(e_1/2 + \varepsilon + B_0)F_1 > 0\,,\end{aligned}$$

thus $\alpha_{1*}\theta \cdot C > 0$, so that M is Kähler by Theorem 2.6

These results can be seen as a generalization of some results in [13], where the projective and Moishezon case is studied.

– Appendix: Ruled surfaces

Let $\pi : E \to C$ be a ruled surface, i.e. C is a smooth curve, E is a smooth surface and every fibre $F \cong \mathbf{P}_1$. We shall refer to [6], ch. V, § 2.

A.1. Any ruled surface is of the form $\mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank two on C; moreover, if $\mathcal{E}$ and $\mathcal{E}'$ are two locally free sheaves of rank two on C, then $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E}')$ are isomorphic ruled surfaces if and only if there exists an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Hence, if $\pi : E \to C$ is a ruled surface, there exists a locally free sheaf $\mathcal{E}$ of rank two on C such that $E \cong \mathbf{P}(\mathcal{E})$ and $H^0(\mathcal{E}) \neq 0$, but $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. In this case, we shall say that $\mathcal{E}$ is normalized. This does not determine $\mathcal{E}$ uniquely, but it determines $e := -\deg \mathcal{E}$, which is an invariant of the ruled surface E.

A.2. Let us fix a section $\sigma : C \to E$ such that its image C^∞ satisfies $\mathcal{L}(C^\infty) = O_E(1)$: then C^∞ has minimal self-intersection among all sections; if moreover F is a generic fibre of $\pi : E \to C$, it holds

$$\begin{cases} C^\infty \cdot C^\infty = -e \\ F \cdot F = 0 \\ C^\infty \cdot F = 1 \end{cases}.$$

We can choose the classes of C^∞ and F as generators of $\operatorname{Pic} E$ and of $\operatorname{Num} E$; but also of the Aeppli group $V^{1,1}_{\mathbb{R}}(E) = H^2(E, \mathbb{R})$ (see [17]: recall that E is Kähler). Hence we shall write in general $C \sim C'$. Notice that if $\sigma' : C \to E$ is a section with image C', it holds: $C' \sim C^\infty + bF$, with $b \geq 0$. In fact, since $\pi(C^\infty) = C = \pi(C')$, the coefficient of C^∞ is 1; moreover $C^\infty \cdot C^\infty \leq C' \cdot C' = C^\infty \cdot C^\infty + 2b$.

A.3. A locally free sheaf $\mathcal{E}$ on a curve C is said to be semistable if for every locally free sheaf $\mathcal{F}, \mathcal{F} \neq 0$, $\mathcal{F} \neq \mathcal{E}$, such that $\mathcal{E} \to \mathcal{F} \to 0$, we have $(\deg \mathcal{F})(rk\mathcal{E}) \geq (\deg \mathcal{E})(rk\mathcal{F})$. Since any locally free sheaf of rank two on a curve C is an extension of invertible sheaves, for such an $\mathcal{E}$ we get that:

a) if $\mathcal{E}$ is semistable, for every exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on C, it holds $\deg \mathcal{L} \leq \deg \mathcal{M}$;

b) if $\mathcal{E}$ is unstable, then there exists a unique (up to isomorphisms) exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on C and $\deg \mathcal{L} > \deg \mathcal{M}$.

A.4. If $\mathcal{E}$ is a normalized locally free sheaf of rank two on C, then $\mathcal{E}$ is semistable if and only if $\deg \mathcal{E} \geq 0$. In particular, let M be a compact complex threefold and C a smooth curve of genus g on M; consider $N^*_{C|M} \cong I_C/I_C^2$, and normalize it by tensoring with a suitable invertible sheaf $\mathcal{L}$. Call $\mathcal{E}_0$ the normalized sheaf of rank two on C, and define $e_1 := -\deg \mathcal{E}_0$. Then the ruled surface $E_1 \to C$ given by $E_1 := \mathbf{P}(\mathcal{E}_0)$ is nothing but the exceptional divisor of the blow-up $\alpha : M' \to M$ of M along C, ([5] p. 604) and:

$$\begin{gathered} N_{C|M} \text{ is unstable } \Leftrightarrow N^*_{C|M} \text{ is unstable } \Leftrightarrow \mathcal{E}_0 \text{ is unstable} \\ \Leftrightarrow e_1 > 0 \Leftrightarrow \text{ there exists on } E_1 \text{ a unique section } C_1^\infty \\ \text{with negative self-intersection, more precisely, } C_1^\infty \cdot C_1^\infty = -e_1 \,. \end{gathered}$$

This section C_1^∞ corresponds to a surjective map $\mathcal{E}_0 \to \mathcal{F} \to 0$, where $\mathcal{F}$ is an invertible sheaf on C, such that, if $\sigma(C) = C_1^\infty$,

$$0 \to O_C \to \mathcal{E}_o \to \mathcal{F} \to 0 \text{ and } \mathcal{F} = \sigma^*(\mathcal{L}(C_1^\infty) \otimes O_{C_1^\infty}) \,.$$

Proposition A.5. *Let $\pi : E \to C$ be a ruled surface.*

a) *Suppose* $D \equiv aC^\infty + bF$ *is an irreducible curve on* $E, D \neq F$ *and* $D \neq C^\infty$*: if* $e \geq 0$*, then* $a > 0$ *and* $b \geq ae$*; if* $e < 0$*, then either* $a = 1$ *and* $b \geq 0$ *or* $a \geq 2$ *and* $b \geq (ae)/2$.

b) *Suppose* $D \equiv aC^\infty + bF$ *is a divisor on* E*: if* $e \geq 0$*,* D *is ample if and only if* $a > 0$ *and* $b > ae$*; if* $e < 0$*,* D *is ample if and only if* $a > 0$ *and* $b > (ae)/2$.

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