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A geometry for Lagrangians on the bundle $\mathbb{R}^n \times T^1_n\mathcal{M}$

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RIASSUNTO: *Questo lavoro tratta di alcune strutture geometriche del fibrato di getti $\mathbb{R}^n \times T^1_n\mathcal{M}$, con il modello geometrico di fibrato tangente non-autonomo. Si costruiscono particolari campi vettoriali che vengono confrontati con il campo tensoriale $\mathbb{K}$, associato alla connessione di Ehresmann. Il sistema di equazioni alle derivate parziali, associato al tensore $\mathbb{K}$ è allora ricondotto al sistema di Eulero-Lagrange per una lagrangiana regolare definita sullo spazio di getti.*

ABSTRACT: *This article is concerned with some geometric structures of the jet-bundle $\mathbb{R}^n \times T^1_n\mathcal{M}$, modelled by the non autonomous tangent bundle geometry. We use this fact to construct similar objects like a particular family of vector fields and we study its relationship with a tensor field $\mathbb{K}$, which defines a type of Ehresmann connection. The integration of the system of partial differential equations associated to $\mathbb{K}$ is then accomplished via the integration of the corresponding Euler-Lagrange equations for a regular Lagrangian defined on this jet bundle.*

1 – Introduction

One of the interesting aspects of the modern formulation of Classical Mechanics is its coordinate-free presentation due to the underlying natural structures: the *symplectic structure* of the Hamiltonian formalism and the so-called *almost tangent structure* of the Lagrangian counterpart. They allowed an elegant exposition as well as a better understanding of

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the contents of the analytical formalism and now this formalism can be considered a topic of Symplectic Geometry. So it would not be an exaggeration to call it *Symplectic Mechanics*.

Further progress in the study of these canonical structures have been made and modern techniques of Fiber Bundles and Differential Geometry has been carried out to generalize the ordinary intrinsic approach to cover also field theories (see the books [17], [24] and [26] and references therein). There are some alternative ways of proceeding with these generalizations and, as far as we know, some of them are known in the literature as: the *multisymplectic* approach [2], [8], [9], the *polysymplectic* approach [11], the *k-symplectic* approach [1], and the *almost k-tangent* and *k-cotangent* approaches [14], [15], the latter being a generalization of the polysymplectic and k-symplectic cases (however in an appropriate setting, polysymplectic and k-symplectic structures turn out to be essentially equivalent and can be identified with an integrable k-almost cotangent structure [13]).

Concerning the tangent bundle geometry many independent publications have appeared since the early works by Klein [12] and generalizations recovering Euler-Lagrange equations for Lagrangian functions in several independent variables were also published (references could be found in [24], [26], [28] for instance).

One of the purposes of this article is to retake the subject for the case where the geometry can be modelled by the canonical tangent bundle geometry of a finite dimensional manifold $\mathcal{M}$. This means that we will be in the context of some geometric objects underlying the bundles $T\mathcal{M}$ as well as $\mathbb{R} \times T\mathcal{M}$ that can be extended to the bundle $\mathbb{R}^n \times T^1_n\mathcal{M}$, where $T^1_n\mathcal{M}$ is the first order jet bundle of smooth maps $f\colon \mathbb{R}^n \to \mathcal{M}$ at the origin. This latter manifold is known as the *tangent bundle of the* $\mathbf{n}$*–velocities* (this terminology is due to Ehresmann [6]). Thus a first task is the study of the parallelism which exists between the almost-tangent geometry characterization of ordinary differential equations of second order (SODE, semispray [17], [26], [28]) and second order partial differential equations. Roughly speaking this relation comes from the fact that "the bundle of 1-jets are a Whitney sum of tangent bundles", i.e.

$$[T^1_n\mathcal{M}] \overset{\text{diffeo}}{\longleftrightarrow} \underbrace{[T\mathcal{M} \oplus_{\mathcal{M}} \cdots \oplus_{\mathcal{M}} T\mathcal{M}]}_{n}. \tag{1}$$

The knowledge of the subject is of geometric interest. For example, it is possible to extend the study of Crampin-Thompson [5], [27] and Nagano [22] (*when a manifold is the tangent bundle of some manifold*?) to the global question of when a manifold is the bundle $T_n^1 Q$ of some manifold Q, Merino [20]. Also, as we shall see, it is possible to establish a relationship among connections with respect to the fibration $\rho_0^1 : \mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$, jet fields and partial differential equations. Indeed the second purpose of the work is to give an application of the subject to the study of connections on $\mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$.

Connections on fibred manifolds is an useful geometric strategy to study partial differential equation systems (for instance, the integrability of the horizontal distribution defined by a connection is equivalent to the integrability of the correspondent system, like integrability of ordinary differential equations systems are geometrically characterized by the corresponding vector field).

A fruitful result of such relationship is that we can characterize integral solutions of some partial differential equations as extremals of the Euler-Lagrange equations. Indeed it is possible to construct a certain type of connection (called *field connection*) on ρ_0^1: $\mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$ such that there is canonically associated a class of vector fields, the *associated semisprays*, whose integrals are by definition the solutions of the connection. We shall show (Theorem 4.1) that if we give an arbitrary family of semisprays on $\mathbb{R}^n \times T_n^1\mathcal{M}$ then there is a field connection such that the original family is precisely the associated field of semisprays of the connection. The use of a general result due to D. Saunders gives the relationship of this abstract situation to the Euler-Lagrange equations, relating its integrals solutions, (assuming that integrability conditions are verified, of course). In fact the extremals can be characterized by integral manifolds of the associated horizontal distribution of $\mathbb{K}$. Thus it suffices to establish a relationship between Lagrangian functions and $\mathbb{K}$ (see Proposition 4.3).

Finally, it remains to observe that our study is strongly inspired in the tangent bundle geometry and restricted to the case of partial differential equations related to Lagrangians depending explicitly on a set of real independent variables. A more general study for the investigation of connections as higher order partial differential equations on a general fibred manifold $(\mathbf{E}, \pi, N)$ needs a more sophisticated treatment as it was shown by A. Vondra in his monograph [28].

The work is structured as follows. In § 2 we give some definitions for clearness and support for the next sections. The section § 3 is concerned with the geometry of $T^1_n\mathcal{M}, T^1_n[T^1_n\mathcal{M}]$ and field connections. An illustration of the preceding study is the object of §4 and a simple example is given at the end of the section.

All objects considered throughout the paper as manifolds, mappings, forms, vector fields, etc are of C^∞ class. The manifolds are finite dimensional, Hausdorff, paracompact, etc. The summation convention on repeated indices is adopted. We would like to acknowledge M. de León for suggestions during the preparation of the first draft.

2 – Field of n-vectors

2.1 – Preliminaries

Let $\mathcal{M}$ be a differentiable manifold of dimension m, $\mathbb{R}^n$ the n-dimensional Euclidean space with coordinates $x = (x_i) = (x_1, \cdots, x_n)$ and $J^1(\mathbb{R}^n, \mathcal{M})$ the $n + (n+1)m$-dimensional manifold of one-jets from $\mathbb{R}^n$ to $\mathcal{M}$, with elements denoted by $\tilde{f}^1(x)$, or $j^1_x f$, with the following canonical bundle structures: the *source projection* $\alpha^1 : J^1(\mathbb{R}^n, \mathcal{M}) \to \mathbb{R}^n$; $\alpha^1(\tilde{f}^1(x)) = x$; the *target projection* $\beta^1 : J^1(\mathbb{R}^n, \mathcal{M}) \to \mathcal{M}$; $\beta^1(\tilde{f}^1(x)) = f(x)$ and the *source-target projection* $\rho^1_0 : J^1(\mathbb{R}^n, \mathcal{M}) \to \mathbb{R}^n \times \mathcal{M}$; $\rho^1_0(\tilde{f}^1(x)) = (x, f(x))$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathcal{M}$ is a smooth mapping.

We recall that one jets of mappings from $\mathbb{R}^n$ to $\mathcal{M}$ may be identified with one jets of sections of the fibration π: $\mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^n$ since its sections are graphs of maps from $\mathbb{R}^n$ to $\mathcal{M}$. Thus, if $J^1\pi$ is the corresponding first order jet manifold of all sections of $\pi : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^n$ then the identification of $J^1\pi$ with $J^1(\mathbb{R}^n, \mathcal{M})$ gives the following commutative diagram

$$\begin{tikzcd} J^1(\mathbb{R}^n, \mathcal{M}) \arrow[rr, "\rho^1_0"] \arrow[dr, "\alpha^1"'] & & \mathbb{R}^n \times \mathcal{M} \arrow[dl, "\pi"] \\ & \mathbb{R}^n & \end{tikzcd}$$

The *tangent bundle of* $\mathbf{n}$*-velocities* $T^1_n\mathcal{M}$ is the $(n+1)m$-dimensional manifold $J^1_0(\mathbb{R}^n,\mathcal{M})$ of the one jets at the origin of $\mathbb{R}^n$. We also denote the target projection by $\beta^1\colon T^1_n\mathcal{M}\to\mathcal{M}$. Locally, if U is a chart of $\mathcal{M}$ with local coordinates y^A, $1\le A\le m$, $(y^A,z^A_{[\mathbf{a}]})=(y^A,z^A_1,\dots,z^A_a,\cdots,z^A_n)$ the corresponding induced coordinates on $(\beta^1)^{-1}(U)$ and $\sigma\colon\mathbb{R}^n\to\mathcal{M}$ a mapping such that $\sigma(0)\in U$ then $\tilde\sigma^1(0)$ is represented by

$$(2)\quad y^A(\tilde\sigma^1(0))=y^A(\sigma(0))\cong\sigma^A(0),\, z^A_a(\tilde\sigma^1(0))=(\partial y^A/\partial x_a)(\sigma(0))\cong\sigma^A_a(0).$$

Let $\mathcal{F}$ be a tensor of type $(1,1)$ on $\mathcal{M}$ such that $\mathcal{F}=\sum_{A,B}F^B_A(\partial/\partial y^B)\otimes dy^A$. Then the $\mathbf{a}$*–lift* $\mathcal{F}^{[\mathbf{a}]}$ of $\mathcal{F}$ to $T^1_n\mathcal{M}$ is the $(1,1)$- tensor with local expression:

$$\mathcal{F}^{[\mathbf{a}]}=\sum_{A,B}F^B_A\left(\frac{\partial}{\partial z^B_a}\right)\otimes dy^A,$$

(see [21] for further details about the intrinsic construction). If $\mathcal{I}_{\mathcal{M}}=\sum_A(\partial/\partial y^A)\otimes dy^A$ is the identity tensor on $\mathcal{M}$ then for each $a\in\{1,2,\dots n\}$, its $\mathbf{a}$–lifting defines the tensor $\mathcal{J}_{\mathbf{a}}=\mathcal{I}^{[\mathbf{a}]}_{\mathcal{M}}$ locally given by

$$(3)\qquad \mathcal{J}_{\mathbf{a}}=\sum_A\frac{\partial}{\partial z^A_a}\otimes dy^A.$$

Let ω be a p–form on $T^1_n\mathcal{M}$. We define

$$(4)\qquad \begin{aligned}&(\iota_{\mathcal{J}_{\mathbf{a}}}\omega)(X_1,\dots,X_p)=\sum_{b=1}^{p}\omega(X_1,\dots,\mathcal{J}_{\mathbf{a}}(X_b),\dots,X_p),\\&(\text{resp.}\,d_{\mathcal{J}_{\mathbf{a}}}=\iota_{\mathcal{J}_{\mathbf{a}}}d-d\iota_{\mathcal{J}_{\mathbf{a}}}),\end{aligned}$$

where $X_1,\dots,X_p$ are vector fields on $(T^1_n\mathcal{M})$ and d is the usual exterior differentiation.

To close this section we recall the following. Let $(\mathbf{E},\pi,N)$ be a fibred bundle where $\dim N=n$, $\dim\mathbf{E}=n+r$ and $\pi:\mathbf{E}\to N$ is the projection. A *connection* on $(\mathbf{E},\pi,N)$ in the sense of Ehresmann is a complementary vector sub-bundle $\mathbf{H}$ of $T\mathbf{E}$ of the vertical sub-bundle $\mathbf{V}=\operatorname{Ker}T\pi$ such that $T\mathbf{E}$ may be decomposed as the Whitney sum $T\mathbf{E}=\mathbf{H}\oplus\mathbf{V}$. If $\mathbf{h}:T\mathbf{E}\to\mathbf{H}$ is the horizontal projector then setting $\Gamma=2\mathbf{h}-\mathrm{Id}_{T\mathbf{E}}$ (resp. $\Gamma=\mathbf{Id}_{T\mathbf{E}}-2\mathbf{v}$) we deduce that Γ is a tensor field of type $(1,1)$ on $\mathbf{E}$ such that $\Gamma\circ\Gamma=\Gamma^2=\mathrm{Id}_{T\mathbf{E}}$. As $\Gamma\circ\mathbf{h}=\mathbf{h}$ (resp. $\Gamma\circ v=-\mathbf{v}$) we have that the eigenvector bundle corresponding to the eigenvalue 1 (resp. -1) is $\mathbf{H}$ (resp. $\mathbf{V}$).

Reciprocally, if Γ is a tensor field of type $(1,1)$ on $\mathbf{E}$ such that $\Gamma\circ\Gamma = \Gamma^2 = \mathbf{Id}_{T\mathbf{E}}$ and in addition if the eigenvector bundle corresponding to the eigenvalue -1 is $\mathbf{V} = \{X \in T\mathbf{E} \,|\, T\pi(X) \equiv 0\}$ at each point of $\mathbf{E}$ then we have a vertical projector $\mathbf{v} = (1/2)(\mathbf{Id}_{T\mathbf{E}} - \Gamma)$ (resp. a horizontal projector $\mathbf{h} = (1/2)(\mathbf{Id}_{T\mathbf{E}} + \Gamma)$). Thus we shall say that Γ is a connection on $(\mathbf{E}, \pi, N)$.

2.2 – Concomitant geometric objects

Let $\sigma\colon \mathbb{R}^n \to \mathcal{M}$ be a mapping, $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ the canonical basis of $\mathbb{R}^n$ and $\sigma_a\colon \mathbb{R} \to \mathcal{M}$ the curve defined by $\sigma_a(t) = \sigma(t\,\mathbf{e_a})$, where $a \in \{1, \cdots, n\}$. Then $\tilde{\sigma}^1_a(0) \in T\mathcal{M}$ and we may define

$$\varphi_{\mathbf{a}}\colon T^1_n\mathcal{M} \to T\mathcal{M}; \quad \varphi_{\mathbf{a}}(\tilde{\sigma}^1(0)) \mapsto \tilde{\sigma}^1_a(0). \tag{5}$$

Using the map $\varphi_{\mathbf{a}}$ we construct the diffeomorphism $\Lambda_{T\mathcal{M}} : T^1_n\mathcal{M} \to T\mathcal{M} \oplus_{\mathcal{M}} \ldots \oplus_{\mathcal{M}} T\mathcal{M}$ from the jet manifold $T^1_n\mathcal{M}$ to the Whitney sum of $T\mathcal{M}$ n-times given by (see [3])

$$\Lambda_{T\mathcal{M}}(\tilde{\sigma}^1(0)) = (\varphi_1(\tilde{\sigma}^1(0)), \ldots, \varphi_n(\tilde{\sigma}^1(0))).$$

Therefore, as for any $\mathbf{v} \in T^1_n\mathcal{M}$ we have $\mathbf{v} = \tilde{\tau}^1(0)$, for some $\tau : \mathbb{R}^n \to \mathcal{M}$, from the above definition of $\varphi_{\mathbf{a}}$ there are n tangent vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ at the same point $\tau(0) \in \mathcal{M}$ uniquely associated to $\mathbf{v}$ (note that $(T^1_n\mathcal{M}, \beta^1, \mathcal{M})$ is endowed with a vector bundle structure with fiber $\mathbb{R}^{nm}$). Thus a section of $T^1_n\mathcal{M} \to \mathcal{M}$ is concomitant with n sections of the n canonical projections $T^1_n\mathcal{M} \to T\mathcal{M}$ and this suggests the following

Definition 2.1. Let K be a finite dimensional manifold. A section $\xi\colon K \to T^1_nK$ of $\beta^1\colon T^1_nK \to K$ is called field of n–vectors. We shall write $\xi = (\xi_{[\mathbf{1}]}, \cdots, \xi_{[\mathbf{a}]}, \cdots, \xi_{[\mathbf{n}]})$.

$$\begin{tikzcd} T^1_nK \arrow[rr] & & TK \\ & K \arrow[ul, "\xi"] \arrow[ur, "\xi_{[\mathbf{a}]}"'] & \end{tikzcd}$$

In particular, if $K = T^1_n\mathcal{M}$ then any section $\xi\colon T^1_n\mathcal{M} \to T^1_n[T^1_n\mathcal{M}]$, is identified with a unique set $(\xi_{[\mathbf{1}]}, \cdots, \xi_{[\mathbf{a}]}, \cdots, \xi_{[\mathbf{n}]})$ of vector fields on $T^1_n\mathcal{M}$, each of which is also uniquely identified with a set $(\xi^1_{[\mathbf{a}]}, \cdots, \xi^b_{[\mathbf{a}]}, \cdots \cdots, \xi^n_{[\mathbf{a}]})$ on $T\mathcal{M}$. In fibred coordinates one has

$$\xi_{[\mathbf{a}]} = \xi^A_a \frac{\partial}{\partial y^A} + \xi^A_{b;a} \frac{\partial}{\partial z^A_b},\ 1 \le a \le n,\ \text{(summation on } b).$$

Sometimes we shall set

$$\xi_{[\mathbf{a}]} = (y^A, z^A_j, \xi^A_a, \xi^A_{b;a}).$$

2.3 – Solutions

Let us return to (5) and let $(y^A, \dot{y}^A)$ be the induced coordinates on a chart of $T\mathcal{M}$. Then the use of $\Lambda_{T\mathcal{M}}$ says that

$$y^A(\tilde{\sigma}^1_a(0)) = y^A(\tilde{\sigma}^1(0)) \cong \sigma^A(0); \dot{y}^A(\tilde{\sigma}^1_a(0)) = z^A_a(\tilde{\sigma}^1(0)) = \frac{\partial y^A}{\partial x_a}(\sigma(0)) \cong$$
$$\cong \frac{\partial \sigma^A}{\partial x_a}(0) \cong \sigma^A_a(0)\,,$$

(see (2)). Next we translate by $x \in \mathbb{R}^n$ the action of σ, and we define the curve $\sigma_{x,a}(t) = \sigma(x + t\,\mathbf{e_a})$. Then we see that $\tilde{\sigma}^1_{x,a}(0) \in T\mathcal{M}$ is locally given by

$$\left(\sigma^A_{x,a}(0), \frac{d\sigma^A_{x,a}}{dt}(0)\right) = \left(\sigma^A(x), \frac{\partial \sigma^A}{\partial x_a}(x)\right)\,, \tag{6}$$

Now if we set $\sigma_x(y) = \sigma(x + y)$ then $j^1_0\sigma_x \in T^1_n\mathcal{M}$ is locally identified with

$$\left(\sigma^A(x), \frac{\partial \sigma^A}{\partial x_1}(x), \cdots, \left(\sigma^A(x), \frac{\partial \sigma^A}{\partial x_n}(x)\right)\right)\,.$$

This suggests the following

Definition 2.2. A solution of a field $\xi\colon \mathcal{M} \to T^1_n\mathcal{M}$ of n–vectors is a map $\sigma\colon \mathbb{R}^n \to \mathcal{M}$ defined on an open domain $U \subset \mathbb{R}^n$ such that $\xi(\sigma(x)) = j^1_0(\sigma_x)$, for all $x \in U$. In particular, if $\xi_{[\mathbf{a}]}(\sigma(x)) = j^1_0\,\sigma_{x,a}$

then σ is said an **a**-*solution* of ξ. Thus a solution is an **a**-solution for all $1 \leq a \leq n$.

If we set $\xi_{[\mathbf{a}]} = (y^A, \xi_a^A)$ then σ is a solution if and only if σ is a solution of the system of partial differential equations

$$\xi_a^A = \frac{\partial y^A}{\partial x_a}, \quad \text{i.e.} \quad \xi_a^A(\sigma(x)) = \frac{\partial \sigma^A}{\partial x_a}(x)\,.$$

In a similar way, if we consider fields ξ: $T_n^1\mathcal{M} \to T_n^1[T_n^1\mathcal{M}]$ on $T_n^1\mathcal{M}$ then **a**-solutions as well as solutions are also characterized by a set of partial differential equations, but we shall consider fields ξ where the solutions verify equations of type

(7) $$\xi_a^A = \frac{\partial y^A}{\partial x_a}, \;\; \xi_{b;a}^A = \frac{\partial^2 y^A}{\partial x_b \partial x_a}\,,$$

given by the system

$$\xi_a^A = z_a^A, \;\; \xi_{b;a}^A = \frac{\partial z_a^A}{\partial x_b}\,.$$

The characterization of such fields is as follows. First recall that there is a well globally defined vector field on the tangent bundle of every finite dimensional manifold called *Liouville* or *dilation* vector field (see [4], [5], [17]). In our context, if we consider the homothetia of ratio e^s, $s \in \mathbb{R}$, then the infinitesimal generator $\mathcal{C}$ of $(s, y^A, z_a^A) \mapsto (y^A, e^s\, z_a^A)$ is locally given by

$$\mathcal{C} = \sum_{A,a} z_a^A \frac{\partial}{\partial z_a^A}\,,$$

Thus there is a well defined field of n vectors $\mathcal{C} = (\mathcal{C}_{[\mathbf{1}]}, \cdots, \mathcal{C}_{[\mathbf{n}]})$ such that each $\mathcal{C}_{[\mathbf{a}]}$ is Liouville. At the local level one has that for each $1 \leq a \leq n$

(8) $$\mathcal{C}_{[\mathbf{a}]} = z_a^A \frac{\partial}{\partial z_a^A}\,,$$

and $\mathcal{C}_{[\mathbf{a}]} = (y^A, z_a^A, 0, z_a^A)$ is so that $\mathcal{J}_{\mathbf{a}}(\mathcal{C}_{[\mathbf{a}]}) = 0$, where $(\mathcal{J}_{\mathbf{1}}, \cdots, \mathcal{J}_{\mathbf{n}})$ is the field of tensors given by (3).

Definition 2.3. A field of n-vectors ξ: $T_n^1\mathcal{M} \to T_n^1[T_n^1\mathcal{M}]$ is said a field of n-semisprays if $\mathcal{J}_{\mathbf{a}}(\xi_{[\mathbf{a}]}) = \mathcal{C}_{[\mathbf{a}]}$, for all $1 \leq a \leq n$.

It is easy to see that for such fields $\xi_a^A = z_a^A$ and so the solutions verify the system (7). We remark also that if $\xi = (\xi_{[1]}, \cdots, \xi_{[n]})$ is a field on $T_n^1\mathcal{M}$ then the $\mathbf{a}$–lift $\xi_{[\mathbf{a}]}^{[\mathbf{a}]}$ of each $\xi_{[\mathbf{a}]}$ to $T_n^1[T_n^1\mathcal{M}]$ is $\mathcal{C}_{[\mathbf{a}]}$ (it follows from [22] that these vector fields are the canonical vector fields on the vector bundle $\pi^a\colon T_n^1\mathcal{M} \to T_{n-1}^1\mathcal{M}$ given by $\pi^a(X_1, \cdots, X_n) = (X_1, \cdots, X_{a-1}, X_{a+1}, \cdots, X_n)$, $1 \le a \le n$).

3 – Field connections

For what follows we shall identify $J^1(\mathbb{R}^n, \mathcal{M})$ with $\mathbb{R}^n \times T_n^1\mathcal{M}$. Fibred coordinates will be also denoted by (x_a, y^A, z_a^A), with $1 \le a \le n, 1 \le A \le m$. It is easy to verify that all canonical structures defined before can be transported from $T_n^1\mathcal{M}$ to $\mathbb{R}^n \times T_n^1\mathcal{M}$. If we preserve the notation for the natural geometric objects ($\mathcal{J}, \mathcal{C}$, etc) then we define the following family on $\mathbb{R}^n \times T_n^1\mathcal{M}$,

$$\tilde{\mathcal{J}}_{\mathbf{a}} = \mathcal{J}_{\mathbf{a}} - \mathcal{C}_{[\mathbf{a}]} \otimes dx_a\,, \tag{9}$$

where now $\mathcal{C} = (\mathcal{C}_{[1]}, \cdots \mathcal{C}_{[n]})$ is the infinitesimal generator of $(s, x_a, y^A, z_a^A) \mapsto (x_a, y^A, e^s\, z_a^A)$, $s \in \mathbb{R}$.

A field of n-vectors $\xi = (\xi_{[1]}, \cdots, \xi_{[\mathbf{a}]}, \cdots, \xi_{[n]})$ is a field of n-semisprays if for all a we have

$$\tilde{\mathcal{J}}_{\mathbf{a}}(\xi_{[\mathbf{a}]}) = 0, \quad dx_b(\xi_{[\mathbf{a}]}) = \delta_{ab}\,.$$

Thus we may express locally

$$\xi_{[\mathbf{a}]} = \frac{\partial}{\partial x_a} + z_a^A \frac{\partial}{\partial y^A} + \xi_{b;a}^A \frac{\partial}{\partial z_b^A}\,.$$

Let $\mathbf{K}$ be a $(1,1)$ tensor field on this manifold such that for all a

$$\left.\begin{aligned} &\mathcal{J}_{\mathbf{a}} \circ \mathbf{K} = \tilde{\mathcal{J}}_{\mathbf{a}} \circ \mathbf{K} = \tilde{\mathcal{J}}_{\mathbf{a}},\\ &\mathbf{K} \circ \tilde{\mathcal{J}}_{\mathbf{a}} = -\tilde{\mathcal{J}}_{\mathbf{a}},\\ &\mathbf{K} \circ \mathcal{J}_{\mathbf{a}} = -\mathcal{J}_{\mathbf{a}}. \end{aligned}\right\} \tag{10}$$

Then from a long but straightforward calculation one has that locally

$$\begin{aligned}
\mathbf{K}(\partial/\partial x_a) &= -z_a^B(\partial/\partial y^B) + \mathbf{K}_{ba}^B(\partial/\partial z_b^B),\\
\mathbf{K}(\partial/\partial y^A) &= \partial/\partial y^A + \mathbf{K}_{Ab}^B(\partial/\partial z_b^B),\\
\mathbf{K}(\partial/\partial z_a^A) &= -\partial/\partial z_a^A,
\end{aligned}$$

that is

$$\mathbf{K} = \Big(-z_a^B\frac{\partial}{\partial y^B} + \mathbf{K}_{ba}^B\frac{\partial}{\partial z_b^B}\Big)\otimes dx_a + \Big(\frac{\partial}{\partial y^A} + \mathbf{K}_{Ab}^B\frac{\partial}{\partial z_b^B}\Big)\otimes dy^A - \frac{\partial}{\partial z_a^A}\otimes dz_a^A\,.$$

Therefore $\mathbf{K}^2 \neq \mathbf{K}$, but $\mathbf{K}^3 - \mathbf{K} = 0$.

PROPOSITION 3.1. *The tensor* $\mathbf{K}$ *defines a connection on* $\rho_0^1 : \mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$.

PROOF. We first define two canonical operators $\mathcal{P}$, $\mathcal{Q}$ associated to $\mathbf{K}$:

$$\mathcal{P} = \mathbf{K}^2,\ \mathcal{Q} = \mathrm{Id} - \mathbf{K}^2\,.$$

Therefore

$$\mathcal{P}^2 = \mathcal{P},\ \mathcal{Q}^2 = \mathcal{Q},\ \mathcal{P}\circ\mathcal{Q} = \mathcal{Q}\circ\mathcal{P} = 0,\ \mathcal{P}+\mathcal{Q} = \mathrm{Id}\,,$$

i.e. $\mathcal{P}$ and $\mathcal{Q}$ are complementary operators. From the above equalities we deduce that locally

$$\begin{aligned}
\mathcal{P}(\partial/\partial x_a) &= -z_a^A(\partial/\partial y^A) - (\mathbf{K}_{ba}^A + z_a^B\mathbf{K}_{cB}^A(\partial/\partial z_c^A)\,,\\
\mathcal{P}(\partial/\partial y^A) &= \partial/\partial y^A\,,\\
\mathcal{P}(\partial/\partial z_a^A) &= \partial/\partial z_a^A\,,\\
\mathcal{Q}(\partial/\partial x_a) &= \partial/\partial x_a + z_a^A(\partial/\partial y^A) + (\mathbf{K}_{ba}^B + z_a^A\mathbf{K}_{bA}^B)(\partial/\partial z_b^B)\,,\\
\mathcal{Q}(\partial/\partial y^A) &= \mathcal{Q}(\partial/\partial z_a^A) = 0\,.
\end{aligned}$$

We put $\mathbf{P} = \mathrm{Im}\,\mathcal{P}$, $\mathbf{Q} = \mathrm{Im}\,\mathcal{Q}$. Therefore $\mathbf{P}$ and $\mathbf{Q}$ are complementary distributions on $\mathbb{R}^n\times T_n^1\mathcal{M}$, i.e., $T(\mathbb{R}^n\times T_n^1\mathcal{M}) = P\oplus\mathbf{Q}$. Note that $\{\partial/\partial y^A, \partial/\partial z_a^A\}$, resp.

$$\xi_{[\mathbf{a}]} = \partial/\partial x_a + z_a^A(\partial/\partial y^A) + (\mathbf{K}_{ba}^B + z_a^A\mathbf{K}_{bA}^B)(\partial/\partial z_b^B)\,,$$

is a local basis of $\mathbf{P}$, resp. $\mathbf{Q}$.

Let

$$\mathfrak{P} = (1/2)(\mathrm{Id} + \mathbf{K})\,\mathcal{P}),\ \ \mathfrak{Q} = (1/2)(\mathrm{Id} - \mathbf{K})\,\mathcal{P}$$

and set

$$\mathbf{H} = \ \mathrm{im}\,\mathfrak{P},\ \mathbf{V} = \ \mathrm{im}\,\mathfrak{Q}\,.$$

Then $\mathbf{V}$ is locally generated in each point by $(\partial/\partial z_a^A)$, and so it is a vertical distribution with respect to the fibration $\rho_0^1 : \mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \ \mathcal{M}$. Therefore $\mathbf{H}$ and $\mathbf{V}$ are complementary distributions in $\mathbf{P}$ $(= \ \mathbf{H} \oplus \mathbf{V})$.

Now, let us set $\mathbf{H}' = \mathbf{H} \oplus \mathbf{Q}$. As $\mathbf{V}, \mathbf{H}'$ are vector bundles over $\mathbb{R}^n \times T_n^1\mathcal{M}$ one has that $\mathbf{K}$ defines a connection on the fibration ρ_0^1: $\mathbb{R}^n \times T_n^1\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$, with horizontal fibration $\mathbf{H}'$, that is one has the decomposition

$$T(\mathbb{R}^n \times T_n^1\mathcal{M}) = \mathbf{H}' \oplus \mathbf{V}. \qquad \square$$

Definition 3.1. $\mathbf{K}$ will be called field connection.

From the considerations developed above one has that at each point of $\mathbb{R}^n \times T_n^1\mathcal{M}$, there is defined a local basis

$$\frac{\partial}{\partial y^A},\ \ \frac{\partial}{\partial z_a^A},\ \ \mathcal{Q}\Big(\frac{\partial}{\partial x_a}\Big) = \frac{\partial}{\partial x_a} + z_a^B\frac{\partial}{\partial y^B} + \xi^C_{b;a}\frac{\partial}{\partial z_b^C} = \xi_{[\mathbf{a}]}\,,$$

where $\xi^C_{b;a} = \mathbf{K}^C_{ba} + z_a^B\mathbf{K}^C_{bB}$. So $\mathcal{Q}(\partial/\partial x_a)$ is an $\mathbf{a}$–semispray. We say that $\mathbf{K}$ and this family of semisprays are *associated.*

Now, to relate the solutions of a field $\xi = (\xi_{[\mathbf{1}]}, \cdots, \xi_{[\mathbf{n}]})$ of n-vectors on $\mathbb{R}^n \times T_n^1\mathcal{M}$ with a field connection $\mathbf{K}$ it is sufficient to recall that a solution of ξ at $x \in \mathbb{R}^n$ is a mapping σ: $\mathbb{R}^n \to \mathcal{M}$ such that

$$\xi(\tilde{\sigma}^1(x)) = j_0^1[\tilde{\sigma}^1(x)]$$

where $\tilde{\sigma}^1$ is the jet prolongation mapping $x \to j_x^1(\sigma)$. A solution of $\mathbf{K}$ is then defined as being a solution of the associated family of semisprays $\xi_{[\mathbf{a}]}$. Therefore σ is a solution of $\mathbf{K}$ if and only if σ satisfies the following system of equations:

$$z_a^B = \frac{\partial\sigma^B}{\partial x_a},\ \ \xi^A_{b;a} = \frac{\partial^2\sigma^A}{\partial x_b\partial x_a} = \mathbf{K}^A_{ba} + z_a^B\mathbf{K}^A_{bB}\,.$$

Note that we are not taking into account the problem of the existence of such solutions.

4 – An application

An application of the above results is that we can characterize integral solutions of some partial differential equations as extremals of the Euler-Lagrange equations. This can be stated in the following form:

THEOREM 4.1. *If* $\mathcal{L}\colon \mathbb{R}^n \times T^1_n\mathcal{M} \to \mathbb{R}$ *is a regular Lagrangian and* $\{\xi_{[\mathbf{a}]}\}$, $a = \{1, \cdots, n\}$ *is a family of* $\mathbf{a}$*–semisprays then there exists a connection* $\mathbb{K}$ *with respect to the fibration* $\rho^1_0\colon \mathbb{R}^n \times T^1_n\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$ *whose solutions are the extremal solutions of the Euler-Lagrange equations if and only if*

$$\iota_{\mathcal{Q}}(d\alpha_{\mathcal{L}}) = (n-1)\, d\alpha_{\mathcal{L}}\,,$$

where $\mathcal{Q} = \mathrm{Id} - \mathbb{K}^2$ *and* $\alpha_{\mathcal{L}}$ *is the Poincaré–Cartan form defined by* (13).

Let us first show the existence of $\mathbb{K}$, that is,

PROPOSITION 4.2. *Let* $\{\xi_{[\mathbf{a}]}\}$ *be a family of* $\mathbf{a}$*–semisprays on* $\mathbb{R}^n \times T^1_n\mathcal{M}$*, locally written as*

$$\xi_{[\mathbf{a}]} = \frac{\partial}{\partial x_a} + z^A_a \frac{\partial}{\partial y^A} + \xi^A_{b;a} \frac{\partial}{\partial z^A_b}\,. \tag{11}$$

Then there exists a field connection $\mathbb{K}$ *with respect to the fibration* ρ^1_0 : $\mathbb{R}^n \times T^1_n\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$ *with associated semisprays being precisely* (11).

PROOF. We have

$$[\xi_{[\mathbf{a}]}, \partial/\partial y^B] = -(\partial\xi^A_{b;a}/\partial y^B)(\partial/\partial z^A_b)\,,$$
$$[\xi_{[\mathbf{a}]}, \partial/\partial z^B_b] = -\partial/\partial y^B - (\partial\xi^A_{c;a}/\partial z^B_b)(\partial/\partial z^A_c)\,.$$

For what follows we recall the following Lie derivation of a tensor by a vector field

$$(L_X\mathbf{S})(Y) = [X, \mathbf{S}(Y)] - \mathbf{S}([X, Y])\,,$$

where X, Y are vector fields, $[\,,\,]$ is the Lie bracket and $\mathbf{S}$ is a $(1,1)$-tensor field.

Consider the following tensor field

$$\mathbf{K}_{[\mathbf{a}]} = -L_{\xi_{[\mathbf{a}]}} \tilde{\mathcal{J}}_{\mathbf{a}}\,.$$

Then

$$\begin{aligned}\mathbf{K}_{[\mathbf{a}]}(\partial/\partial y^B) &= \partial/\partial y^B + (\partial\xi^A_{b;a}/\partial z^B_a)(\partial/\partial z^A_b)\,,\\ \mathbf{K}_{[\mathbf{a}]}(\partial/\partial z^B_a) &= -\partial/\partial z^B_a\,,\end{aligned}$$

Now, consider the canonical volume form ω of $\mathbb{R}^n$, taken as a form on $\mathbb{R}^n \times T^1_n\mathcal{M}$, and the following tensor field of type $(n+1,1)$

$$\mathbf{K}\wedge\omega = \sum_{\mathbf{a}} \mathbf{K}_{[\mathbf{a}]}\wedge\omega\,.$$

Then from the above local computation one has

$$\mathbf{K}\wedge\omega = \Big(\frac{\partial}{\partial y^A} + \frac{\partial\xi^B_{b;a}}{\partial z^A_a}\Big)\frac{\partial}{\partial z^B_b}\otimes dy^A\wedge\omega - \frac{\partial}{\partial z^A_a}\otimes dz^A_a\wedge\omega\,.$$

Let us set

$$\mathbb{Q} = 1/2(\mathrm{Id}\wedge\omega + \mathbf{K}\wedge\omega),\ \mathbb{P} = \iota_\omega\mathbb{Q}\,. \tag{12}$$

where ι_ω denotes contraction. Then one has

$$\mathbb{P} = (\partial/\partial y^A + 1/2\ (\partial\xi^B_{b;a}/\partial z^A_a)\partial/\partial z^B_b)\otimes dy^A\,.$$

To obtain a connection on $\rho^1_0 : \mathbb{R}^n \times T^1_n\mathcal{M} \to \mathbb{R}^n \times \mathcal{M}$, we introduce the tensor field:

$$\mathbb{K} = -\mathrm{Id} + 2\ \mathbb{P} + \sum_a\ \Big(\frac{\partial}{\partial x_a} - [\mathcal{C}, \xi_{[\mathbf{a}]}]\Big)\otimes dx_a,\ \ \mathcal{C} = \sum_{b,A} z^A_b \frac{\partial}{\partial z^A_b}\,.$$

Then $\mathbb{K}$ verifies (10), $\mathbb{K}^3 - \mathbb{K} = 0$ and $\mathbb{K}$ is locally expressed as

$$\begin{aligned}\Big(-z^A_b\frac{\partial}{\partial y^A} - \Big(z^A_b\frac{\partial\xi^B_{b;c}}{\partial z^A_b} - \xi^B_{b;c}\Big)\frac{\partial}{\partial z^B_c}\Big)\otimes dx_b + \Big(\frac{\partial}{\partial y^A} + \frac{\partial\xi^B_{b;a}}{\partial z^A_a}\frac{\partial}{\partial z^B_b}\Big)\otimes dy^A\\ -\frac{\partial}{\partial z^A_a}\otimes dz^A_a\,.\end{aligned}$$

Furthermore, one has the following equalities:

$$\begin{aligned} \mathbb{K}^2(\partial/\partial x_b) &= -z_b^B\,(\partial/\partial y^B) - \xi^B_{b;c}\,(\partial/\partial z^B_c)\,, \\ \mathbb{K}^2(\partial/\partial y^A) &= \partial/\partial y^A \\ \mathbb{K}^2(\partial/\partial z^A_a) &= \partial/\partial z^A_a\,. \end{aligned}$$

Now, adopting the above procedure for the projectors $\mathcal{P} = \mathbb{K}^2$ and $\mathcal{Q} = \mathrm{Id} - \mathbb{K}^2$, one obtains

$$\mathcal{Q}\Big(\frac{\partial}{\partial x_a}\Big) = \frac{\partial}{\partial x_a} + z^B_a \frac{\partial}{\partial y^B} + \xi^B_{b;a}\frac{\partial}{\partial z^B_b} = \xi_{[\mathbf{a}]}\,. \qquad \square$$

Let $\mathcal{L} : \mathbb{R}^n \times T^1_n\mathcal{M} \to \mathbb{R}$ be a Lagrangian function. The *Poincaré–Cartan form* associated to $\mathcal{L}$ is the n–form defined by

$$\alpha_{\mathcal{L}} = (d_{\tilde{\mathcal{J}}_{\mathbf{a}}}\mathcal{L}) \,\wedge\, \omega_{\mathbf{a}} + \mathcal{L}\,\omega\,, \tag{13}$$

(where $\omega_{\mathbf{a}} = \iota_{\partial/\partial x_a}\omega$, $\tilde{\mathcal{J}}_{\mathbf{a}}$ is defined by (9) and $d_{\tilde{\mathcal{J}}_{\mathbf{a}}}$ in a similar way as (4)). We set $\Omega_{\mathcal{L}} = d\alpha_{\mathcal{L}}$.

It remains to show the second assertion of the theorem, i.e.,

PROPOSITION 4.3. *The solutions of* $\mathbb{K}$ *are solutions of the Euler–Lagrange equations if and only if*

$$\iota_{\mathcal{Q}}(\Omega_{\mathcal{L}}) = (n-1)\Omega_{\mathcal{L}}\,.$$

PROOF. The proof follows Saunders ([25], p. 188) and the above results. Indeed, as

$$\mathcal{Q} = \Big(\frac{\partial}{\partial x_a} + z^A_a\frac{\partial}{\partial y^A} + \xi^A_{b;a}\frac{\partial}{\partial z^A_b}\Big) \otimes dx_a\,, \tag{14}$$

and $\mathcal{L}$ is regular, i.e. the matrix

$$\Big(\frac{\partial^2\mathcal{L}}{\partial z^A_a \partial z^B_b}\Big)$$

is invertible, then a direct calculation shows that

$$z_a^A = \xi_a^A$$

$$\iota_{\mathcal{Q}}(\Omega_{\mathcal{L}}) - (n-1)\Omega_{\mathcal{L}} = \Big[\frac{\partial \mathcal{L}}{\partial y^A} - \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial x_a} - z_a^B \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial y^B} - \xi_{b;a}^B \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial z_b^B}\Big] dy^A \wedge \omega.$$

Thus $\iota_{\mathcal{Q}}(\Omega_{\mathcal{L}}) = (n-1)\Omega_{\mathcal{L}}$ if and only if

$$\frac{\partial \mathcal{L}}{\partial y^A} - \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial x_a} - z_a^B \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial y^B} - \xi_{b;a}^B \frac{\partial^2 \mathcal{L}}{\partial z_a^A \partial z_b^B} = 0\,.$$

Therefore, if $\sigma : \mathbb{R}^n \to \mathcal{M}$ is a solution of $\mathbb{K}$ (since it is a solution of the associated semisprays $\xi_{[\mathbf{a}]}$) then σ verifies

$$z_a^B = \frac{\partial \sigma^B}{\partial x_a}, \quad \xi_{b;a}^A = \frac{\partial^2 \sigma^A}{\partial x_b \partial x_a}\,, \tag{15}$$

and so

$$\frac{\partial \mathcal{L}}{\partial \sigma^A} - \frac{\partial^2 \mathcal{L}}{\partial\Big(\dfrac{\partial \sigma^A}{\partial x_a}\Big)\partial x_a} - \frac{\partial^2 \mathcal{L}}{\partial\Big(\dfrac{\partial \sigma^B}{\partial x_a}\Big)\partial \sigma^B} \frac{\partial \sigma^B}{\partial x_a} - \frac{\partial^2 \mathcal{L}}{\partial\Big(\dfrac{\partial \sigma^A}{\partial x_a}\Big)\partial\Big(\dfrac{\partial \sigma^B}{\partial x_b}\Big)} \frac{\partial^2 \sigma^B}{\partial x_b \partial x_a} =$$

$$= \left(\frac{\partial \mathcal{L}}{\partial y^A} - \frac{d}{dx_a}\left(\frac{\partial \mathcal{L}}{\partial\Big(\dfrac{\partial \sigma^A}{\partial x_a}\Big)}\right)\right) = 0\,.$$

i.e. σ solves the Euler-Lagrange equations. ∎

As a simple example let us consider the coordinates (x_1, x_2, y, z_1, z_2) for $\mathbb{R}^2 \times T_2^1\mathbb{R} \cong \mathbb{R}^5$ and the *wave equation*

$$\rho\left(\frac{\partial^2 \Psi}{\partial x_1^2}\right) = \tau\left(\frac{\partial^2 \Psi}{\partial x_2^2}\right). \tag{16}$$

where $\Psi = \Psi(x_1, x_2)$ and ρ, τ are the coefficients of mass density and tension, taken as constants. To avoid unnecessary calculations and to simplify things let us suppose that the physical system is given by a single frequency ω with one degree of freedom and that $\rho = \tau = \omega = 1$. The separation of the variables gives a general solution of the wave equation

$$\Psi(x_1, x_2) = \alpha(x_1)\,\beta(x_2)\,,$$

and so we have the system of equations

$$\frac{\partial^2\alpha}{\partial {x_1}^2} = -\alpha, \qquad \frac{\partial^2\beta}{\partial x_2^2} = -\beta\,,$$

As there is no mixed partial derivatives in the wave equation one has that the projector $\mathcal{Q}$ is given by the semisprays

$$\xi_{[\mathbf{1}]} = \frac{\partial}{\partial x_1} + \frac{\partial\alpha}{\partial x_1}\frac{\partial}{\partial y} - \alpha\,\frac{\partial}{\partial z_1}\,,$$
$$\xi_{[\mathbf{2}]} = \frac{\partial}{\partial x_2} + \frac{\partial\alpha}{\partial x_2}\frac{\partial}{\partial y} - \beta\,\frac{\partial}{\partial z_2}\,,$$

and the $1,2$-solutions give solutions of the equation. As we know, a particular one is

$$\sigma(x_i) = a_i\, e^{\mathbf{i}\, x_i}\,.$$

If we take only the real part, then the projector $\mathcal{P} = \mathrm{Id} - \mathcal{Q}$ define the vector fields

$$a_i\left[\sin(x_i)\,\frac{\partial}{\partial y} + \cos(x_i)\,\frac{\partial}{\partial z_i}\right],\ \ 1 \le i \le 2\,.$$

Note that the Lagrangian is given by $\mathcal{L} = \frac{1}{2}\{\sigma z_1^2 - \tau z_2^2\}$, and (16) is the corresponding Euler-Lagrange equation.

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