Rendiconti di Matematica, Serie VII
Volume 18, Roma (1998), 749-762

Boundedness of solutions to some linear elliptic equations with right hand side in the Morrey space $\mathbf{L}^{1,\lambda}$

A. D'OTTAVIO – F. LEONETTI – C. MUSCIANO

Riassunto: *Si dimostra la limitatezza delle soluzioni delle equazioni lineari ellittiche* (1.1) *con secondo membro in* $L^{1,\lambda}$.

Abstract: *We prove boundedness of distributional solutions* u *to linear elliptic equations* $-\operatorname{div}(a(x)Du(x)) = -\operatorname{div} f(x) + f_0(x)$ *where the right hand side* f, f_0 *is only in* $L^{1,\lambda}$

1 – Introduction

We consider linear equations in divergence form

$$(1.1)\quad -\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} a_{ij}(x) D_j u(x)\right) = -\sum_{i=1}^{n} D_i f_i(x) + f_0(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

where $D_i = \partial/\partial x_i$, the coefficients $a_{ij} : \Omega \to \mathbb{R}$ are measurable, bounded and elliptic, $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,r}$, $r \geq 1$. When

Key Words and Phrases: *Linear elliptic equations – Weak solutions – Regularity – Boundedness.*
A.M.S. Classification: 35J99 – 35D10
We acknowledge the support of MURST, GNAFA-CNR, INdAM, MURST 60%, MURST 40%.

the gradient Du is in L^2 (and the right hand side $f_0, f_1, \dots, f_n$ fulfill suitable assumptions) the solution u enjoys regularity properties which are well known, [7], [18], [26], [28], [34], in particular, u is hölder continuous when $f_0 = f_1 = \cdots = f_n = 0$. On the contrary, when the gradient Du is no longer assumed in L^2 but in some L^r with $r < 2$, then u may not enjoy all the nice properties of the case $r = 2$, in particular, u may be unbounded, even if $f_0 = f_1 = \cdots = f_n = 0$, [32]. Let us name solutions $u \in W^{1,r}$, $r < 2$, weak solutions in order to emphasize that the gradient is not assumed to be in L^2. Recently, a great deal of work has been done in order to understand the behaviour of such weak solutions: higher integrability of the gradient for nonlinear elliptic systems has been studied in [17], [21], [12], [22], [13], [24], [25]; examples of weak solutions are in [32], [23]; uniqueness of weak solutions is studied in [1], [10], [13], [15]. On related topics, we also quote [8], [27]. In the present paper we prove boundedness of weak solutions to (1.1) in dimension 2 and 3, provided the coefficients a_{ij} are θ-hölder continuous, for suitable θ's, see the next section. Let us come back to the linear equation (1.1): bounded measurable coefficients a_{ij} allow solutions $u \in W^{1,r}$, $r < 2$, to be unbounded, even if $f_0, f_1, \dots, f_n = 0$, see [32]. On the contrary, if a_{ij} are hölder continuous, higher integrability on the right hand side $f_0, f_1, \dots, f_n \in L^p$ improves the integrability of the gradient Du as in [5], [16], [33], [9], thus giving continuity of u by Sobolev imbedding theorem, if p is large enaugh. In this paper we show that boundedness of u can be achieved without higher integrability on the right hand side but assuming that it belongs to suitable Morrey spaces $L^{1,\lambda}$. [30] shows in $L^{1,\lambda}$ a function g enjoing no higher integrability than L^1, thus L^p theory cannot be used in such a case, see the example in Section 4 at the end of the paper. Elliptic equations with L^1 data have been also studied in [1], [3], [6], [11], [29], [2].

2 – Notation and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. For $i, j = 1, \dots, n$ we consider functions $a_{ij} : \Omega \to \mathbb{R}$ and we assume that the matrix $\{a_{ij}\}$ is elliptic: for some constants $0 < l \leq L$ we have

$$(2.1) \qquad l|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \leq L|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega.$$

We also assume θ-hölder continuity, that is, for some $\theta \in (0,1]$,

$$a_{ij} \in C^{0,\theta}(\overline{\Omega}), \qquad \forall i,j = 1,\dots,n. \tag{2.2}$$

Now, we consider functions $f_i : \Omega \to \mathbb{R}, i = 0,1,\dots,n$, satisfying

$$f_0 \in L^{1,\lambda-1}(\Omega), \qquad 1 \leq n-1 < \lambda \leq n \tag{2.3}$$

$$f_1,\dots,f_n \in L^{1,\lambda}(\Omega), \tag{2.4}$$

where $L^{1,\nu}(\Omega)$ is the Morrey space, that is the set of all $v \in L^1(\Omega)$ such that

$$\sup \rho^{-\nu} \int\limits_{\Omega\cap B(x,\rho)} |v(y)|dy < \infty,$$

the supremum being taken over all $x \in \Omega$ and all $\rho > 0$, $B(x,\rho)$ being the open ball around x with radius ρ. Let $u : \Omega \to \mathbb{R}$ belong to the Sobolev space $W^{1,r}(\Omega)$ and verify

$$\begin{aligned} \int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) D_j u(x) D_i \phi(x) dx = \int\limits_{\Omega} \sum_{i=1}^{n} f_i(x) D_i \phi(x) dx + \\ + \int\limits_{\Omega} f_0(x)\phi(x)dx, \end{aligned} \tag{2.5}$$

for every $\phi \in C_0^\infty(\Omega)$. We prove the following

THEOREM. *Assume that* (2.1), (2.2), (2.3), (2.4) *hold; if* $u \in W^{1,r}(\Omega)$,

$$\frac{n}{1+\theta} < r \tag{2.6}$$

and u *solves the equation* (2.5), *then*

$$u \in L^\infty_{\mathrm{loc}}(\Omega). \tag{2.7}$$

Remark 1. Note that $1 \leq \frac{n}{1+\theta}$ in (2.6).

Remark 2. Let us come back to (2.6) in our Theorem: $\frac{n}{1+\theta} < 2$ only if $n = 2$ or $n = 3$. Thus, our Theorem deals with weak solutions in dimension 2 or 3.

Remark 3. We note that $\frac{n}{2} = \lim_{\theta \to 1} \frac{n}{1+\theta}$. Thus, our Theorem implies that, if $u \in W^{1,r}(\Omega)$, $\frac{n}{2} < r$, solves (2.5) with (2.1), (2.3), (2.4), then $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$, provided the coefficients a_{ij} are hölder continuous for a suitable exponent θ.

Remark 4. When a_{ij} are constant, a careful inspection of the proof shows that our Theorem holds true with (2.6) replaced by $1 < r$.

Remark 5. The proof of our Theorem collects ideas and techniques contained in [4], [14], [19], [20].

3 – Proof of the Theorem

We split the matrix $a(x)$ into its symmetric part $a^+(x)$ and skewsymmetric one $a^-(x)$:

$$a^+_{ij}(x) = \frac{a_{ij}(x) + a_{ji}(x)}{2}, \qquad a^-_{ij}(x) = \frac{a_{ij}(x) - a_{ji}(x)}{2}.$$

Since

$$\sum_{i,j=1}^{n} a^-_{ij}(x)\xi_i\xi_j = 0,$$

then (2.1) yields

$$l|\xi|^2 \leq \sum_{i,j=1}^{n} a^+_{ij}(x)\xi_j\xi_i \leq L|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega. \tag{3.1}$$

For every $x^0 \in \Omega$ we consider the $n \times n$ real matrix $a^+(x^0) = \{a^+_{ij}(x^0)\}$: because of (3.1), it is symmetric and positive, thus its eigenvalues are positive real numbers $\lambda^1, \dots, \lambda^n$; we select eigenvectors $w^1, \dots, w^n$ such that they are an orthonormal basis in $\mathbb{R}^n$. We define the linear mapping

$$G : \mathbb{R}^n \to \mathbb{R}^n, \qquad G(x) = (G_1(x), \dots, G_n(x)),$$

by means of

$$G_i(x) = \sum_{j=1}^{n} (\lambda^i)^{-1/2} w_j^i x_j \qquad \forall i = 1, \dots, n,$$

where $w^i = (w_1^i, \dots, w_n^i)$ and $x = (x_1, \dots, x_n)$. Let us remark that eigenvalues, eigenvectors depend on x^0, thus G itself depends on x^0; because of ellipticity (3.1) we can give the following estimates independent on x^0:

$$L^{-1}|x - x'|^2 \le |G(x) - G(x')|^2 \le l^{-1}|x - x'|^2, \qquad \forall x, x' \in \mathbb{R}^n, \tag{3.2}$$

$$L^{-n/2} \le |\det JG| \le l^{-n/2} \tag{3.3}$$

$$\sum_{i,j=1}^{n} (JG)_{ij}^2 \le n l^{-1} \tag{3.4}$$

where JG is the Jacobian matrix of G, that is

$$(JG)_{ij} = \frac{\partial G_i}{\partial x_j}. \tag{3.5}$$

The matrix JG diagonalizes $a^+(x^0)$:

$$\sum_{i,j=1}^{n} (JG)_{\alpha i} a_{ij}^+(x^0)(JG)_{\beta j} = \delta_{\alpha\beta}, \qquad \forall \alpha, \beta = 1, \dots, n, \tag{3.6}$$

where $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$, $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$. We set $y^0 = G(x^0)$. Using (3.2) we get

$$L^{-1/2} \operatorname{dist}(x^0, \partial\Omega) \le \operatorname{dist}(y^0, \partial(G(\Omega))), \tag{3.7}$$

and

$$B(x^0, \sqrt{l}R) \subset G^{-1}(B(y^0, R)) \subset B(x^0, \sqrt{L}R), \quad \forall R > 0, \tag{3.8}$$

where $B(z, \rho)$ is the open ball around z, with radius ρ and ∂A is the boundary of the set A. For every σ with $0 < \sigma < L^{-1/2} \operatorname{dist}(x^0, \partial\Omega)$, we have $G^{-1}(B(y^0,\sigma)) \subset \Omega$, thus (2.5) holds true for every smooth function ϕ with compact support in $G^{-1}(B(y^0,\sigma))$:

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}(x^0) D_j u(x) D_i \phi(x) dx =$$

$$= \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} \left(a_{ij}(x^0) - a_{ij}(x)\right) D_j u(x) D_i \phi(x) dx + \tag{3.9}$$

$$+ \int_{G^{-1}(B(y^0,\sigma))} \sum_{i=1}^{n} f_i(x) D_i \phi(x) dx + \int_{G^{-1}(B(y^0,\sigma))} f_0(x)\phi(x) dx.$$

Moreover

$$\int \sum_{i,j} a_{ij}(x^0) D_j u D_i \phi = \int \sum_{i,j} a^+_{ij}(x^0) D_j u D_i \phi + \int \sum_{i,j} a^-_{ij}(x^0) D_j u D_i \phi.$$

Since ϕ is smooth with compact support and $a^-(x^0)$ is skewsymmetric, we have

$$\int \sum_{i,j} a^-_{ij}(x^0) D_j u D_i \phi = -\int u \sum_{i,j} a^-_{ij}(x^0) D_j D_i \phi = 0,$$

thus

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a^+_{ij}(x^0) D_j u(x) D_i \phi(x) dx =$$

$$= \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} \left(a_{ij}(x^0) - a_{ij}(x)\right) D_j u(x) D_i \phi(x) dx + \tag{3.10}$$

$$+ \int_{G^{-1}(B(y^0,\sigma))} \sum_{i=1}^{n} f_i(x) D_i \phi(x) dx + \int_{G^{-1}(B(y^0,\sigma))} f_0(x)\phi(x) dx.$$

Let us remark that (3.10) holds true for every test function ϕ lipschitz continuous, vanishing on the boundary of $(G^{-1}(B(y^0,\sigma)))$, then we may insert

$$\phi(x) = \sigma^2 - |G(x) - y^0|^2 \tag{3.11}$$

into (3.10). From now on, ϕ will be the function in (3.11). Let us treat the left-hand side of (3.10): changing variable, setting $y = G(x)$, $v(y) = u(G^{-1}(y))$, $\psi(y) = \phi(G^{-1}(y))$, using the chain rule and (3.6) yield

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}^{+}(x^0) D_j u(x) D_i \phi(x) dx =$$

$$= |\det JG|^{-1} \int_{B(y^0,\sigma)} \sum_{i,j=1}^{n} a_{ij}^{+}(x^0) \sum_{\beta=1}^{n} \frac{\partial v}{\partial y_\beta}(y)(JG)_{\beta j} \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial y_\alpha}(y)(JG)_{\alpha i} dy = \tag{3.12}$$

$$= |\det JG|^{-1} \int_{B(y^0,\sigma)} \sum_{\alpha=1}^{n} \frac{\partial v}{\partial y_\alpha}(y) \frac{\partial \psi}{\partial y_\alpha}(y) dy = (I).$$

Integration by parts yields

$$(I) = |\det JG|^{-1} \left\{ \int_{\partial B(y^0,\sigma)} v(y) \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial y_\alpha}(y) N_\alpha(y) d\mathcal{H}_{n-1}(y) + \right.$$

$$\left. - \int_{B(y^0,\sigma)} v(y) \sum_{\alpha=1}^{n} \frac{\partial^2 \psi}{\partial y_\alpha^2}(y) dy \right\} = \tag{3.13}$$

$$= |\det JG|^{-1} \left\{ -2\sigma \int_{\partial B(y^0,\sigma)} v(y) d\mathcal{H}_{n-1}(y) + 2n \int_{B(y^0,\sigma)} v(y) dy \right\}.$$

In order to deal with the right-hand side of (3.10), we recall (2.2), (3.11), (3.4) and (3.8), so that

$$|\phi| \le \sigma^2, \quad |D\phi| \le 2\sqrt{n/l}\sigma \qquad \text{in } G^{-1}(B(y^0,\sigma)) \tag{3.14}$$

and

$$
\begin{aligned}
&\left| \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} \left(a_{ij}(x^0) - a_{ij}(x)\right) D_j u(x) D_i \phi(x) dx \right| \leq \\
&\leq 2\sqrt{n/l}\sigma \int_{B(x^0,\sqrt{L}\sigma)} \left(\sum_{i,j=1}^{n} |a_{ij}(x^0) - a_{ij}(x)|^2 \right)^{1/2} |Du(x)| dx \leq \\
&\leq 2\sqrt{n/l}\sigma [a] (\sqrt{L}\sigma)^{\theta} \left(\int_{B(x^0,\sqrt{L}\sigma)} |Du(x)|^r dx \right)^{1/r} |B(x^0, \sqrt{L}\sigma)|^{1-1/r} \leq \\
&\leq 2\sqrt{n/l} [a] (\sqrt{L})^{\theta + n(1-1/r)} \omega_n^{1-1/r} \|Du\|_{L^r(\Omega)} \sigma^{1+\theta+n(1-1/r)},
\end{aligned}
\tag{3.15}
$$

where $\omega_n = |B(0,1)|$, $|.|$ stands for the n-dimensional Lebesgue measure,

$$
[a] = \left(\sum_{i,j=1}^{n} [a_{ij}]^2 \right)^{1/2} \qquad \text{and } [a_{ij}] = \sup |a_{ij}(x) - a_{ij}(x')|/|x - x'|^{\theta},
$$

the supremum being taken over $x, x' \in \overline{\Omega}$. Moreover,

$$
\begin{aligned}
&\left| \int_{G^{-1}(B(y^0,\sigma))} \sum_{i=1}^{n} f_i(x) D_i \phi(x) dx \right| \leq 2\sqrt{n/l}\sigma \int_{B(x^0,\sqrt{L}\sigma)} |f| dx \leq \\
&\leq 2\sqrt{n/l}\sigma \|f\|_{L^{1,\lambda}(\Omega)} \left(\sqrt{L}\sigma\right)^{\lambda} = \\
&= 2\sqrt{n/l} \left(\sqrt{L}\right)^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} \sigma^{1+\lambda},
\end{aligned}
\tag{3.16}
$$

where $f = (f_1, \ldots, f_n)$, and

$$
\begin{aligned}
&\left| \int_{G^{-1}(B(y^0,\sigma))} f_0(x) \phi(x) dx \right| \leq \sigma^2 \int_{B(x^0,\sqrt{L}\sigma)} |f_0| dx \leq \\
&\leq \sigma^2 \|f_0\|_{L^{1,\lambda-1}(\Omega)} \left(\sqrt{L}\right)^{\lambda-1} \sigma^{\lambda-1} = \left(\sqrt{L}\right)^{\lambda-1} \|f_0\|_{L^{1,\lambda-1}(\Omega)} \sigma^{1+\lambda}.
\end{aligned}
\tag{3.17}
$$

Equality (3.10) and the previous estimates merge into

$$-\sigma \int_{\partial B(y^0,\sigma)} v(y) d\mathcal{H}_{n-1}(y) + n \int_{B(y^0,\sigma)} v(y) dy \leq c_1 \sigma^{1+\gamma}, \tag{3.18}$$

where

$$\begin{aligned} c_1 = \frac{l^{-n/2}}{2} \Big\{ & 2\sqrt{n/l}[a](\sqrt{L})^{\theta+n(1-1/r)} \omega_n^{1-1/r} \|Du\|_{L^r(\Omega)} + \\ & + 2\sqrt{n/l} \left(\sqrt{L}\right)^{\lambda} \|f\|_{L^{1,\lambda}(\Omega)} + \left(\sqrt{L}\right)^{\lambda-1} \|f_0\|_{L^{1,\lambda-1}(\Omega)} \Big\} \end{aligned} \tag{3.19}$$

and

$$\gamma = \min \left\{ \theta + n \left(1 - \frac{1}{r}\right), \lambda \right\}; \tag{3.20}$$

inequality (3.18) holds true for almost every $\sigma \in (0,1]$ with $\sqrt{L}\sigma < \operatorname{dist}(x^0, \partial\Omega)$. If we set

$$h(\sigma) = \int_{B(y^0,\sigma)} v(y) dy, \tag{3.21}$$

then inequality (3.18) can be written as

$$\sigma \frac{d}{d\sigma} h(\sigma) \geq n h(\sigma) - c_1 \sigma^{1+\gamma}. \tag{3.22}$$

Set

$$\tilde{h}(\sigma) = \int_{B(y^0,\sigma)} v(y) dy + K \sigma^{1+\gamma} \tag{3.23}$$

and find $K \geq 0$ such that

$$\sigma \frac{d}{d\sigma} \tilde{h}(\sigma) \geq n \tilde{h}(\sigma), \tag{3.24}$$

that is, taking into account (3.22), $K(1+\gamma-n) \geq c_1$. Since we assumed (2.3), (2.4) and (2.6), $1+\gamma-n$ turns out to be positive, thus

$$K = \frac{c_1}{1+\gamma-n} \tag{3.25}$$

is an admissible value and (3.24) holds true. Because of (3.24), $\sigma \to \sigma^{-n}\tilde{h}(\sigma)$ is increasing, thus

$$\rho^{-n} \int_{B(y^0,\rho)} v(y)dy \leq \sigma^{-n} \left(\int_{B(y^0,\sigma)} v(y)dy + K\sigma^{1+\gamma} \right), \tag{3.26}$$

for every ρ, σ such that $0 < \rho \leq \sigma \leq \min\{1, (4L)^{-1/2} \operatorname{dist}(x^0, \partial\Omega)\}$. Now we come back to $u(x) = v(G(x))$: we change variable, we use (3.3) and (3.8) in order to get

$$\fint_{G^{-1}(B(y^0,\rho))} u(x)dx \leq \sigma^{-n}(L/l)^{n/2}\omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\sigma^{1+\gamma} \right), \tag{3.27}$$

where

$$\fint_A u(x)dx = |A|^{-1} \int_A u(x)dx.$$

Inequality (3.27) gives us an estimate from above. In order to get the corresponding estimate from below, we consider $-u$. Since $-u$ solves (2.5) with $-f_0, -f_1, \ldots, -f_n$ instead of $f_0, f_1, \ldots, f_n$, then inequality (3.27) holds for $-u$ too; thus

$$\left| \fint_{G^{-1}(B(y^0,\rho))} u(x)dx \right| \leq \sigma^{-n}(L/l)^{n/2}\omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\sigma^{1+\gamma} \right), \tag{3.28}$$

for every $x^0 = G^{-1}(y^0) \in \Omega$, $0 < \rho \leq \sigma \leq \min\{1, (4L)^{-1/2} \operatorname{dist}(x^0, \partial\Omega)\}$. Because of (3.8), we may use Theorem 8.8 in [31], thus

$$\lim_{\rho \to 0} \fint_{G^{-1}(B(y^0,\rho))} u(x)dx = u(x^0), \tag{3.29}$$

for almost every $x^0 \in \Omega$. Now, it is easy to see that u is locally bounded; this ends the proof, yet, let us write explicitly the estimate for $\sup|u|$. For every $\epsilon \in (0, 2\sqrt{L}]$, set

$$\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}, \tag{3.30}$$

then we can use (3.28) for every $x^0 \in \Omega_\epsilon$ and $0 < \rho \le \sigma = \epsilon(4L)^{-1/2}$: for almost every $x^0 \in \Omega_\epsilon$ we let ρ go to zero and we get

$$|u(x^0)| \le \left(\frac{2L}{\epsilon\sqrt{l}}\right)^n \omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\right),$$

thus

$$\|u\|_{L^\infty(\Omega_\epsilon)} \le \left(\frac{2L}{\epsilon\sqrt{l}}\right)^n \omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\right), \tag{3.31}$$

for every $\epsilon \in (0, 2\sqrt{L}]$. □

4 – An example

Consider $1 \le r < \infty$, $0 < \mu < 1$ and let $g : \mathbb{R} \to \mathbb{R}$, $g \ge 0$ be a measurable function such that

$$\int_{-1}^{1} g^r(x)dx < \infty, \tag{4.1}$$

$$\int_{-\epsilon}^{\epsilon} g^p(x)dx = \infty, \qquad \forall p > r, \qquad \forall \epsilon > 0, \tag{4.2}$$

there exists a positive constant c_2 such that

$$\int_{I_l} g^r(x)dx \le c_2 l^\mu, \tag{4.3}$$

for every interval $I_l = (a, b)$ with $I_l \subset (-1, 1)$ where $l = b - a$. Such a function can be obtained from [30], pages 12, . . . , 20. For the convenience of the reader, we write it at the end of this section. Now we set

$$u : \mathbb{R}^2 \to \mathbb{R}, \qquad u(x, y) = \int_0^x g(t)dt. \tag{4.4}$$

Then

$$(4.5) \qquad u \in C^0([-1,1]\times[-1,1]) \cap W^{1,r}((-1,1)\times(-1,1)),$$

$$(4.6) \qquad Du(x,y) = (g(x),0), \qquad |Du(x,y)| = g(x),$$

$$(4.7) \quad |Du| \in L^{1,\lambda}((-\tfrac{1}{2},\tfrac{1}{2})\times(-\tfrac{1}{2},\tfrac{1}{2})), \quad \lambda = \frac{1+\mu}{r} + 2 - \frac{2}{r}, \quad 1<\lambda<2,$$

$$(4.8) \qquad |Du| \in L^r((-1,1)\times(-1,1)),$$

$$(4.9) \qquad |Du| \notin L^p((-\epsilon,\epsilon)\times(-\epsilon,\epsilon)), \qquad \forall p > r, \qquad \forall \epsilon > 0.$$

Such a function u verifies $-\operatorname{div}(Du) = -\operatorname{div}(f)$, with $f = Du$, thus (2.1), ... ,(2.5) are fulfilled with $a_{ij} = \delta_{ij}$, $l = L = 1$, $\theta = 1$, $n = 2$, $f_0 = 0$, $f_i = \frac{\partial u}{\partial x_i}$, $\lambda = \frac{1+\mu}{r} + 2 - \frac{2}{r}$. (2.6) is fulfilled, provided $1 < r$. Let us write explicitly the function g, taken from [30], pages 12, ... , 20. For $0 < \mu < 1$, $1 \le r < \infty$, set

$$\rho_i = (2^{-i})^{1/\mu},$$
$$s_i = (1 - 2^{-1/\mu})\rho_i^{1-\mu}(e^{-2^i})^{1/(1-\mu)-1},$$
$$\psi_i = \text{ the integer part of } [(\rho_i - \rho_{i+1})/s_i],$$
$$d_i = (e^{-2^i})^{1/(1-\mu)},$$
$$j(\mu) \text{ is a suitable large positive integer, depending on } \mu,$$
$$I_i = \{x \in \mathbb{R} : -\rho_i < x < \rho_i\},$$
$$D_i = \bigcup_{k=0}^{\psi_i - 1} \{x \in I_i : \rho_i - ks_i - d_i < x < \rho_i - ks_i\},$$
$$1_{D_i} \text{ is the characteristic function of the set } D_i,$$

then,

$$g(x) = \sum_{i=j(\mu)}^{\infty} (e^{2^i})^{1/r} 1_{D_i}(x).$$

REFERENCES

[1] P. Bénilan – L. Boccardo – T. Gallouët – R. Gariepy – M. Pierre – J. L. Vazquez: *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa **22** (1995), 241-273.

[2] L. Boccardo: *Problemi ellittici con termine noto L^1 e misura*, preprint S.I.S.S.A. Trieste **114** (1996).

[3] L. Boccardo – T. Gallouët: *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989) 149-169.

[4] S. Campanato: *Fundamental interior estimates for a class of second order elliptic operators*, in: Partial Differential Equations and the Calculus of Variations, Essays in honor of Ennio De Giorgi, Birkhäuser, Boston, 1989, 251-259.

[5] S. Campanato – G. Stampacchia: *Sulle maggiorazioni L^p nella teoria delle equazioni ellittiche*, Boll. Un. Mat. Ital. **20** (1965) 393-399.

[6] A. Dall'Aglio: *Approximated solutions of equations with L^1 data. Application to the H-convergence of quasilinear parabolic equations*, Ann. Mat. Pura Appl. **170** (1996) 207-240.

[7] E. De Giorgi: *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad.Sci. Torino cl. Sci. Fis. Mat. Nat. **3** (1957) 25-43.

[8] G. Di Fazio: *Poisson equations and Morrey spaces*, J. Math. Anal. Appl. **163** (1992) 157-167.

[9] G. Di Fazio: *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital. **10-A** (1996) 409-420.

[10] A. Dolcini: *A uniqueness result for very weak solutions of p-harmonic type equations*, Boll. Un. Mat. Ital. **10-A** (1996) 71-84.

[11] A. Fiorenza – C. Sbordone: *Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1*, preprint.

[12] D. Giachetti – F. Leonetti – R. Schianchi: *On the regularity of very weak minima*, Proc. Roy. Soc. Edinburgh **126 A** (1996) 287-296.

[13] D. Giachetti – F. Leonetti – R. Schianchi: *Boundary regularity and uniqueness for very weak $\mathcal{A}$-harmonic functions*, Atti Sem. Mat. Fis. Univ. Modena, supplemento al vol. **46** (1998), 765-769.

[14] E. Giusti: Private communication, 1986.

[15] L. Greco – T. Iwaniec – C. Sbordone: *Inverting the p-harmonic operator*, Manuscripta Math. **92** (1997) 249-258.

[16] R. A. Hager – J. Ross: *A regularity theorem for linear second order elliptic divergence equations*, Ann. Sc. Norm. Sup. Pisa **26** (1972) 283-290.

[17] T. Iwaniec – C. Sbordone: *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994) 143-161.

[18] O. Ladyzhenskaya – N. Ural'tseva: *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[19] F. Leonetti: *On the regularity of ω-minima*, Boll. Un. Mat. Ital. **5-B** (1991) 21-38.

[20] F. Leonetti: *An integral estimate for weak solutions to some quasilinear elliptic systems*, Comment. Math. Univ. Carolinae **32** (1991) 39-43.

[21] J. Lewis: *On very weak solutions of certain elliptic systems*, Comm. P. D. E. **18** (1993) 1515-1537.

[22] C. Li – A. McIntosh – K. Zhang: *Higher integrability and reverse hölder inequalities*, preprint.

[23] J. Malý: *Examples of weak minimizers with continuous singularities*, Expositiones Mathematicae, to appear.

[24] G. Moscariello: *Weak minima and Quasiminima of variational integrals*, Boll. Un. Mat. Ital., to appear.

[25] G. Moscariello: *On weak minima of certain integral functionals*, preprint D.I.I.M.A., Università di Salerno.

[26] J. Moser: *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960) 457-468.

[27] F. Murat: Private communication, 1996.

[28] J. Nash: *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **8** (1958) 931-954.

[29] L. Orsina: *Weak minima for functionals and elliptic equations with measure data*, C.R.A.S., to appear.

[30] L. C. Piccinini: *Proprietà di inclusione e interpolazione tra spazi di Morrey e loro generalizzazioni*, Scuola Normale Superiore Pisa, 1969.

[31] W. Rudin: *Analisi reale e complessa*, Boringhieri, Torino, 1978.

[32] Serrin: *Pathological solutions of elliptic differential equations*, Ann. Sc. Norm. Sup. Pisa **18** (1964) 385-387.

[33] C. G. Simader: *On Dirichlet's boundary value problem*, Lecture Notes In Mathematics 268, Springer Verlag, 1972.

[34] G. Stampacchia: *Equations elliptiques du second ordre á coefficients discontinues*, Les Presses de l'Univ. de Montréal, 1966.

Lavoro pervenuto alla redazione il 26 novembre 1996
modificato il 29 ottobre 1997
ed accettato per la pubblicazione il 25 novembre 1998.
Bozze licenziate il 30 dicembre 1998

Indirizzo degli Autori:

Anna D'Ottavio – Via Coste 1, 67030 Villetta Barrea (L'Aquila), Italy
email: dottavi@univaq.it

Francesco Leonetti – Dipartimento di Matematica – Università di L'Aquila, 67100 L'Aquila, Italy – email: leonetti@univaq.it

Chiara Musciano – Via Monte Velino 15, 67100 L'Aquila, Italy – email: musciano@univaq.it