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Finite sums and generalized forms of Bernoulli polynomials

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RIASSUNTO: *In questo lavoro si introducono nuove classi di polinomi di Bernoulli, utili per il calcolo di somme parziali di polinomi di Hermite e di Laguerre. Si discute la possibilità di estendere la classe dei numeri di Bernoulli e si discute l'importanza di questi nuovi numeri per lo studio di somme parziali di polinomi di Hermite e di Laguerre di tipo generalizzato.*

ABSTRACT: *We introduce new classes of Bernoulli polynomials, useful to evaluate partial sums of Hermite and Laguerre polynomials. We also comment on the possibility of extending the class of Bernoulli numbers itself, and indicate their importance in the derivation of partial sums involving generalized forms of Hermite and Laguerre polynomials.*

1 – Introduction

The fundamental theorem of the evaluation of finite sums ensures that [1]

$$\sum_{n=1}^{N-1} n^r = \frac{1}{r+1}[B_{r+1}(N) - B_{r+1}] \tag{1.1}$$

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where B_r denotes the r-th Bernoulli number and $B_r(x)$ the Bernoulli polynomial of order r.

Bernoulli numbers and polynomials are defined through the generating functions [2]

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n \tag{1.2}$$

and

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x) \,. \tag{1.3}$$

It is therefore evident that Bernoulli numbers and polynomials are linked by the relation

$$B_n(x) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s} x^s \,. \tag{1.4}$$

The Bernoulli polynomials can be viewed as particular cases of the Appell polynomials [3] and satisfy the recurrence, common to this class of polynomials,

$$\frac{d}{dx} B_n(x) = n \cdot B_{n-1}(x) \,. \tag{1.5}$$

A straightforward expression of the $B_n(x)$ is suggested by the generating function

$$\frac{te^{xt+yt^2}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} {}_H B_n(x, y) \,, \tag{1.6}$$

where ${}_H B_n(x, y)$ denotes the Hermite-Bernoulli polynomials defined as

$${}_H B_n(x, y) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s} \cdot H_s(x, y) \,. \tag{1.7}$$

We have denoted by $H_s(x, y)$ the Kampé de Fériet polynomials [4], a generalization of the ordinary Hermite, whose series definition writes

$$H_n(x, y) = \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r} n!}{r!(n - 2r)!} \,. \tag{1.8}$$

By exploiting the properties of the $H_n(x,y)$ [5] we can infer those of ${}_HB_n(x,y)$, listed below

$$ {}_HB_n(x,0) = B_n(x)\,. \tag{1.9} $$

and

$$ \begin{aligned} &\frac{\partial}{\partial x}({}_HB_n(x,y)) = n({}_HB_{n-1}(x,y)) \\ &\frac{\partial}{\partial y}({}_HB_n(x,y)) = n(n-1)({}_HB_{n-2}(x,y))\,. \end{aligned} \tag{1.10} $$

The last two relations can be combined, to get

$$ \frac{\partial}{\partial y}({}_HB_n(x,y)) = \frac{\partial^2}{\partial x^2}({}_HB_n(x,y))\,, \tag{1.11} $$

which leads to the operational relation

$$ {}_HB_n(x,y) = e^{y\left(\frac{\partial}{\partial x}\right)^2} B_n(x)\,. \tag{1.12} $$

This paper is devoted to the analysis of the usefulness of this and other classes of polynomials involving binomial discrete convolutions of Bernoulli numbers and of ordinary or generalized polynomials.

The paper consists of three sections. In Section 2) we will exploit the ${}_HB_n(x,y)$ to derive finite sums involving Hermite polynomials, in Section 3) we will introduce the Laguerre Bernoulli polynomials and discuss their usefulness within the framework of partial summs, Section 4) is devoted to concluding remarks where we comment on the obtained results introduce generalized forms of Bernoulli numbers and outline the elements of forthcoming investigations.

2 – Hermite-Bernoulli polynomials and finite sums of Hermite polynomials

The proof of the identity (1.1) can be achieved fairly straightforwardly, by noting that

$$ \sum_{n=1}^{N-1} n^r = \Big(\frac{\partial}{\partial\lambda}\Big)^r \Big[\Big(\sum_{n=0}^{N-1} e^{\lambda n}\Big)\Big]_{\lambda=0} = \Big(\frac{\partial}{\partial\lambda}\Big)^r \Big[\frac{e^{\lambda N}-1}{e^{\lambda}-1}\Big]_{\lambda=0}\,. \tag{2.1} $$

We can take advantage from the generating function definitions (1.2), (1.3) to write

$$\begin{aligned}(2.2)\qquad &\Big(\frac{\partial}{\partial\lambda}\Big)^r\Big[\frac{e^{\lambda N}-1}{e^{\lambda}-1}\Big]_{\lambda=0}=\Big(\frac{\partial}{\partial\lambda}\Big)^r\Big[\frac{e^{\lambda N}-1}{\lambda}\frac{\lambda}{e^{\lambda}-1}\Big]_{\lambda=0}=\\ &=\Big(\frac{\partial}{\partial\lambda}\Big)^r\Big[\sum_{q=0}^{\infty}\frac{\lambda^q}{q!}B_q\Big(-\frac{1}{\lambda}+\frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{N^m}{m!}\lambda^m\Big)\Big]_{\lambda=0}=\\ &=\Big(\frac{\partial}{\partial\lambda}\Big)^r\Big[\sum_{q=0}^{\infty}\frac{\lambda^q}{q!}B_q\sum_{m=1}^{\infty}\frac{N^m}{m!}\lambda^{m-1}\Big]_{\lambda=0}\end{aligned}$$

which on account of eq. (1.4) yields the identity (1.1).

The so far outlined procedure can be exploited to derive the further relation

$$(2.3)\qquad \sum_{n=0}^{N-1}(x+ny)^r=\frac{y^r}{r+1}\Big[B_{r+1}\Big(N+\frac{x}{y}\Big)-B_{r+1}\Big(\frac{x}{y}\Big)\Big],\quad y\neq 0\,.$$

This last result and the considerations, developed in the introductory remarks about the Hermite-Bernoulli polynomials, allow to get new identities concerning finite sums of Kampé de Feriét polynomials.

The following relation

$$(2.4)\qquad e^{\eta\left(\frac{\partial}{\partial x}\right)^2}(ax+b)^n=H_n(ax+b,\eta a^2)$$

which is an elementary consequence of the basic properties of the $H_n(x,y)$ polynomials [4], can be exploited along with eqs. (1.12), (2.3) to derive the partial sum

$$(2.5)\quad \sum_{n=0}^{N-1}H_r(x+ny,\eta)=\frac{y^r}{r+1}\Big[{}_HB_{r+1}\Big(N+\frac{x}{y},\frac{\eta}{y^2}\Big)-{}_HB_{r+1}\Big(\frac{x}{y},\frac{\eta}{y^2}\Big)\Big]\,.$$

This last result, not known to the Authors' knowledge, can be viewed as a by product of the operatorial techniques developed in refs. [5].

3 – Laguerre-Bernoulli polynomials

We introduce a further extension of the Bernoulli polynomials, by defining the Laguerre-Bernoulli polynomials, for this reason we remind that the two variable Laguerre polynomials $\mathcal{L}_n(x,y)$ are specified by the generating function [6]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathcal{L}_n(x,y) = e^{yt}\mathcal{C}_0(xt) \tag{3.1}$$

where $\mathcal{C}_0(x)$ is the 0-th order Tricomi function ($\mathcal{C}_0(x) = \sum_0^{\infty} \frac{(-1)^r x^r}{(r!)^2}$) and by the series

$$\mathcal{L}_n(x,y) = \sum_{s=0}^{n} \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2}\,. \tag{3.2}$$

We can therefore conclude that the Laguerre-Bernoulli polynomials, we will denote by ${}_{\mathcal{L}}B_n(x,y)$, can be defined through the generating function

$$\frac{t}{e^t-1}e^{yt}\mathcal{C}_0(xt) = \sum_{n=0}^{n} \frac{t^n}{n!}({}_{\mathcal{L}}B_n(x,y)) \tag{3.3}$$

and the discrete convolution

$${}_{\mathcal{L}}B_n(x,y) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s}\mathcal{L}_s(x,y)\,. \tag{3.4}$$

The properties of the ${}_{\mathcal{L}}B_n(x,y)$ are a direct consequence of those of the two variable Laguerre polynomials and, indeed, they satisfy the recurrence relations

$$\begin{aligned} \frac{\partial}{\partial y}({}_{\mathcal{L}}B_n(x,y)) &= n({}_{\mathcal{L}}B_{n-1}(x,y))\,, \\ -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}({}_{\mathcal{L}}B_n(x,y)) &= n({}_{\mathcal{L}}B_{n-1}(x,y)) \end{aligned} \tag{3.5}$$

and also

$$-\frac{\partial}{\partial x}x\frac{\partial}{\partial x}({}_{\mathcal{L}}B_n(x,y)) = \frac{\partial}{\partial x}({}_{\mathcal{L}}B_n(x,y))\,. \tag{3.6}$$

It is important to emphasize that either $\mathcal{L}_n(x,y)$ and ${}_{\mathcal{L}}B_n(x,y)$ can be considered, with respect to the y variable, Appéll polynomials.

The use of the above relations and of the identity

$$\left[\frac{\partial^n}{\partial\lambda^n} e^{y\lambda}\mathcal{C}_0(x\lambda)\right]_{\lambda=0} = \mathcal{L}_n(x,y) \tag{3.7}$$

allow to establish the further relation

$$\sum_{n=0}^{N-1} \mathcal{L}_r(x, y+nz) = \frac{z^r}{r+1}\left[{}_{\mathcal{L}}B_{r+1}\Big(\frac{x}{z}, N+\frac{y}{z}\Big) - {}_{\mathcal{L}}B_{r+1}\Big(\frac{x}{z},\frac{y}{z}\Big)\right]. \tag{3.8}$$

We must underline that either eqs. (3.8) and (2.5) can be considered particular cases of more general identities, involving the Appéll-Bernoulli polynomials, which will be discussed in the forthcoming section.

4 – Concluding remarks

In the previous sections we have presented a number of identities, involving generalized forms of Bernoulli polynomials. The previous results can be unified by exploiting, as common feature, the often quoted Appéll polynomials $P_n(x,w)$, which satisfy the generating function [4]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x,w) = \mathcal{A}(wt)e^{xt}, \mathcal{A}(0) \neq 0 \tag{4.1}$$

this class of polynomials, usually defined for $w = 1$, can be used as convolution basis to define the Appell-Bernoulli polynomials, namely

$${}_PB_n(x,w) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s}P_s(x,w)\,. \tag{4.2}$$

The use of the methods, discussed in the previous sections, can be used to state the finite sum

$$\sum_{n=0}^{N-1} P_r(x+nz, w) = \frac{z^r}{r+1}\left[{}_PB_{r+1}\Big(N+\frac{x}{z},\frac{w}{z}\Big) - {}_PB_{r+1}\Big(\frac{x}{z},\frac{w}{z}\Big)\right] \tag{4.3}$$

which can be considered the identity unifying those obtained in the previous section.

Before closing this paper, let us note that we can go further in the present analysis by introducing other classes of number defined e.g. as

Hermite convolution of Bernoulli numbers on themselves, namely

$$ {}_H B_n = \sum_{s=0}^{[n/2]} \frac{n! B_{n-2s} B_s}{(n-2s)!s!} \tag{4.4} $$

or as

$$ B_{n,m} = m!n! \sum_{s=0}^{\min(n,m)} \frac{B_{n-s} B_{m-s} B_s}{(n-s)!(m-s)!s!} \,. \tag{4.5} $$

These new types of Bernoulli numbers will be shown to provide a powerful tool to study further classes of partial sums.

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