Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 1-46

The perturbation functor in the calculus of variations

O. AMICI – B. CASCIARO – M. FRANCAVIGLIA

Abstract: *In the framework of second order Calculus of Variations on jet bundles we show that the operator which determines the "First Variation" is a functor which we call "Perturbation Functor". This functor allows us to link the Jacobi morphism for the second variation to the first variation of a new Lagrangian. Its naturality properties are discussed. We also show that it permutes with most of the relevant cohomology functors of the Calculus of Variations and with the de Rham's one.*

0 – Introduction

In the last decades several techniques having a geometrical origin have been developed to deal with partial differential equations in general and, more particularly, for those equations which are the consequence of a variational principle (see, e.g., [1], [2], [3], [4] and references quoted therein). In all these frameworks, which are of course based on the use of the jet-prolongations (possibly of infinite order) of both the bundles and the equations involved, the tools of homological algebra have revealed themselves to be extremely powerful. As a few examples we mention: the work of Anderson and Duchamp ([5], for the introduction of cochain

Key Words and Phrases: *Fibred manifold – Jet space – First variation – Category – Functor.*
A.M.S. Classification: 58A12 – 58A20 – 58E30 – 18F15 – 18G60

complexes in the Calculus of Variations); the work of Bryant and Griffiths ([6] and [7], where the notion of cohomological tower is extensively used); of Tulczyjew and Dedecker ([8], with the introduction of the so-called "Lagrange complex"); of Krupka ([9] and [10], with the introduction of the so-called "variational sequences"; see also [11]).

The Calculus of Variations on jet bundles is a very powerful method in Analysis, Geometry and Matematical Physics. It allows in fact a global perspective on the problems and helps, via Noether's theorem, to provide a general setting for conservation laws (see, e.g., [14]). The fundamental ingredients in this direction are contained in the notion of contact forms, of Poincaré-Cartan forms, of local and global exactness (both at the "strong" level of the bundle or at the "weak" level of the space of critical sections).

In recent investigations of ours ([15], [16], [17]) we have been considering the somehow neglected problem of second variation of a Lagrangian action from the geometrical viewpoint, together with the ensuing notion of (generalized) Jacobi equation. In particular, we have been able to show that the Euler-Lagrange equations together with the Jacobi equations are in fact the Euler-Lagrange equations of a "derived" variational principle in a larger space, governed by a "deformed Lagrangian" which is an algebraic counterpart of the first variation of the original Lagrangian (see [16] for the definition of this new Lagrangian, [15] and [18] for an application to Riemannian Geometry and [19] for a short review).

In the course of our investigations we have realized that most of the relevant constructions entering the first variation, the second variation, the Poincaré-Cartan form and the Jacobi morphism can be alltogether factorized through a functorial operation which can be given the name of "perturbation functor". The perturbation functor, denoted by $\mathcal{P}$, essentially associates to any given Lagrangian $\mathcal{L}$ its first order deformation, in such a way that all relevant quantities of the Calculus of Variations are carried over to the analogous quantities for the new Lagrangian. Such a functor $\mathcal{P}$ is not unique, owing to the well known fact that equivalent Lagrangians and equivalent Jacobi morphisms exist (see, e.g., [10], [14], [18]), although it will be possible to choose "canonical" one.

In this paper we shall develop the basic tools to construct a reasonable (and canonical) perturbation functor in the physically relevant case of Lagrangian theories of order at most two; generalizations to higher orders are of course possible and will be considered elsewhere. We shall then begin (Section 1) with a short account about the basic framework of the Calculus of Variations on jet bundles and the notion of first order deformation of a Lagrangian. Section 2 will be devoted to introduce the fundamental categories of bundles and morphisms which are needed to our purposes, as well as to define the perturbation functor $\mathcal{P}$ and discuss some of its basic features; among them, the most useful comes from a surprising aspect of the procedure which following [16] determines the deformed Lagrangian, which in turn is determined by the existence of a class of immersions (which will be investigated in this Section and which must be taken into account, not only to understand the main properties of the deformation procedure, but also to avoid mistakes which can occur in practical calculations). In Section 3 we shall briefly account on some of the many relations existing between the cohomological interpretation of our functor $\mathcal{P}$ and the existing cohomological tools of [6] and of [9]. Our comparison will be based on the introduction of suitable ideals of forms in the de Rham complex of a convenient jet-prolongation of the relevant bundle. The sub-complex we derive differs in general from the previously existing ones and, in a sense, it is intermediate between the variational complex of [9] and the whole de Rham's complex. We shall investigate how properties of $\mathcal{P}$ reflect in these three cohomological complexes, as well as in the complex introduced in [6].

Among the results of this comparison we quote the construction of a second type of "tower prolongation" (here called "Jacobi tower") obtained by iterating the action of the functor $\mathcal{P}$. This tower prolongation is in a sense the completion of the "tower prolongation" of Bryant and Griffiths and, if applied to the cohomology investigated in [6], it provides informations on the conservation laws of the higher order Jacobi fields, while, if applied to the cohomology introduced in [9], it provides informations on the "Lepagean equivalence" of higher order deformed Lagrangians. Since the notion of "Jacobi tower" applies to any "level" of the tower construction of [6], we obtain a family of cohomological groups, here called "JBG-wall" (where JBG means Jacobi, Bryant and Griffiths). An analogous construction is made for the cohomological groups of [9], since the Bryant-Griffiths tower construction applies to these groups, too. Finally, since closed ideals generate their own cohomological groups, we show that a Jacobi tower construction is possible for both ideals used by Bryant-Griffiths in [6] and by Krupka in [9]. As we said above, we also introduce a new complex in which the Euler-Lagrange form is closed and we show that even for this complex it is possible to perform the "wall construction". More detailed investigations about the interrelationship among these various cohomologies will in fact form the subject of a forthcoming paper ([20]).

Our investigation will pay a continuous attention to the "naturality" properties of the perturbation functor, especially in view of its possible applications to the problem of conservation laws. This intriguing aspect of the theory is still under investigation and will as well form the subject of a further paper ([21]). The present paper contains an appendix, which contains a few remarks about the applications to some relevant partial differential equations of parabolic type (in the sense of [6] and [7], heat equation and KdV equation included).

1 – Preliminaries and notation

In this first Section we shall recall the main framework we need in this paper.

1.1 – Basics on calculus of variations

Let us first list some basic facts about the Calculus of Variations on fibered manifolds. Notation follows closely [2] and [22], to which we refer the reader for further details.

Let $\mathcal{B} = (B, M, \pi)$ be a fibered manifold over a m-dimensional manifold M, with p-dimensional fibers. We will denote by (x^μ), $\mu \in \{1, \dots, m\}$ a local coordinate system on M and by (x^μ, y^a), $a \in \{1, \dots, p\}$ a fibered coordinate system on $\mathcal{B}$ over (x^μ).

The *bundle of vertical vectors of* $\mathcal{B}$ is defined as follows. We set $V\pi \equiv \mathrm{Ker}(T\pi) \subseteq TB$ and we define a bundle over B as $V\mathcal{B} = (V\pi, B, \nu_B)$, where ν_B is the appropriate restriction of the natural projection $\tau_B : TB \to B$. For notational convenience, if there is no danger of confusion, we shall write VB instead of $V\pi$. In the sequel we shall be also concerned with double fibrations $C \xrightarrow{\alpha} B \xrightarrow{\pi} M$. In this case there are two vertical bundles, namely those defined by $\mathrm{Ker}(T\alpha)$ over B and by $\mathrm{Ker}[T(\pi \circ \alpha)]$ over M, respectively; they will be respectively denoted by $V^{\mathcal{B}}C$ (or, more simply, just by VC) and by $V^M C$. Hereafter, for the sake of simplicity, "vertical" will shortly mean "vertical with respect to a given projection" whenever there is no need to specify which projection is being considered (if this is already clear from the context).

For any (regular) domain D (i.e., $D \subseteq M$ is a compact m-dimensional submanifold with sufficiently regular boundary) $\Gamma_D(\pi)$ will denote the set of (local) sections $\lambda : D \to B$. Moreover, $J^k\mathcal{B} \equiv (J^k B, B, \pi^k)$ will denote the k-th order jet-prolongation of $\mathcal{B}$, with naturally induced coordinates $(x^\mu, y^a, y^a_\mu, y^a_{\mu\nu}, \dots)$. If $\lambda \in \Gamma_D(\pi)$ is a local section, locally expressed by $(x^\mu, \lambda^a(x^\rho))$, thence its k-th order jet-prolongation $j^k\lambda$ has local expression $(x^\mu, \lambda^a(x^\rho), \partial_\nu\lambda^a(x^\rho), \partial^2_{\mu\nu}\lambda^a(x^\rho), \dots)$.

A section $\Sigma : D \to J^k B$ is said to be *holonomic* iff there exists a section $\lambda : D \to B$ such that $\Sigma = j^k\lambda$. We denote by $\Lambda M = \bigoplus_{0\leq h\leq m} \Lambda^h M$ the exterior bundle of M and by $\mathbf{\Omega}(M) = \bigoplus_{0\leq h\leq m} \mathbf{\Omega}_h(M)$ the module of its sections, i.e. of differential forms of M. We set:

$$\mathbf{ds} = dx^1 \wedge \dots \wedge dx^m \quad , \quad \mathbf{ds}_\mu \equiv \partial_\mu \rfloor \mathbf{ds} \ , \tag{1.1}$$

where $X\rfloor$ (or, equivalently, sometimes i_X) denotes inner product with respect to a vectorfield X on M; the forms (1.1) determine a (local) basis for m-forms and $(m-1)$-forms, respectively.

A fibered morphism $\mathcal{L} : J^2 B \to \Lambda^m M$ will be called a (*second order*) *Lagrangian*. The Lagrangian $\mathcal{L}$ is locally expressed by:

$$\mathcal{L} = L(x^\mu, y^a, y^a_\mu, y^a_{\mu\nu})\mathbf{ds} \ , \tag{1.2}$$

where L is a scalar density on J^2B with respect to coordinate changes in the base manifold M. The Lagrangian $\mathcal{L}$ defines a *variational problem* (of the second order) on $\mathcal{B}$, through the *action functionals*:

$$\mathcal{A}(\lambda) = \int_D \mathcal{L} \circ (j^2\lambda) \ . \tag{1.3}$$

Critical sections are those sections $\lambda \equiv \lambda_0 \in \Gamma_D(\pi)$ such that

$$\delta\mathcal{A} \equiv \frac{\partial}{\partial\varepsilon}\mathcal{A}(\lambda_\varepsilon)_{|\varepsilon=0} = 0$$

for all homotopic 1-parameter deformations λ_ε (with $\varepsilon \in]-a, a[$, $a > 0$) which strongly fix the boundary (i.e., $j^1\lambda_\varepsilon|_{\partial D} = j^1\lambda|_{\partial D}$, for any ε).

Here and in the sequel the first variation operator δ will shortly denote the ε-derivative $\frac{\partial}{\partial\varepsilon}$ evaluated at $\varepsilon = 0$. It is well known that critical sections are those sections which satisfy the "Euler-Lagrange equations" of $\mathcal{L}$ (see below). From now on we shall consider only homotopic 1-parameter deformations which strongly fix the boundary.

1.2 – Horizontal forms and canonical momenta

For any integer k let $\mathcal{H}or(J^k\mathcal{B}) = \bigoplus_{0\leq q\leq m}\mathcal{H}or^q(J^k\mathcal{B})$ be the tensor algebra of horizontal forms of $J^k\mathcal{B}$ (i.e., those forms which vanish whenever they are evaluated on a set of vectorfields containing at least one vertical vectorfield).

Definition 1.1 (see [23]). The **horizontal differential** is the operator d_H uniquely defined on $\mathcal{H}or(J^k\mathcal{B})$ with values into $\mathcal{H}or(J^{k+1}\mathcal{B})$ and intrinsically expressed by:

$$(d_H\omega)\circ j^{k+1}\lambda = d(\omega\circ j^k\lambda) \quad \forall\lambda\in\Gamma(\pi)\ ,$$

for all $\omega\in\mathcal{H}or(J^k\mathcal{B})$, where d is the exterior differential operator of M.

Locally, d_H is determined by a family of operators d_μ acting on smooth functions, called *formal derivatives*. As an example, if $f : J^4B\to\mathbb{R}$ is a differentiable mapping, then $d_\mu f$ is the function on J^5B defined by:

$$d_\mu f = \frac{\partial f}{\partial x_\mu} + \frac{\partial f}{\partial y^a}y^a_\mu + \frac{\partial f}{\partial y^a_\nu}y^a_{\nu\mu} + \frac{\partial f}{\partial y^a_{\nu\rho}}y^a_{\nu\rho\mu} + \frac{\partial f}{\partial y^a_{\nu\rho\sigma}}y^a_{\nu\rho\sigma\mu} + \frac{\partial f}{\partial y^a_{\nu\rho\sigma\tau}}y^a_{\nu\rho\sigma\tau\mu}\ .$$

Finally, we set $d_V = d - d_H$, where d is now the exterior differential operator in $J^k\mathcal{B}$ (see [14]). It is known that $d_H^2 = 0$ and $d_V^2 = 0$, so that $d_Vd_H = -d_Hd_V$ because of $d^2 = 0$ (in $J^k\mathcal{B}$).

We also recall that, if $\mathcal{B} = (B, M, \pi)$ is a fibered manifold and $B_x \equiv \pi^{-1}(x)$ is its fiber over x, for any $x\in M$, then one defines the *dual vertical bundle* by setting $V^*B = \sqcup_{x\in M}(TB_x)^*$; this vector bundle $V^*\mathcal{B} = (V^*B, B, \mu_B)$ is not a sub-bundle of the cotangent bundle (T^*B, B, π_B). Let us denote by $\otimes_M$ the tensor product of vector bundles over M.

Theorem 1.1 (see [14]). *There exist two global bundle morphisms denoted by* $f^{\mathcal{B}}_{(1)}(\mathcal{L}) : J^3B \to \Lambda^{m-1}M\otimes_M V^*B$ *and* $f^{\mathcal{B}}_{(2)}(\mathcal{L})$:

$J^2B \to \Lambda^{m-1}M \otimes_M V^*J^1B$, and a global bundle morphism $e^{\mathcal{B}}(\mathcal{L}) : J^4B \to \Lambda^m M \otimes_M V^*B$, associated to the Lagrangian $\mathcal{L}$ and to its action (1.3)*, where $V^*J^1\mathcal{B} \cong J^1V^*\mathcal{B}$ is the dual bundle of the vector bundle $VJ^1\mathcal{B} \cong J^1V\mathcal{B}$* (*these last two isomorphisms being canonical*)*, which enter the following expression for the first perturbation of $\mathcal{L}$ under any homotopic variation $\lambda_\varepsilon : D \to B$ of any section $\lambda \equiv \lambda_0$:*

$$\delta(\mathcal{L} \circ j^2\lambda_\varepsilon) = e^{\mathcal{B}}(\mathcal{L}) \circ j^4\lambda + d_H[f^{\mathcal{B}}_{(1)}(\mathcal{L}) + f^{\mathcal{B}}_{(2)}(\mathcal{L})] \circ j^4\lambda \,. \tag{1.4}$$

Equation (1.4) is known as the (*global*) *first variation formula* for $\mathcal{L}$. As we said above, the critical sections of (1.3) satisfy Euler-Lagrange equations:

$$e^{\mathcal{B}}(\mathcal{L}) \circ j^4\lambda = 0 \ .$$

The bundle morphisms entering (1.4) have local expressions given, respectively, by:

$$\begin{aligned} f^\mu_a &\equiv [f^{\mathcal{B}}_{(1)}(\mathcal{L})]^\mu_a = p^\mu_a - d_\nu p^{\mu\nu}_a \ , \\ f^{\mu\nu}_a &\equiv [f^{\mathcal{B}}_{(2)}(\mathcal{L})]^{\mu\nu}_a = p^{\mu\nu}_a \ , \\ e_a &\equiv [e(\mathcal{L})^{\mathcal{B}}]_a = p_a - d_\mu[f^{\mathcal{B}}_{(1)}(\mathcal{L})]^\mu_a = \\ &= p_a - d_\mu p^\mu_a + d_\nu d_\mu p^{\mu\nu}_a \ , \end{aligned} \tag{1.5}$$

having defined the *canonical momenta* $(p_a, p^\mu_a, p^{\mu\nu}_a)$ by setting

$$p_a \equiv p_a(\mathcal{L}) = \frac{\partial L}{\partial y^a}, \quad p^\mu_a \equiv p(\mathcal{L})^\mu_a = \frac{\partial L}{\partial y^a_\mu}, \quad p^{\mu\nu}_a \equiv p(\mathcal{L})^{\mu\nu}_a = \frac{\partial L}{\partial y^a_{\mu\nu}} \ . \tag{1.6}$$

The local components $(f^\mu_a, f^{\mu\nu}_a)$ of the bundle morphisms $f^{\mathcal{B}}_{(1)}(\mathcal{L})$ and $f^{\mathcal{B}}_{(2)}(\mathcal{L})$ are known as the *true momenta*, while $e^{\mathcal{B}}(\mathcal{L})$ is the *Euler-Lagrange morphism*.

Remark. Notice that the bundle morphisms above determine in turn the following tensorfields, which by an abuse of notation will be denoted by the same symbols of the corresponding morphisms:

$$\begin{aligned} f^{\mathcal{B}}_{(1)}(\mathcal{L}) &= f^\mu_a dy^a \wedge \mathbf{ds}_\mu \ , \\ f^{\mathcal{B}}_{(2)}(\mathcal{L}) &= f^{\mu\nu}_a dy^a_\mu \wedge \mathbf{ds}_\nu \ , \\ e^{\mathcal{B}}(\mathcal{L}) &= e_a dy^a \wedge \mathbf{ds} \ . \end{aligned} \tag{1.7}$$

1.3 – Contact forms and symmetries

Definition 1.2. The ideal of **contact forms** $\mathcal{K}(J^2\mathcal{B})$, is the ideal of the exterior algebra $\mathbf{\Omega}(J^3B)$ formed by those forms $\eta \in \mathbf{\Omega}(J^2B)$ which vanish along all holonomic sections $j^2\lambda$ of the bundle $J^2\mathcal{B} = (J^2B, M, \pi)$.

The ideal $\mathcal{K}(J^2\mathcal{B})$ is generated by the following family of local 1-forms:

$$\theta^a = dy^a - y^a_\sigma dx^\sigma \quad , \quad \theta^a_\mu = dy^a_\mu - y^a_{\mu\sigma} dx^\sigma \ , \tag{1.8}$$

by the ring $\mathbf{\Omega}^0(J^2B)$.

Definition 1.3. The **Poincaré-Cartan form** is the m-form along the canonical projection of J^3B onto B, having the following local expression:

$$\mathbf{\Theta} \equiv \Theta^{\mathcal{B}}(\mathcal{L}) = (f^\mu_a \theta^a + f^{\mu\nu}_a \theta^a_\nu) \wedge \mathbf{ds}_\mu + \mathcal{L} \ . \tag{1.9}$$

Finally, the form $\mathbf{\Omega} \equiv \mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) = d\mathbf{\Theta}$ is the multiplectic form of the variational problem (see [24]). This form $\mathbf{\Omega}$ determines the Euler-Lagrange equations, which can in fact be equivalently written as:

$$(j^3\sigma)^*(i_v(\mathbf{\Omega})) = 0 \ , \quad \forall v \in VJ^3B \cong J^3VB \ . \tag{1.10}$$

For more details see, e.g., [22] and [25].

Now we denote by $\mathbf{L}$ the Lie derivative operator, defined on the sections of a bundle $\mathcal{B}$ whenever the bundle is a natural bundle (see [26]) or a gauge-natural bundle (see [27], [28] and [29]).

Definition 1.4 (see [12] and [25]). An **infinitesimal symmetry** is a vectorfield $\Xi \in \mathcal{X}(J^3B)$ is said to be of $\mathcal{L}$ if:

$$\mathbf{L}_\Xi[\Theta^{\mathcal{B}}(\mathcal{L})] = 0 \ . \tag{1.11}$$

Then $E^c(\mathcal{L}, \Xi, \lambda) = (j^3\lambda)^*(\Xi \rfloor \Theta(\mathcal{L}))$ is called the **conserved Noether current associated to** Ξ.

If λ is a solution of the Euler-Lagrange equations of $\mathcal{L}$ one has $d_\mu[E^c(\mathcal{L}, \Xi, \lambda)]^\mu = 0$ (see [12] and [25]).

As we explained in the Introduction, we are here interested into investigating the naturality of the "first order perturbation" procedure, by means of a functor suitably defined on a suitable category. Obviously, the "largest" is the category on which the functor is defined, the strongest will be its naturality properties.

Definition 1.5. A (local) section $\lambda : U \to B$ is said to be **admissible for** Φ if and only if the mapping $\phi_t^\lambda \equiv \phi_t \circ \lambda : U \to \phi_t^\lambda(U) = V_t$ is a local diffeomorphism.

Theorem 1.2. *Equation* (1.11) *is meaningful even if the local* 1-*parameter group* $\Phi = \{\Phi_t\}$ *generated by* Ξ *is not a* (*local*) *group of bundle automorphisms*, *but just a group of diffeomorphisms of the total space.*

Proof. In fact, let us set $\phi_t = \pi \circ \Phi_t : B \to M$. Then the action of Φ on λ is defined by setting

$$\lambda_t(x) \equiv (\Phi_t^* \lambda)(x) = \Phi_t \circ \lambda \circ (\phi_t^\lambda)^{-1}(x) \tag{1.12}$$

for any $x \in V_t$; the family $\{\lambda_t\}_{t\in(-\varepsilon,\varepsilon)}$, with $\varepsilon > 0$, is a homotopic variation of $\lambda \equiv \lambda_0$ and, as in [22] and [26], we have:

$$\mathbf{L}_\xi(\lambda) \equiv \left[\frac{d}{dt}\lambda_t\right]_{|t=0} = T\lambda \circ \xi_\lambda - \Xi \circ \lambda \ , \tag{1.13}$$

where $\xi_\lambda = T\pi \circ \Xi \circ \lambda$ is a vectorfield over the basis M (which depends of course on the section λ). ∎

Remark. As a consequence, the results of [22] and [26] hold true also in this case, which is obtained by restoring the classical definition of the action of a differentiable mapping on a "field". In fact, let $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ be two fiber bundles and $F : B \to B'$ a differentiable mapping between the total spaces of the two bundles (not necessarily a bundle morphism). We set $f_F = \pi' \circ F : B \to M'$ and call it the *basic map associated to* F. We also say that a section $\lambda : M \to B$ is *admissible for* F if and only if $\tilde{f}_F = f_F \circ \lambda : M \to M'$ is a (local) diffeomorphism; in this case, of course, M and M' have to be of the same dimension. Then the classical action of F on the set of admissible sections is given by:

$$F.\lambda = F \circ \lambda \circ (f \circ \lambda)^{-1} \ , \tag{1.14}$$

for any admissible section $\lambda : M \to B$.

1.4 – Second variation of Lagrangians

We will follow [16] for the second variation.

Definition 1.6. The **first order perturbation** $\mathcal{L}_{(1)} : J^2VB \to \Lambda^m M$ of the Lagrangian $\mathcal{L} \equiv \mathcal{L}_{(0)}$ is the (unique and global) morphism with local expression given by:

$$\mathcal{L}_{(1)} \equiv L_{(1)}\mathbf{ds} = \{p_a \rho^a + p_a^\mu \rho_\mu^a + p_a^{\mu\nu} \rho_{\mu\nu}^a\}\mathbf{ds} \;, \tag{1.15}$$

where $(\rho^a, \rho_\mu^a, \rho_{\mu\nu}^a)$ are the local components of an element of VJ^2B (canonically identified with J^2VB).

The action functional associated to the Lagrangian $\mathcal{L}_{(1)}$ is given by:

$$\tilde{\mathcal{A}} = \int_D \mathcal{L}_{(1)} \circ (j^2\lambda \times j^2 v) \;, \tag{1.16}$$

for any local section $\lambda \in \Gamma_D(\pi)$ and any vertical vectorfield v which projects onto the section λ. We also set:

$$e^{\mathcal{B}}(\mathcal{L}_{(1)}) = \tilde{e}_a d\rho^a + E_a dy^a \;. \tag{1.17}$$

Theorem 1.3. *The following holds*:

$$\begin{aligned} \tilde{e}_a &= [e^{\mathcal{B}}(\mathcal{L})]_a = e_a \;, \\ E_a &\equiv [E^{\mathcal{B}}(\mathcal{L})]_a = P_a - d_\mu [F_{(1)}^{\mathcal{B}}]_a^\mu = P_a - d_\mu P_a^\mu + d_\mu d_\nu P_a^{\mu\nu} \;, \end{aligned} \tag{1.18}$$

where

$$\begin{aligned} [F_{(1)}^{\mathcal{B}}]_a^\mu &\equiv [F_{(1)}^{\mathcal{B}}(\mathcal{L})]_a^\mu = P_a^\mu - d_\nu P_a^{\nu\mu} \\ [F_{(2)}^{\mathcal{B}}]_a^{\mu\nu} &\equiv [F_{(2)}^{\mathcal{B}}(\mathcal{L})]_a^{\mu\nu} = P_a^{\mu\nu} \;, \end{aligned} \tag{1.19}$$

being

$$\begin{aligned} P_a &\equiv \frac{\partial L_{(1)}}{\partial y^a} \\ P_a^\mu &\equiv [P^{\mathcal{B}}(\mathcal{L})]_a^\mu = \frac{\partial L_{(1)}}{\partial y_\mu^a} \\ P_a^{\mu\nu} &\equiv [P^{\mathcal{B}}(\mathcal{L})]_a^{\mu\nu} = \frac{\partial L_{(1)}}{\partial y_{\mu\nu}^a} \;, \end{aligned} \tag{1.20}$$

with e_a *defined by* (1.5).

Proof. See [16].

Remark. With the positions of Theorem 1.3 the non-covariant part E_a represents the coefficients of the Jacobi morphism of $\mathcal{L}$ (as defined in our previous paper [15]).

Definition 1.7. The **Hessian mapping** $\mathrm{Hess}_{\mathcal{B}}(\mathcal{L})\colon J^2B\times_B J^2VB\times_B J^2VB\to\Lambda^m M$, where $\times_B$ denotes the fibered product over B, is given by:

$$\begin{aligned}\mathrm{Hess}_{\mathcal{B}}(\mathcal{L})_{(x,y)}(\xi;\rho) = [P_a]_{(x,y)}(\rho)\xi^a + [P^{\mu}_{a}]_{(x,y)}(\rho)\xi^a_\mu +\\ + [P^{\mu\nu}_{a}]_{(x,y)}(\rho)\xi^a_{\mu\nu}\ ,\end{aligned} \tag{1.21}$$

where $\xi = (\xi^a, \xi^a_\mu, \xi^a_{\mu\nu})$ are the local coordinates of a further point belonging to the fiber of $J^2V\mathcal{B}$ over the point of B having local coordinates (x^μ, y^a).

Equation (1.21) gives in fact the Hessian mapping of the variational problem (see [17]).

1.5 – Basic categories

We finally list the basic categories used in this paper. We shall adopt the following standard notation. If τ is any category , we shall denote by $\tau(O,O')$ the set of all morphisms in τ from O into O', being O, O' objects of τ.

i) The category $\mathcal{M}an$ having as objects the (C^∞-differentiable) manifolds and as morphisms the (C^∞-differentiable) mappings between manifolds.
ii) The category $\mathbf{B}un$ whose objects are the fiber bundles $\mathcal{B} = (B, M, \pi)$ over any manifold M (object of $\mathcal{M}an$) and whose morphisms are the usual bundle morphisms (i.e., the fiber preserving differentiable mappings between fiber bundles).
iii) By $\mathbf{VB}un$ we denote the subcategory of $\mathbf{B}un$ having as objects the vector bundles and as morphisms the linear bundle morphisms.
iv) In this last category we will make use of the subcategory $T\mathcal{M}an$ whose objects are the tangent bundles of the manifolds M of $\mathcal{M}an$ and, if M and N are two manifolds of $\mathcal{M}an$, a mapping $F : TM \to TN$ belongs to the set of morphisms $T\mathcal{M}an(TM, TN)$ in this category if and only if $F = Tf$ is the *tangent mapping of the mapping*

$f \in \mathcal{M}an(M, N)$. In the following, by an abuse of notation, we will denote simply by TM the tangent bundle (TM, M, τ_M); moreover T is the so called *tangent functor*.

v) Finally, $\mathbf{V}ec$ will denote the category of real vector spaces whose morphisms are linear mappings between pairs of real vector spaces.

The basic functor we shall need between the category $\mathcal{M}an$ and the category $\mathbf{V}ec$, namely the functor which associates to any manifold M its total de Rham cohomology group $H_{dR}(M)$, will be denoted by H_{dR}. Recently, (see, e.g., [6] and [9]), some new cohomological functors related to the Calculus of Variations and/or to partial differential equations have also been introduced in the literature.

A result which can be easily inferred by comparing [6] with [9] is that the construction needed to obtain the cohomological groups related to these functors is somehow standard. In fact, all these cohomological groups are obtained by first choosing some graded ideal $\mathcal{I}(J^kB)$ of the graded exterior algebra $\widehat{\mathbf{\Omega}}(J^kB) \equiv \mathcal{H}or(J^k(\mathcal{B}) \oplus \mathcal{K}(J^k\mathcal{B})$ ($k = \infty$ is not excluded) having the property

$$d(\mathcal{I}(J^kB)) \subseteq \mathcal{I}(J^kB) \ . \tag{1.22}$$

One then takes the quotient of $\widehat{\mathbf{\Omega}}(J^kB)$ with respect to $\mathcal{I}(J^kB)$, to obtain a cochain complex, and then considers the cohomological groups of this last complex. Notice that $\mathbf{\Omega}(J^kB) \subseteq \widehat{\mathbf{\Omega}}(J^kB)$. Finally, the ideals of $\widehat{\mathbf{\Omega}}(J^kB)$ verifying (1.22) will be called *closed ideals*.

In order to introduce the aforementioned functors (especially for the functor defined in [6], which is far too general with respect to the case considered here) we need some further construction. These will be given in Section 2, where we shall introduce the "perturbation functor", while the relations of our new functor with the functors of [6] and [9] will be shortly discussed in Section 3.

2 – The first order perturbation functor

Physicists make use of many "perturbation techniques", which are quite different among each other. Here we shall consider only those perturbations which were studied in an explicit way in [16], since they are the starting point from which many physical results are obtained through the Calculus of Variations; we just quote [30] and [17] (where applications to gravitational theories can be found, both in the case of General Relativity and in the case of non-linear gravitational Lagrangians) since these two papers are more closely related to our present interest.

2.1 – Definition of the functor $\mathcal{P}$

In a previous paper of ours [18] it was shown that the "complete lift" used in differential geometry (see, e.g., [31]) is nothing but a particular case of the perturbation technique recently introduced in [16]. The functor we shall be dealing with can be deduced by using the perturbation considered in the aforementioned papers and is in fact obtained by composing the tangent functor with some other suitable functors, having the same degree of naturality.

As is well known, adding any divergence to a given Lagrangian $\mathcal{L}$ does not affect Euler-Lagrange equations $e^{\mathcal{B}}(\mathcal{L}) \circ j^4\lambda = 0$, but several constructions suffer changes: e.g., the Poincaré-Cartan form changes, giving then rise to different boundary terms in the action, as well as a different but dynamically equivalent version of equation (1.10) (see, e.g., [32]). Therefore, even in the class of "perturbations" considered here one can define many different "perturbations" for the same "original" set of Euler-Lagrange equations. We shall here propose a kind of a "canonical choice". We think in fact that our functor is the simplest possible one and, as an example, we shall compare it with the one which could be deduced from Taub's paper [30]. In any case, all the functors obtained in this way would be "equivalent in a suitable sense" from the viewpoint of the Calculus of Variations.

Before going on, let us first notice that there is no substantial difference between mappings and sections from the viewpoint of the Calculus of Variations. In fact, if the variational problem is defined on the set of all mappings from M into a further manifold N, one can uniquely identify any mapping $h : M \to N$ with the section $\lambda_h : M \to M \times N$ of the trivial bundle $pr_1 : M \times N \to M$, being pr_1 the natural projection on the first factor, defined by $\lambda_h(x) = (x, h(x))$. The converse is also true, as any section $\lambda : M \to B$ is nothing but a mapping which satisfies the constraint $\pi \circ \lambda = id$. An analogous remark holds also for the groups of diffeomorphisms.

In order to define our perturbation functor we need some new categories and some new functors which are easily obtained from the ones considered in Section 1. Because of our introductory remarks about conservation laws, the category $\mathbf{B}un$ does not contain enough morphisms. Hence, we denote by $\mathbf{B}$ the category having as objects all the bundles $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}un$ and as morphisms all the differentiable mappings between the total spaces of pairs of bundles. If $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are two objects of $\mathbf{B}$, we have then $\mathbf{B}(\mathcal{B}, \mathcal{B}') = \mathcal{M}an(B, B')$. Obviously, the category $\mathbf{B}un$ is a full sub-category of this category $\mathbf{B}$.

Recall that for any pair of objects $\mathcal{B}$, $\mathcal{B}'$ of $\mathbf{B}$ and for any $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ we have set $f_F = \pi' \circ F$ and we have called it the *basic map associated to* F. It is obvious that F belongs to $\mathbf{B}un(\mathcal{B}, \mathcal{B}')$ if and only if the basic map f_F is constant along the fibers of $\mathcal{B}$. In this case f_F defines a map $f'_F : M \to M'$ which is called the “induced map” and is such that $f_F \equiv \pi' \circ F = f'_F \circ \pi$.

The second category we need is denoted by $T\mathbf{B}$. Its objects are the tangent bundles $T\mathcal{B} = (TB, TM, T\pi)$, i.e. the images under the tangent functor T of all bundles $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}$, while its morphisms are the images by T of the morphisms of $\mathbf{B}$.

A third category we shall need, denoted by $R\mathbf{B}$, is defined as follows: its objects are the fiber bundles of the trivial type $R\mathcal{B} = (\mathbb{R} \times B, \mathbb{R} \times M, id_{\mathbb{R}} \times \pi)$, where $\mathcal{B} = (B, M, \pi)$ is any object of $\mathbf{B}$, and $id_{\mathbb{R}} \times \pi : \mathbb{R} \times B \to \mathbb{R} \times M$ is defined by setting $(id_{\mathbb{R}} \times \pi)(t, y) = (t, \pi(y))$ for any $(t, y) \in \mathbb{R} \times B$. In this category a morphism $F \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$, being $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ objects of $\mathbf{B}$, is a pair of mappings $(id_{\mathbb{R}}, \tilde{F}) : \mathbb{R} \times B \to \mathbb{R} \times B'$. Hence we have typical morphisms $(id_{\mathbb{R}}, \tilde{F})(\varepsilon, y) = (\varepsilon, \tilde{F}(\varepsilon, y))$, where $\tilde{F} : \mathbb{R} \times B \to B'$ is the *mapping defining a homotopic variation* $F_\varepsilon \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ of $F_0 : B \to B'$, with $\varepsilon \in \mathbb{R}$; i.e., $\tilde{F}(\varepsilon, x) = F_\varepsilon(x)$, for any $\varepsilon \in \mathbb{R}$ and $x \in M$.

Remark. Since in the Calculus of Variations we are interested only into a neighborhood of $0 \in \mathbb{R}$, we can consider as homotopic variations (modulo a possible reparametrization on ε) only the families $F_\varepsilon \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$, with ε varying in the whole of $\mathbb{R}$, identifying them with the morphisms of the category $R\mathbf{B}$. Since all objects M of the category $\mathcal{M}an$ are objects of $\mathbf{B}$ via the trivial bundle structure (M, M, id_M), also the homotopic variations of local sections can be considered as morphisms in the previous category; in this last case we shall consider only homotopic variations strongly preserving the boundary (see [17]).

THEOREM 2.1. *There exists a natural covariant functor from the category* $\mathbf{B}$ *into the category* $R\mathbf{B}$, *which, with an abuse of notation, will be again denoted by* R. *This functor* R *will be called the* **canonical lift**.

PROOF. Immediate, by defining R as the functor which associates to any bundle $\mathcal{B} = (B, M, \pi)$ of $\mathbf{B}$ the bundle $R\mathcal{B} = (\mathbb{R} \times B, \mathbb{R} \times M, id_{\mathbb{R}} \times \pi)$ of $R\mathbf{B}$ and to any morphism $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ between the objects $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbf{B}$ the morphism $RF = (id_{\mathbb{R}} \times F) \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$. ∎

The canonical lift of F acts on a homotopic variation $\sigma : \mathbb{R} \times M \to \mathbb{R} \times \mathcal{B}$ in the following way. Let $\lambda_\varepsilon : M \to B$ be the family of mappings defining σ, i.e. $\sigma(\varepsilon, x) = \lambda_\varepsilon(x)$. We say that σ is *admissible* for RF if and only if λ_ε is admissible for F, for any $\varepsilon \in \mathbb{R}$. Then we can consider the mapping $\tau : \mathbb{R} \times M' \to B'$ defining the homotopic variation $F.\lambda_\varepsilon = F \circ \lambda_\varepsilon \circ (f_F \circ \lambda_\varepsilon)^{-1} : M' \to B'$, for any $\varepsilon \in \mathbb{R}$, being f_F the basic map associated to F. By these remarks the action of RF is defined as $(RF).\sigma = (id_{\mathbb{R}}, \tau) : \mathbb{R} \times M \to \mathbb{R} \times \mathcal{B}$.

Finally we have the further category $T^R\mathbf{B}$ whose objects are the bundles $T^R\mathcal{B} = (T\mathbb{R} \times TB, T\mathbb{R} \times TM, id_{T\mathbb{R}} \times T\pi)$, with $\mathcal{B} = (B, M, \pi)$ any object of $\mathbf{B}$, and whose morphisms are the mappings $(id_{T\mathbb{R}}, \tilde{F}) \in T^R\mathbf{B}(T^R\mathcal{B}, T^R\mathcal{B}')$, where $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are bundles of $\mathbf{B}$ and $\tilde{F} : \mathbb{R} \times TB \to B'$ is a mapping which defines a homotopic variation $F_\varepsilon : TB \to TB'$ between linear bundle morphisms.

DEFINITION 2.1. The **evaluation functor** $\mathcal{E}$ is the covariant functor from the category $T^R\mathbf{B}$ with values into the category $\mathbf{B}$ defined as follows: the functor $\mathcal{E}$ associates to any object $T^R\mathcal{B}$ of $T^R\mathbf{B}$ the canonical lift $RT\mathcal{B}$ of $T\mathcal{B}$ and to any morphism $(id_{T\mathbb{R}}, \tilde{F}) \in T^R\mathbf{B}(T^R\mathcal{B}, T^R\mathcal{B}')$ the morphism $(id_{\mathbb{R}}, \tilde{F}_0) \in R\mathbf{B}(RT\mathcal{B}, RT\mathcal{B}')$, via the canonical identification $T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ obtained by means of the standard chart $(\mathbb{R}, id_{\mathbb{R}})$, $\tilde{F}_0$ being defined by $\tilde{F}_0(z) = \tilde{F}(0, z)$, for all $z \in TB$.

We set now $\mathcal{E} \circ T = T_{\mathcal{E}}$

DEFINITION 2.2. The **first order perturbation functor**, is defined on the category $\mathbf{B}$ and takes its values into the category $R\mathbf{B}$. It is the covariant functor defined by

$$\mathcal{P} = \mathcal{E} \circ T \circ R = T_{\mathcal{E}} \circ R \tag{2.1}$$

and it associates to any bundle $\mathcal{B} = (B, M, \pi)$ the bundle $\mathcal{P}\mathcal{B} = (\mathbb{R} \times TB, \mathbb{R} \times TM, id_{\mathbb{R}} \times T\pi)$.

It is now easy to see that the following holds:

PROPOSITION 2.2. *For any homotopic variation* $\sigma : \mathbb{R} \times M \to \mathcal{B}$, *which is assumed to be admissible for a morphism* $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$, *with* $\mathcal{B}$ *and* $\mathcal{B}'$ *objects in* $\mathbf{B}$, *one has*:

$$\mathcal{E}(T((RF).\sigma)) = T_{\mathcal{E}}((RF).\sigma) = (\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma) \ .$$

Moreover, we have:

PROPOSITION 2.3. *The functor* $\mathcal{P}$ *is a true perturbation functor.*

PROOF. Let us consider two bundles $\mathcal{B}=(B, M, \pi)$ and $\mathcal{B}'=(B', M', \pi')$ and a morphism $\tilde{F} \in R\mathbf{B}(R\mathcal{B}, R\mathcal{B}')$. We first notice that $T_{\mathcal{E}}\tilde{F}$ belongs to $W \equiv T^*(\mathbb{R} \times B) \otimes T(\mathbb{R} \times B') \cong (T^*\mathbb{R} \otimes T\mathbb{R}) \oplus (T^*B \otimes T\mathbb{R}) \oplus (T^*\mathbb{R} \otimes TB') \oplus (T^*B \otimes TB')$; here $\otimes$ and $\oplus$ generically denote the product bundles over the product of the bases with the natural vector bundle structures given by pairwise operations in the product of the fibers. Since the standard chart $(\mathbb{R}, id_{\mathbb{R}})$ is fixed in $\mathbb{R}$, we have the mapping $w : W \to TB'$, which acts as follows: to any element of W it associates the component belonging to $TB' \otimes T^*\mathbb{R}$, considered as forming a vector of B'. In fact, if X belongs to W, we have:

$$X = a\frac{\partial}{\partial t} \otimes dt + \omega \otimes \frac{\partial}{\partial t} + dt \otimes Y + P \ ,$$

where a is an arbitrary real number, Y a vector of B', ω a 1-form of B and P a tensor on $T^*B \otimes TB'$. Then it follows:

$$w(X) = Y \ .$$

We have thence (with an obvious meaning of the symbols used):

$$\delta\lambda = w((\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma)) \ , \tag{2.2}$$

with the obvious relation between the homotopic deformation λ_ε and the mapping σ. (We write $\delta\lambda$ for the first variation of λ_ε since it refers to the value at $\varepsilon = 0$). This proves our claim. ∎

2.2 – Coordinate expression of $\mathcal{P}$

Some of the properties which will be useful in the sequel can now be easily seen in terms of local coordinates. Hence we consider two fiber bundles $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$, a morphism $F \in \mathbf{B}(\mathcal{B}, \mathcal{B}')$ and a homotopic variation $\sigma : \mathrm{I\!R} \times M \to \mathrm{I\!R} \times B$ admissible for F. We notice that M and M' have the same dimension, since we have assumed that admissible sections exist; we denote by (x^μ, y^a) and (z^μ, y^A) natural coordinate systems in $\mathcal{B}$ and $\mathcal{B}'$, respectively, and by $z^\mu = f^\mu(x^\nu, y^a)$ the local representation of the basic map $f_F : B \to M'$. Then we have:

$$[w((\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma))]^A = \left\{ \frac{\partial F^A}{\partial y^b} - \left[\frac{\partial F^A}{\partial x^\mu} + \frac{\partial F^A}{\partial y^a}\frac{\partial \sigma^a}{\partial x^\mu} \right] C^\mu_\nu \frac{\partial f^\nu}{\partial y^b} \right\} \frac{\partial \sigma^b}{\partial \varepsilon} \ , \tag{2.3}$$

where the matrix $\|C^\mu_\nu\| \equiv \|C^\mu_\nu(j^1\sigma)\|$ is the inverse of the matrix $\|\bar{C}^\nu_\mu\|$ defined by:

$$\bar{C}^\nu_\mu \equiv \bar{C}^\nu_\mu(j^1\sigma) = \frac{\partial f^\nu}{\partial x^\mu} + \frac{\partial f^\nu}{\partial y^a}\frac{\partial \sigma^a}{\partial x^\mu} \ , \tag{2.4}$$

which has maximal rank since σ is an admissible homotopic variation. Now we notice that $T\sigma$ is a section from the basis $\mathrm{I\!R} \times M$ into the total space $T^*(\mathrm{I\!R} \times M) \otimes T(\mathrm{I\!R} \times B) \cong (T^1_1\mathrm{I\!R}) \oplus (T^*\mathrm{I\!R} \otimes TB) \oplus (T^*M \otimes T\mathrm{I\!R}) \oplus (T^*M \otimes TB)$. Since σ is a section of a bundle and many of its derivatives are hence constant, we can replace the previous vector bundle by the simpler vector bundle $(VB \otimes T^*\mathrm{I\!R}) \oplus (T^*M \otimes VB)$. Finally, when the functor $T_{\mathcal{E}}$ is considered, the previous bundle simplifies further to a bundle diffeomorphic to $J^1B \times_B VB$. Hence we can define the new action of $\mathcal{P}_{\mathcal{B}}F$ by simply setting:

$$[(\mathcal{P}_{\mathcal{B}}F)_*(y, v)]^A = \left\{ \frac{\partial F^A}{\partial y^b} - (d_\mu F^A) C^\mu_\nu \frac{\partial f^\nu}{\partial y^b} \right\} v^b \ , \tag{2.5}$$

for any (y, v) not belonging to the closed subset of $J^1B \times_B VB$ having local equation:

$$\det \|\bar{C}^{\nu}_{\mu}\| = \det \|d_{\mu} f^{\nu}\| = 0 \ . \tag{2.6}$$

Equation (2.2) shows that the functor $\mathcal{P}$ defined by (2.1) is a true perturbation functor, acting through the action (2.5) and defined everywhere except a closed subset of the bundle $(J^1B \times_B VB, M, \tau)$, where τ denotes the obvious projection, determined by equation (2.6); the elements of the domain of regularity for $\mathcal{P}_{\mathcal{B}}F$ will be called *admissible for* $\mathcal{P}_{\mathcal{B}}F$. Luckily enough, the use of this complicated form of the first order perturbation functor can be avoided in most cases: we shall need it, in fact, only to study the perturbation of the Noether equation in its classical form, i.e. when the action of the 1-parameter group is defined by (1.12). In the other cases the category $\mathbf{B}un$ is enough for the study of variational problems.

PROPOSITION 2.4. *In the category* $\mathbf{B}un$, *the first order perturbation functor restricted to a simpler functor* $\widehat{\mathcal{P}}$ *which does not depend any longer on* $j^1\sigma$, *but only on the* ε*-derivative of* σ.

PROOF. In fact, in this case (2.5) becomes

$$[w((\mathcal{P}_{\mathcal{B}}F).(T_{\mathcal{E}}\sigma))] = \frac{\partial F^A}{\partial y^a}(\delta\lambda)^a \frac{\partial}{\partial y^A} \tag{2.7}$$

and all the sections become admissible. Hence $(\mathcal{P}_{\mathcal{B}}F)$ can be considered as a fiberpreserving linear mapping defined on VB taking its values into VB'. As a consequence, we can replace $\mathcal{P}$ with a new functor $\widehat{\mathcal{P}}$, which associates to any bundle $\mathcal{B}$ over M the bundle $V\mathcal{B}$ over M endowed with the obvious projection and which transforms morphisms according to (2.7). In other words, $\widehat{\mathcal{P}}$ associates to any mapping $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$ the mapping $\widehat{\mathcal{P}}_{\mathcal{B}}F \in \mathbf{B}un(V\mathcal{B}, V\mathcal{B}')$ defined by:

$$(\widehat{\mathcal{P}}_{\mathcal{B}}F)_{(y,v)} = \frac{\partial F^A}{\partial y^a} v^a \frac{\partial}{\partial y^A} \ , \tag{2.8}$$

for any vertical vector v over $y \in B$, having local components v^a. This ends our proof. ∎

Definition 2.3. The functor $\widehat{\mathcal{P}}$ is called **reduced first-order perturbation functor**.

Remarks.

1). Equation (2.8) shows that in this case $\widehat{\mathcal{P}}_{\mathcal{B}}F : VB \to VB'$ is a bundle morphism also with respect to the bundle structures $VB \to B$ and $VB' \to B'$ and moreover $F : B \to B'$ is the map induced by $\widehat{\mathcal{P}}_{\mathcal{B}}F$. Because of this, in the sequel we shall omit to write the induced maps and diagrams, for the sake of brevity.

2). We remark that in this case $\widehat{\mathcal{P}}$ can be alternatively defined as the unique functor which associates to any object $\mathcal{B}$ in the category $\mathbf{B}un$ its vertical bundle $V\mathcal{B}$ and to any bundle morphism $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$, with $\mathcal{B}$ and $\mathcal{B}'$ objects of $\mathbf{B}un$, the unique bundle morphism $\widehat{\mathcal{P}}(F) : VB \to VB'$ defined by setting:

$$\delta(F \circ \lambda) = \widehat{\mathcal{P}}_{\mathcal{B}}(F)(\delta\lambda) \tag{2.9}$$

for all mappings $\sigma : \mathrm{I\!R} \times M \to B$ which define a homotopic variation of a section $\lambda : M \to B$.

Theorem 2.5. *When the functor $\widehat{\mathcal{P}}$ is restricted to curves, as in the case of Riemannian Geometry, it essentially coincides with the tangent functor T.*

Proof. This follows easily from (2.4) and (2.8). □

Many of the consequences of Theorem (2.5) existence are well known, even if they were never explicitely introduced as a consequence of variational principles (this aspect of Riemannian Geometry includes more properties than what people generally think; as an example of this fact we just quote [17], where the curvature of general variational problems of "harmonic type" is discussed in some detail). The results related to the existence of the perturbation functor for curves are in fact known as consequences of the *complete lift* (see [31]) and are related to our functor in the following way. The fiber bundle $\mathrm{I\!R} \times M \to \mathrm{I\!R}$ can be associated to the Riemannian manifold (M, g) and curves can be thought as sections of this bundle in an obvious way. Since we have $V(\mathrm{I\!R} \times M) = \mathrm{I\!R} \times TM$, to any differentiable mapping $f : M \to M'$ one can associate the mapping $F_f : \mathbb{R} \times M \to \mathbb{R} \times M'$ defined by $F_f(\varepsilon, x) = (\varepsilon, f(x))$. Then $\widehat{\mathcal{P}}_{\mathbb{R}\times M}(F_f) = (id, Tf) : \mathbb{R} \times TM \to \mathbb{R} \times TM'$ and the total differential Tf of f is nothing but the complete lift of f. Most of the constructions related to the variational aspects of Riemannian Geometry, e.g. those discussed in [31], will then coincide with our results (see also [18] for more details).

2.3 – Some Lagrangian properties of the reduced first order perturbation functor $\widehat{\mathcal{P}}$

Let us now consider the tensor bundle $\tau M = (\mathbf{T}M, M, \chi_M)$ with total space $\mathbf{T}^*M = \bigoplus_{(r,s)\in\mathbf{N}^2} T^r_s M$, where $T^r_s M$ is the bundle of tensors of type (r,s), for any $(r, s) \in \mathbb{N}^2$ and $(r, s) \neq (0, 0)$, while $T^0_0 M = M \times \mathbb{R}$. We set $T^0_r M \equiv T_r M$, for any $r \in \mathbb{N}$. We stress that, if $\mathcal{B}$, $\mathcal{B}'$ are two objects in $\mathbf{B}un$ and $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$, then the reduced first order perturbation functor $\widehat{\mathcal{P}}$ determines the map $\widehat{\mathcal{P}}_{\mathcal{B}}(F)$ which associates to any vertical vector of the total space of the bundle $V\mathcal{B}$ a contravariant vector of the total space of $\mathcal{B}'$. Hence, at a first sight, this functor seems to have nothing to do with Lagrangians which are instead determined by mappings from J^2B into $\Lambda^m M$. However, this is not the case, since it easy to see that each vertical bundle VT_rM splits as follows with a natural projection:

$$pr_1 : VT_rM \cong (T_rM) \oplus_M (T_rM) \to T_rM\,, \quad \forall r \in \mathbf{N}\,. \tag{2.10}$$

Proposition 2.6. *Let $\mathcal{L} : B \to \Lambda^m M$ be a Lagrangian. The following holds*:

$$\widehat{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L} = (\mathcal{L}, \mathcal{L}_{(1)}) : VJ^2B \cong J^2VB \longrightarrow V\Lambda^m M \cong \Lambda^m M \oplus_M \Lambda^m M\,, \tag{2.11}$$

where $\mathcal{L}_{(1)}$ is the first order perturbation of the Lagrangian $\mathcal{L}$.

Proof. It is a straightforward consequence of results of [16] together equation (1.15) of Section 1, since both the reduced first order perturbation functor and the identification (2.10) preserve symmetries. ∎

Remarks. A virtual application of a strictly analogous functor is due to Taub, who explicitly introduced a Lagrangian previously used in an implicit way to study the stability of relativistic gaseous masses (see [30] and the papers quoted therein). The perturbed Lagrangian used by Taub is the following:

$$\tilde{\mathcal{L}}_y(v) = \{e^{\mathcal{B}}(\mathcal{L})\}_y(v) + \{d_H[f^{\mathcal{B}}_{(1)}(\mathcal{L}) + f^{\mathcal{B}}_{(2)}(\mathcal{L})]\}_y(v) \ , \tag{2.12}$$

for any $y \in J^2B$ and $v \in VB$, both projecting onto the same point of B (see (1.4) of [30]). As a consequence, between Taub's and our perturbation there is only a "difference in simplicity", since (2.12) is obtained from (2.11) by means of a formal integration by parts, i.e. by adding to the Lagrangian a divergence which does not affect the variational problem (see [32]). This difference might however have some relevance, not only because of the different complication in the calculations; in fact we know that divergences determine those physical quantities which are pushed to the boundary of the region considered and enter the conservation laws through Stokes' theorem, so that they cannot be arbitrarily changed. This is true not only in classical physics, but also in General Relativity (see [32] for an example related to the Komar superpotential).

We conclude this Section by noticing that the morphism $\widehat{\mathcal{P}}(\mathcal{L})$ contains the first order "deformed" Lagrangian $\mathcal{L}_{(1)}$ of $\mathcal{L} \equiv \mathcal{L}_{(0)}$ in the sense of [16] and hence it contains informations on the Jacobi equations of the variational problems.

2.4 – The reduced functor $\overline{\mathcal{P}}$

In order to consider all the other geometric objects related to the Calculus of Variations, we need a more sophisticated construction than (2.10). For this purpose we first recall some results of [31]. Let us denote by $\mathcal{T}^r_s(M)$ the module of tensorfields of type (r,s) on M, being $\mathcal{T}^0_0(M) \equiv \mathbf{\Omega}_0(M) \equiv \mathcal{F}(M)$ the ring of smooth functions, and we set $\mathcal{T}(M) \equiv \bigoplus_{(r,s)\in\mathbb{N}^2} \mathcal{T}^r_s(M)$. We also denote by (x^μ, v^ν) the local coordinates induced on the tangent bundle TM by a local coordinate system (U, x^μ) on M.

Proposition 2.7 (see, e.g., [31] for a proof). *There exists an $\mathcal{F}(M)$-linear isomorphism from $\mathcal{T}(M)$ into $\mathcal{T}(TM)$, denoted by v and called* **vertical lift**, *defined by*:

$$(S\otimes T)^v = S^v \otimes T^v, \quad \forall\, S \in \mathcal{T}^r_s(M), \forall\, T \in \mathcal{T}^h_k(M), \forall\, r,s,h,k \in \mathbb{N} \tag{2.13a}$$

and

$$\left(\frac{\partial}{\partial x^\mu}\right)^v = \frac{\partial}{\partial v^\mu} \quad , \quad (dx^\mu)^v = dx^\mu \ , \tag{2.13b}$$

for any $\mu \in \{1, \dots, m\}$).

We need a further definition:

Definition 2.4. The **complete lift** is the $\mathbb{R}$-linear map $c : \mathcal{T}(M) \to \mathcal{T}(TM)$ defined by:

$$f^c \equiv df : TM \to \mathbb{R} \ , \quad \forall f \in \mathcal{F}(M) \ , \tag{2.14a}$$

$$\begin{gathered}(S \otimes T)^c = S^c \otimes T^v + S^v \otimes T^c \, , \\ \forall S \in \mathcal{T}^r_s(M) \ , \ \forall T \in \mathcal{T}^h_k(M) \ , \ \forall r, s, h, k \in \mathbb{N}\end{gathered} \tag{2.14b}$$

and

$$\left(\frac{\partial}{\partial x^\mu}\right)^c = \frac{\partial}{\partial x^\mu} \quad , \quad (dx^\mu)^c = dv^\mu \quad , \ \forall \mu \in \{1, \dots, m\} \ . \tag{2.14c}$$

Notice that if X and S are a vectorfield and a tensorfield defined on M, respectively, then the following relation between Lie derivatives exists:

$$(\mathbf{L}_X(S))^c = \mathbf{L}_{X^c}(S^c) \ .$$

Proposition 2.8. *Let us fix* $(r, s) \in \mathbb{N}^2$, *with* $r + s \geq 1$, *and let* $S \in \mathcal{T}^r_s(M)$ *be a tensorfield. Consider the total differential* $TS : TM \to T(T^r_s M)$ *and the complete lift* $S^c : TM \to T^r_s(TM)$. *Then there exists an immersion* $\xi \equiv \xi^r_s(M) : T^r_s(TM) \to T(T^r_s M)$ *such that* $\phi^r_s(M) \circ TS = S^c$, *for any* $S \in \mathcal{T}^r_s(M)$.

PROOF. In fact, a tensor $S \in T^r_s M$ belonging to the fiber over $x \in M$ has local expression:

$$S = S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} \ . \tag{2.15}$$

Denoting by $(x^\mu, S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})$ the local coordinate system of $T^r_s M$ induced by the local coordinate system (U, x^μ) of M, we can write the local expression of a vector $X \in T_S(T^r_s M)$, being $S \in T^r_s M$ a tensor over a point $x \in U$, as follows:

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}} \ . \tag{2.16}$$

We can always find a tensorfield $\widetilde{S}$ defined on U such that:

$$(X^\mu \partial_\mu \widetilde{S}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})_x = X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \quad , \quad (\widetilde{S}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})_x = S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \ , \tag{2.17}$$

where $Y = X^\mu \frac{\partial}{\partial x^\mu} \in T_x M$ and $S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}$ are the local components of S. We stress that (2.17) is equivalent to $(T\widetilde{S})_Y = X$. Then we set:

$$\xi^r_s(X) \equiv (\widetilde{S}^c)_Y \ , \tag{2.18}$$

since the tensor on the right hand side does not depend on the local coordinate system nor it depends on the tensorfield $\widetilde{S}$. By using (2.13) and (2.14) one can see that:

$$\begin{aligned} \xi^r_s(X) &= X^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} + \\ &+ \sum_{h=1}^{r} S^{\mu_1 \dots \mu_h \dots \mu_r}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_h}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s} + \\ &+ \sum_{h=1}^{s} S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_h \dots \nu_s} \frac{\partial}{\partial v^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial v^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dv^{\nu_h} \otimes \dots \otimes dx^{\nu_s} \ . \end{aligned} \tag{2.19}$$

This proves our claim. □

REMARK. We stress that, if $\mathcal{B} = (B, M, \pi)$ and $\mathcal{B}' = (B', M', \pi')$ are two fiber bundles and $F \in \mathbf{B}un(\mathcal{B}, \mathcal{B}')$ is a bundle morphism, then equation (2.8) can be equivalently written as:

$$\widehat{\mathcal{P}}_{\mathcal{B}}(F) = [(TF)_{|VB}]^\perp \ , \tag{2.20}$$

where $[\dots]^\perp$ denotes the "vertical part" obtained by projection through the natural projection of TB onto $V^M B$.

When applied to vectorfields, equation (2.20) gives then rise to vectorfields which determine local 1-parameter groups having a trivial action on the Lagrangians obtained by (2.11), since the the "horizontal" components of the original vectorfields are lost. The existence of the family $\xi^r_s \equiv \xi^r_s(B) : T^r_s(TB) \to T(T^r_s B)$ and equation (2.19) allow us to associate to the functor $\widehat{\mathcal{P}}$ a new functor $\overline{\mathcal{P}}$ in all the cases in which $\widehat{\mathcal{P}}$ acts on tensorfields on the manifold B, considered as obvious bundle morphisms. In fact, a tensorfield $S \in \mathcal{T}^r_s(B)$ can be considered as a morphism $S : B \to T^r_s B$, with respect to the bundle structure of $\mathcal{B}$ and the obvious bundle structure $T^r_s B \to M$. Then, by using (2.17), we can set

$$\overline{\mathcal{P}}_{\mathcal{B}}(S) = \xi^r_s(\widehat{\mathcal{P}}_{\mathcal{B}}(S)) \ . \tag{2.21}$$

The local expression of $\overline{\mathcal{P}}_{\mathcal{B}}(S)$ can be easily calculated by using the local expression of ξ^r_s given by (2.16) for any tensorfield S of type (r,s) on B. This gives quite complicated formulae in the general case, since several terms are involved. We shall thence limit ourselves to write these formulae only for vectorfields and 1-forms, because they will be needed below. Hence, we set:

$$\begin{aligned}
&\left(\frac{\partial}{\partial x^\mu}\right)^v = \frac{\partial}{\partial v^\mu} \ , \quad \left(\frac{\partial}{\partial y^a}\right)^v = \frac{\partial}{\partial v^a} \ , \\
&\left(\frac{\partial}{\partial x^\mu}\right)^c = \frac{\partial}{\partial x^\mu} \ , \quad \left(\frac{\partial}{\partial y^a}\right)^c = \frac{\partial}{\partial y^a} \ , \\
&(dx^\mu)^v = dx^\mu \ , \qquad (dy^a)^v = dy^a \ , \\
&(dx^\mu)^c = dv^\mu \ , \qquad (dy^a)^v = dv^a \ .
\end{aligned} \tag{2.22}$$

Let $X = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial y^a}$ be the local expression of a vector field X and $\omega = \omega_\mu dx^\mu + \omega_a dy^a$ be the local expression of a 1-form, defined on B. Then, for any vertical vector $v = v^a \frac{\partial}{\partial y^a}$, we have:

$$\overline{\mathcal{P}}_{\mathcal{B}}(X)_v = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial y^a} + v^b \frac{\partial X^\mu}{\partial y^b} \frac{\partial}{\partial v^\mu} + v^b \frac{\partial X^a}{\partial y^b} \frac{\partial}{\partial v^a} \tag{2.23a}$$

and

$$(2.23b) \qquad [\overline{\mathcal{P}}_{\mathcal{B}}(\omega)]_v = \omega_\mu dx^\mu + \omega_a dy^a + v^b \frac{\partial \omega_\mu}{\partial y^b} dv^\mu + v^b \frac{\partial \omega_a}{\partial y^b} dv^a \ .$$

2.5 – The variational component of $\widetilde{\mathcal{P}}$

In order to determine the last functor into which we are interested we need some further construction. Let $(x^{\mu'}, y^{a'})$ $(\mu, \mu' \in \{1, \ldots, m\})$ be a further local bundle coordinate system whose domain intersects the domain of the coordinate system (x^μ, y^a). We will denote by $\varphi^{\mu'}(x^\mu)$ and $\psi^{a'}(x^\mu, y^a)$ the transition functions, together with their inverses ϕ^μ and ψ^a. Let us consider the tangent bundle TB and let us recall that in the charts induced on this manifold the following transformation laws hold:

$$(2.24) \qquad \begin{aligned} &\text{i)} \quad x^{\mu'} = \varphi^{\mu'}(x^\mu) \ , \\ &\text{ii)} \quad y^{a'} = \psi^{a'}(x^\mu, y^a) \ , \\ &\text{iii)} \quad v^{\mu'} = v^\mu \varphi^{\mu'}_\mu \\ &\text{iv)} \quad v^{a'} = v^\mu \psi^{a'}_\mu + v_a \psi^a_{a'} \ , \end{aligned}$$

for any $v = v^\mu \frac{\partial}{\partial x^\mu} + v^a \frac{\partial}{\partial y^a} \in T_y B$ in a point $y \in B$ belonging to the intersection domain. Here and in the sequel we set $\varphi^\mu_{\mu'} = \partial_{\mu'} \varphi^\mu$, $\psi^a_{\mu'} = \partial_{\mu'} \psi^a$ and so on. Now, we consider the subbundle $\pi^{VB} : (\pi_{TB})^{-1}(VB) = \tau^* VB \to VB$ of the cotangent bundle $(T^*(TB), TB, \pi_{TB})$ and a 1-form $\alpha = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_\mu dv^\mu + \beta_a dv^a \in \tau^* VB$. Then, the transformation laws (2.24) induce the following transformations on the local components of α:

$$(2.25) \qquad \begin{aligned} &\text{i)} \quad \alpha'_{\mu'} = \alpha_\mu \varphi^\mu_{\mu'} + \alpha_a \psi^a_{\mu'} + \beta_a \psi^a_{a'\mu'} \psi^{a'}_b v^b \ , \\ &\text{ii)} \quad \alpha_{a'} = \alpha_a \psi^a_{a'} + \beta_a \psi^a_{a'b'} \psi^{b'}_b v^b \ , \\ &\text{iii)} \quad \beta_{\mu'} = \beta_{\mu'} \varphi^\mu_{\mu'} + \beta_a \psi^a_{\mu'} \ , \\ &\text{iv)} \quad \beta_{a'} = \beta_a \psi^a_{a'} \ . \end{aligned}$$

On the other hand, one obtains from (2.4) the transition functions on the bundle VB by simply setting $v^\mu = 0$. The corresponding transformation laws of the local components of a 1-form $r = \rho_\mu dx^\mu + \rho_a dy^a + \sigma_a dv^a$

defined on VB are then given by:

$$
\text{(2.26)} \qquad
\begin{aligned}
&\text{i)} \quad \rho'_{\mu'} = \rho_\mu \varphi^\mu_{\mu'} + \rho_a \psi^a_{\mu'} + \sigma_a \psi^a_{a'\mu'} \psi^{a'}_b v^b \ , \\
&\text{ii)} \quad \rho_{a'} = \rho_a \psi^a_{a'} + \sigma_a \psi^a_{a'b'} \psi^{b'}_b v^b \ , \\
&\text{iii)} \quad \beta_{a'} = \beta_a \psi^a_{a'} \ .
\end{aligned}
$$

THEOREM 2.9. *Let us consider the vector bundle $T^*B \oplus_B T^*VB \to VB$, in which the fiber over a vector $v \in V_yB$, with $y \in B$, is given by $T^*_y \oplus T^*_v VB$, with the obvious structure of real vector space. There exists a bundle isomorphism $\eta_* : \tau^*VB \to T^*B \oplus_B T^*VB$ which associates to any covariant vector $\alpha = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_\mu dv^\mu + \beta_a dv^a$ of τ^*VB over the vector v of VB the ordered pair (ω, ρ) of $T^*B \oplus_B T^*VB$, being $\omega = \beta_\mu dx^\mu + \beta_a dy^a$ and $\rho = \alpha_\mu dx^\mu + \alpha_a dy^a + \beta_a dv^a$, with the covariant vector ρ belonging to the fiber of T^*VB over v.*

PROOF. Immediate by comparing (2.25), (2.26) together with the transformation laws of T^*B.

REMARK. The bundle $T^*B \oplus_B T^*VB \to VB$ possesses a naturally induced structure of vector bundle. Moreover, the bundle over VB of covariant tensors of order r determined by the vector bundle structure on $T^*B \oplus_B T^*VB$ turns out to be isomorphic to $T_rB \oplus_B T_rVB$, for any $r > 0$. Hence, if $\nu_r VB$ denotes the restriction of the bundle of covariant tensors T_rTB to VB, we can consider the power $(\eta_*)_r : \nu_r VB \to T_rB \oplus_B T_rVB$.

DEFINITION 2.5. We set $\phi_r = \xi^0_r \circ (\eta_*)_r$, for any $r \geq 0$ and $\Phi \equiv (\phi_r)_{r \geq 1}$. Then we have the following covariant functor:

$$
\text{(2.27)} \qquad \widetilde{\mathcal{P}}_{\mathcal{B}} \equiv pr_2 \circ \phi_r \circ \mathcal{P}_{\mathcal{B}} \equiv pr_2 \circ \xi^0_r \circ \overline{\mathcal{P}}_{\mathcal{B}} : \mathcal{T}_rB \to \mathcal{T}_rVB \ ,
$$

where $pr_2 : T_rB \oplus_B T_rVB \to T_rVB$ is the canonical projection. The functor $\widetilde{\mathcal{P}}$ acts on the appropriate categories which can be easily determined and it is called the **variational component of the reduced first order perturbation functor**.

In order to determine the action of the functor $\widetilde{\mathcal{P}}$ on the local components of covariant tensorfields we need some more pieces of notation. Let us denote by $\mathcal{A}_r(h)$ the set of multiple indices

$$
A_r(h) = (\mu_1, \ldots, \mu_h, a_1, \ldots, a_{r-h}) \ ,
$$

with $h \in \{1, \ldots, r\}$. We make the convention that the multiple indices in which the μ's do not appear are of the type $A_r(0)$ and the multiple indices not having the a's are of the type $A_r(r)$. We shall call the previous multiple indices *basic multiple indices*. The standard action of the permutation group G_r on the basic multiple indices determines all the multiple indices needed to study the tensors of B. We consider now the set of local covariant tensors of B defined by:

$$dz^{\sigma(A_r(h))} = dz^{\sigma(\mu_1)} \otimes \cdots \otimes dz^{\sigma(\mu_h)} \otimes dz^{\sigma(a_1)} \otimes \cdots \otimes dz^{\sigma(a_{r-h})} \ , \tag{2.28}$$

for any $A_r(h) \in \mathcal{A}_r$ and $\sigma \in G_r$, having set $dz^\mu = dx^\mu$ and $dz^a = dy^a$, for any $\mu \in \{1, \ldots, r\}$ and any $a \in \{1, \ldots, p\}$, respectively. Then the family $(dz^{\sigma(A_r(h))})$, obtained when σ spans G_r and $A_r(h)$ spans $\mathcal{A}_r$, is a local system of generators of T_rB, which is obtained from the standard local basis of T_rB by repeating exactly $h!(r-h)!$-times each element $dz^{\sigma(A_r(h))}$, for any $\sigma \in G_r$ and $A_r(h) \in \mathcal{A}_r$, for any $r > 0$. Moreover, if $S \in \mathcal{T}_rB$, we have:

$$S = \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} S_{\sigma(A_r(h))} dz^{\sigma(A_r(h))} \ , \tag{2.29}$$

where $S_{\sigma(A_r(h))}$ are the standard local components of S and the Einstein convention on the multiple indices $A_r(h)$ is used without any danger of confusion.

Then, for all sections $\omega : B \to T_rB$, having local expression:

$$\omega = \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} \omega_{\sigma(A_r(h))} dz^{\sigma(A_r(h))} \ , \tag{2.30}$$

we have:

$$\begin{aligned} \widetilde{\mathcal{P}}_{\mathcal{B}}(\omega) = & \sum_{h=0}^{r} \sum_{\sigma \in G_r} \frac{1}{h!(r-h)!} v^a \partial_a [\omega_{\sigma(A_r(h))}] dz^{\sigma(A_r(h)} + \\ & + \sum_{h=0}^{r-1} \sum_{\sigma \in G_r} \frac{1}{h!(r-1-h)!} \omega_{\sigma(A_r(h)\hat{a})} dz^{\sigma(A^1_r(h)\hat{a})} \ , \end{aligned} \tag{2.31}$$

having set

$$A^1_r(h)\hat{a} \equiv (\mu_1, \ldots, \mu_h, a_1, \ldots, a_{r-h-1}, \hat{a})$$

and

$$dz^{\sigma(A^1_r(h)\hat{a})} = dz^{\sigma(\mu_1)} \otimes \cdots \otimes dz^{\sigma(\mu_h)} \otimes dz^{\sigma(a_1)} \otimes \cdots \otimes dz^{\sigma(a_{r-h-1})} \otimes dz^{\sigma(\hat{a})} \ ,$$

being, in this case, $dz^{\hat{a}} = dv^{\hat{a}}$, for any $\hat{a} \in \{1, \ldots, p\}$. Obviously, the functor $\mathcal{P}$ can be easily obtained from $\widetilde{\mathcal{P}}$ in these cases.

Remark. We conclude this part by noticing that, if Ξ is a vectorfield and ω is a covariant tensorfield both defined on B, then from (2.14d) and the definition of $\widetilde{\mathcal{P}}$ we easily obtain:

$$\mathbf{L}_{\overline{\mathcal{P}}_{\mathcal{B}}(\Xi)}(\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega)) = \widetilde{\mathcal{P}}_{\mathcal{B}}(\mathbf{L}_{\Xi}(\omega)) \ . \tag{2.32}$$

3 – Relations of the functor $\mathcal{P}$ with the calculus of variations and with some cohomological functors

3.1 – Action on forms of the perturbation functors

Let $\mathcal{B} = (B, M, \pi)$ be a fiber bundle. We shall denote by $\mathcal{T}^*(B) = \bigoplus_{r\in\mathbb{N}} \mathcal{T}_r(B)$ the direct sum of the modules $\mathcal{T}_r(B)$ of tensorfields of type (0,r), i.e. the sections of the bundle τ^*B. We also recall that, if $\mathcal{B}$, $\mathcal{B}'$ are objects in $\mathbf{B}un$, then the functor $\widehat{\mathcal{P}}$ defines a map $\widehat{\mathcal{P}}_{\mathcal{B},\mathcal{B}'} : \mathbf{B}un(\mathcal{B},\mathcal{B}') \to \mathbf{B}un(\widehat{\mathcal{P}}(\mathcal{B}), \widehat{\mathcal{P}}(\mathcal{B}')) = \mathbf{B}un(V\mathcal{B}, V\mathcal{B}')$, which transforms a morphism $f \in \mathbf{B}(\mathcal{B},\mathcal{B}')$ into the morphism $\widehat{\mathcal{P}}(f) \in \mathbf{B}(V\mathcal{B}, V\mathcal{B}')$, given by (2.8). This holds also for the functor $\widetilde{\mathcal{P}}$.

A number of results holds becuase of (2.14):

Proposition 3.1. *The variational component $\widetilde{\mathcal{P}}$ of the reduced first order perturbation functor $\widehat{\mathcal{P}}$ acts as a derivative on $\mathcal{T}^*(B)$, considered as a $\mathcal{T}^*(M)$-algebra, via the natural identification induced by pull-back along $\pi : B \to M$.*

As a consequence, by using the simplified notation introduced in Section 2, we have:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega \otimes \omega') = \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega) \otimes \omega' + \omega \otimes \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega') \tag{3.1}$$

and

$$\widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\alpha \otimes \omega) = \alpha \otimes \widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}(\omega) \ , \tag{3.1'}$$

for all $\omega \in \mathcal{T}_r(B)$, $\omega' \in \mathcal{T}_s(B)$, $\alpha \in \mathcal{T}_h(M)$ and $r, s, h \in \mathbb{N}$; i.e., in this section we shall consider the vertical lift as an identification morphism.

PROPOSITION 3.2. *The variational component $\widetilde{\mathcal{P}}_{\mathcal{B},\mathbf{T}^*B}$ of the reduced first order perturbation functor preserves the symmetries of tensors.*

Hence, when $\widetilde{\mathcal{P}}_{\mathcal{B}}$ is reduced to the exterior algebra of B, we can replace in (3.1) the tensor product with the exterior product, so that:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(\mathbf{\Omega}(B)) \subseteq \mathbf{\Omega}(VB) \; . \tag{3.2}$$

PROPOSITION 3.3. *The functor $\widetilde{\mathcal{P}}_{\mathcal{B}}$ is localizable*; *i.e., if N is an open submanifold of M and if $\pi : B' \to N$ defines a sub-bundle of $\mathcal{B}$, then:*

$$[\widetilde{\mathcal{P}}_{\mathcal{B}}(\omega)]_{|\mathcal{B}'} = \widetilde{\mathcal{P}}_{\mathcal{B}'}(\omega_{|\mathcal{B}'}) \; , \tag{3.3}$$

for any $\omega \in \mathbf{\Omega}(B)$.

Now we are ready to prove one of the main results of this paper.

THEOREM 3.4. *There exists a morphism $\widetilde{\mathcal{P}}^*_{\mathcal{B},\Lambda B} : H_{dR}B \to H_{dR}VB$, being H_{dR} the de Rham* ($\mathbb{R}$*-valued*) *cohomology functor.*

PROOF. We first recall that a function $f : B \to \mathbb{R}$ can be identified with a section $f : B \to B \times \mathbb{R}$ of the bundle $pr_1 : B \times \mathbb{R} \to B$. Since we have the identification $pr_1 : V^M(B \times \mathbb{R}) \cong (VB) \times \mathbb{R} \to VB$, the mapping $\widetilde{\mathcal{P}}_{\mathcal{B}}(f)$ is a section of this bundle, and hence a function. For this function we have locally:

$$\begin{aligned}
\widetilde{\mathcal{P}}_{\mathcal{B}}\left(\frac{\partial f}{\partial x^\mu}\right) &= \frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial x^\mu} \, , \\
\widetilde{\mathcal{P}}_{\mathcal{B}}\left(\frac{\partial f}{\partial y^a}\right) &= \frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial y^a} \, , \\
\frac{\partial(\widetilde{\mathcal{P}}_{\mathcal{B}} f)}{\partial v^a} &= \frac{\partial f}{\partial y^a} \; ;
\end{aligned} \tag{3.4}$$

the above identities hold since, in virtue of (2.14) we have:

$$[\widetilde{\mathcal{P}}_{\mathcal{B}}(f)](v) = \left(\frac{\partial f}{\partial y^a}(y)\right) v^a \ , \tag{3.5}$$

for any $v \equiv v^a \frac{\partial}{\partial y^a} \in VB$ projecting onto $y \in B$. From (2.14) and (3.4), it also follows:

$$\widetilde{\mathcal{P}}_{\mathcal{B}}(d\omega) = d(\widetilde{\mathcal{P}}_{\mathcal{B}}\omega) \ , \tag{3.6}$$

for all $\omega \in \mathbf{\Omega}_r(B)$ and any integer $r \in \{1, \dots, m+p\}$. Hence, the map $\widetilde{\mathcal{P}}_{\mathcal{B},\Lambda B}$ is a cochain morphism from $\mathbf{\Omega}(B)$ into $\mathbf{\Omega}(VB)$. As a consequence, it defines a morphism $\widetilde{\mathcal{P}}^*_{\mathcal{B},\Lambda B} : H_{dR}D \to H_{dR}(VB)$, as we planed. ∎

3.2 – Fundamental properties of $\widetilde{\mathcal{P}}$

Now, we consider the bundle $J^2V\mathcal{B}$ together with its natural bundle structure $J^2VB \to VB$ and the local basis for the contact 1-forms, given by:

$$\widetilde{\theta}^a = dv^a - v^a_\sigma dx^\sigma \ , \quad \widetilde{\theta}^a_\mu = dv^a_\mu - v^a_{\mu\sigma} dx^\sigma \ . \tag{3.7}$$

The family of 1-forms defined by combining (1.8) together with (3.7) determines a local basis for the contact 1-forms with respect to the bundle structure $J^2VB \to B$. Moreover, from (2.14) we have:

$$\widetilde{\mathcal{P}}_{J^2\mathcal{B}}(\theta^a) = \widetilde{\theta}^a \ , \quad \widetilde{\mathcal{P}}_{J^2\mathcal{B}}(\theta^a_\mu) = \widetilde{\theta}^a_\mu \ . \tag{3.8}$$

We need two technical lemmae:

LEMMA 3.5. *Let $f : J^2B \to \mathbb{R}$ be a function, which induces the mapping $d_V f : J^3B \to T^*M$ and the perturbation $\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f : J^2VB \to \mathbb{R}$, where the obvious identifications with sections are used. Considering the induced morphisms $d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) : J^3VB \to T^*J^3VB$ and $\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V f) : J^3VB \to T^*J^3VB$, the following hold*

$$d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V f) \ , \tag{3.9a}$$

$$d_H(\widetilde{\mathcal{P}}_{J^2\mathcal{B}} f) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_H f) \ . \tag{3.9b}$$

PROOF. The lemma follows easily from (2.14), (3.4) and (3.8). Equation (3.9b) holds because of (3.6) and (3.9a), being $d_H = d - d_V$. ∎

Analogous calculations give the following lemma:

LEMMA 3.6. *Let $\omega \in \mathbf{\Omega}(J^2B)$ be any form the following hold*:

$$d_V(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\omega) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_V\omega) \ , \tag{3.10a}$$

$$d_H(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\omega) = \widetilde{\mathcal{P}}_{J^3\mathcal{B}}(d_H\omega) \ . \tag{3.10b}$$

Using the previous lemmae, by simple calculations one obtains also the following fundamental result:

THEOREM 3.7. *The variational component of the reduced first order perturbation functor $\widetilde{\mathcal{P}}$ satisfies the following "naturality properties":*

$$\begin{aligned}
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(f^{\mathcal{B}}_{(1)}(\mathcal{L})) &= f^{V\mathcal{B}}_{(1)}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(f^{\mathcal{B}}_{(2)}(\mathcal{L})) &= f^{V\mathcal{B}}_{(2)}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L})) &= e^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(\Theta^{\mathcal{B}}(\mathcal{L})) &= \Theta^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ , \\
\widetilde{\mathcal{P}}_{J^3\mathcal{B}}(\mathbf{\Omega}^{\mathcal{B}}(\mathcal{L})) &= \mathbf{\Omega}^{V\mathcal{B}}(\widetilde{\mathcal{P}}_{J^2\mathcal{B}}\mathcal{L}) \ .
\end{aligned} \tag{3.11}$$

Finally, from (1.17), (2.9), (2.11) and (3.11) we deduce that:

THEOREM 3.8. *The morphism $\widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L}))$ is the Jacobi morphism of $\mathcal{L}$ and the following holds*:

$$\delta^2(\mathcal{L}\circ j^2\lambda_\varepsilon) = \widetilde{\mathcal{P}}_{J^4\mathcal{B}}(e^{\mathcal{B}}(\mathcal{L}))\circ j^4 v + \delta[(d_H f^{\mathcal{B}}_{(1)}(\mathcal{L}) + d_H f^{\mathcal{B}}_{(2)}(\mathcal{L}))\circ j^4\lambda_\varepsilon] \tag{3.12}$$

REMARK. Equation (3.12) gives the second variation of the Lagragian $\mathcal{L}$ expressed by the variational component $\widetilde{\mathcal{P}}$ of the first order perturbation functor $\widehat{\mathcal{P}}$.

3.3 – The comparison between cohomologies

Now, we determine some relations among the variational component of the first order perturbation functor $\widetilde{\mathcal{P}}$ and some of the functors defined by other authors in various cohomological theories related to problems involving partial differential equations. Obviously, these relations can be considered as a further measure of the naturality of the functors $\mathcal{P}$, $\widehat{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$ introduced here. To this purpose we consider two different versions of the cohomological theory introduced by ANDERSON and DUCHAMP (see [5]) and developed by many authors, among which we recall VINOGRADOV ([33]; see [6] and [9] for a more detailed bibliography). We shall also introduce a third version of Vinogradov's cohomological theory, which better exploits the naturality of the functors introduceded here and puts forward some problems which apparently were not considered in the previous literature known to us.

The cohomology considered in [6] is not extremely well suited to include the global versions of the Euler-Lagrange equations. In fact, the only case known to us in which this cohomological theory works well for variational problems is the case obtained by taking $B = M \times \mathbb{R}^p$ (with p any integer) and $\pi = pr_1 : B \to M$ (see [34]). Moreover, the "tower construction" of [7] does not seem to be suited to include the differential equations ensuing from variational problems, as we shall shortly see below. We shall thence suggest a "naive" solution for both problems. We recall once again that the construction considered here has the unique purpose of testing the naturality of the variational component of the first order perturbation functor. Accordingly, "better for our purposes" will not in general mean "better" (especially when one considers the important results of [7] and [10]), even if we believe that it could be useful to compare some of the possible constructions together with their applications. Finally, we stress that the variational methods involve many more types of partial differential equations than people generally think, as it will be pointed out by the examples of the Appendix (related to "parabolic" systems of partial differential equations in the sense of [6], heat equations and KdV equations included). This remark can be especially useful for the cohomological groups considered here, since the problems coming from the degeneracy of the Lagrangian and from the signature of its associated Hessian do not seem to play an important role, at least for the moment.

Let again $\mathcal{B} = (B, M, \pi)$ be a bundle. Let us denote by $\pi^h_k : J^h B \to J^k B$ the canonical projections, for any $h, k \in \mathbf{N}$, with $h > k$ and let us set $J^0\mathcal{B} = \mathcal{B}$. Then we have canonical inclusions $(\pi^h_k)^* : \mathbf{T}^* J^k B \to \mathbf{T}^* J^h B$, for any $h, k \in \mathbf{N}$, with $h > k$; we shall use $(\pi^h_k)^*$ as identification morphisms. Then, more or less clearly, the specific construction of [9] suggests to overcome the use of the bundle $J^\infty\mathcal{B}$ of infinite jet prolongations of sections of $\mathcal{B}$ which has better "flatness" properties but has a complicated topology (see, e.g., [34]), by just considering and suitably working on jet bundles of order $k+1$, being k the highest order on which the r-forms used depend. Since in our hypotheses $de^{\mathcal{B}}(\mathcal{L}) = d_V e^{\mathcal{B}}(\mathcal{L})$ depends on the elements of $J^5 B$, for any Lagrangian $\mathcal{L}$ on $J^2 B$, we shall consider $\mathbf{\Omega}(J^k B) \subseteq \mathbf{\Omega}(J^6 B)$, for any $k \leq 5$. We shall also consider $\mathbf{\Omega}(M) \subseteq \mathbf{\Omega}(B) \subseteq \mathbf{\Omega}(J^6 B)$, via the identification morphism $\pi^* : \mathbf{\Omega}(M) \to \mathbf{\Omega}(J^0 B) = \mathbf{\Omega}(B)$. The previous identifications allow us to consider the ring of smooth functions $\mathbf{\Omega}_0(J^k B)$ as a sub-ring of the ring of smooth functions $\mathbf{\Omega}_0(J^6 B)$ which are constant along the fibers of the bundle $\pi^6_k : J^6 B \to J^k B$, with $k < 6$. We shall denote by $\widetilde{\mathbf{\Omega}}^h_r(J^k B)$ the $\mathbf{\Omega}_0(J^k B)$-module of r-forms along the canonical projection $\pi^h_k : J^h B \to J^k B$, for any $h, k \leq 5$, with $h > k$. Finally, we denote by $\widetilde{\mathbf{\Omega}}_r(M)$ the $\mathbf{\Omega}_0(J^k B)$-submodule of r-forms along the canonical projection $\bar{\pi}^k : \pi \circ \pi^k_0 : J^k B \to M$, for any $k \leq 5$; also this module will be considered as a sub-module of $\mathbf{\Omega}_r(J^6 B)$, for all $r \in \{1, \ldots, m\}$.

We shall use the following known results (see [9]):

Proposition 3.9. *The following contact forms*:

$$\theta^a_{\mu\nu\rho} = dy^a_{\mu\nu\rho} - y^a_{\mu\nu\rho\sigma} dx^\sigma, \qquad \theta^a_{\mu\nu\rho\sigma} = dy^a_{\mu\nu\rho\sigma} - y^a_{\mu\nu\rho\sigma\tau} dx^\tau \ ,$$
$$\theta^a_{\mu\nu\rho\sigma\varepsilon} = dy^a_{\mu\nu\rho\sigma\varepsilon} - y^a_{\mu\nu\rho\sigma\varepsilon\tau} dx^\tau \ , \tag{3.13}$$

together with the contact forms defined by (1.8), *the forms dx^μ and the forms $dy^a_{\mu\nu\rho\sigma\varepsilon\eta}$, determine a local basis C of the $\mathbf{\Omega}_0(J^6 B)$-module $\mathbf{\Omega}_1(J^6 B)$ and hence generate $\mathbf{\Omega}(J^6 B)$. Moreover, the subset C' obtained from C by removing only all the forms dx^μ and $dy^a_{\mu\nu\rho\varepsilon\tau}$ generates an ideal of $\mathbf{\Omega}(J^6 B)$, known as the ideal of* contact forms.

One of the most substantial differences beetwen the viewpoint of [6] and the viewpoint of [9] is the definition of "solution of a system of differential equations". In fact, let $\mathcal{I}$ be an ideal of $\mathbf{\Omega}(J^6B)$ and $\sigma \in \Gamma_D(\pi)$ be a local section, being D a domain in M. In [6] the section σ is said to be a solution of the system of partial differential equations defined by $\mathcal{I}$ if and only if $(j^6\sigma)^*(\mathcal{I}_{|J^6E}) = 0$, being $E = \pi^{-1}(D)$ the total space of the bundle over D naturally induced by the bundle structure of $\mathcal{B}$ and $(j^6\sigma)^* : \mathbf{\Omega}(J^6E) \to \mathbf{\Omega}(M)$ the total differential of $j^6\sigma : D \to J^6B$. In the Calculus of Variations a section σ is instead a solution of the system of partial differential equations defined by $\mathcal{I}$ if and only if $\mathcal{I} \circ \sigma \equiv \{\omega \circ j^6\sigma / \omega \in \mathcal{I}\} = 0$. This alternative definition of solution can be easily inferred from the general theory, since if $\omega_i \circ j^6\sigma = 0$ for a family $(\omega_i)_{i\in I}$ where $I \neq \emptyset$ is any set of indices, then $\omega \circ\ j^6\sigma = 0$ for all ω belonging to the ideal $\mathcal{I}$ generated by the family $(\omega_i)_{i\in I}$.

The definition of solution used in [6] cannot be applied immediately to the Euler-Lagrange equations, since they are globally defined by an (m+1)-form which is locally of the type $e_a\theta^a \wedge \mathbf{ds}$, while $(j^6\sigma)^*(\theta^a) = 0$ holds for all $a \in \{1, \dots, p\}$ because of the very definition of the structure forms θ^a. We stress moreover that the solution suggested in [34] for variational problems defined on the trivial bundle $\mathcal{B}$ given by $pr_1 : M \times \mathbb{R}^p \to M$ is however viable, only due to the fact that one can avoid the use of the contact forms θ^a by fixing on $\mathbb{R}^p$ the standard atlas containing the unique chart $(\mathbb{R}^p, id_{\mathbb{R}^p})$. This obstacle can be overcame by first noticing that all general constructions of [6] continue to hold if one replaces the closed ideal $\mathcal{I}_{\text{var}}$ of $\mathbf{\Omega}(J^\infty B)$, used in [6], with any family of closed ideals $\mathcal{I}_{i\in I}$ $(I \neq \emptyset)$. Then we make the following "naive" suggestion: instead of considering the ideal generated by means of the Euler-Lagrange form $e^{\mathcal{B}}(\mathcal{L})$, we consider the family of ideals generated by the family of m-forms $(i_{\mathbf{v}}(\mathbf{\Omega}))_{\mathbf{v}\in\mathcal{V}(J^3B)}$ together with the family of contact forms already considered in [6], where $\mathcal{V}(J^3B)$ is the module of vertical vectorfields defined on $J^3\mathcal{B}$. The elements of $\mathcal{V}(J^3B)$ must be here considered as a *mere parameters*; for this reason we shall use boldface letters to denote them. This construction allows us to use the results of [6] also in the variational case, since (1.10) holds as an equivalent of the Euler-Lagrange equations. This suggestion could be useful out of the context of this paper, since equation (A.5) of the Appendix shows that $i_{\mathbf{v}}(\mathbf{\Omega})$ belongs to the closed ideal $\mathcal{I}_{\text{var}}$ generated by the contact forms of the adapted basis and by the m-form $\mathbf{ds}$, for any $\mathbf{v} \in \mathcal{V}(J^3B)$. Hence, the previous ideal can be replaced by this last one, obtaining cohomological groups which do not depend on the Lagrangian $\mathcal{L}$.

Remark. By using a variant of the construction presented in [9], one can avoid to introduce the notation needed when using the infinite jet bundle $J^\infty\mathcal{B}$ by just noticing that the set $C' \cup \{\mathbf{ds}\}$ generate a closed ideal $\mathcal{I}'_{\text{var}}(J^6B)$ of $\mathbf{\Omega}(J^6B)$. This fact allows us to consider the cohomological groups $H'_{\text{var}}(J^6B)$ of the quotient cochain complex $\widehat{\mathbf{\Omega}}(J^6B)/\mathcal{I}'_{\text{var}}(J^6B)$ (recall that $\mathbf{\Omega}(J^kB) \subseteq \widehat{\mathbf{\Omega}}(J^kB) = \mathcal{H}or(J^k\mathcal{B}) \oplus \mathcal{K}(J^k\mathcal{B})$), having the total differential modulo $\mathcal{I}'_{\text{var}}(J^6B)$ as a coboundary operator. Then the cohomological group H_{var} is obtained by considering the projective limit of $H'_{\text{var}}(J^6B)$, in the obvious way.

Also the "tower construction" of [6] (in the following it will be called BG-*tower construction*, because the variational component of the first order perturbation functor will determine a further tower which will be called here the *Jacobi tower*) is not well suited to include the Euler-Lagrange equations of variational problems. In fact, the (m+1)-form $d_H(i_{\mathbf{v}}(\mathbf{\Omega}))$ vanishes, when $\mathbf{v} \in \mathcal{V}(J^3B)$ is considered as a mere parameter, as in (1.10), and the BG-tower construction coincides essentially with the horizontal derivative. In order to overcome this problem we shall assume for simplicity that M is orientable and that a global volume form $\mathbf{vol}$ is fixed on M. Then, there exists a unique 1-form $\widetilde{\mathbf{\Omega}}$ on J^6B such that:

$$(3.14) \qquad i_{\mathbf{v}}(\mathbf{\Omega}) = i_{\mathbf{v}}(\widetilde{\mathbf{\Omega}} \wedge \mathbf{vol}) \quad , \quad \forall \mathbf{v} \in \mathcal{V}(J^3B) \; .$$

Following an idea first developed in [17] one can now consider the family of Lagrangians $\mathcal{L}^1_{\mathbf{v}} = i_{\mathbf{v}}(d_H\widetilde{\mathbf{\Omega}})\mathbf{vol} : J^4B \times_M TM \to \Lambda^m M$, locally defined by:

$$(3.15) \qquad \mathcal{L}^1_{\mathbf{v}}(j^4\lambda, X) = (j^6\lambda)^*(d_\mu(\widetilde{\mathbf{\Omega}}_a)v^a X^\mu)\mathbf{vol} \; ,$$

being $\widetilde{\mathbf{\Omega}}_a$, v^a and X^μ the local components of $\widetilde{\mathbf{\Omega}}$, $\mathbf{v}$ and X, respectively, where λ is any section, X is a vectorfield defined on M and $\mathbf{v}$ an element of $\mathcal{V}(J^3B)$. The first variation of the family of Lagrangians (3.14) splits into:

$$(3.16a) \qquad (j^6\lambda)^* d_\mu(\widetilde{\mathbf{\Omega}}) = 0 \; ,$$

$$(3.16b) \qquad [\delta(j^6\lambda)^*(d_\mu(\widetilde{\mathbf{\Omega}}_a))]v^a X^\mu = 0 \; .$$

As is well known, the inverse problem of the Calculus of Variations (i.e., the problem of finding a variational principle whose Euler-Lagrange equation is fixed a priori) is not at all trivial, while the same problem becomes in a sense trivial if one allows the possibility of considering new variables which are not a priori restricted to satisfy any further condition (even if, in some cases, one can consider to be more important the advantages coming from the introduction of variational methods than the problems coming from the "triviality" of the new variables; see, e.g., [37]). Replacing the original Euler-Lagrange equations with the new equations (3.16) goes in fact in this "trivial" direction. However, the usefulness of this alternative variational principle is garanteed by the results of [6] and [7], which ensure that the solutions of the equation (3.16a) are of practical importance, while the second equation (3.16b) does not eliminate any solution of the first equation (3.16a), since it is always verified by the vectorfield $X = 0$ and hence it preserves at least a copy of any solution of the first equation. A second question is whether the relation between $\mathbf{\Omega}$ and $\widetilde{\mathbf{\Omega}}$ preserves or not the informations on the variational problem contained in the first (m+1)-form. A positive answer can be obtained by remarking that being M orientable there exists an atlas of M in which the local expression of $\widetilde{\mathbf{\Omega}}$ coincides with the local expression of $\mathbf{\Omega}$. In any case, most of the relations needed between $\mathbf{\Omega}$ and $\widetilde{\mathbf{\Omega}}$ can be easily deduced from the results of [16].

PROPOSITION 3.10. *The following inlcusion holds*:

$$\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{I}'_{\text{var}}(J^6B)) \subseteq \mathcal{I}'_{\text{var}}(J^6VB) \tag{3.17}$$

and hence the corresponding homological construction can be easily iterated.

PROOF. This can be easily seen by using (2.7) together with the appropriate extension of (3.8) to all the involved contact forms. □

REMARK. In particular equations (3.6) and (3.17) entail that $\widetilde{\mathcal{P}}$ induces a morphism between the corresponding cohomological groups, which will be denoted by the same letter (with an abuse of notation).

The new tower construction obtained as in [6] by iterating the application of $\widetilde{\mathcal{P}}$ to the equivalent Euler-Lagrange equations (1.10) of the

Lagrangians which are obtained by iterating the action of $\widetilde{\mathcal{P}}$ on the original Lagrangian $\mathcal{L}$ has a sure meaning, since it determines the "higher order Jacobi fields" (the reasons for which those fields must be considered are strictly analogous to those well explained in [6], whereby they refer to conservation laws rather than to higher order variations, as here).

Let us now notice that according to the method developed in [6] and to (3.15) the other "levels" of the relevant BG-tower also determine variational problems (in a sense "associated" to the original one we are considering). Accordingly, the previous construction can also be iterated for each level. For the other "levels" of the BG-tower construction the problem of their usefulness comes from the "triviality" of the variational principle (3.15); again, because of (3.16), this problem is related to the usefulness of "Jacobi fields" for generic systems of partial differential equations, which does not seem to be clear to us, nor it has been considered in the existing literature. We limit ourselves to remark that, as in the case of variational problems, also for generic differential equations "Jacobi fields" determine the directions in which a homotopic variation of a solution is still determined by means of solutions. This suggests us to give the following definition:

Definition 3.1. We will call **Jacobi tower** the set of cohomological groups so obtained by iterating the action of $\widetilde{\mathcal{P}}$ on $\mathcal{L}$ while the k^{th}-**Jacobi tower** will be the set of cohomological groups obtained by iterating the action of $\widetilde{\mathcal{P}}$ on the Lagrangians constructed by iterating (3.15) till the k-th term of the corresponding BG-tower. We shall call JBG-**wall** the complete set of cohomological groups obtained in this way.

Let us now turn to consider the approach of "variational sequences". Differently from [6], the construction of [9] is explicitly worked out for variational problems, hence it does not present the problems coming from the definition of solutions of a differential partial equation we discussed before. A further observation of [9] is that one does not need the structure of graded exterior algebra on a quotient cochain complex of $\mathbf{\Omega}(J^6B)$ in order to define its cohomological groups, but simply an Abelian group structure. Finally, a last observation can be obtained from the comparison of [6] and [9]. In fact, if $\mathcal{I}$ is a graded complex of closed modules and $(\mathcal{I}^r)_{1\leq r\leq N}$ is its gradation, then $\mathcal{I}^r$ can be obtained by setting $\mathcal{I}^r = \widetilde{\mathcal{I}}^r + d\widetilde{\mathcal{I}}^{r-1}$, for $1 < r \leq N$ and $\mathcal{I}^1 = \widetilde{\mathcal{I}}^1$, where each of the modules $\widetilde{\mathcal{I}}^r$ can be chosen by using different criteria for each $1 \leq r \leq N$. As an example, in [9] the graded module $\mathcal{I}_K$, with gradation $(\mathcal{I}_K^r)$, is obtained by taking the graded family of modules which are the kernels of a suitable family of $\mathbf{\Omega}^0(J^6B)$-linear applications. This family can be easily described in the following way. Let $\mathcal{I}_K^1(J^6U) = \widetilde{\mathcal{I}}_K^1(J^6U)$ be the submodule of $\mathbf{\Omega}^1(J^6U)$, generated by the set C', of all contact forms on J^6U, having set $J^6U \equiv (\pi \circ \pi_0^6)^{-1}(U)$, for any U open set of M which is the domain of a local coordinate system. Let us also set $\widetilde{\mathcal{I}}_K^r(J^6U) = \mathcal{I}_K^1(J^6U) \wedge \mathbf{\Omega}^{r-1}(J^6U)$, for any $r \in \{2, \dots m\}$, for any open set U of M, on which a local coordinate system is defined. Finally, we set $\widetilde{\mathcal{I}}_K^r(J^6U) = (\mathcal{I}_K^1(J^6U))^{r-m+1} \wedge \mathbf{\Omega}^{m-1}(J^6U)$, where $(\mathcal{I}_K^1(J^6U))^p$ denotes the p-th power with respect to the wedge product, for any $r \in \{m+1, \dots N\}$, where N is the dimension of J^6B. Then, $\widetilde{\mathcal{I}}_K^r$ is the submodule of $\mathbf{\Omega}(J^6B)$ of r-forms whose restrictions belong to $\widetilde{\mathcal{I}}_K^r(U)$, for any $r \in \{2, \dots N\}$ and any open subset U of M which is the domain of a local coordinate system and $\mathcal{I}_K^r = d\widetilde{\mathcal{I}}_K^{r-1} + \widetilde{\mathcal{I}}_K^r$, for any $r \in \{2, \dots N\}$. Again, $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{I}_K)$ is contained into the module obtained with the same criteria starting from the variational problem $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\mathcal{L})$. As a consequence the suitable Jacobi tower can be constructed and analogous remarks hold, as in the previous cohomological groups.

Let us now remark that the papers [9] and [10] were published before [6] and [7], so that they present problem analogous to those we already mentioned for the tower construction of [6]. In fact, the Euler-Lagrange morphism $e^{\mathcal{B}}(\mathcal{L})$ of a Lagrangian $\mathcal{L}$ is such that $d_H e^{\mathcal{B}}(\mathcal{L}) = 0$ holds. We overcome this problem by assuming that M is orientable and that a volume form **vol** is fixed on M. Then, there exists a unique 1-form $\tilde{e}^{\mathcal{B}}(\mathcal{L})$ on $J^6(\mathcal{B})$ such that

$$e^{\mathcal{B}}(\mathcal{L}) = \tilde{e}^{\mathcal{B}}(\mathcal{L}) \wedge \mathbf{vol}\ . \tag{3.18}$$

Again, we have the family of Lagrangians $\mathcal{L}^{1\mathbf{v}} = (d_H \tilde{e}^{\mathcal{B}})_{\mathbf{v}} \otimes \mathbf{vol} : J^4B \times_M TM \to \Lambda M$, $\mathbf{v} \in J^6VB$, defined by:

$$\mathcal{L}^{1\mathbf{v}}(j^4\sigma, X) = ((d_\mu e_a)v^a X^\mu)\mathbf{vol}\ . \tag{3.19}$$

Also in this case all the considerations already made for the BG-tower construction will follow, so that at the end we have a second “wall construction" for the Euler-Lagrange differential equation (1.10) which differs from the standard BG-wall and contains other informations on the same class of problems. These informations are obviously related to the "Lepagean (equivalent) forms", i.e. to the m-forms of J^6B which are suitably obtained from $\mathcal{L}$ to determine the same Euler-Lagrange equation (see [10]).

Remarks. Let us finally make a couple of remarks, which in a sense point towards suggestions which could be in contrast with each other. If one chooses the family of modules $\widetilde{\mathcal{I}}$ in such a way that it has a maximum number of null spaces (as we shall suggest below), then the properties of the cohomological groups obtained will be of course "closer" to the properties of the full de Rham groups of the bundle. On the other hand, when $\mathcal{B}$ coincides with the trivial bundle $pr_1 : [0,1] \times M \to [0,1]$, the family $(\theta^a \wedge dt)$, where θ^a is now given by $\theta^a = dy^a - \dot{y}^a dt$, generates a closed ideal of differential forms which determines cohomological groups isomorphic to the de Rham cohomology of M, while the cohomological groups of [6] and [9] considered here are necessarily trivial.

Example. As an example of a way to obtain cohomological groups which are "close" to the de Rham ones, we consider the submodule $\mathcal{I}_1^{m+2}$ of $\mathbf{\Omega}^{m+2}(J^6B)$ locally generated by the (m+2)-forms $\theta^a_{\mu_1\dots\mu_h} \wedge \theta^b_{\nu_1\dots\nu_k} \wedge \mathbf{ds}$ $(h,k \leq 6)$. Then, by taking $\widetilde{\mathcal{I}}_1^r = 0$, for any $r \neq m+2$, one obtains a closed graded module and hence a cohomological graded group. In this complex $(\mathcal{I}_1^r)_{1\leq r\leq N}$, the Euler-Lagrange form $e^{\mathcal{B}}(\mathcal{L})$ is a cochain in $\mathcal{I}_1^{m+1}$ (see (A.5)) and hence it determines a non-trivial cohomology class. What is important here is that the combined action of the k-jet extensions and of the variational components of the first order perturbation functor allows us to construct the corresponding JBG-wall: this possibilty is a further sign of the naturality of the functors considered here. Finally, let us denote by $\widehat{\mathcal{I}}$ anyone of the graded modules $\mathcal{I}_{BG}$, $\mathcal{I}_K$ and $\mathcal{I}_1$. Then, the restriction of the total differential to $\widehat{\mathcal{I}}$ determines a structure of cochain complex, which in turn determines cohomological groups. Again, the "Jacobi tower construction" can be performed for those groups since $\widetilde{\mathcal{P}}_{J^6\mathcal{B}}(\widehat{\mathcal{I}}) \subseteq \widehat{\mathcal{I}}$. These cohomological groups could be useful, as the case of the trivial bundle $[0,1] \times M$ (which is related to the variational aspects of Riemannian Geometry) shows. We conclude this part by remarking that one could try to find the "best" closed submodule $\mathcal{I}$, if it exists, which would contain most of the informations encoded into the cohomological groups considered here. These problems will be considered in [20].

Conclusive Remark. We conclude our paper by stressing that in the general case the first order perturbation functor is compatible with (1.11), via (2.18), so that the conserved Noether currents of the original Lagrangian $\mathcal{L}$ are transformed by $\overline{\mathcal{P}}_{\mathcal{B}}$ into the Noether currents of the deformed Lagrangian $\widetilde{\mathcal{P}}(\mathcal{L}) \equiv \mathcal{L}_{(1)}$. More details will be given in [21].

– Appendix

A.1 – Augmented variational principles and examples

The basic justification for the introduction of the BG-tower comes from the KDV equation (see [6] and [7]), hence it seems important to indicate methods which allow one to write this equation as the Euler-Lagrange equation of a non-trivial variational principle. This problem has an importance of its own for other reasons, which are well explained in the Introduction to Chapter 2 of the book [36]. As a consequence, many methods have been developed to solve the inverse problem of the Calculus of Variations, even in those cases in which it is clear from the beginning that Lagrangians which determine the system of partial differential equations considered do not exist (e.g., the case of heat equations and KdV equations).

Unfortunately, people interested into this "generalized aspect of the inverse problem" have paid more attention to the systems of partial differential equations coming from technical applications rather than from Mathematics and Physics. In this Appendix, instead of applying one of the existing methods to the KdV equation we prefer to suggest a new one, because this choice will require simple calculations and will suggest that, if one does not find the existing Lagrangians to be satisfactory, one can always try to find new ones. The method considered here belongs to a larger class of methods in which the basic tool is the addition of new variables to the original variables of the given system of partial differential equations. The first example we mention is the method known as "method of mirror variables", explicitely introduced by Glansdorff and Prigogine (see [38]), following an earlier example of a hydrodynamical principle stated by Bateman (see [39]) and elaborated by Morse and Feshbach (see [40]) with some contributions (see, e.g., the papers quoted therein and in [37]). This method consists in adding to the variables of the problem, which will be subjected to variations, an identical number of "mirror variables", which are considered as mere parameters. Obviously, this addition implies in many cases that some new solutions are added to the solutions of the system of partial differential equations one started from.

The method proposed here consists instead in adding just one new dependent variable to the original dependent variables of the problem by requiring that a Lagrangian exists so that: (i) among its Euler-Lagrange equations the equation for the new variable has a simple and possibly "canonical" solution; (ii) in corrispondence with this solution, the remaining Euler-Lagrange equations reduce to the original system of the original variables or, at least, have the same set of solutions. In this way, it is easy to control the relations between the geometric objects related to the "associated Lagrangian" with those related to the original system of partial differential equations (e.g., one might require that the group of gauge transformations of the system coincides with the sub-group of gauge transformations of the associated Lagrangian which preserves the chosen solution for the extra variable; and so on). For the heat equation and the KdV equation the obvious choice for the new variable is what we call *admissible time measure*.

Example A.1 - The heat equation: For the case of the heat equation, let us consider the trivial bundle $\mathcal{B} = (\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^m, pr)$, where pr is the canonical projection of $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^2$ onto $\mathbb{R} \times \mathbb{R}^m$ and let us look for a first order Lagrangian $\mathcal{L} = L(t, x^\mu, \tau, q, \dots)\mathbf{ds}$, where t, x^μ, τ and q the time coordinate, the spatial coordinate and the admissible time measure, while $\mathbf{ds}$ is the standard volume form on $\mathbb{R} \times \mathbb{R}^m$ and "dots" replace the remaining variables, i.e. the partial spacetime derivatives of τ and q. Moreover, we require that one of the two Euler-Lagrange equations of $\mathcal{L}$ is satisfied by the solution $\tau = t$ and that in correspondence of this solution the remaining equation coincides with the heat equation. There exists a large class of functions L determining Lagrangians with this property. The whole class can be determined by using a procedure analogous to the one used in [37] to show that the heat equation cannot be determined by a variational principle in the classical sense (see [37] paragraph 2.6, pp. 65-66). The following function seems to be the simplest function belonging to this class:

$$L = q\left(1 - \frac{\partial \tau}{\partial t}\right) + \delta^{ij}\frac{\partial \tau}{\partial x^i}\frac{\partial q}{\partial x^j} \ . \tag{A.1}$$

The Euler-Lagrange equations of the "associated" Lagrangian are in fact:

$$\frac{\partial q}{\partial t} - \delta^{ij}\frac{\partial^2 q}{\partial x^i \partial x^j} = 0 \tag{A.2a}$$

and

$$1 - \frac{\partial \tau}{\partial t} - \delta^{ij}\frac{\partial^2 \tau}{\partial x^i \partial x^j} = 0 \ , \tag{A.2b}$$

with the obvious meaning of the symbols used. One sees immediately that $\tau = t$ makes (A.2b) satisfied, so that (A.2a) reduces to nothing but the heat equation $\frac{\partial q}{\partial t} - \Delta q = 0$ in flat space $\mathbb{R}^m$.

Example A.2 - The KdV Equation: In the case of KdV equation we take $m = 1$, hence $\mathcal{L} = L(t, x, \tau, u \ldots)\mathbf{ds}$, with the obvious meaning of the symbols used. Even in this case, the set of all functions L whose Euler-Lagrange equations allow the solution $\tau = t$ so that in correspondence of this solution the remaining equation becomes the KdV equation, is large. The following function seems to be the simplest one belonging to this class:

$$L = 6u^2\frac{\partial \tau}{\partial x} + u\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x}\frac{\partial^2 u}{\partial x^2} - u \ . \tag{A.3}$$

This is a second order Lagrangian, whose Euler-Lagrange equations are:

$$12u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{A.4a}$$

and

$$6u\frac{\partial \tau}{\partial x} + \frac{\partial \tau}{\partial t} - 1 + \frac{\partial^3 \tau}{\partial x^3} = 0 \ . \tag{A.4b}$$

It is immediate to see that equation (A.2b) is satisfied by $\tau = t$ so that (A.4a) reduces to the standard KdV equation required.

A.2 – Some technical formulae

We list in this Appendix some formulae used in this papers. Let us consider the $\mathbb{R}$-linear mapping $\Theta^{\mathcal{B}} : \widetilde{\mathbf{\Omega}}_2^m(M) \to \widetilde{\mathbf{\Omega}}_3^m(J^1\mathcal{B})$, where $\widetilde{\mathbf{\Omega}}_k^h$ are the module introduced in Section 3, which associates to any second order Lagrangian $\mathcal{L}$ over M its Poincaré-Cartan m-form. Then we can express the Euler-Lagrange (m+1)-form $e^{\mathcal{B}}(\mathcal{L})$ of any Lagrangian $\mathcal{L}$ on M by means of the multiplectic form $\mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) = d\Theta^{\mathcal{B}}(\mathcal{L})$. In fact, a simple calculation shows that:

$$
\begin{aligned}
e^{\mathcal{B}}(\mathcal{L}) = {} & \mathbf{\Omega}^{\mathcal{B}}(\mathcal{L}) + (d_\nu \frac{\partial^2 L}{\partial y^b \partial y^a_{\mu\nu}} - \frac{\partial^2 L}{\partial y^b \partial y^a_\mu})\theta^b \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
& + \left(d_\nu \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b \partial y^a_{\rho\mu}} + \frac{\partial^2 L}{\partial y^a \partial y^b_{\rho\mu}} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\mu} \right) \theta^b_\rho \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
& + \left(d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\sigma\mu}} - \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_\mu} \right) \theta^b_{\rho\sigma} \wedge \theta^a \wedge \mathbf{ds}_\mu + \\
& + \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\tau}} \theta^b_{\rho\sigma\tau} \wedge \theta^a \wedge \mathbf{ds}_\mu - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} \theta^b_\rho \wedge \theta^a_\nu \wedge \mathbf{ds}_\mu + \\
& - \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} \theta^b_{\rho\sigma} \wedge \theta^a_\nu \wedge \mathbf{ds}_\mu \ ,
\end{aligned} \tag{A.5}
$$

where $\mathcal{L} = L\mathbf{ds}$ holds locally. From the previous equation, by standard calculations we get:

$$
d[e^{\mathcal{B}}(\mathcal{L})] = \{\alpha_{ab}\theta^b + \alpha^\rho_{ab}\theta^b_\rho + \alpha^{\rho\sigma}_{ab}\theta^b_{\rho\sigma} + \alpha^{\rho\sigma\mu}_{ab}\theta^b_{\rho\sigma\mu} + \alpha^{\rho\sigma\mu\tau}_{ab}\theta^b_{\rho\sigma\mu\tau}\} \wedge \theta^a \wedge \mathbf{ds} \,, \tag{A.6}
$$

being

$$
\alpha_{ab} = d_\mu \left(d_\nu \frac{\partial^2 L}{\partial y^b \partial y^a_{\mu\nu}} - \frac{\partial^2 L}{\partial y^b \partial y^a_\mu} \right) \ , \tag{A.7}
$$

$$
\alpha^\rho_{ab} = 2 \frac{\partial^2 L}{\partial y^{[a} \partial y^{b]}_\rho} + 2 d_\mu \left(\frac{\partial^2 L}{\partial y^b \partial y^a_{\rho\mu}} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\mu} + d_\nu \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_{\mu\nu}} \right) \ , \tag{A.8}
$$

$$
\begin{aligned}
\alpha^{\rho\sigma}_{ab} = {} & d_\mu d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} + 2 \frac{\partial^2 L}{\partial y^{(b} \partial y^{a)}_{\rho\sigma}} + 2 d_\nu \frac{\partial^2 L}{\partial y^b_r \partial y^a_{\sigma\nu}} + \\
& - d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_\nu} - \frac{\partial^2 L}{\partial y^b_\rho \partial y^a_\sigma} \,,
\end{aligned} \tag{A.9}
$$

$$\alpha_{ab}^{\rho\sigma\mu} = 2d_\nu \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\nu}} + \frac{\partial^2 L}{\partial y^b_{\rho} \partial y^a_{\sigma\mu}} - \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu}} \tag{A.10}$$

and

$$\alpha_{ab}^{\rho\sigma\mu\tau} = \frac{\partial^2 L}{\partial y^b_{\rho\sigma} \partial y^a_{\mu\tau}} \ . \tag{A.11}$$

The coefficients (A.7)-(A.11) are the relevant coefficients which enter the Jacobi form of the given Lagrangian $\mathcal{L}$; see [16], [17] for details.

REFERENCES

[1] J. F. Pommaret: *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach Science Publisher, Inc., NY., 1978.

[2] D. J. Saunders: *The Geometry of Jet Bundles*, London Mathematical Society Lecture Notes, series **142**, Cambridge University Press, Cambridge, U.K., 1989.

[3] I. S. Krasil'shchik – V. V. Lychagin – A. M. Vinogradov: *The Geometry of Jet Bundles*, Gordon and Breach Science Publisher, Inc., NY., 1986.

[4] D. Krupka: *Lectures on Variational Sequences*, Open Education and Science, Opava, Czech Republic, 1995.

[5] I. M. Anderson – T. Duchamp: Amer. J. Math., **102** (1980) 781-868.

[6] R. L. Bryant – P. A. Griffiths: J. Amer. Math. Soc., **8** (1995), 507-596.

[7] R. L. Bryant – P. A. Griffiths: Duke Math. J., **78** (1995), 531-676.

[8] P. Dedecker – W. M. Tulczyjew: In: *Spectral Sequences and the Inverse Problem of the Calculus of Variations*, Coll. on Diff. Geom. Methods in Math. Physics, Aix-en-Provence, Sept. 1978; Lecture Notes in Math., **836** Springer Verlag, Berlin, 1980, pp. 473-495.

[9] D. Krupka: In: *Differential Geometry and its Applications*, Proc. Conf., Aug. 27-Sept 2, 1989, Brno, 1990, pp. 236-254.

[10] D. Krupka: In: *Topics in the Calculus of Variations: Finite Order Variational Sequences*, Diff. Geom. and its Appl., Proc. Conf. Opava, Aug. 1992, Silesian Univ., 1993, pp. 473-495.

[11] R. Vitolo: Math. Proc. Phyl. Cambridge Soc., **124**, No. 3, 1998, to appear

[12] M. Francaviglia – M. Palese – R. Vitolo: Czech. Math. Journal, **52** (127) (2002), pp. 197–213

[13] M. Francaviglia – M. Palese – R. Vitolo: *A geometric formulation of Hassian and Jacobi tensors for higher order Lagrangians*, (2002), (submitted).

[14] M. Ferraris – M. Francaviglia: Journ. Math. Phys., **26** (6) (1985), 1243-1252.

[15] B. Casciaro – M. Francaviglia: Rend. Mat., Serie VII, **16** (1996), 1-32.

[16] B. Casciaro – M. Francaviglia – V. Tapia: In: *Differential Geometry and its Applications*, Proceedings Brno Conference 1985, J. Janiska et al. eds.; Masaryk Univ., Brno, 1996, pp. 353-372.

[17] O. Amici – B. Casciaro – M. Francaviglia: *Rend. Mat., Roma Serie VII*, **16** (1996), 637-669.

[18] B. Casciaro – M. Francaviglia: Ann. Mat. Pura e Appl., VI, **172** (1997), 219-228.

[19] B. Casciaro – M. Francaviglia: In: *Nonlinear Analysis, Theory, Methods & Applications*, Proc. 2nd World Congress of Nonlinear Analysis 1997, Pergamon, Vol. 30, No. 1, London, 1997, pp. 579-604.

[20] O. Amici – B. Casciaro – M. Francaviglia: *Cohomological Constructions in the Calculus of Variations: a Comparative Approach*, in preparation.

[21] O. Amici – B. Casciaro – M. Francaviglia: *From the Poincarè–Cartan Form to Conservation laws in the Calculus of Variations*, in preparation.

[22] M. Francaviglia: *Relativistic Theories (The Variational Structure)*, Ravello Lectures 1988; Quaderni del C.N.R., 1990.

[23] M. Ferraris – M. Francaviglia – M. Mottini: Rend. Mat. Serie VII, **14** (1994), 457-481.

[24] J. Souriau: *Structures des systemès dynamiques*, Dunod, Paris, 1970.

[25] M. Ferraris – M. Francaviglia: Class. Q. Gravity, Supplement 1992, **9** (1992), 579-595.

[26] M. Ferraris – M. Francaviglia – C. Reina: Journ. Math. Phys., **24** (1) (1983), 120-142.

[27] I. Kolàř – P. W. Michor – J. Slovàk: *Natural Operations in Differential Geometry*, Springer–Verlag, NY., 1993.

[28] L. Fatibene – M. Ferraris – M. Francaviglia – M. Godina: In:*Differential Geometry and Applications*, Proc. Conf. Brno, August 1995; J. Janiska et al. eds. Masaryk Univ., Brno, 1996, pp. 549-558

[29] L. Fatibene – M. Ferraris – M. Francaviglia – M. Godina: Journ. Gen. Rel. and Gra., **30** (9) (1998), 1371-1389.

[30] A. H. Taub: Comm. Math. Phys., **15** (1969), 235.

[31] K. Yano – S. Ishihara: *Tangent and Cotangent Bundles*, M. Dekker, NY., 1973.

[32] M. Ferraris – M. Francaviglia: Journ. Gen. Rel. and Grav., **22** (9) (1990), 965-985.

[33] A. M. Vinogradov: Soviet Math. Dokl., **19** (1978), 144-148.

[34] T. Tsuishita: Osaka J. Math., **19** (1982), 311-363

[35] O. Amici – B. Casciaro – M. Francaviglia: Rend. Circ. Mat., Palermo Serie II, **47** (1998), 207-220

[36] B. D. Vujanovic – S. E. Jones: *Variational Methods in Nonconservative Phenomena*, Academic Press, Inc. Boston, 1989.

[37] P. Glandsdorff – I. Prigogine: Physica, **31** (1965), 1242-1254.

[38] H. L. Drayden – F. D. Murnaghan – H. Bateman: *Motion of an Incompressible Viscous Fluid*, Dover Publications, NY., 1956.

[39] P. M. Morse – H. Feshbach: *Methods of Theoretical Physics*, McGraw–Hill Inc., NY., 1953.

Lavoro pervenuto alla redazione il 18 novembre 2000
ed accettato per la pubblicazione il 16 gennaio 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo degli Autori:

Oriella Amici – Biagio Casciaro – Dipartimento di Matematica – Università di Bari – Via Orabona, 5 – 70125 Bari (Italia)
E-mail: amici@dm.uniba.it casciaro@dm.uniba.it

Mauro Francaviglia – Dipartimento di Matematica – Università di Torino – Via C. Alberto, 10 – 10123 Torino (Italia)

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 47-64

Nodal curves and Brill - Noether theory

E. BALLICO

ABSTRACT: *Here we prove some existence theorems for special spanned line bundles on the general nodal curve of genus $g \geq 2$. We give counterexamples to similar questions for curves with seminormal singularities.*

1 – Introduction

In the first 3 sections of this paper we study the Brill - Noether theory of special divisors on the general k-gonal curve with only ordinary nodes as singularities. On an integral projective curve, Y, there are at least 4 quite different Brill - Noether theories: one can study spanned line bundles, line bundles, spanned rank 1 torsion free sheaves or rank 1 torsion free sheaves. The Brill - Noether theory of rank 1 torsion free sheaves is the only one in which the set of the solutions is always a complete scheme. Passing to the spanned subsheaf, one can reduce the Brill - Noether theory of rank 1 torsion free sheaves to the one for spanned torsion free sheaves. The Brill - Noether theory of line bundles is interesting because it concerns important closed subschemes of the non-complete scheme $\mathrm{Pic}^d(Y)$. For the relations between the last two theories for curves with only ordinary nodes or ordinary cusps as sin-

KEY WORDS AND PHRASES: *Nodal curve – Irreducible curve – Brill - Noether theory – Special line bundle – Special divisors – Seminormal singularity.*
A.M.S. CLASSIFICATION: 14H51 – 14H20

gularities, see 2.3. The Brill - Noether theory of spanned line bundles is the more important one because it concerns the morphisms $Y \to \mathbf{P}^r$. But we have an additional problem because we are interested in k-gonal curves and their Brill - Noether theories depend very much on the singularities of the degree k pencil. For any rank 1 torsion free sheaf F on Y, set $\mathrm{Sing}(F) := \{P \in Y : F \text{ is not locally free at } P\}$. Thus $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(Y)$. We introduce the following definition.

Definition 1.1. Fix integers g, x, k and y with $k \geq 2$, $g \geq 2k + x - y + 2$, $g \geq x \geq y \geq 0$ and $x \geq 0$. Let X be a general smooth $(k-y)$-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-y}(X)$ the degree $k-y$ spanned line bundle on X and $h_M : X \to \mathbf{P}^1$ the associated morphism with $\deg(h_M) = k-y$. Take $x+y$ general points P_i, $1 \leq i \leq x-y$, A_j, $1 \leq j \leq y$, and B_j, $1 \leq j \leq y$, on Y. Fix points Q_i, $1 \leq i \leq x-y$, with $h_M(P_i) = h_M(Q_i)$ for every i. Let $\pi : X \to Y$ be the birational morphism obtained gluing together the points P_i and Q_i for $1 \leq i \leq x-y$, and the points A_j and B_j for $1 \leq j \leq y$. Hence Y is a nodal curve with $p_a(Y) = g$ and x nodes. Set $F := \pi_*(M)$. Thus F is a rank 1 torsion free sheaf on Y with $\deg(F) = k$, $\mathrm{Sing}(F) = \{\pi(A_1), \dots, \pi(A_y)\}$ and $h^0(Y, F) = 2$. We will say that Y or the pair (Y, F) is the *general k-gonal curve of genus g with x nodes and a pencil with y singularities* or just a *general nodal k-gonal curve of genus g with type* (x, y).

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. As a sample of our results we state here the following one which will be proved in Section 2.

Theorem 1.2. *Fix integers g, x, y, k and d with $k \geq 2+y$, $x \geq y \geq 0$, $x > 0$, $g \geq 2k + 2x + 1$ and $2d \geq g+2$. Let Y be the general k-gonal nodal curve of genus g with type (x, y) and F the degree k pencil with card $(\mathrm{Sing}(F)) = y$. Then there is an irreducible locally closed subset Z of $\mathrm{Pic}^d(Y)$ with $Z \neq \emptyset$, $\dim(Z) = \rho(g-x+y, d, 1) - x := 2d-g+x-2-y$ such that every $R \in Z$ is spanned. If $d \leq g-x+y-1$, then we may find Z such that $h^0(Y, \mathrm{Hom}(F, R)) = 0$ for every $R \in Z$.*

The case $y = x$ is the easier one. If $y = x$ we obtain an existence result for embeddings of Y into $\mathbf{P}^r$, $r \geq 3$ (see Theorem 3.1).

In the last section we will consider seminormal curves in the sense of [17] and [9], i.e. curves with the simplest singularities compatible with their number of branches: if the singularity has r branches, then it is formally equivalent to the germ at $0 \in \mathbf{K}^r$ of the union of the r coordinate axis. We will show that for non-nodal seminormal curves the usual existence theorem for special line bundles (even non spanned ones) are not always true if one uses only the Brill - Noether number $\rho(g, r, d) := g-(r+1)(g+r-d)$ in its statement as in the case of smooth curves ([14] or [2]).

2 – Proof of 1.2

In the first part of this section we give several preliminary results needed for the proof of 1.2 and of other related results. Let Y be an integral projective curve, $\pi : X \to Y$ its normalization and F a rank 1 torsion free sheaf on Y. The sheaf $G := \pi^*(F)/\operatorname{Tors}(\pi^*(F))$ has rank 1 and no torsion. Hence $G \in \operatorname{Pic}(X)$. We claim that the natural map $\alpha : H^0(Y, F) \to H^0(X, G)$ is injective; set $x := h^0(Y, F)$ and take x general points $P_1, \dots, P_x$ of X; there is $f \in H^0(Y, F)$ with $f(\pi(P_i)) = 0$ for $i < x$ and $f(\pi(P_x)) \neq 0$; hence $\alpha(f)(P_i) = 0$ for $i < x$ and $\alpha(f)(P_x) \neq 0$, proving the claim.We will call the integer $\delta - \deg(F) := \deg(G)$ the δ-degree of F. By [10], Lemma 1, we have $\deg(F) + p_a(X) - p_a(Y) \leq \delta - \deg(F) \leq \deg(F)$ and $\delta - \deg(F) = \deg(F)$ if and only if $F \in \operatorname{Pic}(Y)$. Furthermore, $\deg(F) - \delta - \deg(F) \geq \operatorname{card}(\operatorname{Sing}(F))$. If F is spanned, then $\pi^*(F)$ is spanned and hence G is spanned.

(2.1) Let R be the one-dimensional complete semilocal ring which is either the completion of an ordinary node or an ordinary cusp. Let $\mathbf{m}$ be the maximal ideal of R (cusp case) or the intersection of the two maximal ideals (nodal case). Let M be a torsion free finitely generated R-module with $\operatorname{rank}(M) = 1$; here we assume that if R is the completion of an ordinary node, then M has constant rank on each of the two branches of R. Since $\operatorname{char}(\mathbf{K}) = 0$, there is a complete classification of all such M: there are uniquely determined integers a, b with $a \geq 0$, $b \geq 0$, $a + b = \operatorname{rank}(M)$ such that $M \cong R^{\oplus a} \oplus \mathbf{m}^{\oplus b}$ [11]. We will need only the case $\operatorname{rank}(M) = 1$.

(2.2) Let Y be an integral projective curve with only ordinary nodes and ordinary cusps as singularities, $\pi : X \to Y$ its normalization and F a rank 1 torsion free sheaf on Y. If $P \in \operatorname{Sing}(F)$, then the completion of F at P is isomorphic either to the maximal ideal (the cusp case) or the intersection of the two maximal ideals of the completion of $\mathbf{O}_{Y,P}$ (the nodal case). Thus $\deg(F) - \delta - \deg(F) = \mathrm{card}(\mathrm{Sing}(F))$. Let Y be an integral projective curve with only ordinary nodes and only ordinary cusps as singularities. The following remark shows the relations between the Brill - Noether theory of (not necessarily spanned) line bundles on Y and the Brill - Noether theory of spanned rank 1 torsion free sheaves on Y.

REMARK 2.3. Let Y be an integral projective curve and F a rank 1 torsion free sheaf such that for every $P \in \mathrm{Sing}(F)$ the curve has at P either an ordinary node or an ordinary cusp. By 2.1 for every $P \in \mathrm{Sing}(F)$ the completion of the stalk of F at P is isomorphic to the maximal ideal of the competion of the local ring $\mathbf{O}_{Y,P}$. Thus there is a unique $L \in \mathrm{Pic}(Y)$ with $F \subseteq L$, $\deg(L/F) = \mathrm{card}(\mathrm{Sing}(F))$ and $\mathrm{Supp}(L/F) = \mathrm{Sing}(F)$. We have $h^0(Y,F) \leq h^0(Y,L) \leq h^0(Y,F) + \mathrm{card}(\mathrm{Sing}(F))$. Furthermore, the integer $h^0(Y,L) - h^0(Y,F)$ is the number of points of $\mathrm{Sing}(F)$ at which L is spanned.

REMARK 2.4. Let X be a smooth projective curve of genus q and $h : X \to \mathbf{P}^1$, $f : X \to \mathbf{P}^1$ non-constant morphisms such that the associated morphism $j := (h,f) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational. Set $a := \deg(h)$, $b := \deg(f)$ and assume $q < ab - a - b + 1$. By the genus formula for a divisor of type (a,b) on $\mathbf{P}^1 \times \mathbf{P}^1$ the curve $j(X)$ is singular. Assume that $j(X)$ has only nodal singularities; by [1], Proposition 2.4 and its proof, this is the case if X is a general a-gonal curve and f is general in the set of all degree b pencils on X not composed with h. Assume that the monodromy group of a generic fiber of h is the full symmetric group; since $\mathrm{char}(\mathbf{K}) = 0$ this is the case if the reduction of a fiber of X has exactly $a - 1$ elements; this condition is always satisfied if X is a general a-gonal curve and h is the associated degree a pencil. Set $z := ab - a - b + 1 - q$. By our assumptions there is a non-empty set of $2z$-ples $(P_1, Q_1, \dots, P_z, Q_z) \in X^{2z}$ with $P_i \neq Q_i$ and $j(P_i) = j(Q_i)$ for every i, i.e. $h(P_i) = h(Q_i)$ and $f(P_i) = f(Q_i)$ for every i. Take 3 general points of $\mathbf{P}^1$, say B_1, B_2 and B_3 and fix $A_i \in X$ with $j(A_i) = B_i$, $1 \leq i \leq 3$. Fix an integer w with $0 < w \leq z$. Assume the existence of a quasi-projective integral subvariety T of the scheme $\mathrm{Hom}^b(X, \mathbf{P}^1)$ of degree b morphisms sending each A_i onto B_i, $1 \leq i \leq 3$, with $\dim(T) = w$, $j \in T$ and such that for every $t \in T$ the pair $j_t := (h, f_t)$ associated to the corresponding morphism $f_t : X \to \mathbf{P}^1$ satisfies the previous conditions. We claim that for a general $(P_1, \dots, P_w) \in X^w$ there is $t \in T$ and $(Q_1, \dots, Q_w) \in X^w$ such that $P_i \neq Q_i$ for every i and $j_t(P_i) = j_t(Q_i)$ for every i. Consider the following statement $T(k)$, $0 \leq i \leq w$.

Statement $T(k)$: for a general $(P_1, \dots, P_k) \in X^k$ there are a $(w-k)$-dimensional irreducible subvariety $T(P_1, \dots, P_k) \subseteq T$ and $(Q_1, \dots, Q_k) \in X^w$ with $P_i \neq Q_i$ for every i with $1 \leq i \leq k$ such that for every $t \in T(P_1, \dots, P_k)$ we have $j_t(P_i) = j_t(Q_i)$ for every i with $1 \leq i \leq k$. Furthermore, the set of all $t \in T$ satisfying this condition has codimension k in T.

The first assertion of Statement $T(w)$ is the claim we want to prove. Statement $T(0)$ is empty: just take $T(\varnothing) := T$. Assume proved $T(k)$ for some integer k with $k < w$ and take the corresponding points $Q_1, \dots, Q_j$. Set $J := \{(P, Q, t) \in X^2 \times T(P_1, \dots, P_k)$ with $P \neq Q$, $f_t(P) \notin \{f_t(P_1), \dots, f_t(P_k), B_1, B_2, B_3\}$, $f_t(P) = f_t(Q)$ and $j_t(P_i) = j_t(Q_i)$ for every $i\}$. Call $\pi_1 : J \to X$ and $\pi_3 : J \to T(P_1, \dots, P_k)$ the projections on the first and third factor. Since $w \leq z$ each fiber of π_3 is finite and non-empty. Thus every irreducible component of J has dimension $w - k > 0$. If J contains a slice $\{P\} \times X \times \{t\}$, then $f_t(X) = f_t(P)$ and hence f_t is constant; this is impossible because $\deg(j_t) = b$ by assumption. Since J is not union of slices $\{P\} \times X \times T(P_1, \dots, P_k)$, π_1 is dominant. By the assumption on the monodromy group of the generic fiber of h, for any fixed $t \in T$ and for general $P \in X$ either $h^{-1}(h(P)) \cap f_t^{-1}(f_t(P)) = \{P\}$ or $h^{-1}(h(P))$ is contained in $f_t^{-1}(f_t(P))$, i.e. $h = f_t$. We apply this observation to the general element of $T(P_1, \dots, P_k)$ to obtain the first assertion of $T(k+1)$ and to the general elements of similar codimension k irreducible component of T to obtain the last assertion of $T(k+1)$. Hence we obtain $\dim(T(P_1, \dots, P_k, P_{k+1})) < \dim(T(P_1, \dots, P_k))$ for general P_{k+1}, i.e. we obtain the last assertion of $T(k+1)$. We have $T(P_1, \dots, P_{k+1}) \neq\to$ for general P_{k+1} because of π_1 is dominant. Thus $T(k+1)$ holds. By induction we obtain $T(w)$, proving the claim.

Remark 2.5. Let X be a smooth projective curve of genus q and $h : X \to \mathbf{P}^1$, $f : X \to \mathbf{P}^1$ non-constant morphisms such that the associated morphism $j := (h, f) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational. Set $a := \deg(h)$, $b := \deg(f)$ and assume $q < ab - a - b + 1$. Assume that $j(X)$ has only nodal singularities and that the monodromy group of a generic fiber of h is the full symmetric group. By our assumptions there is a non-empty set of $2z$-ples $(P_1, Q_1, \dots , P_z, Q_z) \in X^{2z}$ with $P_i \neq Q_i$ and $j(P_i) = j(Q_i)$ for every i, i.e. $h(P_i) = h(Q_i)$ and $f(P_i) = f(Q_i)$ for every i. Take 3 general points of $\mathbf{P}^1$, say B_1, B_2 and B_3 and fix $A_i \in X$ with $j(A_i) = B_i$, $1 \leq i \leq 3$. Fix an integer w with $0 < w \leq z$ and an integer $\alpha > w$. Assume the existence a quasi-projective integral subvariety T of the scheme $\mathrm{Hom}^b(X, \mathbf{P}^1)$ of degree b morphisms sending each A_i onto B_i, $1 \leq i \leq 3$, with $\dim(T) = w$, $j \in T$ and such that for every $t \in T$ the pair $j_t := (h, f_t)$ associated to the corresponding morphism $f_t : X \to \mathbf{P}^1$ satisfies the previous conditions. By Remark 2.5 for a general $(P_1, \dots , P_w) \in X^w$ there is $t \in T$ and $(Q_1, \dots , Q_w) \in X^w$ such that $P_i \neq Q_i$ for every i and $j_t(P_i) = j_t(Q_i)$ for every i. Take a general element $(P_{\alpha-w+1}, Q_{\alpha-w+1}, \dots , P_\alpha, Q_\alpha)$ of $X^{2\alpha-2w}$. Let Y be the nodal curve obtained from X gluing together each pair (P_i, Q_i), $1 \leq i \leq \alpha$. By construction Y is a nodal curve with α nodes and with a degree b pencil of type $(\alpha, \alpha - w)$.

Example 2.6. Fix an even integer $g = 2b \geq 6$ and let X be a general smooth curve of genus $g-1$. Thus X has no spanned line bundle, L, with $1 \leq \deg(L) \leq [(g-1+3)/2] = b$ and a finite set, S, of line bundles, R, with $\deg(R) = b + 1$ and $h^0(X, R) = 2$. Furthermore, every $R \in S$ is spanned and $\mathrm{card}(S) = (2b)!/(b - 1)!b!) \neq 0$ ([2], p. 211) . Fix P, $Q \in X$ such that for every $R \in S$ the morphism $h_R : X \to \mathbf{P}^1$ has $h_R(P) \neq h_R(Q)$. Let Y be the curve obtained from X gluing P and Q. Thus Y is a curve with $p_a(Y) = g$, a unique ordinary node as singularities and with X as normalization. Call $\pi : X \to Y$ the normalization. Thus $\pi(P) = \pi(Q)$ is the singular point. For every $R \in S$ the rank 1 torsion free sheaf $\pi_*(R)$ has degree $b+2$ and $h^0(Y, \pi_*(R)) = h^0(X, R) = 2$. The condition $h_R(P) \neq h_R(Q)$ is equivalent to the fact that R is not the pull-back of a spanned line bundle on Y. Thus the condition $h_R(P) \neq h_R(Q)$ is equivalent to the spannedness of $\pi_*(R)$. We claim that there is no $M \in \mathrm{Pic}(Y)$ with $1 \leq \deg(M) \leq b + 1$ and $h^0(Y, M) \geq 2$. Assume the existence of such M. Thus $h^0(X, \pi^*(M)) \geq h^0(Y, M) \geq 2$. If $\deg(M) \leq b$ this is impossible because Y is general. Assume $\deg(M) = b+1$. Then $\pi^*(M) \in S$. We just saw that this is impossible by the choice of the pair $\{P, Q\}$. Notice that if we choose $\{P, Q\}$ general the curve Y is the general nodal curve of genus g with exactly one node. However, if we fix X general of genus $g - 1 = 2b$

and take as Y' the curve obtained from X gluing together two points in the same fiber of one of the morphisms, then we obtain a nodal curve Y' with one node, normalization with general moduli and $L \in \mathrm{Pic}(Y')$ with $\deg(L) = b+1$ and L spanned, while $\rho(g, b+1, 1) = -1 < 0$. Take a rank 1 torsion free sheaf F on Y with $\deg(F) \leq b+2$ and $h^0(Y, F) \geq 2$. Since F is not locally free, we have $\deg(\pi^*(F)/\operatorname{Tors}(\pi^*(F))) < \deg(F)$ and indeed $\deg(\pi^*(F)/\operatorname{Tors}(\pi^*(F))) = \deg(F) - 1$ ([10], Lemma 1). Since $\pi^*(F)/\operatorname{Tors}(\pi^*(F)) \in \mathrm{Pic}(X)$ and $h^0(X, \pi^*(F)/\operatorname{Tors}(\pi^*(F))) \geq 2$, we obtain $\pi^*(F)/\operatorname{Tors}(\pi^*(F)) \in S$. Thus $\deg(F) = b+2$ and there is a natural bijection between S and the set of all such sheaves F.

Example 2.7. Take b, g, X and S as in Example 2.6. Fix a point $A \in X$ such that for every $R \in S$ the morphism h_R is étale at A. Let $\pi' : X \to Y'$ the birational and bijective morphism with $p_a(Y') = g$, Y' with $\pi'(A)$ as unique singular point and an ordinary cusp at $\pi'(A)$. By the choice of A we may apply the proof of Example 2.6 in our situation just with notational modifications. Since as A we may take a general point of X, this description of the rank 1 torsion free sheaves of degree at most $b+2$ is the description of such sheaves for the general cuspidal curve of genus g with a unique singular point.

Examples 2.6 and 2.7 may be generalized in the following way. We omit the easy proof.

Proposition 2.8. *Let X be a smooth projective curve. Fix positive integers r and d such that for every integer $z \leq d-2$ and every $L \in \mathrm{Pic}^z(X)$ we have $h^0(X, L) \leq r$, while the set $S := \{R \in \mathrm{Pic}(X) : \deg(R) = d-1$ and $h^0(X, R) = r+1\}$ is finite. Fix P, $Q \in X$ such that for every $R \in S$ the morphism $h_R : X \to \mathbf{P}^r$ has $h_R(P) \neq h_R(Q)$. Let Y be the curve obtained from X gluing P and Q. Then there is no $M \in \mathrm{Pic}(Y)$ with $1 \leq \deg(M) \leq d$ and $h^0(Y, M) \geq r+1$. Furthermore, every* rank 1 *torsion free sheaf F on Y with* $\deg(F) \leq d$ *and $h^0(Y, F) \geq r+1$ has* $\deg(F) = d$ *and $h^0(Y, F) = r+1$ and there is $R \in S$ such that $F \cong \pi_*(R)$.*

Proposition 2.9. *Let X be a smooth projective curve. Fix positive integers r and d such that for every integer $z \leq d-2$ and every $L \in \mathrm{Pic}^z(X)$ we have $h^0(X, L) \leq r$, while the set $S := \{R \in \mathrm{Pic}(X) : \deg(R) = d-1$ and $h^0(X, R) = r+1\}$ is finite. Fix $A \in X$ such that for* *every $R \in S$ the morphism $h_R : X \to \mathbf{P}^r$ is étale at A. Let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X)+1$ and $\pi(A)$ as an ordinary cusp. Then there is no $M \in \mathrm{Pic}(Y)$ with $1 \le \deg(M) \le d$ and $h^0(Y, M) \ge r + 1$. Furthermore, every* rank 1 *torsion free sheaf F on Y with* $\deg(F) \le d$ *and* $h^0(Y, F) \ge r+1$ *has* $\deg(F) = d$ *and* $h^0(Y, F) = r+1$ *and there is $R \in S$ such that $F \cong \pi_*(R)$.*

Remark 2.10. Fix an integer $y > 0$. Let X' be an integral projective curve and $L \in \mathrm{Pic}(X')$ with $h^0(Y, L) \le y$. Let $\alpha : X' \to Y$ be a birational morphism with Y obtained from X' creating y new nodes gluing together y general pairs of points of X'. The proof of 2.6 and 2.7 shows that there is no $R \in \mathrm{Pic}(Y)$ with $\alpha^*(R) \cong L$ and R spanned.

Remark 2.11. Let X be a smooth curve and $R \in \mathrm{Pic}(X)$ with R spanned and $h^0(X, L) = r + 1 \ge 3$. Let $h_R : X \to \mathbf{P}^r$ be the morphism induced by R. Fix P, $Q \in X$ such that $h_R(P) \ne h_R(Q)$, i.e. such that $h^0(X, R(-P-Q)) = r - 1$; this condition is satisfied for a general pair $(P, Q) \in X \times X$. Let Y be the curve obtained gluing together P and Q, i.e. let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X) + 1$ and $\pi(P) = \pi(Q)$ as an ordinary node. Take a linear space V with $H^0(X, R(-P-Q)) \subset V \subset H^0(X, R)$, $\dim(V) = r$ and V spanning R; since $r-1 > 0$, $R(-P-Q)$ has at most finitely many base points and hence we may take as V a general linear subspace of $H^0(X, R)$ containing $H^0(X, R(-P-Q))$ and different from $H^0(X, R(-P-Q))$; in particular the set of all such linear spaces V is parametrized by an irreducible one-dimensional variety. The morphism h_V associated to V factors through π and hence there is $R_V \in \mathrm{Pic}(Y)$ with $\pi^*(R_V) = R$, $h^0(Y, R_V) = r$, R_V spanned and $\pi^*(H^0(Y, R)) = V$. Hence if $V \ne V'$, then R_V and $R_{V'}$ are not isomorphic.

Remark 2.12. Let X be a smooth curve and $R \in \mathrm{Pic}(X)$ with R spanned and $h^0(X, L) = r + 1 \ge 3$. Let $h_R : X \to \mathbf{P}^r$ be the morphism induced by R. Fix $A \in X$ such that h_R is étale at P, i.e. such that $h^0(X, R(-2P)) = r - 1$; since $\mathrm{char}(\mathbf{K}) = 0$ this condition is satisfied by a general $A \in X$. Let Y be the curve with $\pi : X \to Y$ as normalization map, $p_a(Y) = p_a(X) + 1$ and $\pi(A)$ as an ordinary cusp. Take a linear space V with $H^0(X, R(-2A)) \subset V \subset H^0(X, R)$, $\dim(V) = r$ and V spanning R; since $r - 1 > 0$, $R(-2A)$ has at most finitely many base points and hence we may take as V a general linear subspace of $H^0(X,R)$ containing $H^0(X,R(-2A))$ and different from $H^0(X,R(-2A))$; in particular the set of all such linear spaces V is parametrized by an irreducible one-dimensional variety. The morphism h_V associated to V factors through π and hence there is $R_V \in \mathrm{Pic}(Y)$ with $\pi^*(R_V) = R$, $h^0(Y,R_V) = r$, R_V spanned and $\pi^*(H^0(Y,R)) = V$. Hence if $V \neq V'$, then R_V and $R_{V'}$ are not isomorphic.

From Remarks 2.11 and 2.12 and the existence part of Brill - Noether theory on smooth curves we obtaing at once the following result.

Corollary 2.13. *Let Y be an integral projective curve with only ordinary nodes and only ordinary cusps as singularities. Set $g := p_a(X)$ and $x :=$ card(Sing(Y)). Fix integers r, d with $r \geq 2$ and $\rho(g-x, r+x, d) \geq 0$. Then there exists an integer $b \leq d$ and $L \in \mathrm{Pic}(Y)$ with* $\deg(L) = b$, $h^0(Y,L) \geq r+1$ *and L spanned.*

Proof of Theorem 1.2. Let X be a general smooth $(k-y)$-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-y}(X)$ the degree $k-y$ pencil. First assume $d \leq g-x+y-1$. We apply [8], part (2) of Cor. 1 of Section 1, to X with respect to the following data: $g' := g-x$, $k' := k-y$, $r = f = 1$, $d = \deg(E) = y$, $\gamma = g'+1$. Since $(g-x+y+2)/2 \leq d \leq g-x+y-1$, we obtain the existence of a spanned $T \in W^1_d(X)$ with $h^0(X, T \otimes M^*) = 0$. Alternatively, we could quote here [6], Theorem 2.2.2. Thus there is an irreducible component W of $W^1_d(X)$ with $W \neq \emptyset$, $\dim(W) \geq \rho(g-x,1,d)$ and such that a general $N \in W$ is spanned and with $h^0(X, N \otimes M^*) = 0$. By our numerical assumptions we have $\rho(g-x,1,d) \geq x$. We claim that for a general ordered set of $x+y$ points $(P_1, \ldots, P_{x-y}, A_1, \ldots, B_y)$ there is $(Q_1, \ldots, Q_{x-y}) \in X^{x-y}$ with $Q_i \neq P_i$ for every i and a locally closed irreducible subset Z of W with $Z \neq \emptyset$, $\dim(Z) = \dim(W) - x$ and such that for every $R \in Z$ we have $h_R(P_i) = h_R(Q_i)$ for every $i \leq x-y$. The claim and 1.2 in this range follow from Remark 2.10, the proof of Remark 2.4 (see in particular Statement $T(k)$ and Remark 2.5. Now assume $g-x+y \leq d \leq g-x+y+k-3$. We apply [8], part (2) of Cor. 1 of Section 1, $k-4$ times with respect to the integers $g' := g-x+y$, $r = f$ with $2 \leq f \leq k-3$, $\gamma = g'+r = g-x+y+f$, $d = \deg(E)$ and conclude in the same way. Now assume $d > g-x$. By assumption we have $x-y < g/2$ and hence $\dim(\mathrm{Pic}^d(X)) \geq x-y$. For a general

$R \in \mathrm{Pic}^d(X)$ we have $h^1(X, R) = 0$ and R is spanned. We apply the previous proof taking as Z a non-empty open subset of $\mathrm{Pic}^d(X)$.

3 – Embeddings in $\mathbf{P}^r$ and the Lüroth semigroup

By [5], Section 1, 2 and 3, for all integers d, g and r with $r \geq 3$ and either $d \geq g+r$ or $d-r < g \leq d-r+[(d-r-2)/(r-2)]$ there is an irreducible component $W(d,g,r)$ of the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^r)$ of degree d curves of $\mathbf{P}^r$ with arithmetic genus g such that a general $C \in W(d,g,r)$ is smooth, connected and non-degenerate and with $h^1(C, N_{C,r}) = 0$, where $N_{C,r}$ is the normal bundle of C in $\mathbf{P}^r$. In particular $W(d,g,r)$ is generically smooth and of dimension $h^0(C, N_{C,r}) = (r+1)d - (r-3)(g-1)$. If $\rho(g,r,d) \geq 0$, then $W(d,g,r)$ contains smooth curves with general moduli ([5], Proposition 3.1). If $d \geq g+r$ for a general $C \in W(d,g,r)$ we have $h^1(C, \mathbf{O}_C(1)) = 0$ and hence $h^0(C, \mathbf{O}_C(1)) = d+1-g$. If $d \leq g+r$ for a general $C \in W(d,g,r)$ we have $h^0(C, \mathbf{O}_C(1)) = r+1$.

Theorem 3.1. *Fix integers g, k, x, r with $x > 0$, $k \geq 2+x$ and $r \geq 3$; assume either $d \geq g+r$ or the existence of an integer $t > 0$ and an integer $e \geq 3x$ such that $d = r+2+e+t(r-2)$ and $g = r+2+e-3x+t(r-1)$. Let Y be a general k-gonal nodal curve of genus g and type (x,x). Then there exists a very ample $L \in \mathrm{Pic}^d(Y)$ with $h^0(Y,L) \geq r+1$ and such that for a general embedding $j : Y \to \mathbf{P}^r$ associated to L we have $j(Y) \in W(d,g,r)$.*

Proof. The (omitted) case $x = 0$ is [4], part (a) of Theorem 0.1. The case $d \geq g+r$ is trival, taking non special embeddings. Hence from now on we will assume $d < g+r$. We will modify the proof of [4], Theorem 0.1, to obtain 3.1. For all integers d', g', x' with $0 \leq x' \leq k-2$ and $d' = r+2+e'+t'(r-2)$, $g' = e'-3x'+t'(r-1)$ (as in the statement of 3.1) call $A(d',g',x')$ the following assertion:

Assertion $A(d',g',x')$: there is a pair (C,T) with the following properties:

(i) $C \in W(d',g',r)$ and C satisfies the thesis of 3.1 for the parameters r, k, d', g', x' and $h^1(C, N_{C,r} \otimes \mathbf{I}_Z) = 0$, where Z is the first infinitesimal neighborhood of $\mathrm{Sing}(C)$ in C;

(ii) T is a subset of C_{reg} contained in a positive divisor, D, of the degree $k-x'$ pencil of C with $\mathrm{card}(T)=r+2$ and such that T is in linearly general position, i.e. such that every proper subset T' of T spans a linear subspace $\langle T'\rangle$ of $\mathbf{P}^r$ with $\dim(\langle T'\rangle)=\mathrm{card}(T')-1$.

Notice that Z in condition (i) of $A(d',g',r)$ is an effective Weil divisor of degree $3(\mathrm{card}(\mathrm{Sing}(C))$. Assume $A(d',g',x')$ and take C, D, T satisfying it. Fix an integer t with $0\le t\le r+1$. We want to prove $A(d'+r,g'+t,x')$. Since any $r+3$ points of $\mathbf{P}^r$ in lineraly general position are contained in a unique rational normal curve, it is easy to check the existence of a rational normal curve D with D intersecting quasi-transversally C, $D\cap C\subset T$ and $\mathrm{card}(D\cap C)=t+1$. Set $W:=C\cup E$. By [16], proof of Theorem 5.2, (or [12] and a dimensional count, or [5]) and [5], 2.3 and 3.1, we have $h^1(W,N_{W,r})=0$, $W\in W(d'+r,g'+t,r)_{\mathrm{reg}}$ and the nodal curve W is smoothable. If $t\ge 2$ the nodal curve W is stable, while if $0\le t\le 1$ it is only semistable. Fix a subset A of E with $A\cap C=\emptyset$ and $\mathrm{card}(A)=t+1$. Let V be the pencil of divisors on E generated by $D\cap C$ and A. Using Knudsen - Harris - Mumford theory of admissible coverings ([13], Section 4) we get that the stable reduction of W in the moduli scheme $M^-_{g'+t}$ of stable curves of genus $g'+t$ of the variety of smooth $(k-x')$-gonal curves. We may even assume for general C that W has no non-trivial automorphism, i.e. we may even assume that $M^-_{g'+t}$ is smooth at the point corresponding to the stable reduction of W. By [5], Theorem 3.1, or the proof of [16], 5.2, the rational map τ from $\mathrm{Hilb}(\mathbf{P}^r)$ to $M^-_{g'+t}$ is dominant. A dimensional count shows that near W the fiber of τ over $\tau(W)$ has the smallest a priori possible dimension. Thus τ is flat at W and hence open at W. The proof of [5], Lemma 1.2, gives also $h^1(W,N_{W,r}\otimes\mathbf{I}_Z)=0$ and this means that we may do the previous limit without smoothing the nodes in C, i.e. that W is a flat limit inside $\mathrm{Hilb}(\mathbf{P}^r)$ of a family of nodal gonal curves of type (x',x'). Taking A general, we see how to obtain the last condition of $A(d'+r,g'+t,x')$. Hence we may continue and cover all triples (d,g,x') claimed by 3.1 if we may start the induction with some k-gonal curve of type (x',x'). However, at the beginning we only know the case $x'=0$ (for instance from part (a) of [4], Theorem 0.1). To start this procedure for the first x steps we will increase by one the integer x', i.e. we will pass from x' to $x'+1$. This is possible without modifying the proof of [5], Lemma 1.2, only if

$t \le r-2$. For simplicity we will use it and hence in the first x steps will loose $3x$ in the upper bound of the genus with respect to the degree. This explains the the term "$-3x$" in the expression of g in the statement of 3.1. We fix $P \in C \cap E$ and call B the union of Z and the first infinitesimal neighborhood of P in W. Thus $\deg(B) = 3 + \deg(Z) = 3 + 3x'$. As in [5], Lemma 2.1, using a Mayer - Vietoris exact sequence and the description of $N_{W,r}$ we obtain $h^1(W, N_{W,r} \otimes \mathbf{I}_B) = 0$. Then we may apply a partial smoothing in which we may preserve $x'+1$ nodes (the ones of $\mathrm{Sing}(C) \cup \{P\}$), obtaing the case $x'+1$ needed.

Remark 3.2. In the proof of 3.1 if $d \le g+r$, then we found L with $h^0(Y,L) = r+1$.

Proposition 3.3. *Fix integers g, x, k and d with $k \ge 2+x$, $x > 0$, $g \ge 2k+2x+1$ and $2d \ge 2g+6$. There is a nodal k-gonal curve Y of genus g and type (x,x) with as normalization a general $(k-x)$-gonal curve, X, of genus $g-x$ and with the following property. There is $R \in \mathrm{Pic}^d(Y)$ with $h^0(Y,R) = 3$, R spanned and such that the associated morphism $h_R : Y \to \mathbf{P}^2$ is étale at every point of $\mathrm{Sing}(Y)$, it is birational and the curve $h(Y)$ has only ordinary nodes as singularities except one point, P; P is an ordinary point of multiplicity $\deg(R)-k$, $h_R^{-1}(P) \cap \mathrm{Sing}(Y) = \emptyset$ and the degree $k-x$ pencil on X is induced by the pencil of lines in $\mathbf{P}^2$ passing through P.*

Proof. Let X be a general smooth k-gonal curve of genus $g-x$. Call $M \in \mathrm{Pic}^{k-x}(X)$ the degree $k-x$ pencil. By [15] (or see [8], theorem in part 2 of the introduction, or, for its statement, the introduction of [1] or [7], 2.2) there is an irreducible component W of $W^2_d(X)$ with $W \neq \emptyset$, $\dim(W) = \rho(g-x,d,2)$ such that a general $N \in W$ is spanned, $h^0(X, N \otimes M^*) = 1$, the corresponding morphism is birational, and its image, C, with only ordinary nodes except one point, P, which is an ordinary point of multiplicity $d-k+x$. Furthermore, M is induced by the pencil of lines through P. Fix x of the singular points, say $B_1, \dots, B_x$, of C and let Y be the partial normalization of C in which we normalize all nodes except the ones corresponding to the points $B_1, \dots, B_x$. Y solves our problem.

Definition 3.4 Let Y be an integral projective curve. Set $LS(Y) := \{d \in \mathbf{Z}$: there is a spanned line bundle L on Y with $\deg(L) = d\}$ and

$LS(Y)' := \{d \in \mathbf{Z}$: there is a spanned rank 1 torsion free sheaf F on Y with $\deg(F) = d\}$. $LS(Y)$ will be called the *Lüroth semigroup* of Y. $LS(Y)$ is a semigroup of the set, $\mathbf{N}$, of non-negative integers. $LS(Y)'$ will be called the *singular Lüroth set* of Y. It is very easy to find a nodal curve Y such that $LS(Y)'$ is not a semigroup (see the proof of Example 2.6 and 1.2 for $x = y = 1 \ll k \ll g$).

Proposition 3.5. *Fix integers g, k and x with $x > 0$, $k \geq 2 + x$ and $g \geq 2k - x + 3$. Let Y be a general k-gonal nodal curve of genus g with type (x, x). Then the singular Lüroth set $LS(Y)'$ of Y contains the integers $t(k - x) + x$ for $1 \leq t \leq \min\{x, [(g - x)/(k - x)]\}$, $t(k - x)$ for $\min\{x + 1, [(g - x)/(k - x)]\} < t \leq [(g - x)/(k - x)]$ and all integers $\beta \geq [(g - x + 3)/2] + x$.*

Proof. Let $\pi : X \to Y$ be the normalization map. Thus X is a general smooth $(k-x)$-gonal curve of genus $g-x$. Let $M \in \mathrm{Pic}^{k-x}(X)$ be the degree $k - x$ pencil. By [3] and [6], Theorem 2.2 (see the discussion in [6], 0.2), the Lüroth semigroup $LS(X)$ of X contains the integers $t(k-x)$ (induced by $M^{\otimes t}$) and all the integers α with $[(g-x+3)/2] \leq \alpha \leq g - x$. If $A \in \mathrm{Pic}(X)$, then $\deg(\pi_*(A)) = \deg(A) + x$ and $h^0(Y, \pi_*(A)) = h^0(X, A)$. Hence to show that $t(k-x)+x \in LS(Y)'$ it is sufficient to show that $\pi_*(M^{\otimes t})$ is spanned, while to show that $[(g - x + 3)/2] + x + e \in LS(Y)'$ it is sufficient to find $A \in \mathrm{Pic}^{[(g-x+3)/2]+e}(X)$, A spanned with $\pi_*(A)$ spanned, i.e. with $\pi_*(A)$ spanned at each point of $\mathrm{Sing}(Y)$. Move Y keeping fixed X, i.e. move the $2x$ points $\pi^{-1}(\mathrm{Sing}(Y))$. Since any symmetric product of X is irreducible and the type is (x, x), for a general Y we obtain that either $\pi_*(M^{\otimes t})$ is spanned or it is not spanned at each point of $\mathrm{Sing}(Y)$. Assume that the second possibility occurs and call B the subsheaf of $\pi_*(M^{\otimes t})$ spanned by $H^0(Y, \pi_*(M^{\otimes t}))$. We may even assume that t is the first integer for which this possibility occurs. Again, by the irreducibility of the symmetric product we obtain that $\pi_*(M^{\otimes t})/B$ has the same length, v, at each point of $\mathrm{Sing}(Y)$. By [3] we have $h^0(X, M^{\otimes t}) = t + 1$ for $t \leq [(g - x)/(k - x)]$. First assume $t < x$. X and hence M are fixed. By Remark 2.10 for general Y there is no spanned $R \in \mathrm{Pic}(Y)$ with $M^{\otimes t} \cong \pi^*(R)$. If $t \geq x + 1$ we have $h^0(X, M^{\otimes t}) \geq x + 2$ and hence applying x times Proposition 2.7 we obtain the existence of a spanned $R \in \mathrm{Pic}(Y)$ with $M^{\otimes t} \cong \pi^*(R)$. Now take $A \in \mathrm{Pic}(X)$

computing one of the integers α of $LS(X)$ with $[(g-x+3)/2] \leq \alpha \leq g-x$ and A general. By [6], Theorem 2.2, and the generality of A we have $h^0(X, A) = 2$. Thus using Remark 2.10 we obtain easily that for general Y and general A the corresponding sheaf $\pi_*(A)$ has no subsheaf B with length $(\pi_*(A)/B) \geq x$ and $h^0(Y, B) = 2$, i.e. we obtain the spannedness of $\pi_*(A)$. Every integer $u \geq g+1$ may be realized as an element of $LS(Y)$ and hence of $LS(Y)'$ just taking a general non-special $R \in \mathrm{Pic}^u(Y)$.

Proposition 3.6. *Fix integers g, k, x, y and t with $x > 0$, $x \geq y \geq 0$, $k \geq 2 + x$ and $g \geq 2k - x + 3$ and $t < (g-x)/2 + 1$. Let Y be a general k-gonal nodal curve of genus g with type (x, y). If $t \neq a(k-y)$ for all integers a, then $t \notin LS(Y)$. If $t = a(k-y)$ for some integer a and $y > a$, then $t \notin LS(Y)$.*

Proof. Let $\pi : X \to Y$ be the normalization map. Thus X is a general smooth $(k-y)$-gonal curve of genus $g - x$. Let $M \in \mathrm{Pic}^{k-y}(X)$ be the degree $k - y$ pencil. By [1], Theorem 0.1, there is no $L \in \mathrm{Pic}^t(X)$ with L spanned, unless $t = a(k-y)$ for some integer a and in this case we have $L \cong M^{\otimes a}$. Hence $t \notin LS(Y)$ if $t \neq a(k-y)$ for every integer a. Assume $t = a(k-y)$. By [3] we have $h^0(X, M^{\otimes a}) = a + 1$. Apply Remark 2.10 and the assuption $y > a$.

Lemma 3.7. *Fix integers g, k and x with $x \geq 0$ and $g \geq 2k + x + 3$. Let Y be a general k-gonal nodal curve of type $(x, 0)$ and $M \in \mathrm{Pic}^k(Y)$ the degree k spanned line bundle on Y. Then for all integers t with $0 \leq t \leq [g/(k-1)]$ we have $h^0(Y, M^{\otimes t}) = t + 1T$.*

Proof. Let $\pi : X \to Y$ be the normalization. Since $h^0(Y, M) \geq 2$ the value for $h^0(Y, M^{\otimes t})$ is the minimal a priori possible and hence we may use semicontinuity. By definition of general nodal curve of type $(x, 0)$, X is a general smooth k-gonal curve of type $(x, 0)$. By [3] we have $h^0(X, \pi^*(M^{\otimes t})) = t + 1$ for $t \leq [(g-x)/(k-1)]$. Thus we may assume $[(g-x)/(k-1)] < t \leq [g/(k-1)]$. We modify the proof of [3]. We need to find an integral nodal curve, T, of type (k, a) (some a) on $\mathbf{P}^1 \times \mathbf{P}^1$ with normalization of genus $g - x$, at least x nodes and such that a subset, S, of $\mathrm{Sing}(T)$ with $\mathrm{card}(S) = \mathrm{card}(\mathrm{Sing}(T)) - x$ satisfies a certain cohomological condition (say $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{I}_S(k-2, b)) = 0$ for a suitable b). The existence of an integral nodal curve in $\mathbf{P}^1 \times \mathbf{P}^1$ with that numerical invariants follows from [1], Proposition 3.7 and Proposition 4.1. By semicontinuity we may even choose as x "omitted" nodes for the cohomological condition is any subset of $\mathrm{Sing}(T)$ we prefer and hence it is sufficient to have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{I}_{\mathrm{Sing}(T)}(k-2,b)) \le x$. This is even easier than in [3] (case $x = 0$ with cardinality of the singular set $\mathrm{card}(\mathrm{Sing}(T)) - x$).

Proposition 3.8. *Fix integers g, k and x with $x \ge 0$ and $g \ge 2k + x + 3$. Let Y be a general k-gonal nodal curve of type $(x, 0)$ and $M \in \mathrm{Pic}^k(Y)$ the degree k spanned line bundle on Y. For any integer z with $1 \le z < (g - x + 3)/2$ the following conditions are equivalent:*

(i) *$z = tk$ for some integer t;*
(ii) *$z \in LS(Y)$;*

Furthermore, if $z = tk < (g - x + 3)/2$ the only rank 1 *spanned line bundle, L, with* $\deg(L) = z$ *is $M^{\otimes t}$.*

Proof. Since M is spanned, $tk \in LS(Y)$ for every integer t. Thus it is sufficient to show that every spanned line bundle L with $\deg(L) \le (g - x + 3)/2$ is of the form $M^{\otimes t}$. Let $\pi : X \to Y$ be the normalization. Since $\pi^*(M)$ is a spanned line bundle on the general k-gonal curve X, this is [1], Theorem 2.6.

4 – Seminormal singularities

In this section we will consider seminormal curves in the sense of [17] and [9], i.e. curves with the simplest singularities compatible with their number of branches: if the singularity has r branches, then it is formally equivalent to the germ at $0 \in \mathbf{K}^r$ of the union of the r coordinate axis. A seminormal curve singularity is Gorenstein if and only if it is an ordinary double point. The conductor of a seminormal one-dimensional local ring R is the maximal ideal of R.

Definition 4.1. Let Y be a projective seminormal curve and $\pi : X \to Y$ its normalization. Set $g := p_a(Y)$ and $q := p_a(X)$. For every $P \in \mathrm{Sing}(Y)$, set $s(P) := \mathrm{card}(\pi^{-1}(P))$. We may order the integers $s(P)$, $P \in \mathrm{Sing}(Y)$ in non-decreasing order, allowing repetitions. If $\mathbf{K} = \mathbf{C}$ the topological type of $Y(\mathbf{C})$ is unique determined by the integers

$(g, q; \mathrm{card}(\mathrm{Sing}(Y); s(P)_{P\in \mathrm{Sing}(Y)})$. Notice that $g = q + \sum_{P\in \mathrm{Sing}(Y)} s(P) - \mathrm{card}(\mathrm{Sing}(Y))$. We call this set the *numerical data*. The *weight* $\mathrm{weight}(\tau)$ of the numerical data τ or of the curve Y is the maximum of the integers $s(P)$, $P \in \mathrm{Sing}(Y)$. We will say that Y is *general* or that it is general for a prescribed numerical data if X is a general smooth curve of genus q and the set $\pi^{-1}(\mathrm{Sing}(Y))$ is general in X. We will say that Y is *general for the fixed normalization* X if $\pi^{-1}(\mathrm{Sing}(Y))$ is general in X.

REMARK 4.2. Let Y be a seminormal curve and $\pi : X \to Y$ its normalization. Fix $L \in \mathrm{Pic}(Y)$. For every $f \in H^0(Y, L)$ and $P \in \mathrm{Sing}(Y)$ with f vanishing at P the section $\pi^*(f)$ of $\pi^*(L)$ vanishes at each point of $\pi^{-1}(\mathrm{Sing}(P))$. Fix $h \in H^0(X, \pi^*(L))$ and assume that for every $P \in \mathrm{Sing}(Y)$ h has the same value for a fixed trivialization of L near P and hence of $\pi^*(L)$ around $\pi^{-1}(P)$ at each point of $\pi^{-1}(P)$. Then h is of the form $\pi^*(f)$ for some $f \in H^0(Y, L)$ because conductor of a seminormal one-dimensional local ring R is the maximal ideal of R.

REMARK 4.3. Let Z be an integral projective curve, $L \in \mathrm{Pic}(Z)$, $V \subseteq H^0(Z, L)$ a linear subspace with $\dim(V) \geq 2$. Then for every $P \in Z$ there is subspace $V(P)$ of V with $\dim(V(P)) \geq \dim(V)-1$ and such that every $f \in V(P)$ vanishes at P.

Remarks 4.2 and 4.3 and the definition of general seminormal curve with fixed normalization give at once the following result.

LEMMA 4.4. *Let X be a smooth projective curve of genus $q \geq 0$. Fix an integer d and let x be the maximal dimension of an irreducible component of $G^r_d(X)$. Fix a type τ for seminormal curves with normalization of genus q and* $\mathrm{weight}(\tau) > x$. *Let Y be the general seminormal curve of type τ with X as normalization. Then for every $L \in \mathrm{Pic}(Y)$ with* $\deg(L) \leq d$ *we have $h^0(Y, L) \leq r$.*

REMARK 4.5. Use the notation of Lemma 4.4. Notice that for a fixed q and any genus q curve we may find a type τ with $\mathrm{weight}(\tau) > d$ but $d < g$. In this sense there is no hope just using the Brill - Noether numbers $\rho(g, r, d)$ to have on general seminormal curves the usual Brill - Noether theory using line bundles, even if we do not require that the line bundles considered are spanned.

The proof of Remark 2.4 and Remark 2.5 give the following result.

Proposition 4.6. *Fix integers g, q, d and e with $0 < e \leq g - q$. Let X be a smooth curve of genus q and assume the existence of an irreducible component, T, of $G^1_d(X)$ with $\dim(T) \geq g - q + e$ and such that for a general pair $(R, V) \in T$ the line bundle R is spanned by V. Fix a type τ for seminormal curves with genus g, normalization of genus q and e singular points. Let Y be a general seminormal curve with Y as normalization and type τ. Then there is a spanned line bundle of degree d on Y.*

REFERENCES

[1] E. Arbarello – M. Cornalba: *Footnotes to a theorem of Beniamino Segre*, Math. Ann., **256** (1981) 341-362.

[2] E. Arbarello – M. Cornalba – P. Griffiths – J. Harris: *Geometry of Algebraic Curves, I*, Springer-Verlag, 1985.

[3] E. Ballico: *A remark on linear series on general k-gonal curves*, Boll. UMI, (7), **3-A** (1989) 195-197.

[4] E. Ballico: *On special linear systems on curves*, Comm. in Algebra, **18** (1990) 279-284.

[5] E. Ballico – Ph. Ellia: *On the existence of curves with maximal* rank *in* $\mathbf{P}^n$, J. reine angew. Math., **397** (1989) 1-22.

[6] M. Coppens – Ch. Keem – G. Martens: *The primitive length of a general k-gonal curve*, Indag. Math. (N.S.), **5** (1994) 145-159.

[7] M. Coppens – G. Martens: *Linear series on 4-gonal curves*, Math. Nachr., **213** (2000) 35-55.

[8] M. Coppens – G. Martens: *Linear series on a general k-gonal curve*, Abh. Sem. Univ. Hamburg, **69** (1999) 347-371.

[9] E. D. Davis: *On the geometric interpretation of seminormality*, Proc. Amer. Math. Soc., **68** (1978) 1-5.

[10] D. Eisenbud – J. Harris – J. Koh – M. Stillman: *Appendix to*: *Determinantal equations for curves of high degree, by D. Eisenbud, J. Koh, M. Stillman*, Amer. J. Math., **110** (1988) 513-539.

[11] G.-M. Greuel – H. Knörrer: *Einfache Kurvensingularitäten und torsionfreie Moduln*, Math. Ann., **270** (1985) 417-425.

[12] R. Hartshorne – A. Hirschowitz: *Smoothing algebraic space curves*, in: Algebraic Geometry, Sitges 1983, pp. 98-131, Lect. Notes in Math. 1124, Springer-Verlag, 1985.

[13] J. Harris – D. Mumford: *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67** (1982) 23-86.

[14] S. Kleiman – D. Laksov: *On the existence of special divisors*, Amer. J. Math., **94** (1972) 431-436.

[15] B. Segre: *Sui moduli delle curve poligonali, e sopra un complemento al teorema di esistenza di Riemann*, Math. Ann., **100** (1928) 537-551.

[16] E. Sernesi: *On the existence of certain families of curves*, Invent. Math., **75** (1984) 25-57.

[17] C. Traverso: *Seminormality and Picard groups*, Ann. Sc. Norm. Pisa, **25** (1970) 287-304.

Lavoro pervenuto alla redazione il 30 giugno 2001
ed accettato per la pubblicazione il 16 luglio 2002.
Bozze licenziate il 4 settembre 2003

Indirizzo dell'Autore:

E. Ballico – Dept. of Mathematics – Università di Trento – 38050 Povo (TN) (Italia)
E-mail: ballico@science.unitn.it

This research was partially supported by MURST (Italy).

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 65-81

Einstein's field equations in the light of constrained hyperbolic systems

F. BORGHERO – S. PENNISI

Abstract: *Results previously known in the literature, on the hyperbolicity of Einstein's equations, are here quoted and improved. This aim is reached by applying recent techniques on constrained hyperbolic systems. The symmetric hyperbolic form is obtained, also in the four-dimensional formalism using harmonic coordinates. The case of sources due to the presence of matter is also considered, in particular from the view point of Extended Thermodynamics*

1 – Introduction

The importance of Einstein's equations is outstanding and needs no comments. The study of their hyperbolicity presents also some interesting aspects. Obviously, we don't have here the presumption to diminish previously results obtained on this subject by authoritative experts. *We want only to show how a recent general theory on hyperbolic systems, with differential and algebraic constraints, can be successfully applied also to this important problem*; indeed, the validity of the general theory is strengthened because our results are comparable with those obtained in other ways by the above mentioned experts.

Key Words and Phrases: *Einstein's field equations – Harmonic coordinates – Constrained Hyperbolic systems – Symmetric hyperbolic systems – Extended thermodynamics.*
A.M.S. Classification: 83C05 – 83C55 – 35Q75

Let us start noticing that in [1] Strumia has shown how Einstein's equations can be reduced to a first order system of partial differential equations, but one of the hyperbolicity conditions according to FRIEDRICHS (see ref. [2], [3]) seems to fail, the one referring to the possibility, roughly speaking, to obtain the time derivatives as functions of the other quantities; see also [4] for other details. Although apparently strange, this result is just what we would expect from the covariance property; in fact, if the metric tensor $g_{\mu\nu}$ is a solution of Einstein's equations, then so is $g_{\mu'\nu'} = g_{\alpha\beta}[x^\rho(x^{\lambda'})]\partial_{\mu'}x^\alpha\partial_{\nu'}x^\beta$ i.e. the expression determined from $g_{\mu\nu}$ by a general coordinate transformation $x \to x'$. This consideration can be found in papers such as [5]-[7].

In [8] this failure of Einstein's equations to determine $g_{\mu\nu}$ uniquely is compared to the failure of Maxwell's equations to determine the vector potential uniquely. Here we propose another comparison which will be the thread of our subsequent arguments, i.e., the problem of determining the geodesic curves of a surface Σ; for the sake of simplicity, we shall consider Σ belonging to a 3-dimensional euclidean space. If $P = P(u, v)$ are the parametric equations of Σ and $P(\lambda) = P[u(\lambda), v(\lambda)]$ the equations of a geodesic curve γ, then

$$\frac{P'(\lambda)}{|P'(\lambda)|}$$

is the tangent unit vector and one has

$$\frac{d}{d\lambda}\left[\frac{P'(\lambda)}{|P'(\lambda)|}\right] = \frac{1}{\rho}\,\mathbf{n}\,\frac{ds}{d\lambda},$$

where ρ is the radius of curvature, $\mathbf{n}$ is the normal unit vector and s is its arc-length parameter; therefore, the equations of γ can be obtained from the system

$$\left\{\begin{aligned} &\frac{\partial P}{\partial u}\cdot\frac{d}{d\lambda}\left[\frac{P'(\lambda)}{|P'(\lambda)|}\right] = 0,\\ &\frac{\partial P}{\partial v}\cdot\frac{d}{d\lambda}\left[\frac{P'(\lambda)}{|P'(\lambda)|}\right] = 0. \end{aligned}\right. \tag{1.1}$$

These equations don't determine $u(\lambda)$, $v(\lambda)$, because their linear combination, through the coefficients u' and v', is an identity; as Einstein's equations don't determine $g_{\mu\nu}$ due to the arbitrariness of the coordinates transformation, so the equations (1.1) fail to determine u and v because the parameter λ is arbitrary. One may proceed in one of the following ways:

1. Require $|P'(\lambda)| = 1$, in addition to equations (1.1); in other words, we require that λ is the arc-length parameter.
2. Require $d|P'(\lambda)|/d\lambda = 0, \quad |P'(\lambda_0)| = 1$, with λ_0 initial value of λ. Note that these conditions, together with (1.1), are equivalent to

$$\begin{cases} \dfrac{\partial P}{\partial u} \cdot P''(\lambda) = 0, \\ \dfrac{\partial P}{\partial v} \cdot P''(\lambda) = 0, \\ |P'(\lambda_0)| = 1. \end{cases}$$

   This last condition may also be omitted, being content with a λ which is a linear function of the arc-length parameter, without assuming $\lambda = s$.

In the same manner we will investigate the hyperbolicity of Einstein's equations in one of the following ways:

1. Require that the coordinates x^α aren't the most general ones, but the *harmonic coordinates* defined by $\Gamma_\alpha = 0$ (for the expression of Γ^α see the equation $(1.2)_6$ below). *In this way, the Einstein's equations become equations with differential and algebraic constraints.* In this framework they will be studied in Section 2, by applying the general methods outlined in ref. [4], where they have been successfully applied to the equations of relativistic fluid dynamics. See also refs. [9]-[13] for other examples of physical application, such as the relativistic magneto-fluid dynamics, the Maxwell electrodynamics, the equations of the superfluid and those of ultra relativistic gases.
   In ref. [4] it is shown also a method to eliminate the algebraic constraints, in a manner which corresponds to the following method (2).
2. Require $\partial_t \Gamma_\alpha = 0$, $(\Gamma_\alpha)_\Sigma = 0$, where $(\Gamma_\alpha)_\Sigma$ is the value of Γ_α calculated in the initial manifold Σ. This approach will be followed in Section 3. We will see that these further assumptions are equivalent

to $\partial_{(\beta}\Gamma_{\alpha)} = 0$, $(\Gamma_\alpha)_\Sigma = 0$, which have the advantage to be written in 4-dimensional notation.
Following the general methods of paper [4], we obtain the equations found in [5]-[7] in another way; in ref. [5], Fischer and Marsden have transformed these equations in the symmetric hyperbolic form, but in 3-dimensional notation. *Here we reach the same result, but in* 4-*dimensional notation.*
A third approach, which is present in literature, will be exploited in Section 4.
The case of sources due to the presence of matter will be considered in Section 5, *showing how the symmetric hyperbolic form can be obtained also in this case, and also with the equations of relativistic extended thermodynamics and similar* [14]-[16].

We conclude this section reporting the Einstein's equations.

$$G_{\mu\nu} = \chi T_{\mu\nu}, \tag{1.2}$$

with χ the einsteinian gravitational constant, $T_{\mu\nu}$ the energy tensor,

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R \quad \text{(Einstein tensor)},\\
R &= g^{\alpha\beta} R_{\alpha\beta} \quad \text{(scalar curvature)},\\
R_{\mu\nu} &= \frac{1}{2}\, g^{\alpha\beta}[-\partial^2_{\alpha\beta} g_{\mu\nu} - \partial^2_{\mu\nu} g_{\alpha\beta} + \partial^2_{\alpha\nu} g_{\mu\beta} + \partial^2_{\beta\mu} g_{\nu\alpha}] +\\
&\qquad - g_{\rho\sigma}\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\alpha\beta} g^{\alpha\beta} + g^{\alpha\beta} g_{\rho\sigma}\Gamma^\rho_{\mu\alpha}\Gamma^\sigma_{\nu\beta} =\\
&= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \quad \text{(Ricci tensor)},\\
\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\, g^{\mu\lambda}\left(\partial_\beta g_{\lambda\alpha} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta}\right) \quad \text{(Christoffel symbols)},\\
\Gamma^\mu &= \Gamma^\mu_{\alpha\beta} g^{\alpha\beta} \quad \text{(Lanczos symbols)}.
\end{aligned}$$

Obviously the Einstein's equation (1.2) can also be written in the form

$$R_{\mu\nu} = \chi\left(T_{\mu\nu} - \frac{1}{2}\, T_{\alpha\beta} g^{\alpha\beta} g_{\mu\nu}\right). \tag{1.3}$$

When sources are not present, i.e. $T_{\mu\nu} = 0$, we have the so called "*exterior case*" and equation (1.3) reduce to

$$R_{\mu\nu} = 0. \tag{1.4}$$

2 – The Einstein's equations in harmonic coordinates

The transformation equations of the contracted Christoffel symbols Γ^μ, introduced by Lanczos, are

$$(\Gamma')^\lambda = \Gamma^\rho \partial_\rho (x')^\lambda - g^{\rho\sigma} \partial^2_{\rho\sigma} (x')^\lambda \,.$$

Hence we can always find a coordinate system $(x')^\lambda$ where $(\Gamma')^\lambda$ vanishes; such coordinates are called *harmonic coordinates*.

From now on, in this section, we will impose to be already in harmonic coordinates, so that we have $\Gamma^\mu = 0$. The Einstein's equation (1.4) in the exterior case can be reduced to a first order system by setting $\partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}$; one obtains

$$\begin{cases} \partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \\[4pt] \dfrac{1}{2}\, g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] = F_{\mu\nu}(g_{\alpha\beta}, \omega_{\alpha\beta\gamma}), \\[4pt] \partial_{[\beta}\omega_{\alpha]\mu\nu} = 0, \\[4pt] \dfrac{1}{2}\, g^{\alpha\beta}\,(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}) = 0 \quad \text{(i.e. } \Gamma_\lambda = 0). \end{cases} \tag{2.1}$$

Equations $(2.1)_{1\text{-}3}$ *constitute a system of* 110 *equations in the* 50 *unknowns* $g_{\mu\nu}$, $\omega_{\alpha\mu\nu}$, *restricted by the four algebraic constraints* $(2.1)_4$, *so that we have only* 46 *independent variables; obviously, in the system* $(2.1)_{1\text{-}3}$ *there are also* 66 *differential constraints.*

Now a general method to study the hyperbolicity of systems with algebraic and differential constraints has been proposed in ref. [4] and already applied with success to important physical problems.

Here we find another interesting example of physical application. In a few words the method, applied to the present case, consists in multiplying the system $(2.1)_{1\text{-}3}$ on the left by a suitable matrix of rank 46 so that the resulting system is hyperbolic in the time direction defined by t_α, with $t_\alpha t^\alpha = -1$.

Alternatively, this result may be obtained by taking suitable linear combinations of the equations $(2.1)_{1\text{-}3}$. *A possible choice is to consider the system*

$$
(2.2) \qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ h^{\mu\nu}_{\gamma\delta}\, \dfrac{1}{2}\, g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] = \\ \qquad = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}(g_{\alpha\beta}, \omega_{\alpha\beta\varepsilon}), \\ t^\alpha h^{\mu\nu}_{\gamma\delta}[\partial_{[\beta}\omega_{\alpha]\mu\nu}] = 0, \\ \dfrac{1}{2}\, g^{\alpha\beta}(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta}) = 0, \end{cases}
$$

with

$$
(2.3) \qquad h^{\mu\nu}_{\gamma\delta} = g^{(\mu}_{\gamma} g^{\nu)}_{\delta} - \frac{1}{4} g^{\mu\nu} g_{\gamma\delta}\,.
$$

We prove now that the system $(2.2)_{1\text{-}3}$ *is hyperbolic.* Firstly, we consider the system

$$
(2.4) \qquad \begin{cases} t^\alpha t_\alpha d g_{\mu\nu} = 0, \\ h^{\mu\nu}_{\gamma\delta}\, \dfrac{1}{2}\, g^{\alpha\beta}[-t_\alpha d\omega_{\beta\mu\nu} - t_{(\mu} d\omega_{\nu)\alpha\beta} + t_\alpha d\omega_{\nu\mu\beta} + t_\beta d\omega_{\mu\nu\alpha}] = 0, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}[t_{[\beta} d\omega_{\alpha]\mu\nu}] = 0, \\ d\left[\dfrac{1}{2}\, g^{\alpha\beta}\,(2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta})\right] = 0, \end{cases}
$$

in the unknowns $dg_{\mu\nu}$, $d\omega_{\alpha\beta\gamma}$. It is easy to see that this system has only the solution $dg_{\mu\nu} = 0$, $d\omega_{\alpha\beta\gamma} = 0$. In fact, equations $(2.4)_{1,3}$ yield $dg_{\mu\nu} = 0$,

$$
(2.5) \qquad d\omega_{\beta\mu\nu} = -t_\beta t^\delta d\omega_{\delta\mu\nu} + X_\beta g_{\mu\nu},
$$

for every X_β such that

$$
(2.6) \qquad X_\beta t^\beta = 0.
$$

After that, equation $(2.4)_4$ gives

$$X_\lambda = -t^\beta t^\delta d\omega_{\delta\beta\lambda} + \frac{1}{2} g^{\alpha\beta} t_\lambda t^\delta d\omega_{\delta\alpha\beta}.$$

By substituting in (2.5), (2.6), we obtain

$$g^{\alpha\beta} t^\delta d\omega_{\delta\alpha\beta} = -2 t^\delta t^\beta t^\lambda d\omega_{\delta\beta\lambda} \tag{2.7}$$

and

$$d\omega_{\beta\mu\nu} = -t_\beta t^\delta d\omega_{\delta\mu\nu} - g_{\mu\nu} t^\delta t^\rho d\omega_{\delta\rho\gamma}(g^\gamma_\beta + t_\beta t^\gamma).$$

At last, equation $(2.4)_2$ yields $h^{\gamma\delta}_{\mu\nu} t^\beta d\omega_{\beta\mu\nu} = 0$ from which and (2.7) the relation $d\omega_{\beta\mu\nu} = 0$ follows. This result proves that, from the system (2.2), the time derivatives can be obtained as functions of the other quantities.

To prove the hyperbolicity of the system (2.2) it suffices now to see that the following system

$$\left\{ \begin{aligned} & t^\alpha \varphi_\alpha dg_{\mu\nu} = 0, \\ & h^{\mu\nu}_{\gamma\delta} \, \frac{1}{2} \, g^{\alpha\beta} [-\varphi_\alpha d\omega_{\beta\mu\nu} - \varphi_{(\mu} d\omega_{\nu)\alpha\beta} + \varphi_\alpha d\omega_{\nu\mu\beta} + \varphi_\beta d\omega_{\mu\nu\alpha}] = 0, \\ & t^\alpha h^{\mu\nu}_{\gamma\delta} [\varphi_{[\beta} d\omega_{\alpha]\mu\nu}] = 0, \\ & d\left[\frac{1}{2} \, g^{\alpha\beta} \, (2\omega_{\alpha\beta\lambda} - \omega_{\lambda\alpha\beta})\right] = 0, \end{aligned} \right. \tag{2.8}$$

with $\varphi_\alpha = n_\alpha - \lambda t_\alpha$, has real eigenvalues λ and 46 linearly independent (l.i.) eigenvectors $dg_{\mu\nu}$, $d\omega_{\beta\mu\nu}$, for every n_α such that $n_\alpha t^\alpha = 0$, $n_\alpha n^\alpha = 1$.

Also this condition is satisfied: in fact in correspondence to the eigenvalue $\lambda = 0$, equation $(2.8)_1$ is an identity, while equation $(2.8)_3$ is equivalent to $t^\alpha h^{\mu\nu}_{\gamma\delta} d\omega_{\alpha\mu\nu} = 0$. Therefore, we have 22 equations for 50 unknowns and, consequently, 28 l.i. eigenvectors.

In correspondence to $\lambda = \pm 1$ (from which $\varphi_\beta \varphi^\beta = 0$, $\lambda \neq 0$) we obtain the eigenvectors

$$dg_{\mu\nu} = 0, \qquad d\omega_{\beta\mu\nu} = \frac{1}{\lambda}(\varphi_\beta y_{<\mu\nu>} + \varphi^\delta y_{<\delta\beta>} g_{\mu\nu}),$$

where $y_{<\mu\nu>}$ is an arbitrary symmetric traceless tensor; therefore there are 2×9 l.i. eigenvectors corresponding to $\lambda = \pm 1$. In this way, the hyperbolicity of system (2.2) has been proved.

In ref. [4] we find also a method to get rid of the algebraic constraints; its application to our case leads, as a first step, to the system

$$
(2.9)\qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\[2mm] h^{\mu\nu}_{\gamma\delta}\, \dfrac{1}{2}\, g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] + \\ \quad + g^{\mu\nu} g_{\gamma\delta} t^\alpha \partial_\alpha \omega_{\mu\nu} = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}, \\[2mm] t^\alpha h^{\mu\nu}_{\gamma\delta}(\partial_{[\beta}\omega_{\alpha]\mu\nu}) + t^\alpha h^{\mu\nu}_{\gamma\delta} t_\beta \partial_\alpha \omega_{\mu\nu} + g_{\gamma\delta} t^\alpha \partial_\alpha \psi_\beta = 0, \end{cases}
$$

for the determination of the variables $g_{\mu\nu}$, $\omega_{\beta\mu\nu}$, $\omega_{\mu\nu} = \omega_{\nu\mu}$, ψ_β, constrained by $(2.1)_4$. This system is also hyperbolic and has the advantage to have an equal number of equations and of independent variables; when $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ it reduces to the system (2.2) and, moreover, if $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ on an initial hypersurface Σ, then $\omega_{\mu\nu} = 0$, $\psi_\beta = 0$ will propagate also off Σ.

The system (2.9) has been obtained from (2.2) by considering more equations and more independent variables, an idea somehow similar to that conceived in Extended Thermodynamics.

The second, and last, step leads to the system

$$
(2.10)\qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\[2mm] h^{\mu\nu}_{\gamma\delta}\, \dfrac{1}{2}\, g^{\alpha\beta}[-\partial_\alpha \omega_{\beta\mu\nu} - \partial_{(\mu}\omega_{\nu)\alpha\beta} + \partial_\alpha \omega_{\nu\mu\beta} + \partial_\beta \omega_{\mu\nu\alpha}] + \\ \quad + g^{\mu\nu} g_{\gamma\delta} t^\alpha \partial_\alpha \omega_{\mu\nu} = h^{\mu\nu}_{\gamma\delta} F_{\mu\nu}, \\ t^\alpha h^{\mu\nu}_{\gamma\delta}(\partial_{[\beta}\omega_{\alpha]\mu\nu}) + t^\alpha h^{\mu\nu}_{\gamma\delta} t_\beta \partial_\alpha \omega_{\mu\nu} + \\ \quad + g_{\gamma\delta} t^\alpha \partial_\alpha \left[\dfrac{1}{2}\, g^{\mu\nu}(2\omega_{\mu\nu\beta} - \omega_{\beta\mu\nu})\right] = 0, \end{cases}
$$

in the independent variables $g_{\mu\nu}$, $\omega_{\beta\mu\nu}$, $\omega_{\mu\nu}$; *in this way all the constraints, both differential and algebraic, have been eliminated still maintaining the property to be hyperbolic and to have an equal number of equations and of independent variables.* Obviously, by setting $\omega_{\mu\nu} = 0$ in equations (2.10), the differential constraints arising are only identity and one obtains a system in the "old variables" but without algebraic constraints; this system is hyperbolic and, moreover, if $(2.1)_4$ holds on an initial hypersurface Σ, then it will be satisfied also off Σ.

Another method to obtain this result is exposed in the next section.

3 – The Einstein's equations with the further condition $\partial_{(\alpha}\Gamma_{\beta)} = 0$

We can easily see that the following relations hold

$$(3.1) \qquad \Gamma_\beta = 0 \qquad \Leftrightarrow \qquad \begin{cases} \partial_0\Gamma_\beta = 0 \\ (\Gamma_\beta)_\Sigma = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \partial_{(\alpha}\Gamma_{\beta)} = 0 \\ (\Gamma_\beta)_\Sigma = 0, \end{cases}$$

where $(\Gamma_\beta)_\Sigma$ is the value of Γ_β on an initial space-like hypersurface Σ. The first equivalence in (3.1) is trivial; the second one is based on the fact that $\partial_0\Gamma_\beta = 0, (\Gamma_\beta)_\Sigma = 0$ implies $\partial_0(\partial_i\Gamma_\beta) = 0$, $(\partial_i\Gamma_\beta)_\Sigma = 0$, from which $\partial_i\Gamma_\beta = 0$ follows. Vice versa, if the equations in the right hand side of (3.1) hold, then we have

$$\begin{cases} \partial_0\Gamma_0 = 0 \quad \Rightarrow \quad \partial_0\left(\partial_i\Gamma_0\right) = 0, \\ (\partial_i\Gamma_0)_\Sigma = 0, \end{cases}$$

and, consequently, $\partial_i\Gamma_0 = 0$; this result allows to obtain, from $\partial_{(\alpha}\Gamma_{\beta)} = 0$, for $\alpha, \beta = 0, \ldots, i$, that $\partial_0\Gamma_i = 0$. In this way the second equivalence in (3.1) has been proved.

This suggest to consider the equations

$$(3.2) \qquad \partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \qquad R_{\mu\nu} = 0, \qquad \partial_{[\beta}\omega_{\alpha]\mu\nu} = 0, \qquad \partial_{[\mu}\Gamma_{\nu]} = 0.$$

This system has more differential constraints than the system (2.1), but has no algebraic constraints because $(2.1)_4$ has to be imposed only on the initial manifold. The method in ref. [4] already applied in Section 2 to equations (2.1), can now be applied to the system (3.2). One obtains

$$(3.3) \qquad \begin{aligned} &t^\alpha\partial_\alpha g_{\mu\nu} = t^\alpha\omega_{\alpha\mu\nu}, \qquad R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta}g^\gamma_{(\mu}g^\delta_{\nu)}\partial_{[\beta}\omega_{\gamma]\delta\alpha} = 0, \\ &t^\alpha\left(\partial_{[\beta}\omega_{\alpha]\mu\nu}\right) = 0, \end{aligned}$$

which is the new counterpart of system (2.2).

We note that the equations $(3.3)_2$ substantially coincide with those proposed by Fourès-Bruhat, Fischer and Marsden in refs. [5], [7], i.e.,

$$R_{\mu\nu} - g_{\alpha(\mu}\partial_{\nu)}\Gamma^\alpha - 2g^{\alpha\beta}g^\gamma_{(\mu}g^\delta_{\nu)}\partial_{[\beta}\omega_{\gamma]\delta\alpha} = 0$$

or, equivalently,

$$R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta} g^{\gamma}_{(\mu} g^{\delta}_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha} + \Gamma^{\alpha}\omega_{(\nu\mu)\alpha} = 0;$$

the only difference is in the last term which doesn't involve the derivatives of the variables and, therefore, does not affect the study of hyperbolicity.

We retain very interesting to see how a general method such that of ref. [4] *leads to equations obtained in other ways in literature.* These equations $(3.3)_2$ writes explicitly

$$-\frac{1}{2}\, g^{\alpha\beta}\partial_{\alpha}\omega_{\beta\mu\nu} = F_{\mu\nu} - \frac{1}{2}\, g^{\gamma}_{(\mu} g^{\delta}_{\nu)} \omega^{\alpha\beta}_{\gamma} (2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}), \tag{3.4}$$

where we have used the relation $\partial_{\gamma} g^{\psi\eta} = -\omega_{\gamma}{}^{\psi\eta}$ which comes from $g^{\psi\theta}$ contracted with the derivative with respect to x_{γ} of the relation

$$g_{\theta\delta} g^{\delta\eta} = \delta^{\eta}_{\theta}.$$

To prove the hyperbolicity of the system (3.3) is now an easy task because the system

$$-dg_{\mu\nu} = 0, \qquad -\frac{1}{2}\, t^{\beta} d\omega_{\beta\mu\nu} = 0, \qquad t^{\alpha} t_{[\beta} d\omega_{\alpha]\mu\nu} = 0,$$

imply $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = 0$, while the system

$$\lambda dg_{\mu\nu} = 0, \qquad -\frac{1}{2}\, \varphi^{\beta} d\omega_{\beta\mu\nu} = 0, \qquad t^{\alpha} \varphi_{[\beta} d\omega_{\alpha]\mu\nu} = 0,$$

has 50 l.i. eigenvectors, i.e.

- the 30 l.i. solutions of $t^{\beta} d\omega_{\beta\mu\nu} = 0$, $n^{\beta} d\omega_{\beta\mu\nu} = 0$, corresponding to the real eigenvalue $\lambda = 0$,
- the 20 l.i. solutions of $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = x_{\mu\nu}\varphi_{\beta}$, (with $x_{\mu\nu}$ an arbitrary symmetric tensor) corresponding to the real eigenvalues $\lambda = \pm 1$ (i.e., $\varphi_{\beta}\varphi^{\beta} = 0$).

But a more interesting aspect is that the system (3.3) *can be put in the symmetric form*; this result has been obtained by Fischer and Marsden for their system of equations, but in 3-dimensional formalism. Here we obtain in 4-dimensional notation the same result for our system (3.3). In fact, it is equivalent to

$$(3.5)\qquad \begin{cases} t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\ t_\tau \left(-\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha \omega_{\beta\mu\nu}\right) - t^\alpha \partial_{[\tau} \omega_{\alpha]\mu\nu} = t_\tau K_{\mu\nu}, \end{cases}$$

where we have used the expression (3.4) for the equations $(3.3)_2$ and we have called $K_{\mu\nu}$ the second member of (3.4).(1)

The system (3.5) is symmetric; in fact, if we take a linear combination of its left-hand sides through the coefficients $\lambda^{\mu\nu}$, $\lambda^{\tau\mu\nu}$ and substitute ∂_α with δ, we obtain(2)

$$\begin{aligned} & t^\alpha \left(\lambda^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2}\, \lambda^{\tau\mu\nu} \delta\omega_{\tau\mu\nu}\right) + \\ & -\frac{1}{2} \left(t^\beta g^{\alpha\beta'} \lambda_{\beta'\mu'\nu'} \delta\omega_{\beta\mu\nu} + t^\tau \lambda_{\tau\mu'\nu'} g^{\alpha\beta} \delta\omega_{\beta\mu\nu}\right) g^{\mu'\mu} g^{\nu'\nu}. \end{aligned}$$

This expression doesn't change if we exchange $\lambda_{\mu\nu}$ with $\delta g_{\mu\nu}$ and $\lambda_{\tau\mu\nu}$ with $\delta\omega_{\tau\mu\nu}$, thus proving the symmetric form of (3.5).

The result of this section has been achieved at the cost of dealing with modified Einstein's equations, i.e., $(3.3)_2$. A more elegant result will be obtained in the next section by introducing suitable equations for Γ^α.

4 – The unmodified Einstein's equations

In ref. [7], Fischer et al. obtain equations involving Γ^α, drawing it from a consequence of Bianchi identities, i.e., $\nabla_\alpha G^{\alpha\beta} = 0$ (where ∇ is the operator of covariant derivation) or, in other words, from

$$(4.1)\qquad \partial_\mu G^{\mu\nu} + G^{\rho\nu} \Gamma^\mu_{\rho\mu} + G^{\mu\rho} \Gamma^\nu_{\rho\mu} = 0.$$

It seems strange that an equation may be obtained from an identity!

(1)Note that $(3.5)_2$ contracted with t^τ gives $(3.3)_2$ and, after that, it remains $(3.3)_3$.
(2)Note that

$$t^\alpha \lambda^{\tau\mu\nu} \partial_\tau \omega_{\alpha\mu\nu} = t^\tau \lambda^{\alpha\mu\nu} \partial_\alpha \omega_{\tau\mu\nu} \quad\Rightarrow\quad t^\tau \lambda^{\alpha\mu\nu} \delta\omega_{\tau\mu\nu}.$$

The reason of this apparent paradox is that (4.1) is an identity when applied to the whole Einstein tensor $G^{\mu\nu}$, while Fischer et al. apply it to

$$G^{\mu\nu} = \left(g^{\mu(\beta} g^{\gamma)\nu} - \frac{1}{2}\, g^{\mu\nu} g^{\beta\gamma} \right) g_{\alpha\beta} \partial_\gamma \Gamma^\alpha ,$$

i.e. the expression of $G^{\mu\nu}$ calculated on a solution of $R_{\mu\nu} - g_{\alpha(\mu}\partial_{\nu)}\Gamma^\alpha = 0$. In this way they obtain a system of the form

$$\frac{1}{2}\, g^{\beta\nu} \partial^2_{\beta\nu} \Gamma^\mu + A^{\beta\mu}_\alpha (g_{\gamma\delta}, \partial_\lambda g_{\gamma\delta}) \partial_\beta \Gamma^\alpha = 0. \tag{4.2}$$

Therefore (4.2) is not a consequence of Einstein's equations, but of their "modified" expressions which are equivalent to them only under the further assumption of harmonic coordinates; in this case, it is obvious that also (4.2) is an identity! On the other hand, one may consider (4.2) as further equations to consider jointly with Einstein's ones, disregarding their origin; assuming the validity of (4.2) is less restrictive than assuming harmonic coordinates.

So, let us consider the system

$$\begin{cases} R_{\mu\nu} = 0 \quad \text{(Einstein's equations)}, \\ \dfrac{1}{2}\, g^{\beta\nu} \partial^2_{\beta\nu} \Gamma^\mu = -A^{\beta\mu}_\alpha (g_{\gamma\delta}, \partial_\lambda g_{\gamma\delta}) \partial_\beta \Gamma^\alpha \quad \text{(Fischer's equations)}. \end{cases} \tag{4.3}$$

It is expressed in terms of $g_{\mu\nu}$ and of its first, second and third derivatives; obviously, it can be reduced to a first order system considering $g_{\mu\nu}$, $\partial_\alpha g_{\mu\nu}$, $\partial^2_{\alpha\beta} g_{\mu\nu}$ as independent variables.

But the third derivatives of $g_{\mu\nu}$ intervene only through the second derivatives of Γ^μ; therefore, one can consider $g_{\mu\nu}$, $\partial_\alpha g_{\mu\nu}$, Γ^μ, $\partial_\alpha \Gamma^\mu$ as independent variables except for the algebraic constraints

$$\Gamma^\mu = \frac{1}{2}\, g^{\mu\lambda} g^{\alpha\beta} \left(2\partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta} \right). \tag{4.4}$$

The system (4.3) can now be reduced to a first order one by defining $\omega_{\alpha\beta\gamma} = \partial_\alpha g_{\beta\gamma}$, $S^\mu_\beta = \partial_\beta \Gamma^\mu$, i.e.

$$
(4.5) \qquad \begin{cases}
\partial_\alpha g_{\mu\nu} = \omega_{\alpha\mu\nu}, \\
-\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha \omega_{\beta\mu\nu} = F_{\mu\nu}(g_{\gamma\delta}, \omega_{\lambda\gamma\delta}) + \\
\quad -\frac{1}{2}\, g^\gamma_{(\mu} g^\delta_{\nu)} \omega_\gamma{}^{\alpha\beta} (2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}) - S_{\mu\nu}, \\
\partial_{[\beta} \omega_{\alpha]\mu\nu} = 0, \\
\partial_\alpha \Gamma^\mu = S^\mu_\alpha, \\
\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha S^\mu_\beta = -A^{\beta\mu}_\alpha (g_{\gamma\nu}, \omega_{\lambda\gamma\nu}) S^\alpha_\beta, \\
\partial_{[\beta} S^\mu_{\alpha]} = 0,
\end{cases}
$$

where we have used the identity

$$
R_{\mu\nu} = R_{\mu\nu} - \partial_{(\mu} \Gamma_{\nu)} - 2 g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta} \omega_{\gamma]\delta\alpha} + S_{\mu\nu} + 2 g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta} \omega_{\gamma]\delta\alpha},
$$

transformed by the expression (3.4) for $R_{\mu\nu} - \partial_{(\mu}\Gamma_{\nu)} - 2g^{\alpha\beta} g^\gamma_{(\mu} g^\delta_{\nu)} \partial_{[\beta}\omega_{\gamma]\delta\alpha}$ and by the (4.5) for $\partial_{[\beta}\omega_{\gamma]\delta\alpha}$. Moreover $(4.5)_6$ is the integrability condition on $(4.5)_4$ such as $(4.5)_3$ is the integrability condition on $(4.5)_1$.

Now it can be easily seen that the system (4.5) is hyperbolic, without considering the constraints (4.4); therefore, it is sufficient to impose (4.4) only on the initial manifold Σ and then it will propagate off Σ. Also (4.5) can be written in a symmetric form, i.e.,

$$
(4.6) \qquad \begin{cases}
t^\alpha \partial_\alpha g_{\mu\nu} = t^\alpha \omega_{\alpha\mu\nu}, \\
t_\tau \left(-\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha \omega_{\beta\mu\nu} \right) - t^\alpha \partial_{[\tau} \omega_{\alpha]\mu\nu} = \\
\quad = t_\tau \left[F_{\mu\nu} - \frac{1}{2}\, g^\gamma_{(\mu} g^\delta_{\nu)} \omega_\gamma{}^{\alpha\beta} (2\omega_{\alpha\beta\delta} - \omega_{\delta\alpha\beta}) - S_{\mu\nu} \right], \\
t^\alpha \partial_\alpha \Gamma^\mu = t^\alpha S^\mu_\alpha, \\
t_\tau \left(\frac{1}{2}\, g^{\alpha\beta} \partial_\alpha S^\mu_\beta \right) + t^\alpha \partial_{[\tau} S^\mu_{\alpha]} = -t_\tau A^{\beta\mu}_\alpha S^\alpha_\beta.
\end{cases}
$$

Obviously, the system

$$- dg_{\mu\nu} = 0, \qquad - t_\tau \frac{1}{2} t^\beta d\omega_{\beta\mu\nu} - t^\alpha t_{[\tau} d\omega_{\alpha]\mu\nu} = 0,$$

$$- d\Gamma^\mu = 0, \qquad t_\tau \frac{1}{2} t^\beta dS^\mu_\beta + t^\alpha t_{[\tau} dS^\mu_{\alpha]} = 0,$$

has only the solution $dg_{\mu\nu} = 0$, $d\omega_{\alpha\mu\nu} = 0$, $d\Gamma^\mu = 0$, $dS^\mu_\alpha = 0$. Moreover, the eigenvectors are the solutions of the system

$$\lambda dg_{\mu\nu} = 0, \qquad - t_\tau \frac{1}{2} \varphi^\beta d\omega_{\beta\mu\nu} - t^\alpha \varphi_{[\tau} d\omega_{\alpha]\mu\nu} = 0,$$

$$\lambda d\Gamma^\mu = 0, \qquad t_\tau \frac{1}{2} \varphi^\beta dS^\mu_\beta + t^\alpha \varphi_{[\tau} dS^\mu_{\alpha]} = 0;$$

one obtains the eigenvalues

- $\lambda = 0$, to which correspond, as eigenvectors, the 42 l.i. solutions of $t^\beta d\omega_{\beta\mu\nu} = 0$, $n^\beta d\omega_{\beta\mu\nu} = 0$, $t^\beta dS^\mu_\beta = 0$, $n^\beta dS^\mu_\beta = 0$;
- $\lambda = \pm 1$, and the corresponding 28 l.i. eigenvectors $dg_{\mu\nu} = 0$, $d\omega_{\beta\mu\nu} = x_{\mu\nu}\varphi_\beta$, $d\Gamma^\mu = 0$, $dS^\mu_\alpha = X^\mu \varphi_\alpha$, with $x_{\mu\nu}$ an arbitrary symmetric tensor, and X^μ an arbitrary 4-vector.

In the next section will be considered the case where we have sources due to the presence of matter.

5 – The case of interaction with matter

Let us consider now the expression (1.3) with $\chi \neq 0$, for Einstein's equations. Thanks to the identity $\nabla_\alpha G^{\alpha\beta} = 0$ and to $(1.2)_1$, it yields

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{5.1}$$

Usually, this equation doesn't suffice to include the contribution of matter and we have more equations; they can be written in the form

$$\nabla_\alpha T^{\alpha A} = P^A, \quad \text{for } A = 1, \dots, N. \tag{5.2}$$

Obviously, for some values of A the equation (5.2) coincide with (5.1); $T^{\alpha A}$ and P^A are functions of the independent variables. In particular, in Extended Thermodynamics (see for example, refs. [14]-[16]), the equations (5.2) assume the symmetric hyperbolic form by taking suitable independent variables λ_A which define the so called "*mean field*"; more clearly, the equations (5.2) become

$$\frac{\partial T^{\alpha A}}{\partial \lambda_B} \nabla_\alpha \lambda_\beta = P^A, \tag{5.3}$$

with $\frac{\partial T^{\alpha A}}{\partial \lambda_B} = \frac{\partial T^{\alpha B}}{\partial \lambda_A}$, $\frac{\partial T^{\alpha A}}{\partial \lambda_B} u_\alpha$ being a convex functions of λ_B.

But this result is achieved by considering constant the metric tensor $g_{\mu\nu}$; if we avoid this assumption, let us see how the equations (5.3) modify. The equations (5.2) by taking λ_B and $g_{\mu\nu}$ as independent variables, become

$$\frac{\partial T^{\alpha A}}{\partial \lambda_B} \nabla_\alpha \lambda_\beta = P^A - \frac{\partial T^{\alpha A}}{\partial g_{\mu\nu}} \omega_{\alpha\mu\nu}. \tag{5.4}$$

Therefore, the only difference is in the second members which don't involve the derivatives of the field. We can now consider the system constituted by (3.5) with $t^\alpha = u^\alpha$ and by (5.4)(or, alternatively, by (4.6) with $t^\alpha = u^\alpha$ and by (5.4)) and see that it is symmetric hyperbolic in the time direction u^α; moreover, the characteristic velocities don't exceed the speed of light and therefore, for Strumia's Lemma [1], they are hyperbolic in any other time direction.

For the sake of simplicity, let us consider only the example given by the equations of fluid dynamics

$$\nabla_\alpha(\rho u^\alpha) = 0, \qquad \nabla_\alpha[(e+p)u^\alpha u^\beta + p g^{\alpha\beta}] = 0. \tag{5.5}$$

Here ρ, e, p can be considered functions of the entropy density s and of the temperature T and they satisfy the Gibbs relation

$$T\, ds = \frac{1}{\rho}\, de + (e+p)\, d\left(\frac{1}{\rho}\right). \tag{5.6}$$

If we take $\lambda = -s + \frac{e+p}{\rho T}$, $\lambda^\alpha = \frac{u^\alpha}{T}$ as independent variables, the Gibbs relation gives $d\lambda$ from which one obtains

$$\frac{\partial p}{\partial \lambda} = \rho T, \qquad \frac{\partial p}{\partial T} = \frac{e+p}{T}, \qquad \frac{\partial(\rho T)}{\partial T} = \frac{\partial}{\partial \lambda}\left(\frac{e+p}{T}\right), \tag{5.7}$$

the last of which is the integrability condition on $(5.7)_{1,2}$.

Moreover, the above mentioned functions ρ, e, p satisfy the physical conditions

$$\frac{\partial \rho}{\partial \lambda} > 0, \qquad \begin{vmatrix} \dfrac{\partial \rho}{\partial \lambda} & T\,\dfrac{\partial \rho}{\partial T} \\ T\,\dfrac{\partial \rho}{\partial T} & T\,\dfrac{\partial}{\partial T}\left(\dfrac{e+p}{T}\right) \end{vmatrix} > 0,$$

$$(5.8) \qquad \begin{vmatrix} \dfrac{\partial \rho}{\partial \lambda} & T\,\dfrac{\partial \rho}{\partial T} & \rho \\ T\,\dfrac{\partial \rho}{\partial T} & T\,\dfrac{\partial}{\partial T}\left(\dfrac{e+p}{T}\right) & \dfrac{e+p}{T} \\ \rho & \dfrac{e+p}{T} & \dfrac{e+p}{T} \end{vmatrix} \geq 0.$$

The system (5.4), *for this case, reads*

$$\begin{aligned} &\frac{\partial \rho}{\partial \lambda}\,\lambda^\alpha \partial_\alpha \lambda + \left(\rho g^{\alpha\delta} + T^2 \frac{\partial(\rho T)}{\partial T}\,\lambda^\alpha \lambda^\delta\right) \partial_\alpha \lambda_\delta = \\ &= P(\lambda, \lambda^\gamma, g_{\mu\nu}, \omega_{\delta\mu\nu}), \left(\rho g^{\alpha\beta} + T^2 \frac{\partial(\rho T)}{\partial T}\,\lambda^\alpha \lambda^\beta\right) \partial_\alpha \lambda + \\ &\quad + \left[3(e+p) T \lambda^{(\alpha} g^{\beta\delta)} + T^2 \frac{\partial[(e+p)T^2]}{\partial T} \lambda^\alpha \lambda^\beta \lambda^\delta\right] \partial_\alpha \lambda_\delta = \\ &= P^\beta(\lambda, \lambda^\gamma, g_{\mu\nu}, \omega_{\delta\mu\nu}), \end{aligned} \qquad (5.9)$$

which is manifestly symmetric.

Coupling it with (3.5) *or with* (4.6), *one obtains the whole system of equations, which is also symmetric and hyperbolic.*

Obviously, many other situations may be considered, but here we are satisfied with this one.

REFERENCES

[1] A. Strumia: *Wave propagation and symmetric hyperbolic systems of conservation law with constrained field variables*, Il Nuovo Cimento, **101B** (1988), 1.

[2] K. O. Friedrichs: *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math., **7** (1954), 345.

[3] K. O. Friedrichs – P. D. Lax: *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci., **68** (1971), 1686.

[4] S. Pennisi: *Hyperbolic systems with constrained field variables. A covariant and extended approach*, Bollettino U.M.I. **(7)11-B** (1997), 851-870.

[5] Y. Fourès-Bruhat: *Théorèm d'existence pour certain systèmes d'équations aux dérivées partielles non linéaires*, Acta Math., **88** (1952), 141-225.

[6] A. Lichnerowicz: *Relativistic Hydrodynamics and Magnetohydrodynamics*, Benjamin, New York, 1967.

[7] A. E. Fischer – J. E. Marsden: *The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system*, Commun. Math. Phys., **28** (1972), 1-38.

[8] S. Weinberg: *Gravitation and Cosmology*, John Wiley and Sons, New York, 1972.

[9] S. Pennisi: *A covariant and extended model for relativistic magnetofluiddynamics*, Ann. Inst. Henry Poincarè, **58** (1993), 343.

[10] S. Pennisi: *Extended approaches to covariant maxwell electrodynamics*, Continuum Mech. Thermodyn., **8** (1996), 143.

[11] S. Pennisi: *A new approach to constrained systems with a convex extension*, Note di Matematica, **16** (1996), 173.

[12] S. Pennisi: *A covariant and extended approach to some physical problems with constrained field variables*, Le Matematiche, vol. LIII, (1988).

[13] G. Boillat: *Sur l'elimination des constraints involutives*, C.R. Acad. Sci. Paris, **318** (1994), 1053.

[14] I. S. Liu – I. Müller – T. Ruggeri: *Relativistic thermodynamics of gases*, Ann. Phys., **169** (1986), 191.

[15] P. Amendt – H. Weitzner: *Relativistically covariant warm charged fluid beam modeling*, Phys. Fluids, **28** (1985), 949.

[16] A. M. Anile – S. Pennisi: *Fluid models for relativistic electron beams*, Continuum Mech. Thermodyn., **1** (1989), 267.

Lavoro pervenuto alla redazione il 11 febbraio 2002
ed accettato per la pubblicazione il 16 gennaio 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo degli Autori:

Francesco Borghero – Sebastiano Pennisi – Dipartimento di Matematica – Università di Cagliari – Via Ospedale, 72 – 09124 Cagliari (Italia)
E-mail: borghero@unica.it spennisi@unica.it

This work was partially supported by Gruppo Nazionale di Fisica Matematica (GNFM) of INDAM and MIUR (Italy).

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 83-116

Stability and controllability of an abstract evolution equation of hyperbolic type and concrete applications

S. NICAISE

Abstract: *We consider the stability of an abstract evolution equation using Liu's principle based on the exponential stability of the inverse problem with a linear feedback and on an integral inequality. Russell's principle also yields some exact controllability results. Some concrete examples with new stability and controllability results illustrate the interest of our approach.*

1 – Introduction

Stability of different systems of partial differential equations of hyperbolic type with linear or nonlinear feedbacks has been recently the object of several works. Let us quote the stability of the wave equation [18], [19], [20], [23], [22], [43], [26], [10] and the references cited there, of the Petrovsky system [11], [13], [15], [1], [4], of the elastodynamic system [1], [4], [13], of Mawxell's system [3], [21], [39], [7], [36] or combination of them [17], [37]. We actually remark that the approach of recent works

Key Words and Phrases: *Stability of partial differential equations of hyperbolic type – Integral inequality – Evolution equations – Liu principle.*
A.M.S. Classification: 37D35 – 35L10

cited above has a similar structure, namely the use of Liu's principle and of some integral inequalities. Liu's principle consists in estimating the energy of the direct system by some terms related to the feedbacks using a retrograde system with final data equal to the final data of the direct system. These terms are then estimated using the exponential stability of the inverse (retrograde) problem with a linear feedback (based on Russell's principle) and an appropriated integral inequality. Therefore our goal is to present an abstract setting leading to the stability and controllability (via Russell's principle) of the abstract system, setting as large as possible to include all examples of the aforementioned papers and allowing even new applications.

More precisely we first present an abstract setting of hyperbolic type and including the above systems. General assumptions guarantee existence results as well as dissipativeness of the system. In a second step we show that the exponential decay of the energy of the solution is equivalent to the validity of a stability estimate, estimate that can be checked in some particular cases. In a third step we use the so-called Russell's principle "controllability via stability" to obtain controllability results for the abstract system. Finally using Liu's principle [28] and a new integral inequality from [7] we give sufficient conditions on a class of (quite general) feedbacks which lead to an explicit decay rate of the energy. The strength of our approach lies in the fact that the controllability and stability results (with general feedbacks) are only based on the stability estimate with a linear feedback, estimate that may be checked for an explicit problem by different techniques, like the multiplier method, microlocal analysis or any method entering in a linear framework (like nonharmonic analysis for instance). This approach was successfully initiated in [36] for Maxwell's system and is here extended to an abstract system. We further illustrate our approach by considering different examples for which new stability and controllability results are even obtained.

The schedule of the paper is the following one: the abstract setting and its well-posedness are analysed in Section 2. Section 3 is devoted to the equivalence between the exponential stability and the stability estimate. In Section 4 exact controllability results are deduced from Russell's principle. Section 5 is devoted to the stability results for a class of nonlinear feedbacks using Liu's principle. Some applications are presented in the last section.

2 – Abstract setting

In this section we describe a general abstract setting of hyperbolic type that will be used later on. It is motivated by the examples (and other ones) given in Section 6 which all enter in this setting.

Let us fix two real separable Hilbert spaces $\mathcal{H}$, $\mathcal{V}$ with respective inner products $(.,.)_{\mathcal{H}}$, $(.,.)_{\mathcal{V}}$ and such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have the standard diagram:

$$\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}' \hookrightarrow \mathcal{V}'.$$

We suppose given a bounded linear operator A_1 from $\mathcal{V}$ into $\mathcal{V}'$ and a (nonlinear) mapping B from $\mathcal{V}$ into $\mathcal{V}'$. We now define two (nonlinear) operators $\mathcal{A}^+$ and $\mathcal{A}^-$ as follows

$$D(\mathcal{A}^\pm) = \{v \in \mathcal{V} | (\pm A_1 + B)v \in \mathcal{H}\}, \tag{1}$$

$$\mathcal{A}^\pm = (\pm A_1 + B)v, \forall v \in D(\mathcal{A}^\pm). \tag{2}$$

For shortness we often drop the superscript + at $\mathcal{A}^+$.

Motivated by the examples we introduce the following assumptions:

$$\mathcal{A}^+ \text{ is maximal monotone,} \tag{3}$$

$$\mathcal{A}^- \text{ is maximal monotone,} \tag{4}$$

$$D(\mathcal{A}^+) \text{ is dense in } \mathcal{H}, \tag{5}$$

$$D(\mathcal{A}^-) \text{ is dense in } \mathcal{H}, \tag{6}$$

$$\langle A_1 u, u \rangle = 0, \forall u \in \mathcal{V}, \tag{7}$$

$$\langle Bu, u \rangle \geq 0, \forall u \in \mathcal{V}, \tag{8}$$

where hereabove and below $\langle .,. \rangle$ means the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$.

Lemma 2.1. *Under the assumptions* (3), (5), (7) *and* (8), *the evolution equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} + A_1 u + Bu = 0 \ \textit{in} \ \mathcal{H}, t \geq 0, \\ u(0) = u_0, \end{cases} \tag{9}$$

admits a unique (weak) solution $u \in C(\mathbb{R}_+, \mathcal{H})$ *for any* $u_0 \in \mathcal{H}$. *If moreover* $u_0 \in D(\mathcal{A})$, *the problem* (9) *admits a unique (strong) solution* $u \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(\mathcal{A}))$ *and such that* $u(t) \in D(\mathcal{A})$, *for all* $t \geq 0$.

This system is dissipative since its energy

$$\mathcal{E}(t) = \frac{1}{2}||u(t)||^2_{\mathcal{H}},$$

is non-increasing. Moreover for $u_0 \in D(\mathcal{A})$, *we have*

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \langle Bu(t), u(t)\rangle \, dt, \forall 0 \leq S < T < \infty, \tag{11}$$

$$\frac{d}{dt}\mathcal{E}(t) = -\langle Bu(t), u(t)\rangle, \forall t \geq 0. \tag{12}$$

Under the assumptions (4), (6), (7) *and* (8), *the same results hold for* $\mathcal{A}^-$ *(with the same expression for the energy and the same identities* (11) *and* (12) *for* $u_0 \in D(\mathcal{A}^-)$).

Proof. The first assertions follow from nonlinear semigroup theory [42]. For the second assertions it suffices to show (12) since $D(\mathcal{A})$ is dense in $\mathcal{H}$. For $u_0 \in D(\mathcal{A})$, we have

$$\frac{d}{dt}\mathcal{E}(t) = \left(\frac{\partial u}{\partial t}(t), u(t)\right)_{\mathcal{H}} = -(\mathcal{A}u(t), u(t))_{\mathcal{H}},$$

by (9). From the definition of $\mathcal{A}$ and the fact that $u(t) \in \mathcal{V}$, for all $t \geq 0$, we get

$$\frac{d}{dt}\mathcal{E}(t) = -\langle A_1 u(t), u(t)\rangle - \langle Bu(t), u(t)\rangle.$$

This yields (12) owing to (7). □

Remark 2.2. The identity (11) remains valid for $u_0 \in \mathcal{H}$ indeed for a sequence $u_{0n} \in D(\mathcal{A})$ such that $u_{0n} \to u_0$ in $\mathcal{H}$, let u_n be the solution of (9) with initial datum u_{0n}, then they fulfill

$$\mathcal{E}_n(S) - \mathcal{E}_n(T) = \int_S^T \langle Bu_n(t), u_n(t)\rangle \, dt.$$

Since the left-hand side tends to $\mathcal{E}(S)-\mathcal{E}(T)$ (because $u_n \to u$ in $C(\mathbb{R}_+, \mathcal{H})$), the right-hand side admits also a limit that we denote by $\int_S^T \langle Bu(t), u(t)\rangle dt$. This is the so-called hidden regularity of u.

3 – Exponential stability

In this section we find a necessary and sufficient condition which guarantees the exponential stability of (9). This condition is the validity of a stabilility estimate that will be checked in some particular cases in Section 6. We closely follow the arguments of the beginning of Section 3 of [36] given in the case of Mawxell's system and that can be easily extended to our abstract setting. The proofs are nevertheless given for the sake of completeness.

In the whole section we suppose that (3), (5), (7) and (8) hold.

We start with the following definition.

Definition 3.1. We say that the pair (A_1, B) satisfies the stabilility estimate if there exist $T > 0$ and two non negative constants C_1, C_2 (which may depend on T) with $C_1 < T$ such that

$$\int_0^T \mathcal{E}(t)\, dt \leq C_1 \mathcal{E}(0) + C_2 \int_0^T \langle Bu(t), u(t)\rangle\, dt, \tag{13}$$

for all solution u of (9).

That property admits the following equivalent formulation:

Lemma 3.2. *The pair (A_1, B) satisfies the stabilility estimate if and only if there exist $T > 0$ and a positive constant C (which may depend on T) such that*

$$\mathcal{E}(T) \leq C \int_0^T \langle Bu(t), u(t)\rangle\, dt, \tag{14}$$

for all solution u of (9).

Proof.
$\Rightarrow$: Since $\mathcal{E}(t)$ is non-increasing, the estimate (13) implies that

$$T\mathcal{E}(T) \leq C_1\mathcal{E}(0) + C_2 \int_0^T \langle Bu(t), u(t)\rangle \, dt.$$

By Lemma 2.1 we get

$$T\mathcal{E}(T) \leq C_1\mathcal{E}(T) + (C_1 + C_2) \int_0^T \langle Bu(t), u(t)\rangle \, dt.$$

This yields (14) with $C = \frac{C_1+C_2}{T-C_1}$.
$\Leftarrow$: From the monotonicity of $\mathcal{E}$ we may write

$$\int_0^T \mathcal{E}(t)\, dt \leq T\mathcal{E}(0).$$

Again Lemma 2.1 yields

$$\int_0^T \mathcal{E}(t)\, dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}\left(\mathcal{E}(T) + \int_0^T \langle Bu(t), u(t)\rangle \, dt\right).$$

Using the assumption (14) we obtain

$$\int_0^T \mathcal{E}(t)\, dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}(1+C) \int_0^T \langle Bu(t), u(t)\rangle \, dt,$$

which is nothing else than (13). $\square$

Examples of pairs (A_1, B) satisfying the stabilility estimate may be found in Section 6 below (see also Section 3 of [36]).

We now show that the stabilility estimate is equivalent to the exponential stability of (9).

Theorem 3.3. *The pair (A_1, B) satisfies the stabilility estimate if and only if there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{15}$$

for all solution u of (9).

Proof. Assume that the stabilility estimate holds, i.e., by the previous Lemma, (14) equivalently holds. The identity (11) of Lemma 2.1 then yields

$$\mathcal{E}(T) \leq C(\mathcal{E}(0) - \mathcal{E}(T)).$$

This estimate is equivalent to

$$\mathcal{E}(T) \leq \gamma \mathcal{E}(0),$$

with $\gamma = \frac{C}{1+C}$ which is < 1.

Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, \cdots$ (which is valid since our system is invariant by a translation in time), we will get

$$\mathcal{E}(mT) \leq \gamma \mathcal{E}((m-1)T) \leq \cdots \leq \gamma^m \mathcal{E}(0), m = 1, 2, \cdots$$

Therefore we have

$$\mathcal{E}(mT) \leq e^{-\omega m T} \mathcal{E}(0), m = 1, 2, \cdots$$

with $\omega = \frac{1}{T} \ln \frac{1}{\gamma} > 0$. For an arbitrary positive t, there exists $m = 1, 2, \cdots$ such that $(m-1)T < t \leq mT$ and by the nonincreasing property of $\mathcal{E}$, we conclude

$$\mathcal{E}(t) \leq \mathcal{E}((m-1)T) \leq e^{-\omega(m-1)T} \mathcal{E}(0) \leq \frac{1}{\gamma} e^{-\omega t} \mathcal{E}(0).$$

Let us now show the converse implication: from Lemma 2.1, for any $T > 0$, we may write

$$\int_0^T \langle Bu(t), u(t) \rangle \, dt = \mathcal{E}(0) - \mathcal{E}(T).$$

With the help of (15), we get

$$\int_0^T \langle Bu(t), u(t) \rangle \, dt \geq \mathcal{E}(0)(1 - M e^{-\omega T}). \tag{16}$$

The exponential decay (15) also implies

$$\int_0^T \mathcal{E}(t) dt \leq M \mathcal{E}(0) \frac{1 - e^{-\omega T}}{\omega}.$$

Consequently for all $C_1 > 0$, we may write

$$\int_0^T \mathcal{E}(t)dt \le C_1\mathcal{E}(0) + \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)\mathcal{E}(0). \tag{17}$$

Choosing T large enough so that $1-Me^{-\omega T}>0$ and $C_1<\min\{\frac{M(1-e^{-\omega T})}{\omega}, T\}$, (16) and (17) yield (13) with

$$C_2 = \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)(1-Me^{-\omega T})^{-1}. \qquad \square$$

4 – Exact controllability results

Using the results of the previous section and Russell's principle we deduce exact controllability results for the evolution equation associated with the operator $-A_1$ with controls in $L^2(]0,T[;U)$, the control space U being a given real Hilbert space such that $\mathcal{V}$ is continuously embedded into U. We then denote by I_U the embedding from $\mathcal{V}$ into U and $\mathcal{I}_U$ the mapping identifying U as a subspace of $\mathcal{V}'$, i.e.,

$$\langle \mathcal{I}_U u, v\rangle := (I_U u, I_U v)_U, \forall u, v \in \mathcal{V}.$$

The exact controllability problem may be formulated as follows: for all $u_0 \in \mathcal{H}$, we are looking for a time $T > 0$ and a control $J \in L^2(]0,T[;U)$ such that the solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t} - A_1 u = J \text{ in } \mathcal{V}', t \ge 0, \\ u(0) = u_0, \end{cases} \tag{18}$$

satisfies

$$u(T) = 0. \tag{19}$$

Theorem 4.1. *If the assumptions* (3) *to* (8) *hold for the pair* $(A_1, \mathcal{I}_U)$ *and if the pair* $(A_1, \mathcal{I}_U)$ *satisfies the stabilility estimate, then for* $T > 0$ *sufficiently large, for all* $u_0 \in \mathcal{H}$ *there exist a control* $J \in L^2(]0,T[;U)$ *such that the solution* $u \in C([0,T],\mathcal{H})$ *of* (18) *is at rest a time* T, *i.e., satisfies* (19).

Proof. For concrete problems the proof is quite standard. We adapt it to our abstract setting as follows. For further purposes we prefer to solve the inverse problem (so that the asumption "$(A_1, \mathcal{I}_U)$ satisfies the stabilility estimate" is replaced by "$(-A_1, \mathcal{I}_U)$ satisfies the stabilility estimate"): Given $p_0 \in \mathcal{H}$, we are looking for $K \in L^2(]0,T[;U)$ such that the solution $p \in C([0,T],\mathcal{H})$ of

$$\begin{cases} \dfrac{\partial p}{\partial t} + A_1 p = K \ \text{ in } \ \mathcal{V}', t \geq 0, \\ p(T) = p_0, \end{cases} \tag{20}$$

satisfies

$$p(0) = 0. \tag{21}$$

Indeed if the above problem has a solution the conclusion follows by setting

$$u(t) = -p(T-t).$$

We solve problem (20) and (21), using a backward and an inward system with linear boundary feedbacks $\mathcal{I}_U$: First given f_0 in $\mathcal{H}$, we consider $f \in C([0,T],\mathcal{H})$ the unique solution of

$$\begin{cases} \dfrac{\partial f}{\partial t} + A_1 f - \mathcal{I}_U f = 0 \text{ in } \mathcal{H}, t \geq 0, \\ f(T) = f_0. \end{cases} \tag{22}$$

Its existence following from Lemma 2.1 by setting $\tilde{u}(t) = f(T-t)$. Moreover applying Theorem 3.3 to $\tilde{u}(t)$ we get

$$\mathcal{E}(f(t)) \leq M e^{-\omega(T-t)} \mathcal{E}(f_0). \tag{23}$$

Second we consider $g \in C([0,T],\mathcal{H})$ the unique solution of (whose existence and uniqueness still follow from Lemma 2.1)

$$\begin{cases} \dfrac{\partial g}{\partial t} + A_1 g + \mathcal{I}_U g = 0 \text{ in } \mathcal{H}, t \geq 0, \\ g(0) = f(0). \end{cases} \tag{24}$$

We now take $p = g - f$. From (22) and (24), p satisfies (20) with

$$K = -\mathcal{I}_U g - \mathcal{I}_U f. \tag{25}$$

Let us further consider the mapping Λ from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\Lambda(f_0) = g(T).$$

We show that for $T > 0$ such that $d := Me^{-\omega T} < 1$, the mapping $\Lambda - I$ is invertible by proving that $\|\Lambda\|_{\updownarrow L(\mathcal{H},\mathcal{H})} = \sqrt{d}$. Indeed using successively the definition of Λ, Lemma 2.1, the initial condition of problem (24) and the estimate (23) we have

$$\begin{aligned}\|\Lambda f_0\|^2_{\mathcal{H}} = 2\mathcal{E}(g(T)) \leq 2\mathcal{E}(g(0)) \leq \\ \leq 2\mathcal{E}(f(0)) \leq 2Me^{-\omega T}\mathcal{E}(f_0) = d\|f_0\|^2_{\mathcal{H}}.\end{aligned}$$

Since $\Lambda - I$ is invertible for any $p_0 \in \mathcal{H}$, there exists a unique $f_0 \in \mathcal{H}$ such that

$$p_0 = p(T) = g(T) - f(T) = (\Lambda - I)f_0. \tag{26}$$

The proof will be complete if we can show that $K \in L^2(]0,T[;U)$. For that purpose, we remark that Lemma 2.1 (identity (11) applied to $\tilde{u}$ and g which has a meaning thanks to the hidden regularity) yields

$$\begin{aligned}\mathcal{E}(f(T)) - \mathcal{E}(f(0)) &= \int_0^T \|I_U f(t)\|^2_U \, dt,\\ \mathcal{E}(g(0)) - \mathcal{E}(g(T)) &= \int_0^T \|I_U g(t)\|^2_U \, dt.\end{aligned}$$

Summing these two identities and using the initial condition of problem (24), the final condition of (22) and the definition of Λ, we obtain

$$\int_0^T (\|I_U f(t)\|^2_U + \|I_U g(t)\|^2_U)\, dt = \mathcal{E}(f(T)) - \mathcal{E}(g(T)) \leq \frac{1}{2}\|f_0\|^2_{\mathcal{H}}.$$

Using the identity (26) and the boundedness of $(I-\Lambda)^{-1}$ we finally arrive at the estimate

$$\int_0^T(\|I_U f(t)\|^2_U + \|I_U g(t)\|^2_U)dt \leq \frac{1}{2}\|(I-\Lambda)^{-1}p_0\|^2_{\mathcal{H}} \leq \frac{1}{2(1-\sqrt{d})^2}\|p_0\|^2_{\mathcal{H}}. \tag{27}$$

This proves that K given by (25) belongs to $L^2(]0,T[;U)$. $\square$

Remark 4.2. Thanks to the assumptions (5) and (6) the (weak) solution $p \in C([0,T];\mathcal{H})$ of (20) and (21) can be approximated (in$C([0,T];\mathcal{H})$) by a sequence $p_\epsilon \in W^{1,\infty}(\mathbb{R}_+,\mathcal{H}) \cap L^\infty(\mathbb{R}_+,\mathcal{V})$, $\epsilon > 0$, of (strong) solution of (20) with $K_\epsilon \in L^2(]0,T[;U)$ and $p_{0\epsilon} \in \mathcal{V}$ such that

$$K_\epsilon \to K \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0, \tag{28}$$

$$I_U p_\epsilon \to I_U p \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0. \tag{29}$$

Indeed as $f_0 = (\Lambda - I)^{-1} p_0$, by (5), there exists $f_{0\epsilon} \in D(\mathcal{A})$ such that

$$\|f_0 - f_{0\epsilon}\|_{\mathcal{H}} \leq \epsilon. \tag{30}$$

Consider f_ϵ the strong solution of (22) with final datum $f_{0\epsilon}$. By the dissipativeness of the energy, we get

$$\|f(t) - f_\epsilon(t)\|_{\mathcal{H}} \leq \|f_0 - f_{0\epsilon}\|_{\mathcal{H}} \leq \epsilon, \forall t \in [0,T]. \tag{31}$$

Similarly since $f_\epsilon(0)$ belongs to $\mathcal{H}$, by (6), there exists $g_{0\epsilon} \in D(\mathcal{A}^-)$ such that

$$\|g_{0\epsilon} - f_\epsilon(0)\|_{\mathcal{H}} \leq \epsilon. \tag{32}$$

We then consider g_ϵ the strong solution of (24) with initial datum $g_{0\epsilon}$. The dissipativeness of the energy yields

$$\begin{aligned}\|g(t) - g_\epsilon(t)\|_{\mathcal{H}} &\leq \|g(0) - g_{0\epsilon}\|_{\mathcal{H}} \leq \\ &\leq \|f(0) - f_\epsilon(0)\|_{\mathcal{H}} + \|f_\epsilon(0) - g_{0\epsilon}\|_{\mathcal{H}} \leq 2\epsilon, \forall t \in [0,T],\end{aligned}$$

by (31) and (32).

The estimates (31) and (33) show that $p_\epsilon := g_\epsilon - f_\epsilon$ tends to $p = g - f$ in $C([0,T];\mathcal{H})$ as ϵ goes to 0. Finally by Lemma 2.1 we may write

$$\int_0^T \|I_U(f(t) - f_\epsilon(t))\|_U^2 \, dt \leq 2\|f_0 - f_{0\epsilon}\|_{\mathcal{H}}^2$$
$$\int_0^T \|I_U(g(t) - g_\epsilon(t))\|_U^2 \, dt \leq 2\|g(0) - g_\epsilon(0)\|_{\mathcal{H}}^2.$$

These two estimates, the estimates (30), (33) and the definitions of $K_\epsilon := -\mathcal{I}_U g_\epsilon - \mathcal{I}_U f_\epsilon$, of p_ϵ, K and p lead to the properties (28) and (29). ∎

5 – Stability in the nonlinear case

Here we use Liu's principle [28] and an integral inequality from [7] to deduce decay rates of the energy using appropriate nonlinear feedbacks. In view of the examples below we assume that the control space U is of the form

$$U = \prod_{j=1}^{J} U_j, \tag{34}$$

where for all $j = 1, \cdots, J \in \mathbb{N}^\star := \mathbb{N} \setminus \{0\}$, U_j is a closed subspace of $L^2(X_j, \mu_j)^{N_j}$, when $(X_j, \updownarrow A_j, \mu_j)$ is a measure space such that $\mu_j(X_j) < \infty$ and $N_j \in \mathbb{N}^\star$. For all $j = 1, \cdots, J$, we suppose given a mapping $g_j : \mathbb{R}^{N_j} \to \mathbb{R}^{N_j}$ such that

$$(g_j(x) - g_j(y)) \cdot (x - y) \geq 0, \forall x, y \in \mathbb{R}^{N_j} \text{ (monotonicity)}, \tag{35}$$

$$g_j(0) = 0, \tag{36}$$

$$|g_j(x)| \leq M(1 + |x|), \forall x \in \mathbb{R}^3, \tag{37}$$

for some positive constant M. We finally suppose that B is given by

$$\langle Bu, v\rangle = \sum_{j=1}^{J} \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j), \tag{38}$$

where we recall that I_U is the embeding from $\mathcal{V}$ to U and therefore $(I_U u)_j$ is the j^{th} component of $I_U u$.

Remark that the conditions (35) and (36) guarantee the assumption (8) on B, while (37) guarantees that B is well defined. In most examples these conditions guarantee that the assumptions (3) and (4) hold (see Section 6 for some illustrations). We further remark that these conditions always hold for $g_j(x) = x$, corresponding to linear controls, i.e., $B = \mathcal{I}_U$.

We now recall the integral inequality obtained in [7] (compare with Theorem 9.1 of [22] or its extension by P. Martinez [31], [32]).

THEOREM 5.1. *Let $\mathcal{E} : [0, +\infty) \to [0, +\infty)$ be a non-increasing mapping satisfying*

$$\int_S^\infty \phi(\mathcal{E}(t))\, dt \leq T\mathcal{E}(S), \forall S \geq 0, \tag{39}$$

for some $T > 0$ *and some strictly increasing convex mapping* ϕ *from* $[0, +\infty)$ *to* $[0, +\infty)$ *such that* $\phi(0) = 0$. *Then there exist* $t_1 > 0$ *and* c_1 *depending on* T *and* $\mathcal{E}(0)$ *such that*

$$\mathcal{E}(t) \leq \phi^{-1}\left(\frac{\psi^{-1}(c_1 t)}{c_1 T t}\right), \forall t \geq t_1, \tag{40}$$

where ψ *is defined by*

$$\psi(t) = \int_t^1 \frac{1}{\phi(s)}\, ds, \forall t > 0. \tag{41}$$

Remark 5.2. Theorem 5.1 yields exactly the same decay rate as in Theorem 9.1 of [22] when $\phi(t) = t^{1+\alpha}$ for some $\alpha > 0$ (case leading to polynomial decay). Note furthermore that the integral inequality of P. Martinez [31], [32] is different from our integral inequality but gives similar asymptotic behaviour for the energy. $\square$

We now give the consequence of this result to our system (9).

Theorem 5.3. *Assume that the assumptions* (3) *to* (8) *hold for the pairs* (A_1, B) *and* $(A_1, \mathcal{I}_U)$. *Let* g_j, $j = 1, \cdots, J$ *satisfy* (35) *to* (37) *as well as*

$$g_j(x) \cdot x \geq m|x|^2, \forall x \in \mathbb{R}^{N_j} : |x| \geq 1, \tag{42}$$

$$|x|^2 + |g_j(x)|^2 \leq G(g_j(x) \cdot x), \forall x \in \mathbb{R}^{N_j} : |x| \leq 1, \tag{43}$$

for some positive constant m *and a concave strictly increasing function* $G : [0, \infty) \to [0, \infty)$ *such that* $G(0) = 0$. *If the pair* $(-A_1, \mathcal{I}_U)$ *satisfies the stabilility estimate, then there exist* $c_2, c_3 > 0$ *and* $T_1 > 0$ *(depending on* T, $\mathcal{E}(0)$, $\mu_j(X_j)$, $j = 1, \cdots, J$*) such that*

$$\mathcal{E}(t) \leq c_3 G\left(\frac{\psi^{-1}(c_2 t)}{c_2 T t}\right), \forall t \geq T_1, \tag{44}$$

for all solution u *of* (9), *where* ψ *is given by* (41) *for* ϕ *defined by*

$$\phi(s) = T\mu G^{-1}\left(\frac{s}{c_3}\right), \tag{45}$$

where $\mu = \min_{j=1,\cdots,J} \mu_j(X_j)$.

PROOF. By the density of $D(\mathcal{A})$ into $\mathcal{H}$, it suffices to prove (44) for data in $D(\mathcal{A})$. In that case let u be the (strong) solution of (9) and consider p the solution of problem (20) and (21) with $p_0 = u(T) \in D(\mathcal{A})$ with $T > 0$ sufficiently large (whose existence was established in Theorem 4.1). Consider further a sequence p_ϵ of strong solution of (20) with final data $p_{0\epsilon}$ tending to p in $C([0,T],\mathcal{H})$ as ϵ goes to zero and satisfying (28) and (29) (see Remark 4.2).

By (9) and (20) we may write

$$\langle \partial_t u + A_1 u + Bu, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} + \langle \partial_t p_\epsilon + A_1 p_\epsilon - K_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}} = 0.$$

This may be written equivalently

$$\begin{aligned}(\partial_t u, p_\epsilon)_{\mathcal{H}} + (\partial_t p_\epsilon, u)_{\mathcal{H}} + \langle A_1 u, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} + \langle A_1 p_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}} + \\ + \langle Bu, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}} = 0\end{aligned}$$

As the assumption (7) yields

$$\langle A_1 u, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} + \langle A_1 p_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}} = 0,$$

the above identity reduces to

$$(\partial_t u, p_\epsilon)_{\mathcal{H}} + (\partial_t p_\epsilon, u)_{\mathcal{H}} + \langle Bu, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}} = 0$$

Integrating this identity for $t \in (0,T)$, we get

$$(u(T), p_\epsilon(T))_{\mathcal{H}} - (u(0), p_\epsilon(0))_{\mathcal{H}} + \int_0^T (\langle Bu, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, u\rangle_{\mathcal{V}',\mathcal{V}})\, dt = 0.$$

By the definitions of K_ϵ and B we arrive at

$$\begin{aligned}(u(T), p_\epsilon(T))_{\mathcal{H}} - (u(0), p_\epsilon(0))_{\mathcal{H}} = \int_0^T \Bigg((K_\epsilon, I_U u)_U + \\ - \sum_{j=1}^{J} \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U p_\epsilon)_j(x_j)\, d\mu_j(x_j) \Bigg)\, dt\end{aligned}$$

Passing to the limit in ϵ and using the initial and final conditions on p, we have obtained

$$2\mathcal{E}(T)=\int_0^T\left((K,I_Uu)_U-\sum_{j=1}^J\int_{X_j}g_j((I_Uu)_j(x_j))\cdot(I_Up)_j(x_j)\,d\mu_j(x_j)\right)dt$$

Cauchy-Schwarz's inequality leads finally to

$$\begin{aligned}2\mathcal{E}(T)&\leq\|K\|_{L^2(0,T;U)}\|I_Uu\|_{L^2(0,T;U)}+\\&+\|I_Up\|_{L^2(0,T;U)}\left(\sum_{j=1}^J\int_0^T\int_{X_j}|g_j((I_Uu)_j(x_j))|^2\,d\mu_j(x_j)dt\right)^{1/2}.\end{aligned}\tag{46}$$

Let us remark that the estimate (27) and the final conditions on p yield

$$\int_0^T(\|I_Uf(t)\|_U^2+\|I_Ug(t)\|_U^2)\,dt\leq\frac{1}{(1-\sqrt{d})^2}\mathcal{E}(T).$$

This estimate, the definition of K and $p=g-f$ lead to

$$\int_0^T\|K(t)\|_U^2\,dt\leq\frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T)$$
$$\int_0^T\|I_Up(t)\|_U^2\,dt\leq\frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T).$$

Inserting these estimates in (46) we arrive at

$$\begin{aligned}\mathcal{E}(T)&\leq\frac{1}{(1-\sqrt{d})^2}\times\\&\times\left(\sum_{j=1}^J\int_0^T\int_{X_j}\{|(I_Uu)_j(x_j)|^2+|g_j((I_Uu)_j(x_j))|^2\}\,d\mu_j(x_j)dt\right).\end{aligned}\tag{47}$$

We now estimate the right-hand side of (47) as follows: For all $j=1,\cdots,J$ introduce

$$\Sigma_j^+=\{(x,t)\in X_j\times(0,T)||(I_Uu)_j(x,t)|>1\},$$
$$\Sigma_j^-=\{(x,t)\in X_j\times(0,T)||(I_Uu)_j(x,t)|\leq 1\}.$$

Let us split up

$$\int_0^T \int_{X_j} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt = I_j^+ + I_j^-,$$

where

$$\begin{aligned} I_j^+ &:= \int_{\Sigma_j^+} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt, \\ I_j^- &:= \int_{\Sigma_j^-} \{|(I_U u)_j(x_j)|^2 + |g_j((I_U u)_j(x_j))|^2\} \, d\mu_j(x_j) \, dt. \end{aligned}$$

The assumptions (42) and (37) lead to

$$I_j^+ \le c_4 \int_{\Sigma_j^+} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) \, d\mu_j(x_j) dt,$$

for some positive constant c_4 (depending on m and M). By (11) and the property

$$g_j(x) \cdot x \ge 0, \forall x \in \mathbb{R}^{N_j}, \tag{48}$$

following from (35) and (36) we arrive at

$$I_j^+ \le c_4(\mathcal{E}(0) - \mathcal{E}(T)). \tag{49}$$

Similarly by the assumption (43) and the monotonicity of G we have

$$\begin{aligned} I_j^- &\le \int_{\Sigma_j^-} G((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))) \, d\mu_j(x_j) dt \le \\ &\le \int_0^T \int_{X_j} G((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))) \, d\mu_j(x_j) dt. \end{aligned}$$

Jensen's inequality then yields

$$I_j^- \le T\mu_j(X_j) G\left(\frac{1}{T\mu_j(X_j)} \int_0^T \int_{X_j} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) d\mu_j(x_j) dt \right).$$

By (11), we arrive at

$$I_j^- \leq T\mu_j(X_j) G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu_j(X_j)}\right). \tag{50}$$

The estimates (49) and (50) into the estimate (47) and the monotonicity of G give

$$\mathcal{E}(T) \leq c_5 \left\{\mathcal{E}(0)-\mathcal{E}(T) + G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right)\right\},$$

for some positive constant c_5 (depending on T and $\max_j \mu_j(X_j)$), where we recall that $\mu = \min_j \mu_j(X_j)$. This finally leads to

$$\mathcal{E}(0)=\mathcal{E}(0)-\mathcal{E}(T)+\mathcal{E}(T)\leq \max\{1,c_5\}\left\{(\mathcal{E}(0)-\mathcal{E}(T))+G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right)\right\}.$$

As $\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu} \leq \frac{\mathcal{E}(0)}{T\mu}$, the concavity of G yields a constant c_6 (depending continuously on T, $\mathcal{E}(0)$ and μ) such that

$$\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu} \leq c_6 G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right).$$

These two estimates lead to

$$\mathcal{E}(0) \leq c_3 G\left(\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T\mu}\right),$$

for some $c_3 > 0$ (depending on T, $\mathcal{E}(0)$, $\max_j \mu_j(X_j)$, and $\min_j \mu_j(X_j)$).

Using this argument in $[t, t+T]$ instead of $[0, T]$ we have shown that

$$\mathcal{E}(t) \leq c_3 G\left(\frac{\mathcal{E}(t)-\mathcal{E}(t+T)}{T\mu}\right) = \phi^{-1}(\mathcal{E}(t)-\mathcal{E}(t+T)), \forall t \geq 0, \tag{51}$$

when we recall that ϕ was defined by (45).

We conclude by Theorem 5.1 since Lemma 5.1 of [7] shows that the estimate (51) guarantees that $\mathcal{E}$ actually satisfies (39). $\square$

The assumption (42) forbids the use of bounded functions g_j which could be a drawback for some applications. Our next purpose is to obtain a variant of the above result when some mappings g_j do not satisfy (42) adapting the arguments of Theorem 9.10 of [22]. The price to pay is to assume some regularity results for elements of $D(\mathcal{A})$.

THEOREM 5.4. *Assume that the assumptions* (3) *to* (8) *hold for the pairs* (A_1, B) *and* $(A_1, \mathcal{I}_U)$. *Let* g_j, $j = 1, \cdots, J$ *satisfy* (35) *to* (37) *as well as* (43) *for some concave strictly increasing function* $G : [0,\infty) \to [0,\infty)$ *such that* $G(0) = 0$. *Assume further that* $J = J_1 \cup J_2$ *with* $J_1 \cap J_2 = \emptyset$, *that for all* $j \in J_1$, g_j *satisfies* (42) *and there exists* $c_7 > 0$ *and* $\alpha > 2$ *such that for all* $j \in J_2$ *and all* $u \in D(\mathcal{A})$, $(I_U u)_j$ *belongs to* $L^\alpha(X_j, \mu_j)$ *with the estimate*

$$\left(\int_{X_j} |(I_U u)_j(x_j)|^\alpha \, d\mu_j(x_j)\right)^{1/\alpha} \leq c_7 \|u\|_{D(\mathcal{A})}, \tag{52}$$

where we recall that $\|u\|_{D(\mathcal{A})} = \|\mathcal{A}u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}$. *If the pair* $(-A_1, \mathcal{I}_U)$ *satisfies the stabilility estimate, then for every* $u_0 \in D(\mathcal{A})$, *the solution* u *of* (9) *satisfies*

$$\mathcal{E}(t) \leq c_3 G_1\left(\frac{\psi_1^{-1}(c_2 t)}{c_2 T t}\right), \forall t \geq T_1, \tag{53}$$

for some $c_2, c_3 > 0$ *and* $T_1 > 0$ *(depending on* T, $\mathcal{E}(0)$, $\mu_j(X_j)$, $j = 1, \cdots, J$, α *and* $\|u_0\|_{D(\mathcal{A})}$*), where* ψ_1 *is given by* (41) *for* ϕ_1 *defined by* (45) *with* G_1 *instead of* G, *the function* G_1 *being defined by*

$$G_1(x) = G(x) + x^s, \forall x \geq 0,$$

with $s = \frac{\alpha-2}{\alpha-1} \in (0,1)$.

PROOF. We repeat the proof of Theorem 5.3 except for the estimation of I_j^+ when $j \in J_2$, where we now obtain the following estimation: First by (37) we remark that

$$I_j^+ \leq (1 + 4M^2) J_j^+. \tag{54}$$

where

$$J_j^+ := \int_{\Sigma_j^+} |(I_U u)_j(x_j)|^2 \, d\mu_j(x_j).$$

So it remains to estimate J_j^+. For that estimation we remark that the assumption (43) yields

$$g_j(x) \cdot x \geq m_j |x|, \forall x \in \mathbb{R}^{N_j} : |x| \geq 1, \tag{55}$$

for some positive constant m_j. Indeed we notice that (43) and the property $G(0) = 0$ directly imply that

$$g_j(\xi) \cdot \xi > 0, \forall |\xi| = 1.$$

Denoting by $m_j = \min_{|\xi|=1}(g_j(\xi) \cdot \xi)$ we have already proved (55) for $|x| = 1$. For $|x| > 1$ let $\xi = x/|x|$, then by the monotonicity of g_j we have

$$(g_j(x) - g_j(\xi)) \cdot (|x| - 1)\xi \geq 0,$$

which implies

$$g_j(x) \cdot \xi \geq g_j(\xi) \cdot \xi \geq m_j.$$

Multiplying this inequality by $|x|$, we arrive at (55).

Now using (55) we may write

$$J_j^+ \leq m_j^{-s} \int_{\Sigma_j^+} |(I_U u)_j(x_j)|^{2-s} ((I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j))^s \, d\mu_j(x_j).$$

By Hölder's inequality we get

$$\begin{aligned} J_j^+ \leq m_j^{-s} &\left(\int_{\Sigma_j^+} |(I_U u)_j(x_j)|^{\frac{2-s}{1-s}} \, d\mu_j(x_j) \right)^{1-s} \times \\ &\times \left(\int_{\Sigma_j^+} (I_U u)_j(x_j) \cdot g_j((I_U u)_j(x_j)) \, d\mu_j(x_j) \right)^s . \end{aligned}$$

By (11) and the assumption (52) (since $\alpha = \frac{2-s}{1-s}$) we conclude that

$$J_j^+ \leq c_8(\mathcal{E}(0) - \mathcal{E}(T))^s, \tag{56}$$

where $c_8 > 0$ depends on T, α and $\|u_0\|_{D(\mathcal{A})}$ (since Komura-Kato's theorem (see for instance Proposition IV.3.1 of [42] and Lemma 2.1 guarantee that $\|u(t)\|_{D(\mathcal{A})} \leq \|u_0\|_{D(\mathcal{A})}$).

As before the estimates (50), (54) and (56) into the estimate (47) and the monotonicity of G give

$$\mathcal{E}(T) \leq c_9 \left\{ \mathcal{E}(0) - \mathcal{E}(T) + G\left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T\mu}\right) + (\mathcal{E}(0) - \mathcal{E}(T))^s \right\},$$

for some positive constant c_9 depending on T, $\mu_j(X_j)$, $j = 1, \cdots, J$, α and $\|u_0\|_{D(\mathcal{A})}$. The concavity of G and of the mapping $x \to x^s$ yields

$$\mathcal{E}(0) \leq c_3 G_1 \left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T\mu}\right).$$

The conclusion follows as previously. □

Remark 5.5. In (42) (resp. (43)) the proviso $|x| \geq 1$ (resp. $|x| \leq 1$) may be replaced by $|x| \geq \eta$ (resp. $|x| \leq \eta$), for some $\eta > 0$ without changing the conclusion of Theorem 5.3 or Theorem 5.4. □

Examples of functions g_j leading to an explicit decay rate (44) or (53) are given in [7]. Let us give the following illustrations.

Example 5.6. Suppose that g_j satisfies (35) to (37) and (42) as well as

$$x \cdot g_j(x) \geq c_0 |x|^{p+1}, \ |g_j(x)| \leq C_0 |x|^{\alpha}, \forall |x| \leq 1, \tag{57}$$

for some positive constants c_0, C_0, $\alpha \in (0,1]$ and $p \geq \alpha$. Then g_j satisfies (43) with $G(x) = x^{\frac{2}{q+1}}$ and $q = \frac{p+1}{\alpha} - 1$ (which is ≥ 1). If $p = \alpha = 1$ (then $q = 1$) and under the other assumptions of Theorem 5.3 we get an exponential decay (since $\psi^{-1}(t) = e^{-t}$). On the contrary if $p + 1 > 2\alpha$ then we get the decay $t^{-\frac{2\alpha}{p+1-2\alpha}}$ (since $\psi^{-1}(t) = t^{\frac{2}{1-q}}$). A function g satisfying all these assumptions is given by

$$g(x) = \begin{cases} |x|^{\alpha-1}x & \text{if } |x| \leq 1, \\ x & \text{if } |x| \geq 1, \end{cases}$$

for some $\alpha \in (0,1]$. In that case (57) holds for $p = \alpha$.

In the setting of Theorem 5.4 it suffices to take g_j satisfying (35) to (37) and (57) to get the decay rate $t^{-\frac{2}{q'-1}}$ with $q' = \min\{q, \frac{2}{s} - 1\}$. Such a g is given by

$$g(x) = \begin{cases} |x|^{\alpha-1}x & \text{if } |x| \leq 1, \\ \dfrac{x}{|x|} & \text{if } |x| \geq 1, \end{cases}$$

for some $\alpha \in (0, 1]$, which satisfies (57) for $p = \alpha$.

Example 5.6 (Logarithmic decay). Take $g_j(\xi) = \exp(-\frac{1}{|\xi|^{2p_j}})\frac{\xi}{|\xi|^2}$ for $|\xi|$ small enough and for $p_j > 0$. Then by Example 2.4 of [7] (43) holds with

$$G(x) = \frac{C}{|\log x|^{\frac{1}{p}}}$$

and $p = \max_j p_j$ and some constant $C > 0$. In the setting of Theorem 5.3 or Theorem 5.4 we will get the decay

$$\mathcal{E}(t) \leq \frac{C}{|\log t|^{\frac{1}{p}}},$$

since ψ^{-1} is bounded from below.

Example 5.8 (Log-Log decay). Take $g_j(\xi) = \exp(-\exp(1/|\xi|^{2p}))\frac{\xi}{|\xi|^2}$ for $|\xi|$ small enough and for $p > 0$. Then by Example 2.5 of [7] (43) holds with

$$G(x) = \frac{C}{|\log|\log x||^{\frac{1}{p}}}$$

and some constant $C > 0$. In the setting of Theorem 5.3 or Theorem 5.4 we will get the decay

$$\mathcal{E}(t) \leq \frac{C}{|\log|\log t||^{\frac{1}{p}}}.$$

Note that combinations of the above examples give rise to the worse decay rate.

6 – Examples

6.1 – Second order evolution equations

Some examples given below enter in the following framework: Let H and V be two real separable Hilbert spaces such that V is densely and continuously embedded into H. Define the linear operator A_2 from V into V' by

$$\langle A_2 u, v\rangle_{V'-V} = (u, v)_V, \forall u, v \in V, \tag{58}$$

and suppose given a (nonlinear) mapping B_2 from V into V'.

Consider now the second order evolution equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + A_2 u + B_2 \dfrac{\partial u}{\partial t} = 0 \text{ in } V', t \geq 0, \\ u(0) = u_0, \dfrac{\partial u}{\partial t}(0) = u_1. \end{cases}$$

This system is reduced to the first order system (9) using the standard argument of reduction of order: setting $\mathcal{H} = V \times H$, $\mathcal{V} = V \times V$ with natural inner products,

$$x = (u, z),$$

with $z = \frac{\partial u}{\partial t}$ (from now on we use the letter x for generic elements of $\mathcal{H}$ since the letter u is already used in (59) as usual) and introducing the operators

$$A_1 x = (-z, A_2 u), Bx = (0, B_2 z).$$

Under appropriate assumptions on B_2, we can prove the

Theorem 6.1. *If B_2 is monotone, hemicontinuous, bounded and satisfies $B_2 0 = 0$, then the assumptions (3) to (8) hold for the pair (A_1, B).*

Proof. In the above setting we see that

$$D(\mathcal{A}^{\pm}) = \{x = (u, z) \in \mathcal{V} | \pm A_2 u + B_2 z \in H\}.$$

To check the assumptions (3) and (4), from the definitions of A_1, A_2 and the inner product in $\mathcal{H}$ we easily verify that

$$(\mathcal{A}^{\pm}(u, z) - \mathcal{A}^{\pm}(u', z'), (u, z) - (u', z'))_{\mathcal{H}} = \langle B_2 z - B_2 z', z - z'\rangle_{V'-V}.$$

The monotonicity of $\mathcal{A}^{\pm}$ then follows from the same property on B_2.

Let us pass to the maximality of $\mathcal{A}^{\pm}$: for all $(f,g) \in \mathcal{H}$ we are looking for $(u,z) \in D(\mathcal{A}^{\pm})$ such that

$$u \mp z = f \text{ in } V,$$
$$z \pm A_2 u + B_2 z = g \text{ in } H.$$

The first identity is equivalent to

$$u = \pm z + f \text{ in } V,$$

and eliminating u in the second identity we obtain

$$z + A_2 z + B_2 z = g \mp f \text{ in } V'.$$

The solvability of this problem is equivalent to the surjectivity of the operator

$$A : V \to V' : z \to z + A_2 z + B_2 z.$$

For that purpose we make use of Corollary 2.2 of [42] which proves that A is surjective if A is monotone, hemicontinuous, bounded and coercive. The first three properties easily follows from the same property of B_2. The coercivity also easily follows from the fact that

$$\langle Az, z\rangle_{V'-V} = \|z\|_H^2 + \|z\|_V^2 + \langle B_2 z, z\rangle_{V'-V} \geq \|z\|_V^2,$$

this last inequality following from the property $\langle B_2 z, z\rangle_{V'-V} \geq 0$ consequence of the monotonicity of B_2 and the property $B_2 0 = 0$.

The assumptions (5) and (6) are reduced to the density of $D(\mathcal{A})$ since we easily check that $(u,z) \in D(\mathcal{A})$ if and only if $(-u,z) \in D(\mathcal{A}^{-})$. Let us now fix (u,z) in $\mathcal{H}$, then let $\tilde{u} \in V$ be the unique solution of

$$A_2 \tilde{u} = -B_2 z,$$

whose existence follows from Lax-Milgram's lemma. Applying Theorem III.2.B of [41] there exists a sequence of $u_n \in D(\mathcal{A}_2)$ such that

$$u_n \to u - \tilde{u} \text{ in } V, \text{ as } n \to \infty,$$

where $\mathcal{A}_2$ is the Friedrichs extension of A_2. We conclude by remarking that $(\tilde{u}+u_n, z)$ belongs to $D(\mathcal{A})$ and tends to (u, z) in $\mathcal{H}$.

The assumption (7) follows from the identity

$$\langle A_1 x, x\rangle = -(z, u)_V + \langle A_2 u, z\rangle_{V'-V},$$

and the definition of A_2. Finally the assumption (8) follows from the identity

$$\langle Bx, x\rangle = \langle B_2 z, z\rangle_{V'-V},$$

and the positiveness of B_2. ∎

In view of this theorem the assumptions (3) to (8) are reduced to the verification of the above properties of B_2 that we now check for different systems.

In the rest of the section Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a Lipschitz boundary Γ. Some restrictions will be specified later on when they will be necessary. We further denote by ν the unit outward normal vector along Γ.

6.2 – Nonlinear stabilization of the wave equation

Consider the wave equation

$$\text{(60)}\qquad \begin{cases} \partial_t^2 u - \Delta u + f(\partial_t u) = 0 \text{ in } Q := \Omega\times]0,+\infty[, \\ u = 0 \text{ on } \Sigma_0 := \Gamma_0\times]0,+\infty[, \\ \partial_\nu u + au + g(\partial_t u) = 0 \text{ on } \Sigma_1 := \Gamma_1\times]0,+\infty[, \\ u(0) = u_0, \partial_t u(0) = u_1 \text{ in } \Omega, \end{cases}$$

where Γ_0 is a open subset of Γ and $\Gamma_1 = \Gamma \setminus \bar{\Gamma}_0$ is the remainder. The functions f and g are two nondecreasing continuous functions from $\mathbb{R}$ into itself such that $f(0) = g(0) = 0$ and finally a is a nonnegative real number. For the sake of simplicity we suppose that

$$\text{(61)}\qquad \text{either } \Gamma_0 \text{ is not empty or } a > 0,$$

and that

$$\text{(62)}\qquad \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset.$$

The stability of this problem was extensively studied in the litterature, let us cite the papers [18], [19], [20], [23], [22], [43], [26], [10] and the references cited there. Both papers are restricted to some particular choices of Γ_0, a, f and g leading to some exponential or polynomial decay rates of the energy of the solution of (60). In [25], [29], [31], [32], [33], [34], some arbitrary decay rates are obtained for different f and g (even with degenerate or local dissipations). Using the results of the previous sections, we also obtain arbitrary decay rates for a large class of f and g.

The first point is that problem (60) enters in the framework of problem (59) from Subsection 6.1 once we take:

$$\begin{aligned}
H &= L^2(\Omega),\\
V &= \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\},\\
(u, v)_V &= \int_\Omega \nabla u \cdot \nabla v \, dx + a \int_{\Gamma_1} u \cdot v \, d\sigma,\\
\langle B_2 u, v\rangle_{V'-V} &= \int_\Omega f(u) v \, dx + \int_{\Gamma_1} g(u) v \, d\sigma, \forall u, v \in V.
\end{aligned}$$

Let us remark that the assumption (61) implies that the inner product $(\cdot, \cdot)_V$ induces a norm on V equivalent to the usual one. In order to give a meaning to B_2 we simply require

$$|f(x)| \leq C(1 + |x|^\alpha), \forall x \in \mathbb{R}, \tag{63}$$

$$|g(x)| \leq C(1 + |x|^\beta), \forall x \in \mathbb{R}, \tag{64}$$

for some positive constant C, where $\alpha = \frac{n+2}{n-2}$ and $\beta = \frac{n}{n-2}$ if $n \geq 3$ and $\alpha, \beta \geq 1$ if $n = 2$.

Now we readily check that these assumptions guarantee that B_2 fulfils all the assumptions of Theorem 6.1. Consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). In order to deduce stability results for our system (60) we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate (note that we just check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the assumptions (3) to (8)), where the control space U is clearly defined by

$$U = L^2(\Omega) \times L^2(\Gamma_1).$$

This stability estimate was proved in Theorem 1.2 of [10] under the assumption that there exists $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu > 0 \text{ on } \Gamma_1, m \cdot \nu \leq 0 \text{ on } \Gamma_0, \tag{65}$$

$$\frac{1}{R^2} \max\{n-2, n/3\} \leq a(m \cdot \nu) < \frac{n}{R^2} \text{ on } \Gamma_1, \tag{66}$$

where as usual m is the standard multiplier defined by

$$m(x) = x - x_0, \forall x \in \mathbb{R}^n,$$

and $R = \max_{x \in \Omega} |m(x)|$. Under these assumptions, appropriated conditions on f and g lead to exponential, polynomial, logarithmic or other decays. Note that bounded feedbacks are allowed since $D(\mathcal{A}) \hookrightarrow H^1(\Omega) \times H^1(\Omega) \hookrightarrow L^\alpha(\Omega) \times L^\alpha(\Gamma_1)$, for some $\alpha > 2$ consequently Theorem 5.4 may be applied.

For $f = 0$ or $g = 0$ similar results hold (changing the control space U) with less restrictions on Γ_0 and Γ_1, using the exponential decay with linear feedbacks established in [18], [19], [20], [23], [22], [43], [26].

6.3 – Nonlinear stabilization of the elastodynamic system

With the notation of the above subsection, we consider the following elastodynamic system:

$$\begin{cases} \partial_t^2 u - \nabla \sigma(u) + F(\partial_t u) = 0 \text{ in } Q, \\ u = 0 \text{ on } \Sigma_0, \\ \sigma(u) \cdot \nu + au + G(\partial_t u) = 0 \text{ on } \Sigma_1, \\ u(0) = u_0, \partial_t u(0) = u_1 \text{ in } \Omega. \end{cases} \tag{67}$$

As usual $u(x,t)$ is the displacement field at the point $x \in \Omega$ at time t and $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^3$ is the stress tensor given by (here and in the sequel we shall use the summation convention for repeated indices)

$$\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u),$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))_{i,j=1}^3$ is the strain tensor given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the tensor $(a_{ijkl})_{i,j,k,l=1,2,3}$ is made of $W^{1,\infty}(\Omega)$ entries such that

$$a_{ijkl} = a_{jikl} = a_{klij},$$

and satisfying the ellipticity condition

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij},$$

for every symmetric tensor (ε_{ij}) and some $\alpha > 0$. Hereabove and below $\nabla\sigma(u)$ is the vector field defined by

$$\nabla\sigma(u) = (\partial_j\sigma_{ij}(u))_{i=1}^3.$$

The mappings F and G from $\mathbb{R}^n$ into itself satisfy the assumptions (35) to (37). Finally a is a nonnegative real number.

As before we suppose that (61) and (62) hold, but here we further assume that

$$F = 0 \text{ or } G = 0. \tag{68}$$

This last assumption means that we stabilizate our system either by boundary feedback or by internal feedback.

The stability of the system (67) was considered in [11], [13], [15], [1], [4] under some particular hypotheses on Γ_0, Γ_1, a, F and G leading to exponential or polynomial decay of the energy of the solution of (67).

As in the above subsection problem (67) may be expressed in the form (59) from Subsection 6.1 with the choices:

$$\begin{aligned}
H &= L^2(\Omega)^n,\\
V &= \{v \in H^1(\Omega)^n | v = 0 \text{ on } \Gamma_0\},\\
(u,v)_V &= \int_\Omega \nabla u \cdot \nabla v\,dx + a\int_{\Gamma_1} u\cdot v\,d\sigma,\\
\langle B_2u, v\rangle_{V'-V} &= \int_\Omega F(u)\cdot v\,dx + \int_{\Gamma_1} G(u)\cdot v\,d\sigma, \forall u, v \in V.
\end{aligned}$$

The assumptions made on F and G imply that B_2 fulfils the assumptions of Theorem 6.1, consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). For the stability results we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate, where the control space U is defined by

$$U = L^2(\Gamma_1)^n \text{ if } F = 0,$$
$$U = L^2(\Omega)^n \text{ if } G = 0.$$

In the first case the stability estimate was proved in [4] under the assumption (65) (a similar estimate was proved in [11], [1] under stronger assumptions on Γ_0 and Γ_1). If the tensor (a_{ijkl}) corresponds to the Lamé system, then the stability estimate was proved in Lemma 3.2 of [15] under the weaker assumption

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0.$$

In the second case (i.e. $G=0$), the stability estimate for the pair $(-A_1, \mathcal{I}_U)$ was proved in Lemma 3.6 of [13].

As in the previous subsection, these conditions (on Γ_0, Γ_1 and the coefficients (a_{ijkl})) and appropriated conditions on F and G lead to exponential, polynomial, logarithmic or other decays. Bounded feedbacks are also allowed due to the embedding $H^1(\Omega) \times H^1(\Omega) \hookrightarrow L^\alpha(\Omega) \times L^\alpha(\Gamma_1)$, for some $\alpha > 2$.

6.4 – Nonlinear stabilization of a coupled system

We consider the following coupled system in a bounded domain Ω with a C^4-boundary:

$$\left\{ \begin{array}{l} \partial_t^2 u_1 + \Delta^2 u_1 + a u_2 + g_1(\partial_t u_1, \partial_t u_2) = 0 \text{ in } Q, \\ \partial_t^2 u_2 - \Delta u_2 + a u_1 + g_2(\partial_t u_1, \partial_t u_2) = 0 \text{ in } Q, \\ u_1 = \partial_\nu u_1 = u_2 = 0 \text{ on } \Sigma = \Gamma \times]0, \infty[, \\ u_i(0) = u_{0i}, \partial_t u_i(0) = u_{1i} \text{ in } \Omega, i = 1, 2. \end{array} \right. \tag{69}$$

Here g_i are mappings from $\mathbb{R}^2$ into $\mathbb{R}$ such that the mapping G from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$G(x, y) = (g_1(x, y), g_2(x, y)),$$

satisfies the assumptions (35) to (37). Finally a is a scalar function that we assume to be in $L^\infty(\Omega)$.

The above system was considered in [14] when g_1 (resp. g_2) only depends on $\partial_t u_1$ (resp. $\partial_t u_2$). In that case this author proves exponential or polynomial decay rates under appropriated conditions on a, g_1 and g_2. Let us notice that if $a = 0$ and if g_1 (resp. g_2) only depends on $\partial_t u_1$ (resp. $\partial_t u_2$), then the above system is splitted up into the wave equation considered in Subsection 6.2 and the standard Petrovsky system studied in [12]. Our subsequent analysis then covers the analysis of this last Petrovsky system.

First problem (69) is in the form (59) with the definitions (see [14]):

$$\begin{aligned}
H &= L^2(\Omega)^2,\\
V &= H_0^2(\Omega) \times H_0^1(\Omega),\\
((u_1, u_2), (v_1, v_2))_V &= \int_\Omega (\Delta u_1 \Delta u_2 + \nabla u_2 \cdot \nabla v_2)\, dx +\\
&\quad + \int_\Omega a\,(u_1 v_2 + u_2 v_1)\, d\sigma,\\
\langle B_2(u_1, u_2), (v_1, v_2)\rangle_{V'-V} &= \int_\Omega (g_1(u_1, u_2) v_1 + g_2(u_1, u_2) v_2)\, dx,\\
&\quad \forall (u_1, u_2), (v_1, v_2) \in V.
\end{aligned}$$

The assumptions made on g_1 and g_2 imply that B_2 fulfils the assumptions of Theorem 6.1, consequently the corresponding pair (A_1, B) satisfies the assumptions (3) to (8). For the stability results we need to check that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate when the control space U is given by $U = L^2(\Omega)^2$. This stability estimate was proved in Lemma 3.1 of [14] under the assumption

$$\|a\|_{L^\infty(\Omega)} < \frac{1}{c' c"},$$

where $c', c" > 0$ are the constants appearing in the above Poincaré type inequalities:

$$\|u\|^2_{H^2(\Omega)} \le c' \int_\Omega (\Delta u)^2\, dx, \forall u \in H_0^2(\Omega),$$

$$\|u\|^2_{H^1(\Omega)} \le c" \int_\Omega |\nabla u|^2\, dx, \forall u \in H_0^1(\Omega).$$

This condition and appropriated conditions on g_1 and g_2 lead to exponential, polynomial, logarithmic or other decays. As before bounded feedbacks are also allowed.

6.5 – Nonlinear stabilization of Maxwell's equations

We consider Maxwell's equations in $\Omega \subset \mathbb{R}^3$ with a smooth boundary and a nonlinear internal feedback:

$$\tag{70} \begin{cases} \varepsilon \dfrac{\partial E}{\partial t} - \mathbf{curl}\, H + g(E) = 0 \text{ in } Q := \Gamma \times]0, +\infty[, \\ \mu \dfrac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ E \times \nu = 0, H \cdot \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\ E(0) = E_0, H(0) = H_0 \text{ in } \Omega. \end{cases}$$

As usual ε and μ are real, positive functions of class $C^\infty(\bar{\Omega})$. The function g from $\mathbb{R}^3$ into itself is assumed to satisfy the properties (35) to (37).

The stability of this system was studied in [39] with a linear feedback $g(E) = \sigma E$, with $\sigma \geq 0$. In particular the exponential decay was shown in that paper if $\sigma \geq \sigma_0 > 0$.

Contrary to the above examples this system is not a second order system but (compare with [7]) it enters in the setting of (9) once we set

$$\begin{aligned} \mathcal{H} &= L^2(\Omega)^3 \times \hat{J}(\Omega, \mu), \\ \hat{J}(\Omega, \mu) &= \{H \in L^2(\Omega)^3 : \operatorname{div}(\mu H) = 0 \text{ in } \Omega, H \cdot \nu = 0 \text{ on } \Gamma\}, \\ ((E, H), (E', H'))_{\mathcal{H}} &= \int_\Omega (\epsilon E \cdot E' + \mu H \cdot H')\, dx, \\ \mathcal{V} &= V \times \hat{J}(\Omega, \mu), \\ V &= \{E \in L^2(\Omega)^3 : \mathbf{curl}\, E \in L^2(\Omega)^3, E \times \nu = 0 \text{ on } \Gamma\}, \\ \langle A_1(E, H), (E', H') \rangle &= \int_\Omega (\mathbf{curl}\, E \cdot H' - H \cdot \mathbf{curl}\, E')\, dx, \\ \langle B(E, H), (E', H') \rangle &= \int_\Omega g(E \times \nu) \cdot (E' \times \nu)\, d\sigma. \end{aligned}$$

One readily checks (as in [7, Section 3]) that the assumptions (3) and (4) hold since the bilinear form

$$\int_\Omega (\mu^{-1}\mathbf{curl}\, E \cdot \mathbf{curl}\, E' + \epsilon E \cdot E')\, dx$$

is clearly coercive on V. Moreover Lemma 2.3 of [35] implies that (5) and (6) hold. Finally from the definition of A_1 (7) clearly holds, while from the definition of B and the properties (35) and (36) satified by g, (8) holds. As the results of Section 5 of [39] imply that the pair $(-A_1, \mathcal{I}_U)$ satisfies the stability estimate when the control space U is given by $U = L^2(\Omega)^3$, we may conclude exponential, polynomial, logarithmic or other decays under appropriated conditions on g. Here bounded feedbacks are not allowed since V is not embedded into $L^\alpha(\Omega)^3$ for some $\alpha > 2$.

Let us finally notice that Maxwell's equations with a nonlinear boundary feedback

$$\begin{cases} \varepsilon \dfrac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q := \Gamma \times]0, +\infty[, \\ \mu \dfrac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ H \times \nu + g(E \times \nu) \times \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\ E(0) = E_0, H(0) = H_0 \text{ in } \Omega, \end{cases} \tag{71}$$

was studied in [3], [21], [39], [7], [36]. Different decay rates are avalaible under different conditions on ϵ, μ and Γ and appropriated assumptions on g. It was shown in [7] that (71) enters in the setting of (9), where the assumptions (3) and (5) are also checked under some conditions on Ω, ϵ and μ (similar arguments actually imply that (4) and (6) hold as well). The stability analysis following the point of view of our paper is given in [36]. We then refer to that paper for the details.

REFERENCES

[1] F. Alabau – V. Komornik: *Boundary observability, controllability and stabilization of linear elastodynamic systems*, SIAM J. Control Optim., **37** (1999), 521-542.

[2] C. Amrouche – C. Bernardi – M. Dauge – V. Girault: *Vector potentials in three-dimensional nonsmooth domains*, Math. Meth. Applied Sc., **21** (1998), 823-864.

[3] H. Barucq – B. Hanouzet: *Étude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller*, II, C. R. Acad. Sci. Paris, Série I, **316** (1993), 1019-1024.

[4] R. Bey – A. Heminna – J.-P. Lohéac: *A new approach for boundary stabilisation of the linear elastodynamic system*, Revista Matematica de la Universidad Complutense de Madrid, submitted.

[5] F. Conrad – B. Rao: *Decay of solutions of wave equations in a star-shaped domain with nonlinear feedback*, Asymptotic Analysis, **7** (1993), 159-177.

[6] G. Duvaut – J.-L. Lions: *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[7] M. Eller – J. E. Lagnese – S. Nicaise: *Decay rates for solutions of a Maxwell system with nonlinear boundary damping*, Comp. and Apppl. Math., **21** (2002), 135-165.

[8] V. Girault – P.-A. Raviart: *Finite element methods for Navier-Stokes equations*, Springer Series in Comp. Math., **5** Springer-Verlag, 1986.

[9] P. Grisvard: *Elliptic problems in nonsmooth Domains*, Monographs and Studies in Mathematics, **21** Pitman, Boston, 1985.

[10] A. Guesmia: *On the nonlinear stabilization of the wave equation*, Annales Polonici Mathematici, **68** (1998), 191-198.

[11] A. Guesmia: *Existence globale et stabilisation interne non linéaire d'un système d'élasticité*, Portugaliae Mathematica, **55** (1998), 333-347.

[12] A. Guesmia: *Existence globale et stabilisation interne non linéaire d'un système de Petrovsky*, Bull. Belg. Math. Soc. S. Stevin, **5** (1998), 583-594.

[13] A. Guesmia: *Existence globale et stabilisation frontière non linéaire d'un système d'élasticité*, Portugaliae Mathematica, **56** (1999), 361-379.

[14] A. Guesmia: *Energy decay for a damped nonlinear coupled system*, J. Math. Analysis Appl., **239** (1999), 38-48.

[15] M. A. Horn: *Implications of sharp trace regularity results on boundary stabilization of the system of linear elasticity*, J. Math. Analysis Appl., **223** (1998), 126-150.

[16] B. V. Kapitanov: *Stabilization and exact boundary controllability for Maxwell's equations*, SIAM J. Control Optim., **32** (1994), 408-420.

[17] B. V. Kapitanov – M. A. Raupp: *Exact boundary controllability in problems of transmission for the system of electromagneto-elasticity*, Math. Meth. Appl. Sci., **24** (2001), 193-207.

[18] V. Komornik: *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim., **29** (1991), 197-208.

[19] V. Komornik: *Decay estimates for the wave equation with internal damping*, in: Proc. Conf. Control Theory, Vorau, 1993, Int. Ser. Numer. Anal., **118** (1994), 253-266.

[20] V. Komornik: *On the nonlinear boundary stabilization of the wave equation*, Chinese Ann. Math. Ser. B, **14** (1993), 153-164.

[21] V. Komornik: *Boundary stabilization, observation and control of Maxwell's equations*, PanAm. Math. J., **4** (1994), 47-61.

[22] V. Komornik: *Exact controllability and stabilization, the multiplier method*, **RMA 36**, Masson, Paris, 1994.

[23] V. Komornik – E. Zuazua: *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl., **69** (1990), 33-54.

[24] J. E. Lagnese: *Exact controllability of Maxwell's equations in a general region*, SIAM J. Control Optim., **27** (1989), 374-388.

[25] I. Lasiecka – D. Tataru: *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, J. Diff. Integr. Eq., **6** (1993), 507-533.

[26] I. Lasiecka – R. Triggiani: *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim., **25** (1992), 189-224.

[27] J.-L. Lions: *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, tome 1, **RMA 8** Masson, Paris, 1988.

[28] W. J. Liu: *Partial exact controllability and exponential stability of higher dimensional linear thermoelasticity*, ESAIM: COCV, **3** (1998), 23-48.

[29] W. J. Liu – E. Zuazua: *Decay rates for dissipative wave equations*, Ric. Mat., **48** Suppl. (1999), 61-75.

[30] P. Martinez: *Uniform stabilization of the wave equation in almost star-shaped domains*, SIAM J. Control and Opt., **37** (1999), 673-694.

[31] P. Martinez: *A new method to obtain decay rate estimates for dissipative systems*, ESAIM: COCV, **4** (1999), 419-444.

[32] P. Martinez: *A new method to obtain decay rate estimates for dissipative systems with localized damping*, Rev. Mat. Complut., **12** (1999), 251-283.

[33] P. Martinez: *Decay of solutions of the wave equation with a local highly degenerate dissipation*, Asymptot. Anal., **19** (1999), 1-17.

[34] P. Martinez: *Precise decay rate estimates for time-dependent dissipative systems*, Israel J. Math., **119** (2000), 291-324.

[35] S. Nicaise: *Exact boundary controllability of Maxwell's equations in heteregeneous media and an application to an inverse source problem*, SIAM J. Control and Opt., **38** (2000), 1145-1170.

[36] S. Nicaise – M. Eller – J. E. Lagnese: *Stabilization of heteregeneous Maxwell's equations by nonlinear boundary feedbacks*, EJDE, **21** (2002), 1-26.

[37] S. Nicaise: *Stability and controllability of the electromagneto-elastic system*, Portugaliae Mathematica, **60** (2003), 37-70.

[38] C. Pignotti: *Observavility and controllability of Maxwell's equations*, Rendiconti di Mat., **19** (1999), 523-546.

[39] K. D. Phung: *Contrôle et stabilisation d'ondes électromagnétiques*, ESAIM: COCV, **5** (2000), 87-137.

[40] D. L. Russell: *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region*, SIAM J. Control Optim., **24** (1986), 199-229.

[41] R. E. Showalter: *Hilbert space methods for partial differential equations*, Math. and Studies in Math., **1** Pitman, 1977.

[42] R. E. Showalter: *Monotone operators in Banach space and nonlinear partial differential equations*, Math. Surveys and Monographs, **49** AMS, 1997.

[43] E. Zuazua: *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM J. Control Optim., **28** (1990), 466-477.

Lavoro pervenuto alla redazione il 23 maggio 2002
ed accettato per la pubblicazione il 16 gennaio 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo dell'Autore:

S. Nicaise – Université de Valenciennes et du Hainaut Cambrésis – MACS – Institut des Sciences et Techniques de Valenciennes – F-59313 – Valenciennes Cedex 9 (France)
E-mail: snicaise@univ-valenciennes.fr

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 117-129

Asymptotic behavior of convolution powers of a probability measure on harmonic extensions of *H*-type groups

F. TOLLI

Abstract: *We give a local (central) limit theorem and a renewal theorem for radial probability measures on AN-groups.*

– Introduction

Solvable extensions of H-type groups have been objects of intensive studies in recent years, since the discovery made by E. Damek and F.Ricci [4] of a counterexample to the Lichnerowicz conjecture. Indeed, after E. Damek and F. Ricci have shown [5] that, despite the lack of symmetry, it is possible to develop on these groups a harmonic analysis similar to the one developed by Harish-Chandra for semisimple groups, several authors have investigated the possibility to extend to these groups analogous results known for rank one symmetric spaces: multipliers problems [2], Paley-Wiener theorems [6], asymptotic behavior of the heat kernel [1], to mention just a few. In this paper we follow the mainstream, but with a more probabilistic flavor. We first recall two well known results for

Key Words and Phrases: *Local limit theorem – AN-groups.*
A.M.S. Classification: 43A80 – 60BIS

the convolution powers of a nonarithmetic probability measure μ on $\mathbb{R}$. The local (central) limit theorem states that if μ has mean zero and variance σ then the sequence $n^{1/2}\mu^{*n}$ converges weakly to a multiple, which depends only on σ, of the Lebesgue measure on $\mathbb{R}$. The renewal theorem deals with the potential $U(f)(x) = \sum_{n=0}^{+\infty} \mu^{*n} * f(x)$, where f is a continuous function with compact support. We have that $\lim_{x\to-\infty} Uf(x)$ is different from zero if and only if μ has finite positive mean and the mass of f is different from zero; moreover, if this is the case, the value of the limit is the reciprocal of the mean of μ multiplied by the mass of f.

Both these results have been extended to symmetric spaces by Bougerol in [3] and in this paper we will show how Bougerol's method can be easily adapted to the setting of harmonic extensions of H-type groups. In particular we consider a radial probability measure on an AN-group with support not concentrated at the origin and we prove that the sequence $\rho^{-n}n^{3/2}\mu^{*n}$, where $0 < \rho < 1$, converges weakly to a multiple of the spherical function ϕ_0. In particular, for any compact set K of the origin $\mu^{*n}(K)$ decays exponentially. We should recall that the local asymptotic behavior of the convolution powers of a probability measure on any (amenable) connected Lie group has been determined by N. Th. Varopoulos [10]. According to Varopoulos' classification our AN-groups are in the category of NC groups and if μ is a *symmetric* (i.e. $\mu(A) = \mu(A^{-1})$ for every measurable set A) probability measure on such groups then $\mu^{*n}(K) \approx n^{-3/2}$; thus we do not have an exponential decay as for the the radial measures. To clarify the reason of this difference consider the case when the measure μ has a density f. Then μ is symmetric if and only if $f(x^{-1}) = f(x)m(x)$ a.e., where m denotes the modular function. Since radial densities are symmetric in the usual sense, i.e. $f(x^{-1}) = f(x)$, and the modular function is trivial only at the origin, we have that probability measures associated with radial densities are not symmetric.

– Preliminaries

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra endowed with a scalar product $\langle\cdot,\cdot\rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. Let $J_Z : \mathfrak{p} \to \mathfrak{z}$ the linear map defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \quad (X, Y \in \mathfrak{p}; Z \in \mathfrak{z}).$$

Then $\mathfrak{n}$ is a Heisenberg algebra if

$$J_Z^2 = -|Z|^2 I \quad \forall\, Z \in \mathfrak{z},$$

and the corresponding simply connected group N is called of Heisenberg-type or simply H-type group. If $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{p}$ we have that m is always even so that $Q = \frac{m}{2} + k$ is a positive integer called the homogeneous dimension of N. Identifying the group with its Lie algebra via the exponential map we have that the product on N is given by

$$(X, Z)(X', Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right).$$

The semidirect product $G = N \rtimes \mathbb{R}_+$ defined by

$$(X, Z, a)(X', Z', a) = \left(X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X, X'], aa'\right)$$

is a solvable Lie group with Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{z} + \mathbb{R}$. It is equipped with the left invariant Riemannian metric induced by the scalar product

$$\langle (X, Z, l), (X, Z', l') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + ll'$$

on $\mathfrak{g}$. The associated left Haar measure is given by

$$d^L g = dg = a^Q dX dZ \frac{da}{a}$$

while the right Haar measure is given by

$$d^R g = a dX dZ da$$

so that the group is not unimodular. We recall that if $S = KAN$ is the Iwasawa decomposition of semisimple connected Lie groups of real rank one, then the solvable group $AN = NA$ is an example of a harmonic extension of an H-type group.

If $g = an, a \in \mathbb{R}_+, n \in N$ we denote by $a(g)$ the element a and by $r(g) = d(g, e)$ the geodesic distance of g from the identity e. Furthermore we denote by $S_r = \{g \in G : d(g, e) = r\}$ the geodesic sphere of radius r.

A function is said to be radial if it depends only on the geodesic distance or equivalently if it is constant on every geodesic sphere. The space of the continuous (resp. smooth) radial functions with compact support is denoted by $C_c(G)^{\#}$ (resp. $C_c^{+\infty}(G)^{\#}$). A (probability) measure is said to be radial if $\chi_r * \mu = \mu * \chi_r = \mu$,where, for every $r > 0$, χ_r denotes the normalized surface measure induced on S_r by the Haar measure dg. Obviously if μ has a density f we have that μ is radial if and only if f is a radial function. The spherical functions are the radial eigenfunctions of the Laplace-Beltrami operator Δ on G, normalized at the origin. They are real analytic, since Δ is elliptic, and have the following properties [5]:

- all the spherical functions are of the form
$$\phi_z(g) = \phi_z(r(g)) = \int_{S_r} a(y)^{Q/2-z} d\chi_r(y), \quad z \in \mathbb{C};$$
- $\phi_z(r) = \phi_{-z}(r)$;
- $\phi_z(r)$ are holomorphic function of z uniformly bounded in z and r for $-\frac{Q}{2} \leq \Re(z) \leq \frac{Q}{2}$.

The Fourier transform of a radial measure μ is defined as

$$\mathcal{F}\mu(z) = \int_G \phi_z(g) d\mu(g)$$

and obviously the Fourier transform of a radial function is defined as the Fourier transform of the associated measure. If $f \in C_c^{+\infty}(G)^{\#}$ then its Fourier transform $\mathcal{F}f$ is a symmetric entire function that decays exponentially on every vertical line. Moreover the following inversion formula holds true [9], [1]:

$$f(r) = \frac{2^{k-3}\Gamma\left(\dfrac{m+k+1}{2}\right)}{\pi^{(m+k+3)/2}} \int_{\mathbb{R}} \mathcal{F}f(is)\phi_{is}(r)|c(is)|^{-2} ds$$

where c denotes the Harish-Chandra function i.e.

$$c(z) = \frac{2^{Q-2z}\Gamma(2z)\Gamma\left(\dfrac{m+k+1}{2}\right)}{\Gamma\left(\dfrac{Q+2z}{2}\right)\Gamma\left(\dfrac{m+4z+2}{4}\right)}, \quad z \in \mathbb{C}.$$

In the following we will denote by C_1 the constant in front of the integral in the inversion formula.

– The Fourier transform of a measure

Lemma 1. *Let μ be a nonsingular radial probability measure on G. Then the Fourier Transform $\mathcal{F}\mu$ has the following properties*:

1. *$\mathcal{F}\mu(t+is)$ is continuous in the strip $S = \{t+is \in \mathbb{C} : s \in \mathbb{R}, -\frac{Q}{2} \le t \le \frac{Q}{2}\}$ and holomorphic in its interior;*
2. $|\mathcal{F}\mu(t+is)| < \mathcal{F}\mu(t),\ s \neq 0, -\frac{Q}{2} \le t \le \frac{Q}{2}$ *and* $\mathcal{F}\mu(t) < \mathcal{F}\mu(\frac{Q}{2}) = 1,\quad -\frac{Q}{2} < t < \frac{Q}{2}$;
3. $\limsup\limits_{s\to\infty} |\mathcal{F}\mu(t+is)| < \mathcal{F}\mu(t), \quad -\frac{Q}{2} \le t \le \frac{Q}{2}$.

Proof.

1) The spherical functions $\phi_z(g)$ are holomorphic functions of $z \in \mathbb{C}$ that are uniformly bounded in the strip S. This on the one hand implies that the Fourier transform of μ is continuous on S and on the other hand, by Cauchy's formula, that also the derivatives of $\phi_z(g)$ are uniformly bounded in any substrip $-\frac{Q}{2}+\epsilon \le \Re(z) \le \frac{Q}{2}-\epsilon$. Thus the integral $\int_G |\frac{d^l}{dz}\phi_z(g)| d\mu(g)$ is convergent and this guarantees that the function $\mathcal{F}\mu(z)$ is smooth in the interior of S and that $\frac{d^l}{dz}\mathcal{F}\mu(z) = \int_G \frac{d^l}{dz}\phi_z(g) d\mu$.
2) We will first show that analogous inequalities hold for the spherical functions. This has been proved in [5], but for us it is essential to check that the inequalities are strict. If $s \in \mathbb{R}\backslash\{0\},\ -\frac{Q}{2} < t < \frac{Q}{2}$ and $|g| = r \neq 0$,

$$
\begin{aligned}
|\phi_{t+is}(g)| &= \left| \int_{S_r} a(y)^{\frac{Q}{2}-t-is} d_{\chi_r}(y) \right| < \\
&< \int_{S_r} a(y)^{\frac{Q}{2}-t} d_{\chi_r}(y) < \\
&< \left(\int_{S_r} a(y)^Q d_{\chi_r}(y) \right)^{\frac{Q/2-t}{Q}} = \\
&= \phi_{-\frac{Q}{2}}(g)^{\frac{Q/2-t}{Q}} = \phi_{\frac{Q}{2}}(g)^{\frac{Q/2-t}{Q}} = 1,
\end{aligned}
$$

where the first inequality follows from the passage of the absolute value under the integral and the second one by Jensen's inequality. They are strict because the function $a(y)$ is not constant on S_r, for $r>0$ and the first inequality also holds for $t=\pm\frac{Q}{2}$. Then 2) follows from the fact that the support of μ is not concentrated at the origin.

3) This is an immediate consequence of the analogous property of the spherical functions which, in turn, follows from the classical Riemann-Lebesgue lemma. ∎

LEMMA 2. *Let μ be as in the previous lemma and consider the function of real variable $h(s)=\mathcal{F}\mu(is)$. Then*

1. *The first derivative of h vanishes at zero*;
2. *The second derivative of h at zero is strictly negative.*

PROOF. The first statement follows from the symmetry of the spherical functions, namely $\phi_{is}(g)=\phi_{-is}(g)$. By the proof of the previous lemma we have

$$\frac{d^2}{ds}h(0)=\int_G \frac{d^2}{ds}a(y)^{\frac{Q}{2}-is}\bigg|_{s=0} d\mu(y)=-\int_G \ln(a(y))^2 a(y)^{\frac{Q}{2}}d\mu(y)$$

which is clearly nonpositive. It is equal to zero if and only if $a(y)=1$ μ a.e. and this is not the case since the support of μ is not concentrated at the identity. ∎

LEMMA 3. *The Harish-Chandra function has the following properties*:

1. *For $s\in\mathbb{R}$ we have*

$$\lim_{n\to+\infty} n|c(is/\sqrt{n})|^{-2}=4s^2\left|\frac{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{m+2}{4}\right)}{\Gamma\left(\frac{m+k+1}{2}\right)2^Q}\right|^2=4s^2C_2;$$

2. *$c(z)^{-1}$ is holomorphic in the region $S_Q=\{z\in\mathbb{C}:\Re(z)>-\frac{Q}{2}\}$ and there exists k such that*

$$|c(z)|^{-1}\le k(1+|z|)^k \quad z\in S_Q.$$

Proof. Both statements are immediate consequence of the definition of c and well known properties of the Gamma function. For instance 1 follows from the fact the $\Gamma(z)$ is holomorphic and different from zero for $\Re(z) > 0$ and that $\frac{1}{\Gamma(2is)} \approx 2is$ for s in a neighborhood of the origin. $\square$

– The local limit theorem

Theorem 1. *Let μ be a nonsingular radial probability measure on G and f a continuous function with compact support. Then*

$$\lim_{n\to+\infty} n^{3/2}\mathcal{F}\mu(0)^{-n}\int_G f(g)d\mu^{*n}(g) = C\int_G f(g)\phi_0(g)dg$$

where $C = C_1C_2\int_{\mathbb{R}} s^2\exp(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)})ds$.

Proof. Let $\mathcal{R}$ be the operator of radialization [4], then $\int_G f(g)d\mu^{*n} = \int_G \mathcal{R}f(g)d\mu^{*n}(g)$, so we can suppose that f is radial. Since $C_c^\infty(G)^\#$ is dense in $C_c(G)^\#$ [6], we can suppose that f is also smooth. Then, by the Paley-Wiener theorem, the function f has an integrable Fourier transform and thus, by the Fourier inversion formula, we have

$$\begin{aligned}\mathcal{F}\mu(0)^{-n}n^{3/2}\int_G f(g)d\mu^{*n}(g) =&\mathcal{F}\mu(0)^{-n}n^{3/2}f*\mu^{*n}(e) =\\ &=C_1\mathcal{F}\mu(0)^{-n}n^{3/2}\int_{\mathbb{R}}\mathcal{F}f(is)\mathcal{F}\mu(is)^n|c(is)|^{-2}ds.\end{aligned}$$

Notice that for any positive η

$$\lim_{n\to+\infty} C_1n^{3/2}\int_{|s|>\eta}\left(\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}\right)^n\mathcal{F}f(is)|c(is)|^{-2}ds = 0$$

since, by Lemma 1, there exists $0 < \epsilon < 1$ such that $|\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}|^n \le \epsilon^n$, for $|s| \ge \eta$ and the integrand is uniformly dominated by the integrable function $C|\mathcal{F}f(is)(1+|s|^k)|$. Lemma 2 and Taylor's formula give

$$\mathcal{F}\mu(is) = \mathcal{F}\mu(0) + \frac{1}{2}\frac{d^2}{ds^2}\mathcal{F}\mu(0)s^2 + o(s^2),\quad |s| \le \eta,$$

which implies that for $|s| \leq \eta\sqrt{n}$, $|\mathcal{F}\mu(is/\sqrt{n})| \leq \exp(-cs^2/n)$ and

$$\lim_{n\to+\infty}\left(\frac{\mathcal{F}\mu(is/\sqrt{n})}{\mathcal{F}\mu(0)}\right)^n = \lim_{n\to+\infty}\left(1+\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)n}\right)^n = \exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right).$$

Performing a change of variable $s \to s/\sqrt{n}$ and taking in account Lemma 3 we have

$$\lim_{n\to+\infty} C_1 n^{3/2}\int_{|s|\leq\eta}\left(\frac{\mathcal{F}\mu(is)}{\mathcal{F}\mu(0)}\right)^n \mathcal{F}f(is)|c(is)|^{-2}ds =$$

$$= \lim_{n\to+\infty} C_1\int_{|s|\leq\eta\sqrt{n}}\left(\frac{\mathcal{F}\mu\left(\frac{is}{\sqrt{n}}\right)}{\mathcal{F}\mu(0)}\right)^n \mathcal{F}f\left(\frac{is}{\sqrt{n}}\right) n\left|c\left(\frac{is}{\sqrt{n}}\right)\right|^{-2} ds =$$

$$= C_1C_2\int_{\mathbb{R}}\exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right)s^2\mathcal{F}f(0)ds =$$

$$= C_1C_2\int_{\mathbb{R}}\exp\left(\frac{\mathcal{F}\mu''(0)s^2}{2\mathcal{F}\mu(0)}\right)s^2ds\int_G f(g)\phi_0(g)dg. \qquad \square$$

– The renewal theorem

We first recall the classical renewal theorem for the potential of a probability measure on $\mathbb{R}$. Let μ be a nonarithmetic noncentered probability measure on $\mathbb{R}$, define the potential measure $\gamma = \sum_{n=0}^{\infty}\mu^{*n}$ and set $U(f)(x) = \gamma * f(x)$ for $f \in C_c(\mathbb{R})$. Notice that if $\hat{\mu}$ denotes the Euclidean Fourier transform, and f is also smooth, we have, using the Fourier inversion formula, that

$$(1)\qquad U(f)(x) = \sum\mu^{*n} * f(x) = \frac{1}{2\pi}\lim_{b\uparrow 1}\int_{\mathbb{R}}\frac{1}{1-b\hat{\mu}(\xi)}\hat{f}(\xi)e^{-i\xi x}d\xi.$$

The asymptotic behavior of the function $U(f)(x)$ is well known [8]:

- If μ does not have first moment, then $\lim_{x\to\pm\infty} Uf(x) = 0$;
- if μ has first moment and its mean m is positive, then $\lim_{x\to-\infty} Uf(x) = \int f(y)dy/m$ and $\lim_{x\to+\infty} Uf(x) = 0$;

- if μ has first moment and its mean m is negative, then $\lim_{x\to+\infty} Uf(x) = \int f(y)dy/m$ and $\lim_{x\to-\infty} Uf(x) = 0$.

We consider the map $\pi_2 : G \to \mathbb{R}$ that sends $g = na(g)$ to $\ln(a(g))$. This induces a map on the space of measures by setting $\pi_2(\nu)(B) = \nu(\pi_2^{-1}(B))$ for any Borel set in $\mathbb{R}$. In the following we will denote by μ_A the real measure $\pi_2(\mu)$. Notice that $\mathcal{F}(\mu)(\frac{Q}{2} + is) = \hat{\mu}_A(-s)$. In particular if μ is a nonsingular probability measure on G we have, by Lemma 1, that μ_A is a nonarithmetic probability measure on $\mathbb{R}$. We say that μ has first moment if $\int_G |\ln(a(g))| d\mu(g) < +\infty$ and, if this is the case, we call $m = \int_G \ln(a(g)) d\mu(g) > 0$ the mean of μ. Obviously μ has first moment if and only if μ_A has (classical) first moment and if this is the case the mean of μ coincides with the (classical) mean of μ_A.

Theorem 2. *Let μ be a nonsingular radial probability measure on G. Then if μ has mean m*

$$\lim_{r\to+\infty} e^{Qr} \sum_{n=0}^{+\infty} \mu^{*n} * f(r) = \frac{4\pi C_1 \displaystyle\int_G f dg}{c\left(\dfrac{Q}{2}\right) m} \qquad \forall\, f \in C_c(G)^{\#},$$

while the above limit is zero if μ does not have first moment.

Proof. By the density of $C_c^\infty(G)^{\#}$ in $C_c(G)^{\#}$ we can suppose f smooth. Then the Fourier inversion formula gives

$$e^{Qr} \sum_{n=0}^{+\infty} \mu^{*n} * f(r) = C_1 e^{Qr} \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1 - \mathcal{F}\mu(is)} \phi_{is}(r) |c(is)|^{-2} ds \tag{2}$$

Recalling the asymptotic expansion of the spherical functions [2]

$$\phi_{is}(r) = c(is) \sum_{l=0}^{+\infty} \Gamma_l(is) e^{(is-l-\frac{Q}{2})r} + c(-is) \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l-\frac{Q}{2})r}$$

where $\Gamma_0 \equiv 1$ and $\Gamma_l(\cdot)$ are holomorphic functions on $\{z \in \mathbb{C} : \Re(z) < \frac{1}{2}\}$ that satisfy the estimates [2]

$$\sup_{\Re(z)\le 0} |\Gamma_l(z)| \le d(1+l)^d \tag{3}$$

for some constant d independent of z. Then, since $|c(is)|^2 = c(is)c(-is)$, (2) equals

$$C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(-is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(is) e^{(is-l+\frac{Q}{2})r} ds +$$

$$+ C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds.$$

The change of variable $is \to -is$ in the first integral allows us to write the above sum as

$$2C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds.$$

Note that, by (3), $\lim_{r\to+\infty} \sum_{l=\frac{Q}{2}+1}^{+\infty} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} = 0$ for all $s \in \mathbb{R}$ and thus, applying the Lebesgue dominated convergence theorem, we are left to estimate

$$\begin{aligned} &\lim_{r\to+\infty} 2C_1 \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-\mathcal{F}\mu(is)} c(is)^{-1} \sum_{l=0}^{\frac{Q}{2}} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds = \\ &= 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{\mathbb{R}} \frac{\mathcal{F}f(is)}{1-b\mathcal{F}\mu(is)} c(is)^{-1} \Gamma_l(-is) e^{(-is-l+\frac{Q}{2})r} ds \end{aligned} \tag{4}$$

If $0 < b < 1$ the function $F_l(z) = \frac{\mathcal{F}f(z)}{1-\mathcal{F}\mu(z)} c(z)^{-1} \Gamma_l(-z) e^{(-z-l+\frac{Q}{2})r}$ is holomorphic in $\{z \in \mathbb{C} : 0 < \Re(z) < \frac{Q}{2}\}$ and continuous on its closure. Since $F_l(t+is)$ is rapidly decreasing for $s \to \infty$, we can use the Cauchy integral formula to shift the contour of integration obtaining that (4) is equal to

$$2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{\mathbb{R}} \frac{\mathcal{F}f\left(is+\dfrac{Q}{2}\right)}{1-b\mathcal{F}\mu\left(is+\dfrac{Q}{2}\right)} c\left(is+\frac{Q}{2}\right)^{-1} \Gamma_l\left(-is-\frac{Q}{2}\right) e^{(-is-l)r} ds.$$

If $|s| > \eta$ for a fixed positive number η the quantity $|1 - b\mathcal{F}\mu(is+\frac{Q}{2})| = |1 - b\hat{\mu_A}(-s)|$ is bounded from below uniformly in b and thus, by the classical Riemann Lebesgue lemma, we are reduced to estimate

$$
\begin{aligned}
(5)\qquad & 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{\mathcal{F}f\left(is+\dfrac{Q}{2}\right)}{1-b\hat{\mu_A}(-s)} c\left(is+\frac{Q}{2}\right)^{-1} \times \\
& \times \Gamma_l\left(-is-\frac{Q}{2}\right) e^{(-is-l)r} ds
\end{aligned}
$$

The function $g_l(s) = \mathcal{F}f(is+\frac{Q}{2})c(is+\frac{Q}{2})^{-1}\Gamma_l(-is-\frac{Q}{2})$, for η sufficiently small, can be written, by Taylor's formula, as

$$
(6)\qquad g_l(s) = g_l(0) + \frac{d}{ds}g_l(0)s + s^2 M_l(s), \quad |s|\le\eta
$$

where $M_l(s)$ are bounded. On the other hand $|1-b\hat{\mu_A}(is)| \ge cb|s|^2$, $\forall |s|\le\eta$ and thus, using again the Riemann Lebesgue lemma, we have

$$
(7)\qquad \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{s^2 M(s)}{1-b\hat{\mu_A}(is)} e^{(-is-l)r} ds = 0.
$$

We can find smooth functions h_l with compact support whose Fourier transforms satisfy

$$
\hat{h_l}(-s) = g_l(0) + \frac{d}{ds}g_l(0)s + o(s), \quad |s|\le\eta,
$$

so that, taking in account (6) and (7), we have that (5) is equal to

$$
\begin{aligned}
(8)\qquad & 2C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} \lim_{b\uparrow 1} \int_{|s|\le\eta} \frac{\hat{h_l}(-s)}{1-b\hat{\mu_A}(-s)} e^{(-is-l)r} ds = \\
& = 4\pi C_1 \sum_{l=0}^{\frac{Q}{2}} \lim_{r\to+\infty} e^{-lr} \lim_{b\uparrow 1} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{h_l}(s)}{1-b\hat{\mu_A}(s)} e^{isr} ds.
\end{aligned}
$$

By (1) the limit in b is nothing but the potential $Uh_l(-r)$ associated with the measure μ_A. By the classical renewal theorem we have that if μ and thus μ_A, does not have first moment then

$$
\lim_{r\to+\infty} e^{-rl} U h_l(-r) = 0, \quad \forall\, l = 0, 1, \ldots, \frac{Q}{2}.
$$

If the measure μ and thus μ_A has mean m then the above limit is zero for all $l \neq 0$ while

$$\lim_{r\to+\infty} Uh_0(-r) = \frac{\displaystyle\int_{\mathbb{R}} h_0(x)dx}{m} = \frac{\hat{h}_0(0)}{m} =$$

$$= \frac{g_0(0)}{m} = \frac{\mathcal{F}\mu\left(\dfrac{Q}{2}\right)\Gamma_0\left(-\dfrac{Q}{2}\right)c\left(\dfrac{Q}{2}\right)^{-1}}{m} = \frac{\displaystyle\int_G f(g)dg}{c\left(\dfrac{Q}{2}\right)m}$$

which, in virtue of (8), concludes the proof of the theorem. □

REFERENCES

[1] J. Ph. Anker – E. Damek – C. Yacoub: *Spherical analysis on harmonic AN groups*, Ann. Sc. Sup., **22** (1996), 643–679.

[2] F. Astengo: *A class of L^p-convolutors on harmonic extensions of H-type groups*, J. Lie Theory, **5** (1995), 147-164.

[3] Ph. Bougerol: *Comportemente asymptotique des puissances de convolution d'une probabilité sur un espace symétrique*, Asterisque, Soc. Math. France, **74** (1980), 21-42.

[4] E. Damek – F. Ricci: *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc., **27** (1992), 139-142.

[5] E. Damek – F. Ricci: *Harmonic analysis on solvable extensions of H-type groups*, J. Geometric Anal., **2** (1992), 213-248.

[6] B. Di Blasio: *Paley-Wiener type theorems on harmonic extensions of H-type groups*, Monatsh. Math., **123** (1997), 21-42.

[7] B. Di Blasio: *Positive definite functions on harmonic spaces NA*, Boll. Un. Mat. Ital., A (7) **11** (1997), 759-767.

[8] D. Revuz: *Markov Chains*, North - Holland, 1975.

[9] F. Ricci: *The spherical transform on harmonic extensions of H-type groups*, Rend. Sem. Mat. Politec. Torino, **50** (1992), 381-392.

[10] N. Th. Varopoulos – S. Mustapha: *Analysis on Lie Groups*, Cambridge University Press, 2001.

Lavoro pervenuto alla redazione il 4 settembre 2002
ed accettato per la pubblicazione il 9 giugno 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo dell'Autore:

Filippo Tolli – Dipartimento di Matematica – Università di Roma Tre – Largo San Leonardo Murialdo, 1 – 00146 Roma (Italia)
E-mail: tolli@mat.uniroma3.it

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 131-161

Fonctions finement biharmoniques dans un espace biharmonique

C. BENSOUDA – M. EL KADIRI – I. ROUCHDI

Abstract: *We define and study a theory of finely biharmonic functions in a fine domain of a biharmonic space in the sense of Smyrnelis satisfying the axion D*

1 – Introduction

En théorie classique du Potentiel dans $\mathbb{R}^n$, la topologie fine a été définie par H. Cartan en 1940 comme étant la moins fine des topologies rendant continues les fonctions surharmoniques. Cette topologie a été ensuite étendue au cadre des diverses théories axiomatiques du Potentiel et aux théories du Potentiel des processus de Markov.

La théorie du balayage des mesures a permis à Fuglede de développer et étudier dans [12] une théorie des fonctions finement harmoniques dans un ouvert fin (i.e., ouvert au sens de la topologie fine) d'un espace harmonique de Bauer X vérifiant l'axiome de domination (ou Axiome (D)), généralisant la notion classique de fonction harmonique dans un ouvert ordinaire de X.

Smyrnelis [19], [20] a développé une théorie axiomatique des fonctions biharmoniques s'appliquant à un opérateur obtenu par couplage de

Key Words and Phrases: *Harmonic and biharmonic space – Finely harmonic and superharmonic function – Finely biharmonic function – Fine open set.*
A.M.S. Classification: 31B30 – 31D05

deux opérateurs différentiels L_1 et L_2 du second ordre elliptiques ou paraboliques dans un ouvert Ω de $\mathbb{R}^n$, et a montré qu'on peut étendre à ce nouveau cadre les méthodes et les résultats de la théorie classique ou axiomatique des fonctions harmoniques. Dans cette théorie un espace biharmonique $(X, \mathcal{H})$ est la donnée d'un espace localement compact X muni d'un faisceau d'espaces vectoriels de couples de fonctions réelles continues sur les ouverts de X vérifiant certains axiomes. A un tel espace sont associés deux espaces harmoniques de Bauer $(X, \mathcal{H}_1)$ et $(X, \mathcal{H}_2)$.

Bouleau [4] a ensuite montré que dans la théorie de Smyrnelis les couples hyperharmoniques ≥ 0 sont exactement les couples excessifs d'une résolvante triangulaire de noyaux boréliens sur l'espace de base. Boboc et Bucur [2] ont montré que ces couples s'identifient aussi aux fonctions excessives d'une famille résolvante de noyaux sur l'espace $X \oplus X$.

Dans [11], le deuxième auteur a introduit et étudié la notion de fonction finement biharmonique dans un ouvert fin de la théorie classique du Potentiel dans $\mathbb{R}^n$. Outre le fait qu'elle soit l'extension naturelle de la notion de fonction biharmonique aux ouverts fins, l'intérêt de cette notion réside aussi dans le problème d'approximation des fonctions continues sur un compact K par les restrictions à K de fonctions biharmoniques aux voisinages de K.

Notre but dans ce travail est d'étendre les résultats de [11] au cadre d'un espace biharmonique de Smyrnelis dont les espaces harmoniques associés admettent la même topologie fine.

Les notations utilisées dans tout ce travail seront identiques à celles des travaux de Fuglede et Smyrnelis cités dans la bibliographie, auquels on renvoie pour plus de détails.

2 – Mesures biharmoniques

Tout au long de ce travail nous utilisons la théorie locale des fonctions biharmoniques telle qu'elle est présentée par Smyrnelis dans [19] et [20], dont nous rappelons ici quelques résultats nécessaires aux développements qui suivent.

Soit $(X, \mathcal{H})$ un espace biharmonique au sens de [19] d'espaces harmoniques associés $(X, \mathcal{H}_1)$ et $(X, \mathcal{H}_2)$. On note $\mathcal{U}^+(X)$ le cône des couples hyperharmoniques positifs sur X. Par le mot fonction on entendra toujours, sauf mention expresse du contraire, une fonction à valeurs dans $\overline{\mathbb{R}}$. L'ordre sur l'ensemble des couples de fonctions sur un ensemble M est l'ordre produit usuel:

$$(f,g) \leq (h,k) \iff f \leq h \text{ et } g \leq k,$$
$$(f,g) < (h,k) \iff f < h \text{ et } g < k;$$

on écrira aussi $(h,k) \geq (f,g)$ (resp. $(h,k) > (f,g)$) au lieu de $(f,g) \leq (h,k)$ (resp. $(f,g) < (h,k)$). Si $(f,g) \geq (0,0)$, (resp. $(f,g) > (0,0)$) on écrira tout simplement $(f,g) \geq 0$ (resp. $(f,g) > 0$). Soient $F = (f,g)$ et $G = (h,k)$ deux couples de fonctions; on pose $\min(F,G) = (\min(f,h), \min(g,k))$ (resp. $\max(F,G) = (\max(f,h), \max(g,k))$), où, pour deux fonctions u et v, la fonction habituellement notée $\min(u,v)$ (resp. $\max(u,v)$) est définie par $\min(u,v)(x) = \min(u(x), v(x))$ (resp. $\max(u,v)(x) = \max(u(x), v(x))$).

Pour tout couple $\Phi = (f,g)$ de fonctions sur X, et toute partie E de X, on note Φ^E le couple réduit du couple Φ sur E. On rappelle que ce couple est défini par

$$\Phi^E = \inf\{(u,v) \in \mathcal{U}^+(X); (u,v) \geq \Phi \text{ sur } E\},$$

où l'inf est pris au sens de l'ordre produit. Le couple balayé de Φ sur E est noté $\widehat{\Phi}^E$ et défini par $\widehat{\Phi}^E = (\widehat{\Phi}_1^E, \widehat{\Phi}_2^E)$, où, pour une fonction h sur X, $\widehat{h}$ désigne la régularisée s.c.i. de h, i.e. la plus grande minorante s.c.i. de h dans X. On remarquera que l'on a $\Phi^E = (\Phi^+)^E$, où $\Phi^+ = \max(\Phi, 0)$.

Si f est une fonction définie sur une partie A de X, on note ${}^jR_f^A$ et ${}^j\widehat{R}_f^A$, $j = 1, 2$, respectivement la réduite et la balayée de f sur A dans l'espace harmonique $(X, \mathcal{H}_j)$.

Si A est une partie de X, on note $\overline{A}$ l'adhérence de A dans le compactifié d'Alexandroff de X.

Comme en théorie des espaces harmoniques, c'est la notion de balayée d'un couple de mesures qui va nous permettre de définir la notion de couples finement hyperharmoniques, surharmoniques ou harmoniques. A cet effet, nous rappelons le résultat suivant [20, Théorème 7.11 et Théorème 7.12]:

Théorème 2.1. *Pour tout couple (σ, τ) de mesures de Radon positives sur X et toute partie E de X, il existe trois mesures de Radon* *positives σ^E, ς^E et τ^E sur X telles que, pour tout $\mathcal{H}$-potentiel $P = (p, q)$, on ait*

$$\int^* \widehat{P}_1^E d\sigma = \int^* p d\sigma^E + \int^* q d\varsigma^E,$$
$$\int^* \widehat{P}_2^E d\tau = \int^* q d\tau^E,$$

où $\widehat{P}^E = (\widehat{P}_1^E, \widehat{P}_2^E)$.

Lorsque $\sigma = \tau = \epsilon_x, x \in X$, on notera les mesures σ^E, ς^E et τ^E correspondantes dans le théorème précédent par $\sigma_x^E, \varsigma_x^E$ et τ_x^E respectivement. Ce sont ces mesures qui permettent de définir les notions de couples finement harmoniques et finement hyperharmoniques.

On note $\mathcal{P}(X)$ (resp. $\mathcal{P}_1(X)$, resp. $\mathcal{P}_2(X)$) le cône des $\mathcal{H}$- (resp. $\mathcal{H}_1$-, resp. $\mathcal{H}_2$-) potentiels et on pose

$$\mathcal{P}_2'(X) = \{q \in \mathcal{P}_2(X) | \exists p \in \mathcal{P}_1(X) : (p, q) \in \mathcal{P}(X)\}.$$

Lemme 2.2. *Soit $(X, \mathcal{H})$ un espace biharmonique fort. Alors, tout $\mathcal{H}_2$-potentiel q est l'enveloppe supérieure d'une suite croissante (q_n) d'éléments de $\mathcal{P}_2'(X)$.*

Démonstration. Soit (p', q') un $\mathcal{H}$-potentiel tel que $p' > 0$ et $q' > 0$. Il est facile de vérifier que la suite (q_n) définie par $q_n = \min(q, nq')$ répond aux conditions du lemme.

Dans toute la suite de ce travail $(X, \mathcal{H})$ est un espace biharmonique fort au sens de Smyrnelis dont les espaces harmoniques associés $(X, \mathcal{H}_1)$ et $(X, \mathcal{H}_2)$ vérifient l'axiome (D) et admettent la même topologie fine, qu'on appelera topologie fine de X. Les ouverts de cette topologie seront appelés les ouverts fins de X. On utilisera le mot fin (finement) pour distinguer les notions relatives à la topologie fine de celle relatives à la topologie initiale. Pour toute partie A de X, on note $\widetilde{A}$ et $\partial_f A$ l'adhérence fine de A et la frontière fine de A, c'est-à-dire au sens de la topologie fine.

Exemple. Soit Ω un domaine de $\mathbb{R}^n$, $n \geq 1$. On note $\mathcal{H}_\Delta$ le faisceau biharmonique défini sur Ω par le Laplacien:

$$\mathcal{H}_\Delta(\omega) = \{(u, v) \in [\mathcal{C}^2(\omega)]^2 : \Delta u = -v, \Delta v = 0\},$$

pour tout ouvert ω de Ω. Le couple $(\Omega, \mathcal{H}_\Delta)$ est un espace biharmonique dont les espaces harmoniques associés sont identiques à l'espace harmonique classique défini par l'opérateur de Laplace sur Ω. On rappelle d'après [10] que l'espace biharmonique $(\mathbb{R}^n, \mathcal{H}_\Delta)$ est fort si et seulement si $n \geq 5$. Par contre, si Ω est un domaine borné de $\mathbb{R}^n$, l'espace biharmonique $(\Omega, \mathcal{H}_\Delta)$ est fort pour tout $n \geq 1$.

Proposition 2.3. *Pour tout ouvert fin ω de X, et tout $x \in \omega$, on a $\sigma_x^{C\omega} = \epsilon_x^{1,C\omega}$ et $\tau_x^{C\omega} = \epsilon_x^{2,C\omega}$, où $\epsilon_x^{j,C\omega}$, $j = 1, 2$, est la balayée de la mesure ϵ_x sur $C\omega$ dans l'espace harmonique $(X, \mathcal{H}_j)$.*

Démonstration. En appliquant le théorème précédent aux couples $P = (p, 0)$, où p est un $\mathcal{H}_1$-potentiel quelconque sur X, on voit que $\sigma_x^{C\omega} = \epsilon_x^{1,C\omega}$. Pour établir l'égalité $\tau_x^{C\omega} = \epsilon_x^{2,C\omega}$, il suffit d'utiliser le lemme précédent en observant que pour tout $\mathcal{H}$-potentiel $P = (p, q)$, la fonction $\widehat{P}_2^E$ n'est autre que la balayée de q sur E dans l'espace harmonique $(X, \mathcal{H}_2)$.

Remarque. Plus généralement, si σ et τ sont deux mesures de Radon ≥ 0 sur X et si $E \subset X$, les mesures σ^E et τ^E ne sont autres que les balayées des mesures σ et τ relativement aux espaces harmoniques associés $(X, \mathcal{H}_1)$ et $(X, \mathcal{H}_2)$.

Il est bien connu que pour tout ouvert fin ω de X, les mesures $\epsilon_x^{1,C\omega}$ et $\epsilon_x^{2,C\omega}$ sont portées par $\partial_f \omega$ (voir [12]). Donc, d'après la proposition précédente, les mesures $\sigma_x^{C\omega}$ et $\tau_x^{C\omega}$, $x \in \omega$, sont portées par $\partial_f \omega$.

Soit ω un ouvert fin de X. Pour tout $x \in \omega$, on pose $\mu_x^\omega = \sigma_x^{C\omega}$, $\nu_x^\omega = \varsigma_x^{C\omega}$ et $\lambda_x^\omega = \tau_x^{C\omega}$.

Définition 2.4. Soient ω un ouvert fin de X. Pour tout $x \in \omega$, le triplet de mesures $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ est appelé le triplet des mesures biharmoniques de ω au point x.

Remarque. Si ω est un ouvert $\mathcal{H}$-régulier de X et si $x \in \omega$, les mesures μ_x^ω, ν_x^ω et λ_x^ω, ne sont rien d'autres que les mesures biharmoniques habituelles de ω au point x (voir [19]).

Pour quelques propriétés des mesures biharmoniques, utiles pour la suite, nous avons besoin de rappeler deux résultats dûs à Bouleau [4]:

Théorème 2.5. *Il existe un noyau borélien unique $\mathcal{V}$ sur X ayant les propriétés suivantes*:

i) *Pour toute fonction finie continue $\varphi \geq 0$ à support compact sur X, la fonction $\mathcal{V}(\varphi)$ est $\mathcal{H}_1$-surharmonique ≥ 0, finie, continue et $\mathcal{H}_1$-harmonique dans le complémentaire du support de φ.*

ii) *Pour toute fonction $\mathcal{H}_2$-hyperharmonique $v \geq 0$ sur X, $\mathcal{V}(v)$ est la fonction hyperharmonique pure d'ordre 2 associée à v.*

On rappelle (voir [21]) que la fonction hyperharmonique pure d'ordre 2 associée à une fonction $\mathcal{H}_2$-hyperharmonique $v \geq 0$ dans X est la plus petite fonction $\mathcal{H}_1$-hyperharmonique $u \geq 0$ telle que le couple $(u, v) \in \mathcal{U}^+(X)$; elle est donnée par

$$u = \widehat{\inf}\{s : (s, v) \in \mathcal{U}^+(X)\}.$$

Remarque. Si f est une fonction borélienne ≥ 0 sur X, alors $\mathcal{V}(f)$ est $\mathcal{H}_1$-hyperharmonique ≥ 0 dans X.

Théorème 2.6. *Avec les notations du théorème précédent, on a, pour tout couple $(u, v) \in \mathcal{U}^+(X)$,*

$$\mathcal{V}(v) \prec u,$$

i.e. il existe une fonction $\mathcal{H}_1$-hyperharmonique $t \geq 0$ telle que $u = \mathcal{V}(v) + t$.

On va maintenant appliquer les Théorèmes 2.5 et 2.6 pour le calcul du couple balayé sur une partie A de X d'un couple $\Phi = (u, v) \in \mathcal{U}^+(X)$ au moyen du balayage relativement aux espaces harmoniques $(X, \mathcal{H}_1)$ et $(X, \mathcal{H}_2)$.

Dans la suite, si s et t sont deux fonctions $\mathcal{H}_1$- surharmoniques ≥ 0 telles que $s \prec t$, on note $t - s$ la fonction u $\mathcal{H}_1$-surharmonique ≥ 0 telle que $t = u + s$.

Proposition 2.7. *Pour tout couple $\mathcal{H}$-surharmonique $(s, t) \geq 0$ dans X et toute partie A de X, on a*

$$\widehat{(s, t)}^A = ({}^1\widehat{R}^A_{s-\mathcal{V}({}^2\widehat{R}^A_t)} + \mathcal{V}({}^2\widehat{R}^A_t), {}^2\,\widehat{R}^A_t).$$

Démonstration. Remarquons d'abord que l'on a

$$\widehat{(s,t)}^A = (\widehat{\inf}\{u : u \geq s \text{ sur } A, (u, {}^2\widehat{R}_t^A) \in \mathcal{U}^+(X)\}, {}^2\widehat{R}_t^A).$$

Or, pour tout couple $(u, {}^2\widehat{R}_t^A) \in \mathcal{U}^+(X)$, on a, d'après le Théorème 2.6,

$$u = \mathcal{V}({}^2\widehat{R}_t^A) + r$$

où r est une fonction $\mathcal{H}_1$-hyperharmonique ≥ 0. D'autre part, comme $(s, {}^2\widehat{R}_t^A) \in \mathcal{U}^+(X)$, on a, d'après le Théorème 2.6,

$$s = \mathcal{V}({}^2\widehat{R}_t^A) + k,$$

où k est une fonction $\mathcal{H}_1$-surharmonique ≥ 0 dans X. On en déduit que

$$\widehat{(s,t)}^A = (\mathcal{V}({}^2\widehat{R}_t^A) + {}^1\widehat{R}_k^A, {}^2\widehat{R}_t^A),$$

d'où le résultat.

Corollaire 1. *Pour tout couple $\mathcal{H}$-surharmonique $(s,t) \geq 0$ dans X et toutes les parties A et B de X telles que $A \subset B$, on a*

$$\widehat{(\widehat{(s,t)}^A)}^B = \widehat{(s,t)}^A.$$

Démonstration. On a, d'après la Proposition 2.7,

$$\widehat{(s,t)}^A = ({}^1\widehat{R}^A_{s-\mathcal{V}({}^2\widehat{R}_t^A)} + \mathcal{V}({}^2\widehat{R}_t^A), {}^2\widehat{R}_t^A).$$

En utilisant les relations ${}^1\widehat{R}^B_{{}^1\widehat{R}_u^A} = {}^1\widehat{R}_u^A$ et ${}^2\widehat{R}^B_{{}^2\widehat{R}_v^A} = {}^2\widehat{R}_v^A$ pour u $\mathcal{H}_1$-hyperharmonique ≥ 0 et v $\mathcal{H}_2$-hyperharmonique ≥ 0 qui découlent aussitôt du Théorème 9.1.1 et du Corollaire 9.2.3 de [7], on obtient

$$\widehat{(\widehat{(s,t)}^A)}^B = \widehat{(s,t)}^A.$$

Corollaire 2. *Soient ω un ouvert fin de X et $x \in \omega$. Alors la mesure ν_x^ω est portée par la base $b(C\omega)$.*

Démonstration. D'après le Théorème 2.1, on a, pour tout $\mathcal{H}$-potentiel $P = (p, q)$ dans X et tout $x \in \omega$,

$$\widehat{P}_1^{C\omega}(x) = \int^* p d\mu_x^\omega + \int^* q d\nu_x^\omega.$$

Comme $\widehat{\widehat{P}^{C\omega}}^{C\omega} = \widehat{P}^{C\omega}$, on a aussi, toujours d'après le Théorème 2.1,

$$\widehat{P}_1^{C\omega}(x) = \int^* \widehat{P}_1^{C\omega} d\mu_x^\omega + \int^* \widehat{P}_2^{C\omega} d\nu_x^\omega.$$

Comme $\widehat{P}_2^{C\omega} = {}^2\widehat{R}_q^{C\omega}$ en vertu du corollaire précédent, on en déduit que

$$\int (q - {}^2\widehat{R}_q^{C\omega}) d\nu_x^\omega = 0,$$

soit

$$\int^* q d\nu_x^\omega = \int^* {}^2\widehat{R}_q^{C\omega} d\nu_x^\omega,$$

pour tout $q \in \mathcal{P}_2'(X)$. D'où, d'après le Lemme 2.2,

$$\int^* q d\nu_x^\omega = \int^* {}^2\widehat{R}_q^{C\omega} d\nu_x^\omega$$

pour tout $\mathcal{H}_2$-potentiel q dans X. En prenant q strict, ceci montre bien que ν_x^ω est portée par $b(C\omega)$, grâce à [7, Proposition 7.2.2].

Comme $b(C\omega) \subset \partial_f \omega$, on déduit du corollaire précédent que, pour tout $x \in \omega$, la mesure ν_x^ω est portée par $\partial_f \omega$.

Proposition 2.8. *Soit ω un ouvert fin de X. Alors, pour tout $x \in \omega$, la mesure ν_x^ω ne charge pas les ensembles $\mathcal{H}$-polaires.*

Démonstration. Soit $x \in \omega$. Comme la mesure ν_x^ω est portée par $\partial_f \omega$, il suffit de montrer que ν_x^ω ne charge pas les $\mathcal{H}$-polaires contenus dans $\partial_f \omega$. Soit A un ensemble $\mathcal{H}$-polaire $\subset \partial_f \omega$, on peut trouver un $\mathcal{H}$-potentiel $P = (p, q)$ dans X tel que $p = q = +\infty$ sur A et $p(x) < +\infty$. On a alors $\int q d\nu_x^\omega \leq p(x) < +\infty$, d'où $\nu_x^\omega(A) \leq \nu_x^\omega(\{q = +\infty\}) = 0$.

3 – Couples finement hyperharmoniques et couples finement biharmoniques

On désigne par f-lim et f-lim inf respectivement les limites fine et fine inférieure, c'est-à-dire au sens de la topologie fine. Pour un couple $F = (u, v)$ de fonctions sur U, on note $\text{f-}\liminf_{x \to y} F(x)$ le couple (f-$\liminf_{x\to y} u(x)$,f-$\liminf_{x\to y} v(x)$).

On rappelle d'abord qu'une fonction u sur un ouvert fin U de X est dite $\mathcal{H}_j$-finement hyperharmonique dans U, $j = 1, 2$, si u est finement s.c.i., à valeurs dans $] - \infty, +\infty]$, et si la topologie fine induite sur U admet une base $\mathcal{B}$ formée d'ouverts fins ω tels que $\widetilde{\omega} \subset U$ et

$$u(x) \geq \int^* u d\epsilon_x^{j, C\omega}$$

pour tout $x \in \omega$ ([12, p. 67]).

Par analogie avec cette définition on pose la

Définition 3.1. Un couple (u, v) de fonctions sur un ouvert fin U de X est dit finement $\mathcal{H}$-hyperharmonique dans U si u et v sont finement s.c.i. à valeurs dans $]-\infty, +\infty]$, et si on peut trouver une base $\mathcal{B}$ d'ouverts de la topologie fine dans U formée d'ouverts fins ω de X tels que $\widetilde{\omega} \subset U$ et

$$u(x) \geq \int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega, \; v(x) \geq \int^* v d\lambda_x^\omega$$

pour tout $x \in \omega$.

Ces inégalités sont appelées inégalités de la moyenne.

La définition a bien un sens puisque, pour tout $x \in \omega$, les mesures μ_x^ω, ν_x^ω et λ_x^ω sont portées par $\partial_f \omega$ et ne chargent pas les polaires.

On note $\mathcal{U}_f(U)$ l'ensemble des couples finement $\mathcal{H}$-hyperharmoniques dans un ouvert fin U de X, et $\mathcal{U}_f^+(U)$ celui des couples finement $\mathcal{H}$-hyperharmoniques ≥ 0 dans U.

Un couple (u, v) de fonctions sur un ouvert fin U de X est dit finement $\mathcal{H}$-hypoharmonique dans U si le couple $(-u, -v)$ est finement $\mathcal{H}$-hyperharmonique dans U.

Définition 3.2. Un couple (u, v) de fonctions sur un ouvert fin U de X à valeurs dans $\mathbb{R}$ est dit finement $\mathcal{H}$-harmonique (ou simplement finement biharmonique) dans U si (u, v) est à la fois finement $\mathcal{H}$-hyperharmonique et finement $\mathcal{H}$-hypoharmonique.

On dit qu'un $\mathcal{H}$-potentiel $P = (p, q)$ dans X est semi-borné si les potentiels p et q sont semi-bornés.

Théorème 3.3. *Soit (f, g) un couple de fonctions finement s.c.i. sur $\widetilde{U}$ et finement $\mathcal{H}$-hyperharmonique dans U. Si de plus il existe un $\mathcal{H}$-potentiel semi-borné $P = (p_1, p_2)$ dans X tel que $(f, g) \geq -P$, alors on a $(f(x), g(x)) \geq (\int^* f d\mu_x^U + \int^* g d\nu_x^U, \int^* g d\lambda_x^U)$ pour tout $x \in U$ où P est fini.*

Démonstration. Il suffit d'adapter aux couples la démonstration du Théorème 9.4 de [12].

Corollaire 1. *Un couple (u, v) de fonctions sur un ouvert fin U de X est finement $\mathcal{H}$-hyperharmonique dans U si et seulement si u et v sont finement s.c.i. $> -\infty$, et si pour tout ouvert fin relativement compact ω tel que $\widetilde{\omega} \subset U$, sur lequel u et v sont bornées inférieurement, et tout $x \in \omega$, on a*

$$u(x) \geq \int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega, \ v(x) \geq \int^* v d\lambda_x^\omega.$$

Corollaire 2. *Un couple (u, v) de fonctions finement continues sur U à valeurs dans $\mathbb{R}$ est finement harmonique dans U si et seulement si pour tout ouvert fin relativement compact ω tel que $\widetilde{\omega} \subset U$, sur lequel u et v sont bornées, et tout $x \in \omega$, on a*

$$u(x) = \int u d\mu_x^\omega + \int v d\nu_x^\omega, \ v(x) = \int v d\lambda_x^\omega.$$

On déduit aussi du Théorème 3.3 les propriétés suivantes des couples finement hyperharmoniques:

1. L' ensemble $\mathcal{U}_f(U)$ est un cône convexe de sommet 0:
   i) $\forall u, v \in \mathcal{U}_f(U)$, $u + v \in \mathcal{U}_f(U)$,
   ii) $\forall u \in \mathcal{U}_f(U), \forall \lambda \geq 0$ fini, $\lambda u \in \mathcal{U}_f(U)$.
   iii) De plus, le cône $\mathcal{U}_f(U)$ est inf-stable, c'est à dire,
$$\forall F, G \in \mathcal{U}_f(U), \min(F, G) \in \mathcal{U}_f(U).$$

$\mathcal{U}_f^+(U)$ a les mêmes propriétés.

2. Si U_1 et U_2 sont des ouverts fins de X tels que $U_1 \subset U_2$ et si $F = (u,v) \in \mathcal{U}_f(U_2)$, alors $F|_{U_1} = (u|_{U_1}, v|_{U_1}) \in \mathcal{U}_f(U_1)$.
3. Si $(U_i)_{i\in I}$ est une famille d'ouverts fins de X et si F est un couple de fonctions sur $U = \cup_{i\in I} U_i$ tel que $F|_{U_i} \in \mathcal{U}_f(U_i)$ pour tout $i \in I$, alors $F \in \mathcal{U}_f(U)$.

Ces propriétés de faisceau en topologie fine, vraies aussi pour les couples finement biharmoniques, permettent, quitte à se restreindre aux composantes finement connexes de l'ouvert fin U, de se ramener au cas où U est un domaine fin que l'on fixera dans la suite. On rappelle que la topologie fine est localement connexe (voir [12, corollaire du Théorème 9.11]). Quitte à ajouter à U l'ensemble polaire des points irréguliers de sa frontière fine, on le supposera, grâce au principe de prolongement par continuité fine, régulier (donc un K_σ de X).

Lemme 3.4. *Soit (h,k) un couple biharmonique ≥ 0 dans X et ω un ouvert fin relativement compact de X. Alors on a*

$$h(x) = \int h d\mu_x^\omega + \int k d\nu_x^\omega,$$

et

$$k(x) = \int k d\lambda_x^\omega,$$

pour tout $x \in \omega$.

Démonstration. D'après la Proposition 2.7, on a

$$\widehat{(h,k)}^{C\omega} = ({}^1\widehat{R}^{C\omega}_{h-\mathcal{V}({}^2\widehat{R}^{C\omega}_k)} + \mathcal{V}({}^2\widehat{R}^{C\omega}_k), {}^2\widehat{R}^{C\omega}_k).$$

Or les fonctions $h - \mathcal{V}(k)$ et k sont respectivement $\mathcal{H}_1$-harmonique et $\mathcal{H}_2$-harmonique dans X, donc ${}^2\widehat{R}^{C\omega}_k = k$ et ${}^1\widehat{R}^{C\omega}_{h-\mathcal{V}({}^2\widehat{R}^{C\omega}_k)} = h - \mathcal{V}(k)$, d'où $\widehat{(h,k)}^{C\omega} = (h,k)$ et le lemme découle alors du Théorème 2.1 et de la Proposition 2.3.

Corollaire. *Soit (h,k) un couple biharmonique dans un ouvert Ω de X pour la topologie initiale. Alors (h,k) est finement biharmonique dans Ω.*

Théorème 3.5. *Soient Ω un ouvert de X pour la topologie initiale et (u,v) un couple $\mathcal{H}$-hyperharmonique dans Ω. Alors (u,v) est finement $\mathcal{H}$-hyperharmonique dans Ω.*

Démonstration. Les fonctions u et v sont s.c.i., donc finement s.c.i. dans Ω. Ces fonctions sont aussi localement bornées inférieurement. Quitte à se placer localement on peut donc supposer qu'il existe un couple biharmonique $(h,k) > 0$ dans Ω tel que $(u,v) + (h,k) \geq 0$. Soit ω un ouvert fin tel que $\widetilde{\omega} \subset \Omega$ et sur lequel u et v sont bornées inférieurement et soit $x \in \omega$, alors on a

$$\begin{aligned} u(x) + h(x) &= \int^* (u+h) d\epsilon_x \geq \\ &\geq \int^* (u+h) d\mu_x^\omega + \int^* (v+k) d\nu_x^\omega. \end{aligned}$$

Or on a d'après le lemme précédent

$$h(x) = \int h d\mu_x^\omega + \int k d\nu_x^\omega,$$

et

$$k(x) = \int k d\lambda_x^\omega,$$

d'où

$$u(x) \geq \int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega.$$

La fonction v est finement $\mathcal{H}_2$-hyperharmonique d'après [12, Théorème 9.8]. On en déduit que le couple (u,v) est finement $\mathcal{H}$-hyperharmonique dans Ω.

Les quatres propositions qui suivent résultent immédiatement de la Définition 3.1 et de celle des fonctions finement $\mathcal{H}$-hyperharmoniques.

Proposition 3.6. *Soit $\{(u_n, v_n)\}$ une suite croissante d'éléments de $\mathcal{U}_f(U)$. Alors $(\sup_n u_n, \sup_n v_n) \in \mathcal{U}_f(U)$.*

Proposition 3.7. *Soit $(u,v) \in \mathcal{U}_f(U)$, et soit v' une fonction finement $\mathcal{H}_2$-hyperharmonique dans U. Si $v' \leq v$, alors $(u,v') \in \mathcal{U}_f(U)$.*

Proposition 3.8. *Soient u et v deux fonctions respectivement finement $\mathcal{H}_1$-hyperharmonique et $\mathcal{H}_2$-hyperharmonique ≥ 0 dans U. Alors $(u,0) \in \mathcal{U}_f^+(U)$, et $(+\infty, v) \in \mathcal{U}_f^+(U)$.*

Proposition 3.9. *Soit $(u,v) \in \mathcal{U}_f^+(U)$. Alors les fonctions u et v sont respectivement finement $\mathcal{H}_1$-hyperharmonique et $\mathcal{H}_2$-hyperharmonique dans U. En particulier le couple $(u,0) \in \mathcal{U}_f^+(U)$.*

Corollaire. *Pour tout couple $(u,v) \in \mathcal{U}_f(U)$, les fonctions u et v sont finement continues dans U.*

Démonstration. Quitte à se placer finement localement et ajouter à (u,v) un couple biharmonique $(h,k) > 0$, on peut supposer $(u,v) \geq 0$. D'après la Proposition 3.9, les fonctions u et v sont respectivement finement $\mathcal{H}_1$-hyperharmonique et $\mathcal{H}_2$- hyperharmonique, donc finement continues.

Proposition 3.10. *Soient V un ouvert fin contenu dans U, $(u_1, v_1) \in \mathcal{U}_f(U)$ et $(u_2, v_2) \in \mathcal{U}_f(V)$ tels que*

$$\text{f}-\liminf_{x \to y}(u_2, v_2)(x) \geq (u_1(y), v_1(y)), \forall y \in \partial_f V \cap U.$$

Alors le couple (u,v) défini par

$$(u,v)(x) = \begin{cases} \min((u_1,v_1),(u_2,v_2))(x) \text{ si } x \in V, \\ (u_1,v_1)(x) \text{ si } x \in U \setminus V \end{cases}$$

est finement $\mathcal{H}$-hyperharmonique dans U.

Démonstration. On adapte aux couples la démonstration du Lemme 10.1 de [12].

Soit $\mathcal{S}_2'(U)$ le cône des fonctions finement $\mathcal{H}_2$-surharmoniques positives majorées par un élément de $\mathcal{P}_2'(X)$.

Lemme 3.11. *Tout $v \in \mathcal{S}_2'(U)$ est l'enveloppe supérieure d'une suite croissante $(p_n - {}^2\widehat{R}_{p_n}^{CU})$, où (p_n) est une suite d'éléments de $\mathcal{P}_2'(X)$.*

Démonstration. Le lemme résulte immédiatement du Théorème 3 de [14].

Lemme 3.12. *Toute fonction v finement $\mathcal{H}_2$-hyperharmonique ≥ 0 dans U est l'enveloppe supérieure d'une suite d'éléments de $\mathcal{S}_2'(U)$.*

Proposition 3.13. *Pour toute fonction v finement $\mathcal{H}_2$-hyperharmonique ≥ 0 dans U et tout ouvert fin $\omega \subset \widetilde{\omega} \subset U$, le couple $(\int^* v d\nu_.^\omega, \int^* v d\lambda_.^\omega)$ est finement $\mathcal{H}$-hyperharmonique dans ω.*

Démonstration. D'après les lemmes précédents, il suffit de démontrer la proposition lorsque v est de la forme $(q - {}^2\widehat{R}_q^{CU})$ où q est un élément de $\mathcal{P}_2'(X)$. Soit $q \in \mathcal{P}_2'(X)$ et p un $\mathcal{H}_1$-potentiel fini continu tel que $P = (p, q)$ soit un $\mathcal{H}$-potentiel fini continu dans X. On a, dans ω,

$$(\widehat{\widehat{P}_1^{CU}, \widehat{P}_2^{CU}})^{C\omega} = \left(\int^* \widehat{P}_1^{CU} d\mu_.^\omega + \int^* \widehat{P}_2^{CU} d\nu_.^\omega, \int^* \widehat{P}_2^{CU} d\lambda_.^\omega\right).$$

Pour compléter la preuve, il suffit d'utiliser les identités $(\widehat{\widehat{P}_1^{CU}, \widehat{P}_2^{CU}})^{C\omega} = (\widehat{P}_1^{C\omega}, \widehat{P}_2^{C\omega})$ et $\widehat{P}_2^{CU} = {}^2\widehat{R}_q^{CU}$, et le fait que ${}^2\widehat{R}_q^{CU}$ est finement $\mathcal{H}_2$-harmonique dans ω.

Corollaire 1. *Soient $(u, v) \in \mathcal{U}_f^+(U)$ et ω un ouvert fin régulier tel que $\widetilde{\omega} \subset U$. Alors le couple de fonctions*

$$\left(\int^* u d\mu_.^\omega + \int^* v d\nu_.^\omega, \int^* v d\lambda_.^\omega\right)$$

est finement $\mathcal{H}$-hyperharmonique dans ω.

Corollaire 2. *Soient $(u, v) \in \mathcal{U}_f^+(U)$ et ω un ouvert fin régulier tel que $\widetilde{\omega} \subset U$. Alors le couple de fonctions $(u, v)_\omega$ défini par*

$$(u, v)_\omega = \begin{cases} \left(\int^* u d\mu_.^\omega + \int^* v d\nu_.^\omega, \int^* v d\lambda_.^\omega\right) & \text{dans } \omega, \\ (u, v) & \text{dans } U \setminus \omega \end{cases}$$

est finement $\mathcal{H}$-hyperharmonique dans U.

Démonstration. Comme le couple (u, v) est ≥ 0, les fonctions u et v sont finement $\mathcal{H}$-hyperharmoniques ≥ 0 dans U d'après la Proposition 3.9. Il en résulte alors d'après [12, Théorème 14.6 et Théorème 14.7] que l'on a $f\text{--}\liminf_{x\to y}(\int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega) \geq u(x)$ et $f\text{--}\liminf_{x\to y}(\int^* v d\lambda_x^\omega) \geq v(y)$ pour tout $y \in \partial_f\omega$ (on rappelle que $\mu_x^\omega = \epsilon_x^{1,C\omega}$ et $\lambda_x^\omega = \epsilon_x^{2,C\omega}$), d'où le résultat en vertu du Corollaire 1 et de la Proposition 3.10.

4 – Réduction et balayage des couples finement hyperharmoniques

Si f une fonction sur U, on note $\widehat{f}$ sa régularisée finement s.c.i. C'est la plus grande minorante de f qui soit finement s.c.i. dans U, et elle est donnée par
$$f(x) = \mathrm{f} - \liminf_{y\to x} f(y), \ \forall x \in U.$$
Si $F = (f, g)$ est un couple de fonctions sur U, on note $\widehat{F}$ le couple $(\widehat{f}, \widehat{g})$. Ce couple est applelé le couple régularisé finement s.c.i. de F dans U.

Définition 4.1. Soient $A \subset U$ et $F = (f, g)$ un couple de fonctions sur U. Le couple réduit de F sur A, noté F^A, est le couple défini par
$$F^A = \inf\{(u, v) \in \mathcal{U}_f^+(U); (u, v) \geq (f, g) \text{ sur } A\}.$$
Le couple balayé de F sur A est le couple $\widehat{F}^A$ régularisé finement s.c.i. de F^A.

Proposition 4.2. *Soient $A \subset U$ et $F = (f, g)$ un couple de fonctions sur U. Alors le couple $\widehat{F}^A$ est finement $\mathcal{H}$-hyperharmonique dans U.*

Pour toute partie A de U, on note ${}^{j,U}\widehat{R}_f^A$, $j = 1, 2$, la balayée sur A d'une fonction f relativement à U dans l'espace harmonique $(X, \mathcal{H}_j)$.

Proposition 4.3. *Soient (f, g) un couple de fonctions sur U, et $A \subset U$. Posons $\widehat{F}^A = (\widehat{F}_1^A, \widehat{F}_2^A)$. On a alors $\widehat{(f,0)}^A = ({}^{1,U}\widehat{R}_f^A, 0)$ et $\widehat{F}_2^A = {}^{2,U}\widehat{R}_g^A$.*

Démonstration. Le couple $({}^{1,U}\widehat{R}_f^A, 0)$ est finement $\mathcal{H}$-hyperharmonique ≥ 0 dans U et majore $(f,0)$ q.p. sur A, donc $({}^{1,U}\widehat{R}_f^A, 0) \geq \widehat{(f,0)}^A$. D'autre part si $(u,v) \in \mathcal{U}_f^+(U)$ majore le couple $(f,0)$ sur A, alors u est une fonction finement $\mathcal{H}_1$-hyperharmonique ≥ 0 qui majore f sur A, donc $\widehat{(f,0)}^A \geq ({}^{1,U}\widehat{R}_f^A, 0)$, et par suite $\widehat{(f,0)}^A = ({}^{1,U}\widehat{R}_f^A, 0)$. Soit maintenant v une fonction finement $\mathcal{H}_2$-hyperharmonique ≥ 0 sur U telle que $v \geq g$ sur A. Alors le couple $(+\infty, v)$ est finement $\mathcal{H}$-hyperharmonique ≥ 0 et majore (f,g) sur A, d'où ${}^{2,U}\widehat{R}_g^A \geq \widehat{F}_2^A$. L'inégalité inverse découle facilement du fait que pour tout couple finement $\mathcal{H}$-hyperharmonique $(u,v) \geq 0$ tel que $(u,v) \geq (f,g)$ sur A, la fonction v est finement $\mathcal{H}_2$-hyperharmonique ≥ 0 et majore g sur A.

Proposition 4.4. *Soient $(u,v) \in \mathcal{U}_f^+(U)$ et $A \subset U$. Alors on a $\widehat{(u,v)}^A = (u,v)$ q.p. sur A.*

Démonstration. Cela résulte en effet du fait que, pour tout couple finement $\mathcal{H}$-hyperharmonique $(u,v) \geq 0$, les fonctions u et v sont respectivement finement $\mathcal{H}_1$-hyperharmonique et $\mathcal{H}_2$-hyperharmonique positives et de [12, Théorème 11.8].

Remarque. Si $(u,v) \in \mathcal{U}_f^+(U)$ majore q.p. un couple F de fonctions sur $A \subset U$, alors $(u,v) \geq \widehat{F}^A$.

5 – Ordre spécifique dans $\mathcal{U}_f^+(U)$

Remarquons d'abord que le cône $\mathcal{U}_f^+(U)$ est réticulé pour l'ordre naturel, ce qui se démontre comme en théorie des fonctions finement harmoniques.

L'ordre spécifique, noté $\prec$, est défini sur $\mathcal{U}_f^+(U)$ par

$$(u,v) \prec (s,t) \iff \exists (u_1,v_1) \in \mathcal{U}_f^+(U) : (s,t) = (u_1,v_1) + (u,v).$$

Proposition 5.1. *Soient $F_1 = (u_1,v_1), F_2 = (u_2,v_2) \in \mathcal{U}_f^+(U)$. On a alors $[\widehat{(F_1 - F_2)^+}]^U \prec F_1$.*

Démonstration. La proposition se démontre exactement comme dans le cas des fonctions finement hyperharmoniques (voir [12], p.p. 129, 130 et 131).

Corollaire (Propriété de décomposition de Riesz). *Soient F, G et H trois couples de $\mathcal{U}_f^+(U)$ tels que $F \leq G + H$. Il existe alors deux couples $F_1, F_2 \in \mathcal{U}_f^+(U)$ tels que $F = F_1 + F_2$, $F_1 \leq G$ et $F_2 \leq H$.*

Théorème 5.2. *Soient $S, T \in \mathcal{U}_f^+(U)$ et $A \subset U$. On a alors $\widehat{(S+T)}^A = \widehat{S}^A + \widehat{T}^A$.*

Démonstration. L'inégalité $\widehat{(S+T)}^A \leq \widehat{S}^A + \widehat{T}^A$ découle immédiatement de la définition du balayage; l'inégalité inverse s'en déduit en appliquant le corollaire précédent et en utilisant les propriétés des couples et des fonctions finement hyperharmoniques.

On déduit aussi de la Proposition 5.1 que le cône $\mathcal{U}_f^+(U)$ vérifie les axiomes du Chapitre 4 de [7] quand U est muni de la topologie fine, d'où le résultat suivant:

Proposition 5.3. *Le cône $\mathcal{U}_f^+(U)$ est un treillis complètement réticulé pour l'ordre spécifique.*

Si $F, G \in \mathcal{U}_f^+(U)$, on note $F \vee G$ et $F \wedge G$ respectivement le max et le min au sens de l'ordre spécifique. Si $\{F_i; i \in I\}$ est une famille d'éléments de $\mathcal{U}_f^+(U)$, on note $\bigwedge_i F_i$ (resp. $\bigvee_i F_i$) l'enveloppe inférieure (resp. supérieure) au sens de l'ordre spécifique de la famille $\{F_i; i \in I\}$.

Remarquons que, comme dans le cas harmonique, on a

$$F \wedge G + F \vee G = F + G$$

et, pour une famille filtrante croissante (resp. décroissante), au sens de l'ordre spécifique, $\{F_i; i \in I\}$, d'éléments de $\mathcal{U}_f^+(U)$, on a

$$\bigwedge_i F_i = \widehat{\inf_i} F_i \quad \left(\text{resp.} \bigvee_i F_i = \sup_i F_i\right).$$

6 – Couples finement surharmoniques et couples potentiels fins

Dans ce paragraphe on va se contenter d'énoncer seulement les définitions et quelques propriétés essentielles des couples finement surharmoniques et des couples potentiels fins.

Définition 6.1. Un couple (u, v) finement $\mathcal{H}$-hyperharmonique dans un ouvert fin V de X est dit finement $\mathcal{H}$-surharmonique dans V si les fonctions u et v sont finies sur un ensemble dense dans V.

Il n'est pas difficile de voir que, d'après [12, Théorème 12.9], pour qu'un couple $(u, v) \in \mathcal{U}_f(V)$ soit finement $\mathcal{H}$-surharmonique, il faut et il suffit que les fonctions u et v soient finies en au moins un point de chaque composante finement connexe de V.

On note $\mathcal{S}_f(U)$ l'ensemble des couples finement $\mathcal{H}$-surharmoniques dans U. Il est clair que cet ensemble est un cône convexe. On note également $\mathcal{S}_f^+(U)$ le sous-cône de $\mathcal{S}_f(U)$ formé des couples finement $\mathcal{H}$-surharmoniques ≥ 0, et $\mathcal{S}_f^{j,+}(U)$, $j = 1, 2$, le cône des fonctions finement $\mathcal{H}_j$-surharmoniques ≥ 0 dans U.

Définition 6.2. Un couple $P = (p_1, p_2) \in \mathcal{S}_f^+(U)$ est appelé un $\mathcal{H}$-potentiel fin si tout couple (u, v) finement $\mathcal{H}$-hypoharmonique dans U qui le minore au sens de l'ordre naturel produit est ≤ 0.

On note $\mathcal{P}_f(U)$ l'ensemble des $\mathcal{H}$-potentiels fins dans U. Alors $\mathcal{P}_f(U)$ est un sous-cône de $\mathcal{S}_f^+(U)$. C'est aussi une bande de $\mathcal{S}_f^+(U)$, i.e.,

$$\forall P, Q \in \mathcal{S}_f^+(U) : P + Q \in \mathcal{P}_f(U) \Rightarrow P, Q \in \mathcal{P}_f(U).$$

Proposition 6.3. *Soit (s_1,s_2) un couple finement $\mathcal{H}$-surharmonique dans U de X. Alors*

i) *si $s_2 \geq 0$, la fonction s_1 est finement $\mathcal{H}_1$-surharmonique*;
ii) *le couple (s_1, s_2) est un potentiel fin si et seulement si, pour tout $j = 1, 2$, s_j est un $\mathcal{H}_j$-potentiel fin.*

Démonstration. Le i) résulte aussitôt de la Définition 6.1 et de celle des fonctions finement $\mathcal{H}_2$-surharmoniques. Montrons le ii). Supposons que s_j soit un $\mathcal{H}_j$-potentiel fin dans U pour tout $j = 1, 2$, et soit (u, v) un couple finement $\mathcal{H}$-hypoharmonique dans U tel que $(u, v) \leq (s_1, s_2)$. Comme v est finement $\mathcal{H}_2$-hypoharmonique dans U, on a $v \leq 0$, il résulte alors de la définition des couples finement $\mathcal{H}$-hypoharmoniques que u est finement $\mathcal{H}_1$-hypoharmonique dans U, et donc $u \leq 0$. Inversement, supposons que le couple (s_1, s_2) soit un $\mathcal{H}$-potentiel fin. Si u est une fonction finement $\mathcal{H}$-hypoharmonique dans U, telle que $u \leq s_1$, alors le couple $(u, 0)$ est finement $\mathcal{H}$-hypoharmonique $\leq (s_1, s_2)$, d'où $(u, 0) \leq 0$, et par suite $u \leq 0$. Donc s_1 est un $\mathcal{H}_1$-potentiel fin dans U. De même, si v est une fonction finement $\mathcal{H}_2$-hypoharmonique dans U telle que $v \leq s_2$, le couple $(-\infty, v)$ est finement $\mathcal{H}$-hypoharmonique et on a $(-\infty, v) \leq (s_1, s_2)$, d'où $v \leq 0$. Donc s_2 est un $\mathcal{H}_2$-potentiel fin dans U.

Proposition 6.4 (Principe du maximum). *Soient ω un ouvert fin $\subset U$ et $(u, v) \in \mathcal{U}_f(\omega)$ tel que* $\liminf_{x \in \omega, x \to y} (u, v)(y) \geq 0$, *pour tout $y \in \partial_f \omega \cap U$. S'il existe un $\mathcal{H}$-potentiel fin P dans U tel que $(u, v) \geq -P$ dans ω, alors on a $(u, v) \geq 0$ dans ω.*

On signale que la réstriction de l'ordre spécifique dans $\mathcal{U}_f^+(U)$ à $\mathcal{S}_f^+(U)$ fait de ce dernier un treillis complètement réticulé.

Comme en théorie des fonctions finement harmoniques, un couple $H \in \mathcal{S}_f^+(U)$ sera dit $\mathcal{H}$-invariant (ou tout simplement invariant) s'il est orthogonal, pour l'ordre spécifique, à la bande des $\mathcal{H}$-potentiels fins. On note $\mathcal{H}_i(U)$ l'ensemble des couples $\mathcal{H}$-invariants. Il est facile de voir que $\mathcal{H}_i(U)$ est un cône convexe. C'est aussi une bande de $\mathcal{S}_f^+(U)$. On a donc,

$$\forall S \in \mathcal{S}_f^+(U), \exists! P \in \mathcal{P}_f(U), \exists! H \in \mathcal{H}_i(U) : S = P + H.$$

Il est clair aussi que tout couple finement biharmonique est invariant, mais la réciproque est fausse en général. En effet, si h une fonction invariante dans U qui ne soit pas finement $\mathcal{H}_1$-harmonique, le couple $(h, 0)$ est invariant qui n'est pas finement biharmonique.

Question. Le problème de savoir si une fonction invariante dans U est la somme d'une suite de fonctions finement harmoniques positives dans U a été posé par Fuglede dans [17]. A notre connaissance ce problème demeure toujours ouvert. Par analogie avec ce problème on peut poser la question suivante:

Est-ce que tout couple invariant dans U est la somme d'une suite de couples finement $\mathcal{H}$-harmoniques ≥ 0?

Même si la réponse au problème de Fuglede est positive, il semble qu'il n'est pas évident qu'il en soit de même pour les couples $\mathcal{H}$-invariants. En effet, comme on va le voir dans la suite, si $H = (h, k)$ est un couple $\mathcal{H}$-invariant, on a $h = h_1 + \mathcal{V}(k)$, où $\mathcal{V}$ est un noyau borélien sur U, et h_1 est invariante. On voit donc que, même si h_1 et k sont des sommes de suites de fonctions finement $\mathcal{H}_1$-harmoniques ≥ 0 et $\mathcal{H}_2$-harmoniques ≥ 0 respectivement, il ne semble pas facile d'affirmer que le couple $(\mathcal{V}(k), k)$ est la somme d'une suite de couples finement biharmoniques dans U.

7 – Couples finement hyperharmoniques purs

Proposition 7.1. *Soit v une fonction finement $\mathcal{H}_2$-hyperharmonique ≥ 0 dans un ouvert fin V de X. Alors la fonction*

$$u_v = \widehat{\inf}\{u \geq 0; (u, v) \in \mathcal{U}_f^+(V)\}$$

est finement $\mathcal{H}_1$-hyperharmonique dans V et l'on a $(u_v, v) \in \mathcal{U}_f^+(V)$.

Comme en théorie des fonctions biharmoniques, nous adoptons la définition suivante:

Définition 7.2. La fonction u_v de la proposition précédente est appelée la fonction finement hyperharmonique pure d'ordre 2 associée à v.

Soit (u, v) un couple finement $\mathcal{H}$-hyperharmonique ≥ 0 dans U. On dit que ce couple est pur si u est la fonction finement hyperharmonique pure d'ordre 2 associée à v.

Pour alléger les écritures, on notera dans la suite $\mathcal{V}_0(v)$ la fonction finement hyperharmonique d'ordre 2 associée à une fonction finement $\mathcal{H}_2$-hyperharmonique $v \geq 0$ dans U. Cette notation sera justifiée par la suite.

Proposition 7.3. *Soit $(u, v) \in \mathcal{S}_f^+(U)$ un couple pur. Si la fonction v est finement $\mathcal{H}_2$-harmonique dans un ouvert fin $V \subset U$, et si u est finie dans V, alors le couple (u, v) est finement biharmonique dans V.*

Démonstration. Soit ω un ouvert fin relativement compact régulier tel que $\widetilde{\omega} \subset V$; alors le couple $(u, v)_\omega$ est finement $\mathcal{H}$-hyperharmonique dans U d'après le Corollaire 2 de la Proposition 3.13, et l'on a

$$(u, v)_\omega = \Big(\int u d\mu_\cdot^\omega + \int v d\nu_\cdot^\omega, v\Big)$$

dans ω, d'où $u \leq \int u d\mu_\cdot^\omega + \int v d\nu_\cdot^\omega$, et, par suite, $u = \int u d\mu_\cdot^\omega + \int v d\nu_\cdot^\omega$ dans ω. Comme le couple (u, v) est finement continu, on en déduit qu'il est finement biharmonique dans V.

Proposition 7.4. *Soient v_1, v_2 deux fonctions finement $\mathcal{H}_2$-hyperharmoniques ≥ 0 dans U. Alors, si $v_1 \leq v_2$, on a $\mathcal{V}_0(v_1) \leq \mathcal{V}_0(v_2)$.*

Démonstration. La proposition résulte immédiatement de la Définition 7.2 et de la Proposition 3.7.

Proposition 7.5. *Soit $(s_1, s_2) \in \mathcal{S}_f^+(U)$. On a alors $\mathcal{V}_0(s_2) \prec s_1$, i.e. il existe $t \in \mathcal{S}_f^{1,+}(U)$ tel que $\mathcal{V}_0(s_2) + t = s_1$.*

Démonstration. Posons $s = \mathcal{V}_0(s_2)$. Soit ω un ouvert fin relativement compact régulier tel que $\overline{\omega} \subset U$ et soient $v \in -\mathcal{S}_f^{1,+}(\omega)$, bornée supérieurement, et $u \in \mathcal{S}_f^{1,+}(\omega)$, bornée inférieurement, telles que $\text{f-}\limsup_{x\to y, x\in\omega} v(x) \leq s_1(y)$ et $\text{f-}\liminf_{x\to y, x\in\omega} u(x) \geq s(y)$ pour tout $y \in \partial_f \omega$. Considérons la fonction

$$w = \begin{cases} \inf(s_1 + u - v, s) & \text{dans } \omega, \\ s & \text{dans } U \setminus \omega. \end{cases}$$

Alors, d'après les conditions ci-dessus sur u et v et la Proposition 3.10, le couple (w, s_2) est finement $\mathcal{H}$-surharmonique dans U. On en déduit $s_1 + u - v \geq s$ dans ω. Les fonctions u et v étant arbitraires, on en déduit donc d'après [12, Théorème 14.6] que, pour tout $x \in \omega$, tel que $s(x) < +\infty$, on a

$$s_1(x) - s(x) \geq \int (s_1 - s) d\epsilon_x^{1, C\omega}.$$

Le théorème de prolongement par continuité fine [12, Théorème 9.14] nous assure alors que la fonction $s_1 - s$ se prolonge en une fonction finement $\mathcal{H}_1$-surharmonique $t \geq 0$ dans U et l'on a donc $s_1 = s + t$.

PROPOSITION 7.6. *Pour tout* $j = 1, 2$, *soit* v_j *une fonction finement* $\mathcal{H}_j$*-hyperharmonique* ≥ 0 *dans* U. *On a alors*

$$\mathcal{V}_0(v_1 + v_2) = \mathcal{V}_0(v_1) + \mathcal{V}_0(v_2).$$

DÉMONSTRATION. L'inégalité $\mathcal{V}_0(v_1 + v_2) \leq \mathcal{V}_0(v_1) + \mathcal{V}_0(v_2)$ découle simplement de la Définition 7.1 et du fait que le couple $(\mathcal{V}_0(v_1)+\mathcal{V}_0(v_2), v_1 + v_2) = (\mathcal{V}_0(v_1), v_1) + (\mathcal{V}_0(v_2), v_2)$ est finement $\mathcal{H}$-hyperharmonique ≥ 0 dans U. Montrons l'inégalité inverse. Le résultat est trivial si $\mathcal{V}_0(v_1) \equiv +\infty$ ou $\mathcal{V}_0(v_2) \equiv +\infty$. Supposons donc que les fonctions $\mathcal{V}_0(v_1)$ et $\mathcal{V}_0(v_2)$ soient finement $\mathcal{H}_1$-surharmoniques (on rappelle que l'ouvert fin U est supposé finement connexe). Alors, d'après la propriété de décomposition de Riesz des couples finement $\mathcal{H}$-surharmoniques ≥ 0 appliquée à l'inégalité $(\mathcal{V}_0(v_1 + v_2), v_1 + v_2) \leq (\mathcal{V}_0(v_1), v_1) + (\mathcal{V}_0(v_2), v_2)$, on peut trouver deux couples finement $\mathcal{H}$-surharmoniques dans U, $(s_1, t_1) \geq (0, 0)$ et $(s_2, t_2) \geq (0, 0)$, tels que $(\mathcal{V}_0(v_1 + v_2), v_1 + v_2) = (s_1, t_1) + (s_2, t_2)$, et $(s_1, t_1) \leq (\mathcal{V}_0(v_1), v_1)$ et $(s_2, t_2) \leq (\mathcal{V}_0(v_2), v_2)$, ce qui entraîne $t_1 = v_1$ et $t_2 = v_2$ et donc $s_1 = \mathcal{V}_0(v_1)$ et $s_2 = \mathcal{V}_0(v_2)$, ce qui achève la démonstration de la proposition.

PROPOSITION 7.7. *Soient* (v_n) *une suite croissante de fonctions finement* $\mathcal{H}_2$*-hyperharmoniques* ≥ 0 *dans* U, *et soit* $v = \sup_n v_n$. *On a alors* $\mathcal{V}_0(v) = \sup_n \mathcal{V}_0(v_n)$.

DÉMONSTRATION. On a $(\mathcal{V}_0(v), v) \in \mathcal{U}_f^+(U)$, et $v \geq v_n$, donc $(\mathcal{V}_0(v), v_n) \in \mathcal{U}_f^+(U)$ pour tout n d'après la Proposition 7.4, donc $\mathcal{V}_0(v) \geq \mathcal{V}_0(v_n)$ pour tout n, et par suite $\mathcal{V}_0(v) \geq \sup_n \mathcal{V}_0(v_n)$. D'autre part, on a $(\sup_n \mathcal{V}_0(v_n), \sup_n v_n) \in \mathcal{U}_f^+(U)$ pour tout n, d'après la Proposition 3.6, donc $(\sup_n \mathcal{V}_0(v_n), v) \in \mathcal{U}_f^+(U)$, d'où $\mathcal{V}_0(v) \leq \sup_n \mathcal{V}_0(v_n)$.

Pour toute fonction $\mathcal{H}_1$-surharmonique $s \geq 0$ sur X, la fonction $s - {}^1\widehat{R}_s^{CU}$, est bien définie et finement $\mathcal{H}_1$-surharmonique dans le complémentaire dans U d'un ensemble $\mathcal{H}_1$-polaire. Elle se prolonge donc, en vertu du principe du prolongement par continuité fine, en une fonction de $\mathcal{S}_f^{1,+}(U)$, notée s_U.

Soit $\mathcal{V}_U$ le noyau borélien sur U défini par

$$\mathcal{V}_U(f) = \mathcal{V}(\overline{f}) - {}^1\widehat{R}^{CU}_{\mathcal{V}(\overline{f})}|_U$$

pour toute fonction borélienne $f \geq 0$ sur U, où $\overline{f}$ est le prolongement de f à X, nul dans CU, et $\mathcal{V}$ est le noyau du Théorème 2.5. On remarquera que si $U = X$, alors $\mathcal{V}_U = \mathcal{V}$.

Si ω est un ouvert fin de U, on notera $\mathcal{V}_\omega$ le noyau égal à $\mathcal{V}_\delta$ dans chaque composante finement connexe δ de ω.

On note $\mathcal{S}_j^+(X)$, $j = 1, 2$, le cône des fonctions $\mathcal{H}_j$-surharmoniques ≥ 0.

Posons

$$\mathcal{S}_2'(X) = \{t \in \mathcal{S}_2^+(X) : \mathcal{V}(t) \in \mathcal{S}_1^+(X)\}.$$

Remarquons que si $t \in \mathcal{S}_2'(X)$, alors $\mathcal{V}(t)_U = \mathcal{V}_U(t|_U)$.

Proposition 7.8. *Soit $t \in \mathcal{S}_2'(X)$. Alors la fonction finement hyperharmonique pure d'ordre 2 associée à la restriction de t à U est égale à $\mathcal{V}(t)_U$.*

Démonstration. Soit q_0 un $\mathcal{H}_2$-potentiel > 0 tel que $\mathcal{V}(q_0) < +\infty$. On a $\mathcal{V}(t)_U = \sup_n \mathcal{V}(t \wedge nq_0)_U$. D'après la Proposition 7.7, il suffit de montrer que la fonction hyperharmonique pure d'ordre 2 associée à $(t \wedge nq_0)|_U$ est égale à $\mathcal{V}(t \wedge nq_0)_U$, ce qui permet de se ramener au cas où $\mathcal{V}(t)$ est finie. Le couple $(\mathcal{V}(t)_U, t)$ est $\mathcal{H}$-finement hyperharmonique ≥ 0 dans U. En effet, on a, pour tout ouvert fin $\delta \subset \overline{\delta} \subset U$ et tout $x \in \delta$,

$$\begin{aligned}\int \mathcal{V}(t)_U d\mu_x^\delta + \int t d\nu_x^\delta \leq \mathcal{V}(t)(x) - \int {}^1\widehat{R}^{CU}_{\mathcal{V}(t)}|_U d\mu_x^\delta =\\ = (\mathcal{V}(t) - {}^1\widehat{R}^{CU}_{\mathcal{V}(t)})(x),\end{aligned}$$

car le couple $(\mathcal{V}(t), t)$ est finement $\mathcal{H}$-hyperharmonique dans U et la fonction ${}^1\widehat{R}^{CU}_{\mathcal{V}(t)}$ est finement $\mathcal{H}_1$-harmoniques dans U d'après [12, Théorème 10.2]. Soit u une fonction finement $\mathcal{H}_1$-surharmonique ≥ 0 dans U

telle que le couple (u,t) soit finement $\mathcal{H}$-surharmonique dans U, et soit u_1 la fonction définie par

$$u_1 = \begin{cases} \min(u + {}^1\widehat{R}^{CU}_{\mathcal{V}(t)}, \mathcal{V}(t)) & \text{dans } U, \\ \mathcal{V}(t) & \text{dans } X \setminus U. \end{cases}$$

Alors, d'après le Théorème 3.10, le couple (u_1,t) est finement $\mathcal{H}$-surharmonique dans X. Or, comme $t \geq 0$, la fonction u_1 est finement $\mathcal{H}_1$-surharmonique dans X, donc elle est $\mathcal{H}_1$-surharmonique dans X en vertu du Théorème 9.8 de [12]. Il en résulte, d'après la définition des couples finement $\mathcal{H}$-surharmoniques que (u_1,t) est $\mathcal{H}$-surharmonique dans X, d'où $u + {}^1\widehat{R}^{CU}_{\mathcal{V}(t)} \geq \mathcal{V}(t)$ et donc le résultat.

Lemme 7.9. *Pour toute fonction $s \in \mathcal{S}^{2,+1}_f(U)$ majorée par un élément de $\mathcal{S}'_2(X)$, il existe une suite croissante (t_n) de fonctions de $\mathcal{V}(X)$ telle que $s = \sup_n (t_n)_U$.*

Démonstration. Le lemme résulte aussitôt du Théorème 3 de [14].

Théorème 7.10. *Pour toute fonction finement $\mathcal{H}_2$-hyperharmonique $v \geq 0$ dans U, $\mathcal{V}_U(v)$ est la fonction hyperharmonique pure d'ordre 2 associée à v.*

Démonstration. Soit $s \in \mathcal{S}'_2(X)$. On a alors

$$s|_U = s_U + {}^2\widehat{R}^{CU}_t|_U,$$

d'où, d'après la Proposition 7.6,

$$\mathcal{V}_0(s|_U) = \mathcal{V}_0(s_U) + \mathcal{V}_0({}^2\widehat{R}^{CU}_s|_U),$$

et, par suite, d'après la Proposition 7.8,

$$\begin{aligned} \mathcal{V}_0(s_U) &= \mathcal{V}_0(s|_U) - \mathcal{V}_0({}^2\widehat{R}^{CU}_s|_U) = \\ &= \mathcal{V}(s)_U - \mathcal{V}({}^2\widehat{R}^{CU}_s)_U, \end{aligned}$$

q.p. dans U. D'autre part, un calcul facile donne

$$\mathcal{V}(s)_U - \mathcal{V}({}^2\widehat{R}^{CU}_s)_U = \mathcal{V}_U(s_U) \text{ q.p.},$$

d'où $\mathcal{V}_0(t_U) = \mathcal{V}_U(t_U)$. En vertu de la Proposition 7.7, le théorème découle maintenant du Lemme 7.9 et du fait que tout élément de $\mathcal{U}_f^{2,+}(U)$ est l'enveloppe supérieure d'une suite croissante de fonctions de $\mathcal{S}_f^{2,+}(U)$ majorées par des éléments de $\mathcal{S}_2'(X)$.

Remarque. Si v est une fonction finement $\mathcal{H}_2$-hyperharmonique ≥ 0 dans un ouvert fin ω de U, alors la fonction finement hyperharmonique pure d'ordre 2 associée à v est égale à $\mathcal{V}_\omega(v)$.

Le théorème suivant est une application du précédent:

Théorème 7.11. *Si (u, v) est un couple finement $\mathcal{H}$-surharmonique localement borné inférieurement dans X, alors (u, v) est un couple surharmonique dans X.*

Démonstration. Quitte à se placer localement, on peut supposer que $(u, v) \geq 0$ dans X. Alors la fonction v est finement $\mathcal{H}_2$-hyperharmonique ≥ 0, donc $\mathcal{H}_2$-hyperharmonique dans X d'après [12, Théorème 9.8]. D'autre part on a, d'après ce qui précède, $u = \mathcal{V}(v) + t$, où t est une fonction finement harmonique ≥ 0, donc hyperharmonique dans X toujours d'après [12, Théorème 9.8]. Maintenant le théorème résulte du fait que, dans le cas où $U = X$, le noyau $\mathcal{V}_U$ coïncide avec le noyau $\mathcal{V}$ du Paragraphe 2.

Proposition 7.12. *Soit $(u, v) \in \mathcal{S}_f^+(U)$ un couple pur. Si v est un $\mathcal{H}_2$-potentiel fin, alors (u, v) est un $\mathcal{H}$-potentiel fin.*

Démonstration. Soit (h, k) un couple finement $\mathcal{H}$-hypoharmonique dans U tel que $0 \leq (h, k) \leq (u,v)$. Alors k est finement $\mathcal{H}_2$-hypoharmonique ≥ 0 et minore v, donc $k = 0$. On en déduit que h est finement $\mathcal{H}_1$-hypoharmonique, donc le couple $(u-h,v)$ est finement $\mathcal{H}$-hyperharmonique ≥ 0, de sorte que $u - h \geq u$, et par suite $h = 0$.

Maintenant on peut donner également quelques applications de la Proposition 7.13. aux couples invariants.

Proposition 7.13. *Si (h, k) est un couple invariant dans U, alors k est une fonction $\mathcal{H}_2$-invariante dans U.*

Démonstration. En effet, si p est un $\mathcal{H}_2$-potentiel fin qui minore spécifiquement k, alors le couple (h, p) est finement $\mathcal{H}$-surharmonique dans U et donc le couple $(\mathcal{V}_0(p), p)$ est, d'après la proposition précédente, un $\mathcal{H}$-potentiel fin qui minore spécifiquement (h, k) d'après la Proposition 7.5, donc il est nul.

Comme en théorie des fonctions finement harmoniques, nous avons la

Proposition 7.14. *Soit $H = (h, k)$ un couple invariant dans U. Alors H est finement $\mathcal{H}$-harmonique dans le domaine fin $\omega = \{x \in U; h(x) + k(x) < +\infty\}$.*

Démonstration. En effet, comme la fonction k est $\mathcal{H}_2$-invariante d'après la Proposition 7.13, elle est finement harmonique dans ω d'après [12, Théorème 10.2]. La proposition découle maintenant de la Proposition 7.3. appliquée au couple $(\mathcal{V}(k), k)$ et du fait que la fonction finement $\mathcal{H}_1$-hyperhamonique $u \geq 0$ dans U telle que $h = u + \mathcal{V}(k)$, qui est évidemment $\mathcal{H}_1$-invariante, est finement $\mathcal{H}_1$-harmonique dans ω puisqu'elle est finie dans ω.

Proposition 7.15. *Soit $(u, v) \in \mathcal{S}_f^+(U)$ un couple pur. Alors, si v est $\mathcal{H}_2$-invariante dans U, le couple (u, v) est $\mathcal{H}$-invariant. En particulier si v est finement $\mathcal{H}_2$-harmonique dans U, et si u est finie dans U, alors le couple (u, v) est finement $\mathcal{H}$-harmonique dans U.*

Démonstration. Soit (p, q) un $\mathcal{H}$-potentiel fin tel que $(p, q) \prec (u, v)$. On a alors $q \prec v$, et comme v est $\mathcal{H}_2$-invariante et q est un $\mathcal{H}$-potentiel fin, on a $q = 0$, et par suite $(u, v) = (u_1, v) + (p, 0)$, où $(u_1, v) \in \mathcal{S}_f^+(U)$. Mais alors on aura $u_1 \geq u$ et donc $p = 0$ et le couple (u, v) est invariant. Le reste de la proposition est évident.

8 – Problème de Riquier fin

Soit ω un ouvert fin de X. On note $\mathcal{U}_f^i(\omega)$ l'ensemble des couples finement hyperharmoniques (u, v) dans ω tels qu'il existe un $\mathcal{H}$-potentiel semi-borné fini $P = (p, q)$ tel que $(u, v) \geq -P$.

Si (f, g) un couple de fonctions sur $\partial_f \omega$, on pose

$$\overline{H}^{\omega}_{(f,g)} = \inf\{(u, v) \in \mathcal{U}_f^i(\omega) : \mathrm{f}-\liminf_{x \in \omega, x \to y} (u, v)(x) \geq (f(y), g(y)), \forall y \in \partial_f(\omega)\}.$$

On pose aussi $\overline{H}^{\omega}_{(f,g)} = (\overline{H}^{\omega,1}_{(f,g)}, \overline{H}^{\omega,2}_{(f,g)})$ et $\underline{H}^{\omega}_{(f,g)} = -\overline{H}^{\omega}_{(-f,-g)}$.

Il est clair que le couple $\overline{H}^{\omega}_{(f,g)}$ (resp. $\underline{H}^{\omega}_{(f,g)}$) est un couple finement $\mathcal{H}$-hyperharmonique (resp. finement $\mathcal{H}$-hypoharmonique) dans ω.

On dit qu'un couple (f, g) de fonctions sur $\partial_f \omega$ est résolutif (pour le problème de Riquier fin) si on a $\underline{H}^{\omega}_{(f,g)} = \overline{H}^{\omega}_{(f,g)}$ et si ces couples sont finement $\mathcal{H}$-harmoniques dans ω; on les notera alors par $H^{\omega}_{(f,g)}$.

Pour tout $j = 1, 2$, et toute fonction f sur $\partial_f \omega$, on note ${}^j\overline{H}^{\omega}_f$ (resp. ${}^j\underline{H}^{\omega}_f$) la sursolution (la sousolution) du problème de Dirichlet fin dans l'espace harmonique $(X, \mathcal{H}_j)$ pour la donnée frontière f sur $\partial_f \omega$.

Il résulte de la définition et des propriétés des couples finement $\mathcal{H}$-hyperharmoniques que l'on a $\overline{H}^{\omega,1}_{(f,0)} = {}^1\overline{H}^{\omega}_f$ et $\overline{H}^{\omega,2}_{(f,g)} = {}^2\overline{H}^{\omega}_g$, avec les notations de [12], p. 173, relatives au problème de Dirichlet fin.

Théorème 8.1. *Soit (f, g) un couple de fonctions sur $\partial_f \omega$. On a alors*

$$\overline{H}^{\omega,1}_{(f,g)} = \int^* f d\mu^{\omega}_{\cdot} + \int^* g d\nu^{\omega}_{\cdot} \text{ et } \overline{H}^{\omega,2}_{(f,g)} = \int^* g d\lambda^{\omega}_{\cdot}.$$

Démonstration. Le théorème se démontre comme dans le cas finement harmonique ([12], preuve du Théorème 14.6), en utilisant le Théorème 3.3.

Corollaire 1. *Pour tout couple (f, g) de fonctions sur $\partial_f \omega$, on a*

$$\overline{H}^{\omega}_{(f,g)} = ({}^1\overline{H}^{\omega}_f + \overline{H}^{\omega,1}_{(0,g)}, {}^2\overline{H}^{\omega}_g).$$

Corollaire 2. *Un couple (f, g) de fonctions sur $\partial_f \omega$ est résolutif si, et seulement si, pour tout $x \in \omega$, f est μ^{ω}_x-intégrable et g est ν^{ω}_x-intégrable et λ^{ω}_x-intégrable.*

Corollaire 3. *Soit $(u, v) \in \mathcal{U}^+_f(U)$ et ω un ouvert fin de X tel que $\widetilde{\omega} \subset U$, alors on a*

$$\overline{H}^{\omega}_{(u,v)} = (u, v)_{\omega}|_{\omega}.$$

On dit qu'un couple (f, g) de fonctions sur une partie A de X est borné par un $\mathcal{H}$-potentiel P si on a $(|f|, |g|) \leq P$ sur A.

Corollaire 4. *Soit (f, g) un couple de fonctions boréliennes, borné sur $\partial_f \omega$ par un $\mathcal{H}$-potentiel semi-borné fini. Alors (f, g) est résolutif.*

Théorème 8.2. *Supposons que l'ouvert fin ω est $\mathcal{H}$-régulier et soit (f, g) un couple de fonctions finement continues sur $\partial_f \omega$, borné par un $\mathcal{H}$-potentiel semi-borné fini. On a alors*

$$\text{f}-\lim_{x\in\omega\to y} H^{\omega}_{(f,g)}(x) = (f(y), g(y))$$

pour tout $y \in \partial_f \omega$.

Démonstration. Quitte à ajouter à (f, g) un $\mathcal{H}$-potentiel semi-borné fini, on peut supposer que le couple (f, g) est ≥ 0. D'après le théorème précédent et le Théorème 14.6 de [12], on a $H^{\omega}_{(f,g)} = ({}^1H^{\omega}_f + H^{\omega,1}_{(0,g)}, {}^2H^{\omega}_g)$. Or, on sait d'après [12, Théorème 14.7], que pour tout $y \in \partial_f \omega$, f-$\lim_{x\in\omega, x\to y} {}^1H^{\omega}_f(x) = f(y)$ et f-$\lim_{x\in\omega,x\to y} {}^2H^{\omega}_g(x) = g(y)$, donc f-$\liminf_{x\in\omega,x\to y} H^{\omega}_{(f,g)}(x) \geq (f(y), g(y))$. Soit sirt P un $\mathcal{H}$-potentiel semi-borné fini tel que $(f, g) \leq P$ sur $\partial_f \omega$, alors en appliquant ce qui précède au couple $P - (f, g)$, on obtient f-$\limsup_{x\in\omega,x\to y} H^{\omega}_{(f,g)}(x) \leq (f(y), g(y))$, et le théorème est donc démontré.

Proposition 8.3. *Si g est une fonction ≥ 0 sur $\partial_f \omega$, alors $\overline{H}^{\omega,1}_{(0,g)}$ est la fonction hyperharmonique pure d'ordre 2 associée à ${}^2\overline{H}^{\omega}_g$ dans ω.*

Démonstration. Si $(u, v) \in \mathcal{U}^i_f(\omega)$ tel que f-$\liminf(u, v) \geq (0, g)$ sur $\partial_f \omega$, alors $v \geq {}^2\overline{H}^{\omega}_g$, et donc $u \geq \mathcal{V}_\omega({}^2\overline{H}^{\omega}_g)$, d'où l'inégalité $\overline{H}^{\omega,1}_{(0,g)} \geq \mathcal{V}_\omega({}^2\overline{H}^{\omega}_g)$. D'autre part, on peut trouver une suite décroissante (v_n) de fonctions de $\mathcal{S}^{2,+}_f(\omega)$ telle que $\inf_n v_n = {}^2\overline{H}^{\omega}_g$. On a alors $\overline{H}^{\omega}_{(0,g)} \leq \inf_n(\mathcal{V}_\omega(v_n), v_n) = (\mathcal{V}_\omega({}^2\overline{H}^{\omega}_g), {}^2\overline{H}^{\omega}_g)$, donc $\overline{H}^{\omega,1}_{(0,g)} \leq \mathcal{V}_\omega({}^2\overline{H}^{\omega}_g)$.

Théorème 8.4. *Soient ω un ouvert fin régulier tel que $\widetilde{\omega} \subset U$, et $(u, v) \in \mathcal{U}^+_f(U)$. On a alors $(u, v)_\omega = \widehat{(u, v)}^{C\omega}$.*

Démonstration. Soit $(s,t) \in \mathcal{U}_f^+(U)$ tel que $(s,t) \geq (u,v)$ sur $C\omega$. Alors, pour tout $x \in \omega$, on a, d'après le Théorème 3.3, $s(x) \geq \int^* s d\mu_x^\omega + \int^* t d\nu_x^\omega$ et $t(x) \geq \int^* t d\lambda_x^\omega$, et donc $s(x) \geq \int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega$ et $t(x) \geq \int^* v d\lambda_x^\omega$, car les mesures μ_x^ω, ν_x^ω et λ_x^ω sont portées par $\partial_f \omega$. Donc $\widehat{(u,v)}^{C\omega} \geq (u,v)_\omega$. L'inégalité inverse résulte du Corollaire 2 de la Proposition 3.13.

9 – Fonctions finement biharmoniques

Lemme 9.1. *Pour tout ouvert fin ω de X tel que $\overline{\omega} \subset X$ et tout $x \in \omega$, on a $\int d\nu_x^\omega > 0$.*

Démonstration. D'après le Théorème 8.1, le couple $(\int d\nu_\cdot^\omega, 1)$ est finement surharmonique ≥ 0, non identiquement nul dans toute composante finement connexe de ω, donc $\int d\nu_x^\omega > 0$ pour tout $x \in \omega$.

Considérons maintenant la famille $D(U)$ des fonctions f finies finement continues sur U telles que la limite

$$Lf(x) = \lim_{\omega \downarrow x} \frac{f(x) - \displaystyle\int f(y) d\mu_x^\omega(y)}{\displaystyle\int d\nu_x^\omega(y)}$$

existe et soit finie pour tout $x \in U$ (la fraction $\frac{f(x) - \int f(y) d\mu_x^\omega(y)}{\int d\nu_x^\omega(y)}$ est bien définie lorsque $\overline{\omega} \subset X$ d'après le lemme précédent).

Definition 9.2. Une fonction f finement continue sur U est dite finement $\mathcal{H}$-biharmonique (ou simplement finement biharmonique) dans U si $f \in D(U)$ et si Lf est finement $\mathcal{H}_2$-harmonique dans U.

La proposition suivante met en évidence le lien qui existe entre la notion de fonction finement harmonique au sens de la Définition 9.1 et la notion de couple finement biharmonique:

Proposition 9.3. *Soit (u,v) un couple finement biharmonique dans un ouvert fin U. Alors $u \in D(U)$ et $Lu = v$.*

Démonstration. Soient $x \in U$ et $\epsilon > 0$. Comme v est finement continue, il existe un ouvert fin $\omega_0 \subset U$, $x \in \omega_0$, tel que $|v(x) - v(y)| < \epsilon$ pour tout $y \in \omega_0$. Alors, pour tout ouvert fin $\omega \subset \overline{\omega} \subset \omega_0$, $x \in \omega$, on a

$$|u(x) - \int u d\mu_x^\omega - v(x) \int d\nu_x^\omega| < \epsilon \int d\nu_x^\omega,$$

donc $u \in D(U)$ et $Lu = v$.

Corollaire. *Soit (u, v) un couple finement biharmonique dans un ouvert fin U. Alors u et finement biharmonique dans U.*

REFERENCES

[1] N. Boboc – Gh. Bucur – A. Cornea: *Order and convexity in Potential theory: H-cones*, Lecture Notes in Math., **853** Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[2] N. Boboc – Gh. Bucur: *Perturbations in excessive structures*, Lecture Notes in Math., **1014** Springer-Verlag, Berlin, Heidelberg, New York (1983), 155-187.

[3] A. Boukricha: *Espaces biharmoniques*, Lecture Notes in Math., **1096** Springer-Verlag, Berlin, Heidelberg, New York (1983), 116-148.

[4] N. Bouleau: *Espaces biharmoniques et couplages de processus de Markov*, J. Math. Pures Appl., **58** (1979), 187-204.

[5] M. Brelot: *Éléments de la théorie classique du potentiel*, C.D.U, 1969.

[6] G. Choquet: *Lectures on Analysis, Vol. 2*, Mathematics Lect. Notes Series, Benjamen, Inc. New York, 1969.

[7] C. Constantinescu – A. Cornea: *Potential Theory on Harmonic Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[8] C. Dellacherie – P.A. Meyer: *Probabilités et Potentiel*, chap. I à IV, Hermann, Paris, 1975.

[9] C. Dellacherie – P.A. Meyer: *Probabilités et Potentiel*, Chap. XII à XVI, Hermann, Paris, 1987.

[10] M. El Kadiri: *Représentation intégrale dans le cadre de la théorie des fonctions biharmoniques*, Rev. Roumaine Math. Pures Appl. (1997), 579-589.

[11] M. El Kadiri: *Fonctions finement biharmoniques*, Rend. Accad. Sci., **40** (5) (2000), 43-61.

[12] B. Fuglede: *Finely harmonic functions*, Lecture Notes in Math., **289** Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[13] B. Fuglede: *Sur la fonction de Green dans un domaine fin*, Ann. Inst. Fourier, **25** (1975), 201-206.

[14] B. Fuglede: *Localization in Fine Potential Theory and Uniform Approximation by Subharmonic Functions*, J. Funct. Anal., **49** (1982), 52-72.

[15] B. Fuglede: *Integral Representation of Fine Potential*, Math. Ann., **262** (1983), 191-214.

[16] B. Fuglede: *Représentation intégrale des potentiels fins*, C.R. Acad. Sci. Paris, **300** Série I, (1985), 129-132.

[17] B. Fuglede: *On the Riesz representation of finely superharmonic functions*, Lecture Notes in Math., **1344** Springer-Verlag, Berlin, Heidelberg, New York, (1987), 199-201.

[18] R.M. Hervé: *Rechreches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier, **12** (1962), 415-517.

[19] E.P. Smyrnelis: *Axiomatique des fonctions biharmoniques*, 1e section, Ann. Inst. Fourier, **25** (1975), 35-97.

[20] E.P. Smyrnelis: *Axiomatiques des fonctions biharmoniques*, 2e section, Ann. Inst. Fourier, **26** (1976), 1-47.

[21] E.P. Smyrnelis: *Sur les fonctions hyperharmoniques d'ordre 2*, Lecture Notes in Math., **681** Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[22] E.P. Smyrnelis: *Support biharmonique et support harmonique associé*, Lecture Notes in Math., **787**, Springer-Verlag Berlin Heidelberg New York, 1980.

Lavoro pervenuto alla redazione il 4 novembre 2002
ed accettato per la pubblicazione il 9 giugno 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo degli Autori:

Charaf Bensouda – Département de Mathématiques – Faculté des Sciences – Kénitra (Marocco)

Mohamed El Kadiri – B.P. 726 – Salé-Tabriquet – Salé (Marocco)
E-mail: elkadiri@fsr.ac.ma

Ilham Rouchdi – Département de Mathématiques – Faculté des Sciences – Rabat (Marocco)

Rendiconti di Matematica, Serie VII
Volume 23, Roma (2003), 163-188

A kinetic approach to studying the asymptotic behaviour of convection-diffusion equations

R. CAVAZZONI

ABSTRACT: *We present a new approach to the study of the large time behaviour of solutions to the Cauchy problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u^{p-1}\dfrac{\partial u}{\partial x} & x \in \mathbb{R}, \qquad t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

where $p \geq 2$ and $u_0(x) \geq 0$.

1 – Introduction and main results

The aim of this paper is to show how kinetic methods can be used in the study of the long time behaviour of the solution to Cauchy problems for convection-diffusion equations having the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - u^{p-1}\dfrac{\partial u}{\partial x} & x \in \mathbb{R}, \qquad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

where $p \geq 2$ and u_0 is a nonnegative function from $L^1(\mathbb{R})$. More precisely,

KEY WORDS AND PHRASES: *Large-time behaviour – Convection-diffusion equations – Kinetic theory – Diffusion wave.*
A.M.S. CLASSIFICATION: 35K55 – 35B40

we prove that there exists a function u_∞ such that, for every $q \in [1, \infty)$,

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \|u(\cdot,t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0. \tag{2}$$

In 1950, Hopf [7] found an explicit regular solution to the viscous Burgers equation, i.e. equation (1) with $p = 2$. His proof makes use of the so called Hopf-Cole transformation, which turns the viscous Burgers equation into a linear heat equation. More recently, useful estimates on the L^r-norm of the solution to (1) have been derived (see [11] or [5]). In particular, the $L^r(\mathbb{R})$ to $L^q(\mathbb{R})$ smoothing properties of (1) have been shown to be exactly the same as the standard heat equation. In 1987, Chern and Liu [4] studied the large time behaviour of solutions of viscous conservation laws. In 1991, Escobedo and Zuazua [5] analysed the large time behaviour of solutions to the Cauchy problem for convection-diffusion equations. The main result of [5] tells us that if $p = 2$, then the general solution $u = u(x,t)$ to (1) behaves like the self-similar solution as $t \to +\infty$.

In the case where $p > 2$, it is proved in [5] that for every $r \in [1, \infty]$

$$\| u(\cdot,t) - G(\cdot,t) \|_{L^r} \longrightarrow 0 \ \text{ as } \ t \to +\infty, \tag{3}$$

where G is the heat kernel. These results have been obtained by a direct application of standard estimates for the heat kernel and by decay estimates in the integral equation associated with (1).

Related results on the long time behaviour of nonnegative solutions of nonlinear diffusion equations are contained e.g. in [6], [8], [11], [12], [13].

Our main purpose here is the use of a completely different approach to investigate the asymptotic behaviour of diffusion equations. The underlying idea of our approach is derived from the H-theorem of kinetic theory of rarefied gases [3]. In the last years the derivation of diffusion equations as a hydrodynamic limit of particles models has been a well studied subject in kinetic theory [3]. In this paper we shall look for a suitable functional to describe the evolution of the solution to problem (1), in a similar way as for the solution of the Boltzmann equation (see [3] and references cited in [3]). The kinetic approach has been recently applied by Carrillo and Toscani [2] for the N-dimensional porous medium equation: they have obtained the rate of convergence to equilibrium, by an analysis of the time evolution of the entropy production.

In the present paper we consider the asymptotic behaviour of the solution to (1) by techniques different from those of [5]. In particular, we do not make use of L^q-estimates for the derivatives of the solution. We study separately the cases $p = 2$ and $p > 2$. In the case where $p = 2$, we construct a suitable functional to prove the convergence to equilibrium by using the monotonicity in time of the functional. The result on the large time behaviour of equation (1) is proved in the following theorem.

Theorem 1.1. *Assume that* $p = 2$ *and*

$$\begin{aligned} &u_0 \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 1; \\ &u_0(x) \geq 0 \quad a.e. \quad x \in \mathbb{R}. \end{aligned} \tag{4}$$

Let u be the solution to the Cauchy problem (1). *Then for every* $q \in [1, \infty)$

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \, \|u(\cdot, t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0,$$

where

$$u_\infty(x, 2t+1) = \frac{1}{(2t+1)^{1/2}} \frac{e^{-\frac{x^2}{2(2t+1)}}}{(2\pi)^{\frac{1}{2}} \, A - \dfrac{1}{2} \displaystyle\int_{-\infty}^{\frac{x}{(2t+1)^{1/2}}} e^{-y^2/2} dy},$$

and $A = \frac{e^{1/2}}{2(e^{1/2}-1)}$.

A variant of the techniques involved in the proof of Theorem 1.1 enables us to describe the large time behaviour of the solution to (1) when $p > 2$. Indeed, we perform the same time dependent scaling. Let us emphasize that the equation obtained after scaling has coefficients depending on t. The convex functional, which represents the physical entropy for the viscous Burgers equation, will be used to study the large time behaviour also for $p > 2$. However, in this case the functional is not monotone in time. Nevertheless, the specific form of the time derivative allows us to identify the limit as a stationary solution to the Fokker-Planck equation.

Theorem 1.2. *Assume that* $p > 2$ *and*

$$u_0 \in L^1(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 1; \tag{5}$$
$$u_0(x) \geq 0 \quad a.e. \quad x \in \mathbb{R}.$$

Let u *be the solution to the Cauchy problem* (1)*. Then for every* $q \in [1, \infty)$

$$\lim_{t \to +\infty} (2t+1)^{1/2(1-1/q)} \|u(\cdot, t) - u_\infty(\cdot, 2t+1)\|_{L^q(\mathbb{R})} = 0,$$

where

$$u_\infty(x, 2t+1) = \frac{1}{(2t+1)^{1/2}} \gamma \exp\left(-\frac{x^2}{2(2t+1)}\right),$$

and $\gamma = (\int_{\mathbb{R}} \exp(-\frac{x^2}{2})dx)^{-1}$.

The paper is organized as follows. In Section 2, we recall some known results on the initial value problem (1) and we derive some bounds on the solution. Section 3 is devoted to the study of the functional to which we alluded above and to the proof of convergence results. In Section 4, we describe the asymptotic behaviour of (1) when $p \geq 2$, and conclude the proofs of Theorems 1.1 and 1.2. In the last section, we prove that our method can be applied to study the long time behaviour of the solution to a class of convection-diffusion equations in $\mathbb{R}^N$, with $N > 1$.

2 – Preliminaries

In the present section we recall some known results on the solution to the Cauchy problem (1) and we prove some technical Lemmas.

Let us consider the Cauchy problem (1). Thanks to the results by Hopf [7], we know that the viscous Burgers equation (i.e. equation (1) with $p = 2$) admits the following regular solution.

Theorem 2.1. *Let* $u_0 \in L^1(\mathbb{R})$*. Then*

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\, d\eta\right]\right\} dy}{\displaystyle\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\, d\eta\right]\right\} dy}, \tag{6}$$

is a solution to the viscous Burgers equation for $t > 0$ *and satisfies the initial condition:*

$$\int_0^x u(\xi, t)\, d\xi \longrightarrow \int_0^a u_0(\xi)\, d\xi \quad as \quad x \to a \quad and\, t \to 0; \tag{7}$$

for every $a \in \mathbb{R}$*. If* $u_0 \in C(\mathbb{R})$*, then* $u(x,t) \longrightarrow u_0(a)$ *if* $x \to a,\ t \to 0$*. Moreover, given any* $T > 0$*, the function* u *defined by* (6) *is the unique regular solution to the viscous Burgers equation in the strip* $0 < t < T$*, satisfying* (7) *for every* $a \in \mathbb{R}$*.*

The solution to (1) given by (6) belongs to the space $C((0,\infty); L^1(\mathbb{R}))$. We shall make use of the following result of [5], on the existence of the solution together with decay rates, for the initial value problem (1).

THEOREM 2.2. *Given* $u_0 \in L^1(\mathbb{R})$*, there exists a unique classical solution* $u \in C([0,\infty); L^1(\mathbb{R}))$ *to* (1)*, which satisfies the following properties:*

(i) *for every* $q \in (1,\infty)$, $\quad u \in C((0,\infty); W^{2,q}(\mathbb{R})) \bigcap C^1((0,\infty); L^q(\mathbb{R}))$.

(ii) *For every* $q \in [1,\infty)$*, there exists a constant* $C_q = C(q, \|u_0\|_1)$ *such that for every* $t > 0$*:*

$$\begin{cases} \|u(t)\|_q \le C_q t^{-1/2(1-1/q)}, \\ \|u(t)\|_1 \le \|u_0\|_1. \end{cases} \tag{8}$$

(iii) *Let* t_0 *be a nonnegative real number. Then there exists a positive constant* C_∞ *such that for every* $t \ge t_0$*:*

$$\|u(t)\|_\infty \le C_\infty t^{-1/2}. \tag{9}$$

If $p = 2$*, then*

$$\|u(t)\|_\infty \le C_\infty t^{-1/2},$$

for every $t > 0$*.*

One can easily verify that the unique solution in $C\left([0,\infty);L^1(\mathbb{R})\right)$ provided by Theorem 2.2 satisfies condition (6) of Theorem 2.1. Then the unique solution to the viscous Burgers equation in $C\left([0,\infty);L^1(\mathbb{R})\right)$ is the function given by (6).

Remark 2.1. Integrating equation (1) over all of $\mathbb{R}$, we obtain that the total mass of solutions is preserved for every $t>0$:

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx. \tag{10}$$

With no loss of generality we assume that $\displaystyle\int_{\mathbb{R}} u_0(x)dx = 1$.

Lemma 2.1. *Consider equation* (1) *with* $p=2$. *Let* u *be defined by* (6).

Then a real constant δ *exists such that*

$$u(x,t) \geq \frac{e^{-1/2}}{2t^{1/2}} \exp\left(-\frac{\delta^2}{2t}\right)\exp\left(-\frac{x^2}{2t}\right), \tag{11}$$

for every $t>0$.

Proof. Since $\int_{\mathbb{R}} u_0(y)dy = 1$, there exists a compact interval $I\subset\mathbb{R}$ such that $\int_I u_0(y)dy \geq \frac{1}{2}$.

We have

$$\begin{aligned}&\int_{-\infty}^{+\infty} u_0(y)\exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t}+\int_0^y u_0(\eta)\,d\eta\right]\right\}dy \geq\\ &\geq \int_I u_0(y)\exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t}+\int_0^y u_0(\eta)\,d\eta\right]\right\}dy.\end{aligned} \tag{12}$$

Set $I=[-\delta,\delta]$, with $\delta\in\mathbb{R}$.

Then:

$$\begin{aligned}&\int_I u_0(y)\exp\left\{-\frac{1}{2}\left[\frac{(x-y)^2}{2t}+\int_0^y u_0(\eta)\,d\eta\right]\right\}dy \geq\\ &\geq \exp\left(-\frac{1}{2}\right)\int_I u_0(y)\exp\left\{-\frac{1}{2}\frac{(x-y)^2}{2t}\right\}dy \geq\\ &\geq \frac{1}{2}\exp\left(-\frac{1}{2}-\frac{\delta^2}{2t}\right)\exp\left(-\frac{x^2}{2t}\right).\end{aligned} \tag{13}$$

Thus:

$$u(x,t) \geq \frac{A}{t^{1/2}} \exp\left(-\frac{\delta^2}{2t}\right) \exp\left(-\frac{x^2}{2t}\right); \tag{14}$$

where $A = \frac{1}{2}\exp(-\frac{1}{2})$. ∎

In the case where $p > 2$, we derive a similar estimate under an additional assumption on the initial value.

Lemma 2.2. *Let $p > 2$. Assume that $u_0(x) \geq M \exp(-\frac{|x|^2}{2})$ for some positive constant M and for a.e. $x \in \mathbb{R}$. Then there exist positive constants B and C such that*

$$u(x,t) \geq \frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right), \tag{15}$$

for every $t > 1$.

Proof. Let us define $f : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ as

$$f(x,t) = \frac{M}{(2(t+1))^{1/2}} \exp(-\alpha t) \exp\left(-\frac{|x|^2}{2(t+1)}\right),$$

for $(x,t) \in \mathbb{R} \times [0, +\infty)$, where α is a positive constant to be chosen later. We prove that the function f is a subsolution to the equation (1) in $\mathbb{R} \times (0,1)$. We have

$$f(x,0) \leq u_0(x),$$

for every $x \in \mathbb{R}$. It is not difficult to see that

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - f^{p-1}\frac{\partial f}{\partial x} \leq \left(\frac{1}{2(t+1)} - \alpha + \beta e^{-(p-1)\alpha t}\frac{M^{p-1}}{(t+1)^{p/2}}\right) f; \tag{16}$$

where $\beta = 2^{\frac{1}{2}} \max_{z\in\mathbb{R}}[-\sqrt{2}ze^{-z^2(p-1)}]$. Then, if $\alpha > \frac{1}{2} + \beta M^{p-1}$,

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - f^{p-1}\frac{\partial f}{\partial x} \leq 0,$$

for $(x,t) \in \mathbb{R} \times (0,1)$. As a consequence of the comparison principle proved in [10], we obtain that for every $x \in \mathbb{R}$, $u(x,1) \geq B\exp(-\frac{|x|^2}{4})$, with $B = \frac{M}{2}e^{-\alpha}$.

Let us define $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ as

$$g(x,t) = \frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right),$$

where the positive constant C will be fixed later and $(x,t) \in \mathbb{R} \times [1, \infty)$. We have that for every $x \in \mathbb{R}$, $u(x,1) \geq g(x,1)$. Moreover,

$$\begin{aligned} &\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} - g^{p-1}\frac{\partial g}{\partial x} = \\ &= \left(-\frac{C(p-2)}{2t^{\frac{p-2}{2}+1}} - \frac{x}{2t}\left(\frac{Be^{-C}}{t^{1/2}} \exp\left(\frac{C}{t^{\frac{p-2}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right)\right)^{p-1}\right) g \leq \\ &\leq \left(-\frac{C(p-2)}{2t^{\frac{p}{2}}} + \frac{B^{p-1}\bar{\beta}}{t^{p/2}}\right) g; \end{aligned} \tag{17}$$

where $\bar{\beta} = 2^{\frac{p-2}{2}}\beta$.

If we choose C sufficiently large, then we obtain the last expression is negative.

Therefore, due to the comparison principle, inequality (15) follows. □

3 – Convergence results

We now look for solutions to (1) having the form:

$$u(x,t) = \frac{1}{R(t)} v\left(\frac{x}{R(t)}, L(t)\right) = \frac{1}{R(t)} v\,(y,\tau)\,; \tag{18}$$

where $R(t), L(t)$ are unknown functions. Let us impose that the previous function satisfies (1) and determine what functions $R(t), L(t)$ are admissible, in such a way that the initial values of u and v are the same.

The time-dependent scaling and the use of a suitable functional are the main novelty of our approach. Instead of working directly with equation (1), we analyse the asymptotic behaviour of the solution v to the problem

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial v}{\partial x} + yv - e^{-(p-2)\tau} v^p\right] & y \in \mathbb{R}, \quad \tau > 0, \\ v(y,0) = v_0(y) \quad y \in \mathbb{R}. \end{cases} \tag{19}$$

Indeed, it is easily seen that the solutions u and v to problems (1) and (19), respectively, are related by

$$(20)\qquad \begin{aligned} u(x,t) &= \frac{1}{(2t+1)^{1/2}} v\left(\frac{x}{(2t+1)^{1/2}}, \log(2t+1)^{1/2}\right), \\ v(y,\tau) &= e^{\tau} u\left(ye^{\tau}, \frac{e^{2\tau}-1}{2}\right). \end{aligned}$$

Let us notice that $u_0 = v_0$.

We shall prove below convergence results on studying separately the following two cases: $p = 2$ and $p > 2$.

In Section 5, we shall extend the results for a class of convection-diffusion equations in $\mathbb{R}^N$.

3.1 – Case $p = 2$

We introduce a suitable functional for the Fokker-Planck type equation (19). We shall prove the time monotonicity of the functional and its decay to zero as $\tau \to +\infty$, in order to study the asymptotic decay to a fixed equilibrium state v_∞ of the solution to (19).

We first derive the equilibrium state v_∞, by looking for a stationary solution to (19):

$$(21)\qquad \frac{\partial v_\infty}{\partial y} + y v_\infty - v_\infty^2 = 0,$$

i.e.:

$$(22)\qquad v_\infty(y) = \frac{e^{-y^2/2}}{(2\pi)^{\frac{1}{2}} \dfrac{e^{1/2}}{2(e^{1/2}-1)} - \dfrac{1}{2}\displaystyle\int_{-\infty}^{y} e^{-s^2/2} ds},$$

where v_∞ is positive and $\displaystyle\int_{\mathbb{R}} v_\infty(y)dy = 1$.

Notice that $v_\infty = v_1$, where, in accordance with the result proved in [1], v_1 is the unique self-similar solution to (1) with a smooth profile verifying

$$(23)\qquad \frac{\partial}{\partial x'}\left[\frac{\partial v_1}{\partial x'} + x' v_1 - v_1^2\right] = 0.$$

Since $u \in C\left((0,\infty);W^{2,q}(\mathbb{R})\right)\bigcap C^1\left((0,\infty);L^q(\mathbb{R})\right)$ for every $q \in (1,\infty)$, by Theorem 2.2, the solution v to the Cauchy problem (19), belongs to the same class.

Moreover, $\|v(\cdot,\tau)\|_\infty \le C_\infty$ for every $\tau > 0$. Thanks to Lemma 2.1, we have, as $\tau > 0$,

$$
(24) \quad \begin{cases} \text{(P1)} \quad \displaystyle\int_{\mathbb{R}} v(y,\tau)dy = 1; \\ \text{(P2)} \quad v(y,\tau) \ge \dfrac{2^{\frac{1}{2}} e^{\tau} e^{-1/2}}{2\left(e^{2\tau}-1\right)^{\frac{1}{2}}} \exp\left(-\dfrac{2\delta^2}{e^{2\tau}-1}\right) \exp\left(-\dfrac{y^2 e^{2\tau}}{e^{2\tau}-1}\right). \end{cases}
$$

Let us prove the following preliminary result.

LEMMA 3.1. *Let v be the solution to* (19). *Then*

$$
(25) \qquad \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log\frac{v(y,\tau)}{v_\infty(y)}\right) e^{-y^2/2} dy < +\infty,
$$

for every $\tau > 0$.

PROOF. Thanks to (P2), fixed any $\delta > 0$, we have:

$$
(26) \qquad \frac{v(y,\tau)}{v_\infty(y)} \ge C(\delta)\exp\left[\left(-\frac{e^{2\tau}}{e^{2\tau}-1} + \frac{1}{2}\right) y^2\right] = C(\delta)\exp(-\gamma y^2),
$$

for every $\tau > \delta > 0$, for some positive constant γ. Moreover,

$$
(27) \qquad -\log\frac{v(y,\tau)}{v_\infty(y)} < -\log\left(C(\delta)\exp(-\gamma y^2)\right) < \gamma y^2 - \log C(\delta).
$$

Thus,

$$
(28) \qquad \begin{aligned} 0 &\le \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log\frac{v(y,\tau)}{v_\infty(y)}\right) e^{-y^2/2} < \\ &< \bar{C} v(y,\tau) + \left(-1 + \gamma y^2 - \log C(\delta)\right) e^{-y^2/2}, \end{aligned}
$$

where $\bar{C}$ is a positive constant. Thus (25) follows. $\square$

Let us introduce now the following functional, which represents the physical relative entropy for the viscous Burgers equation.

Definition 3.1. For every solution v to (19), let $L : \mathbb{R}^+ \to \mathbb{R}$ be defined as

$$L(\tau) = \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{v_\infty(y)} - 1 - \log \frac{v(y,\tau)}{v_\infty(y)} \right) e^{-y^2/2} dy. \tag{29}$$

The main property of L in connection with our discussion is that L is a monotone non increasing function of τ when v is the solution to (19).

Lemma 3.2. *Let v be the solution to* (19)*. Then for every $\tau > 0$,*

$$\lim_{|y|\to+\infty} \left(\frac{1}{\tilde{v}_\infty} - \frac{1}{\tilde{v}} \right) \left(\frac{\partial v}{\partial y} + yv - v^2 \right) = 0, \tag{30}$$

where $\tilde{v}_\infty(y) = e^{y^2/2} v_\infty(y)$ and $\tilde{v}(y,\tau) = e^{y^2/2} v(y,\tau)$.

Proof.
Let us divide the proof in two steps.
1) We prove that

$$\lim_{|y|\to+\infty} \frac{1}{\tilde{v}_\infty} \left(\frac{\partial v}{\partial y} + yv - v^2 \right) = 0. \tag{31}$$

Since $v(\cdot,\tau) \in W^{2,q}(\mathbb{R})$, for every $\tau > 0$ and for every q, with $1 < q < \infty$, then:

$$\lim_{|y|\to+\infty} v(y,\tau) = \lim_{|y|\to+\infty} \frac{\partial v}{\partial y}(y,\tau) = 0.$$

Moreover, $\tilde{v}_\infty^{-1}$ is bounded. Thanks to formula

$$u(t) = G(t) * u_0 - \int_0^t \nabla G(t-s) * u^2(s) ds, \tag{32}$$

we have

$$\lim_{|x|\to+\infty} xu(x,t) = 0,$$

for $t > 0$.

Hence,

$$\lim_{|y|\to+\infty} yv(y,\tau) = 0.$$

2) Let us prove now that

$$\lim_{|y|\to+\infty} \frac{1}{\tilde{v}}\left(\frac{\partial v}{\partial y} + yv - v^2\right) = 0. \tag{33}$$

Making use of the previous integral formula (32) for the solution to (1) yields $\frac{\partial u}{\partial x}$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{(4\pi t)^{1/2}}\int_{\mathbb{R}} -\frac{x-y}{2t}e^{-\frac{(x-y)^2}{4t}}u_0(y)dy+ \\ &+\int_0^t \frac{1}{4\pi(t-s)}\int_{\mathbb{R}} \frac{1}{2(t-s)}e^{-\frac{(x-y)^2}{4(t-s)}}u^2(y,s)ds+ \\ &-\int_0^t \frac{1}{4\pi(t-s)}\int_{\mathbb{R}} \frac{(x-y)^2}{4(t-s)^2}e^{-\frac{(x-y)^2}{4(t-s)}}u^2(y,s)ds. \end{aligned} \tag{34}$$

Owing to the lower bound of Lemma 2.1, we have that

$$\frac{1}{u}\left|\frac{\partial u}{\partial x}\right|\exp\left(-\frac{x^2}{2(2t+1)}\right) \le \left|\frac{\partial u}{\partial x}\right| t^{1/2}e^{1/t}\exp\left(\frac{x^2}{4t(2t+1)}\right). \tag{35}$$

Therefore,

$$\lim_{|x|\to+\infty} \frac{1}{u}\left|\frac{\partial u}{\partial x}\right|\exp\left(-\frac{x^2}{2(2t+1)}\right) = 0. \tag{36}$$

Thus, on performing the time dependent scaling, we get the conclusion. ∎

Let us now prove the time monotonicity of the relative entropy L.

Lemma 3.3. *Let v be the solution to* (19) *and let L be defined by* (29)*. Then*

$$\frac{d}{d\tau}L(\tau) \le 0,$$

for $\tau > 1$.

Proof. It is easily seen from formula (32) that $\frac{\partial v}{\partial \tau} \in L^1(\mathbb{R})$. Integration by parts yields:

$$
\begin{aligned}
&\frac{d}{d\tau}\int_{\mathbb{R}}\left(\frac{v}{v_\infty}-1-\log\frac{v}{v_\infty}\right)e^{-y^2/2}dy = \\
&\quad=\int_{\mathbb{R}}\left(\frac{1}{v_\infty}-\frac{1}{v}\right)\frac{\partial v}{\partial \tau}e^{-y^2/2}dy = \\
&\quad=\int_{\mathbb{R}}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\frac{\partial \tilde{v}}{\partial \tau}e^{-y^2/2}dy = \\
&\quad=\int_{\mathbb{R}}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\frac{\partial}{\partial y}\left[\tilde{v}^2e^{-y^2/2}\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right]dy = \\
&\quad=\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\left(\frac{\partial v}{\partial y}+yv-v^2\right)\Big|_{-\infty}^{+\infty}+ \\
&\qquad-\int_{\mathbb{R}}\tilde{v}^2e^{-y^2/2}\left(\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right)^2dy.
\end{aligned} \tag{37}
$$

Thus, thanks to Lemma 3.2, we have

$$
\frac{dL}{d\tau}=-\int_{\mathbb{R}}\tilde{v}^2e^{-y^2/2}\left(\frac{\partial}{\partial y}\left(\frac{1}{\tilde{v}_\infty}-\frac{1}{\tilde{v}}\right)\right)^2dy\le 0, \tag{38}
$$

for $\tau > 1$. ∎

Let I be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$
I(\tau)=-\frac{d}{d\tau}L(\tau). \tag{39}
$$

Remark 3.1. Thanks to Lemma 3.3,

$$
\int_\delta^{+\infty} I(s)ds = L(\delta)-L(\infty)<+\infty,
$$

for any $\delta > 1$. Since $I(\tau) \ge 0$ for $\tau > 1$, then there exists a sequence $\tau_k \to +\infty$ such that $I(\tau_k) \to 0$ as $k \to +\infty$.

Let us define $v_k : \mathbb{R} \to \mathbb{R}$ as

$$v_k(y) = v(y, \tau_k), \quad k \in N.$$

Proposition 3.1. *A real constant c exists such that if w_∞ is the function defined by $\frac{1}{w_\infty} = \frac{1}{v_\infty} + c$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}$ to w_∞ as $k \to +\infty$.*

Proof. Thanks to the previous remark and to (P2)

$$\lim_{k\to+\infty} \int_{\mathbb{R}} e^{-2y^2} \left(\frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_\infty} - \frac{1}{\tilde{v}_k} \right) \right)^2 dy = 0.$$

Consequently,

$$\frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_k} - \frac{1}{\tilde{v}_\infty} \right) \longrightarrow 0, \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}).$$

Since $\frac{1}{\tilde{v}_k}$ is a bounded sequence of functions in $L^2_{\text{loc}}(\mathbb{R})$, then there exists a function w_∞ such that

$$\frac{1}{\tilde{v}_k} \longrightarrow \frac{1}{\tilde{w}_\infty}, \text{ strongly in } W^{1,2}_{\text{loc}}(\mathbb{R});$$

as $k \to +\infty$ and

$$\frac{\partial}{\partial y} \left(\frac{1}{\tilde{w}_\infty} \right) = \frac{\partial}{\partial y} \left(\frac{1}{\tilde{v}_\infty} \right), \ a.e.\, y \in \mathbb{R}.$$

Hence,

$$\frac{1}{\tilde{w}_\infty} = \frac{1}{\tilde{v}_\infty} + c;$$

for some constant $c \in \mathbb{R}$; therefore

$$w_\infty(y) = \frac{e^{-\frac{y^2}{2}}}{c + (2\pi)^{\frac{1}{2}} \dfrac{e^{1/2}}{2\,(e^{1/2} - 1)} - \dfrac{1}{2} \displaystyle\int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds}. \tag{40}$$

Since $v_k > 0$, we have $w_\infty > 0$. $\square$

3.2 – Case $p > 2$

After performing the time-dependent scaling (20) in equation (1) as $p > 2$, we study the large time behaviour of the solution to (1).

As a consequence of the results in Theorem 2.2, we have that $\|v(\cdot,\tau)\|_{L^\infty} \leq C_\infty$, for every $\tau \geq \tau_0$ and thanks to Lemma 2.2,

$$(41)\quad \begin{cases} (\mathrm{P1}') & \displaystyle\int_{\mathbb{R}} v(y,\tau)dy = 1; \\ (\mathrm{P2}') & v(y,\tau) \geq \dfrac{2^{\frac{1}{2}} e^{\tau} B e^{-C}}{(e^{2\tau}-1)^{\frac{1}{2}}} \exp\left(\dfrac{2^{(p-2)/2} C}{(e^{2\tau}-1)^{(p-2)/2}}\right) \exp\left(-\dfrac{|y|^2 e^{2\tau}}{e^{2\tau}-1}\right); \end{cases}$$

where the constants B, C have been defined in Lemma 2.2.

We shall prove in this case that the large time behaviour of the solution v is determined by the following equation:

$$(42)\qquad \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial y}(yv).$$

We refer to (42) as the Fokker-Planck equation associated to (1) in the case where $p > 2$.

A stationary solution to (42) is given by the function:

$$(43)\qquad \bar{v}_\infty(y) = \gamma \exp\left(-\frac{y^2}{2}\right),$$

where γ is a constant. Let us choose γ in such a way

$$(44)\qquad \int_{\mathbb{R}} \gamma \exp\left(-\frac{y^2}{2}\right) dy = 1.$$

In the present section we prove that the large time behaviour of (1) is given by the function $\bar{v}_\infty$.

Similarly as the case of the viscous Burgers equation, we define the following convex nonnegative functional.

Definition 3.2. Let v be the solution to (19). Let L be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$(45)\qquad L(\tau) = \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{\bar{v}_\infty(y)} - 1 - \log \frac{v(y,\tau)}{\bar{v}_\infty(y)}\right) e^{-|y|^2/2} dy.$$

Lemma 3.4. *If v is a solution to* (19), *then*

$$L(\tau) < +\infty.$$

Moreover

Lemma 3.5. *Let v be the solution to* (19). *Then*

$$\lim_{|y|\to+\infty}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\left(\frac{\partial v}{\partial y}+yv-e^{-(p-2)\tau}v^p\right)e^{-|y|^2/2}=0, \tag{46}$$

for every $\tau > 0$.

The proofs of Lemmas 3.4 and 3.5 follow the same lines as those of Lemmas 3.1 and 3.2, respectively and will be omitted for brevity.

Lemma 3.6. *Let L be defined as in* (45). *Then*

$$\begin{aligned}&\frac{dL}{d\tau}=\\ &=-\int_{\mathbb{R}}\left(v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)^2-v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)e^{-(p-2)\tau}v^p\right)dy,\end{aligned} \tag{47}$$

for $\tau > 0$.

Proof. Thanks to formula

$$u(t)=G(t)*u_0-\frac{1}{p}\int_0^t\nabla G(t-s)*u^p(s)ds, \tag{48}$$

we have on integrating by parts,

$$\begin{aligned}&\frac{d}{d\tau}\int_{\mathbb{R}}\left(\frac{v}{\bar v_\infty}-1-\log\frac{v}{\bar v_\infty}\right)e^{-|y|^2/2}dy=\\ &=\int_{\mathbb{R}}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\frac{\partial v}{\partial\tau}e^{-|y|^2/2}dy=\\ &=e^{-|y|^2/2}\left(\frac{1}{\bar v_\infty}-\frac{1}{v}\right)\left(\frac{\partial v}{\partial y}+yv-e^{-(p-2)\tau}v^p\right)\Big|_{-\infty}^{+\infty}+\\ &\quad-\int_{\mathbb{R}}\left(v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)^2+v^{-2}e^{-|y|^2/2}\left(\frac{\partial v}{\partial y}+yv\right)e^{-(p-2)\tau}v^p\right)dy.\end{aligned} \tag{49}$$

∎

Lemma 3.7. *Set $v(y,\tau_j) = v_j(y)$, where $j \in \mathbb{N}$. Let v be the solution to* (19). *Then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ such that:*

$$\lim_{k\to+\infty} \int_{\mathbb{R}} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y} + y v_k\right)^2 dy = 0. \tag{50}$$

Proof. Let us suppose by contradiction it does not exist a sequence $(\tau_k)_k$ in such a way that

$$\lim_{k\to+\infty} \int_{\mathbb{R}} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y} + y v_k\right)^2 dy = 0.$$

Let I be the function $I : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\begin{aligned} I(\tau) = {} & \int_{\mathbb{R}} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y} + y v\right)^2 dy + \\ & + \int_{\mathbb{R}} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y} + y v\right) e^{-(p-2)\tau} v^p dy. \end{aligned} \tag{51}$$

On integrating by parts and making use of the L^∞-norm estimates for the function v, we deduce that the second integral in (51) tends to 0 as $\tau \to +\infty$. Therefore, there exists $T > 0$ such that for every $\tau > T$

$$I(\tau) > 0.$$

Moreover, $\frac{dL}{d\tau} \leq 0$ as $\tau > T$; then

$$\int_T^\infty I ds = L(T) - L(\infty) < \infty.$$

Thus we can find a sequence $(\tau_j)_{j\in\mathbb{N}}$ such that $I(\tau_j) \to 0$ as $j \to +\infty$ and we have a contradiction. ∎

The proof of the following result follows the same lines as the proof of Proposition 3.1.

Proposition 3.2. *A real constant C exists in such a way that if $\bar{w}_\infty$ is the function defined by $\bar{w}_\infty = C\bar{v}_\infty$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}$ to $\bar{w}_\infty$.*

4 – Proofs of Theorems 1.1 and 1.2.

Let us begin with the proof of the following inequality, which allows us to obtain the decay of the function v towards the equilibrium v_∞ as $\tau \to +\infty$.

LEMMA 4.1. *Let v be the solution to* (19) *and w_∞ be a positive function. Assume there exist positive constants θ_1, θ_2 such that $\theta_1 e^{-y^2/2} < w_\infty(y) < \theta_2 e^{-y^2/2}$, a.e. $y \in \mathbb{R}$. Then for every $\tau > 0$*

$$\|v(\cdot,\tau)-w_\infty(\cdot)\|^2_{L^1(\mathbb{R})} \le \tilde{B} \int_{\mathbb{R}} \left(\frac{v(y,\tau)}{w_\infty(y)} - 1 - \log \frac{v(y,\tau)}{w_\infty(y)} \right) e^{-|y|^2/2} dx, \tag{52}$$

where $\tilde{B} = \tilde{B}(\|v\|_1, \theta_1, \theta_2)$ is a suitable positive constant.

PROOF. Let $\alpha \in \mathbb{R}$, $\alpha > 2e^2$. Let us fix $\tau > 0$ and define the following set:

$$A_\tau = \left\{ x \in \mathbb{R} : \frac{v(y,\tau)}{w_\infty(y)} > \frac{\alpha}{2} \right\}.$$

Let us denote by f the function:

$$f : [1, +\infty) \to \mathbb{R}, \ : f(z) = z - 1 - 2\log(z).$$

We have that $f(z) > 0$ for every $z > \frac{\alpha}{2}$.

Thus,

$$\begin{aligned} \int_{A_\tau} (v - w_\infty)\, dy &= \int_{A_\tau} \left(\frac{v}{w_\infty} - 1 \right) w_\infty dy \le \\ &\le \int_{A_\tau} \left(\frac{v}{w_\infty} - 1 \right) \theta_2 e^{-y^2/2} dy \le \\ &\le \int_{A_\tau} 2\theta_2 \left(\frac{v}{w_\infty} - 1 - \log \frac{v}{w_\infty} \right) e^{-y^2/2} dy\,. \end{aligned} \tag{53}$$

Since

$$\begin{aligned} 0 &\le \int_{A_\tau} \left(\frac{v}{w_\infty} - 1 - \log \frac{v}{w_\infty} \right) e^{-y^2/2} dy \le \\ &\le \int_{A_\tau} \frac{v}{w_\infty} e^{-y^2/2} dy \le \frac{1}{\theta_1}, \end{aligned} \tag{54}$$

we have that:

$$\left(\int_{A_\tau} (v - w_\infty)\, dy\right)^2 \leq \frac{2\theta_2}{\theta_1} \int_{A_\tau} \left(\frac{v}{w_\infty} - 1 - \log\frac{v}{w_\infty}\right) e^{-y^2/2} dy. \tag{55}$$

Set $B_\tau = \mathbb{R}\backslash A_\tau$ and define the function

$$g : \mathbb{R}^+ \to \mathbb{R}, \quad g(z) = (z-1)^2 - \alpha(z - 1 - \log z).$$

One can verify that $g(z) \leq 0$ if $0 < z \leq \frac{\alpha}{2}$. Hence, for any given $\tau > 0$,

$$\left(\frac{v}{w_\infty} - 1\right)^2 e^{-y^2/2} \leq \alpha\left(\frac{v}{w_\infty} - 1 - \log\frac{v}{w_\infty}\right) e^{-y^2/2}, \tag{56}$$

for $y \in B_\tau$, and

$$\begin{aligned}\left(\int_{B_\tau} (v - w_\infty)\, dy\right)^2 &\leq \int_{B_\tau} \theta_2^2 e^{-y^2/2} dy \int_{B_\tau} \left(\frac{v}{w_\infty} - 1\right)^2 e^{-y^2/2} dy \leq \\ &\leq \theta_2^2\, (2\pi)^{1/2} \int_{B_\tau} \left(\frac{v}{w_\infty} - 1\right)^2 e^{-y^2/2} dy.\end{aligned} \tag{57}$$

Then

$$\begin{aligned}&\|v(\cdot,\tau) - w_\infty(\cdot)\|^2_{L^1(\mathbb{R})} \leq \\ &\leq \left(\theta_2^2\, (2\pi)^{1/2}\,\alpha + \frac{2\theta_2}{\theta_1}\right) \int_{\mathbb{R}} \left(\frac{v}{w_\infty} - 1 - \log\frac{v}{w_\infty}\right) e^{-y^2/2} dy,\end{aligned} \tag{58}$$

if $\tau > 0$. □

Let us prove now Theorem 1.1 to state the large time behaviour of the solution to (1) in the case where $p = 2$.

Proof of Theorem 1.1. As a consequence of Proposition 3.1, we obtain by Lebesgue theorem that as $k \to +\infty$,

$$\int_{\mathbb{R}} \left(\frac{v(y,\tau_k)}{w_\infty(y)} - 1 - \log\frac{v(y,\tau_k)}{w_\infty(y)}\right) e^{-y^2/2} dy \to 0. \tag{59}$$

Thanks to the result of Lemma 4.1 and Proposition 3.1, we have that $\|v(\cdot,\tau_k) - w_\infty(\cdot)\|_{L^1(\mathbb{R})} \longrightarrow 0$, as $k \to +\infty$. It follows that $\|w_\infty(\cdot)\|_{L^1(\mathbb{R})} = 1$

and $w_\infty = v_\infty$. Since for every $\tau > 0$, $\|v(\cdot,\tau)\|_\infty \le C_\infty$, we have by interpolation that

$$\lim_{\tau\to+\infty} \|v - v_\infty\|_{L^p(\mathbb{R})} = 0;$$

as $1 < p < \infty$.

After performing the time-dependent scaling, we get the conclusion. ∎

We prove now the main result on the large time behaviour of the solution to (1) as $p > 2$.

In contrast to the case where $p = 2$, we are not able to define a suitable functional L, which is monotone non increasing in time. Nevertheless, one can prove that $\int_{\mathbb{R}} (\frac{v(y,\tau)}{\bar v_\infty(y)} - 1 - \log \frac{v(y,\tau)}{\bar v_\infty(y)}) e^{-y^2/2} dy$ converges to zero as $\tau \to +\infty$.

Proof of Theorem 1.2. Thanks to the inequality of Lemma 4.1 and to Proposition 3.2, we deduce that $\bar w_\infty = \bar v_\infty$. Let us prove now that the function $L(\tau)$ converges to zero as $\tau \to +\infty$.

Step 1. The function I is continuous in τ, thanks to the results of Theorem 2.1. We have to consider the following three cases:

1) there exists $T > 0$ such that $I(\tau) \ge 0$ for every $\tau \ge T$; then L is a Lyapunov functional and the conclusion follows as for the case where $p = 2$.
2) There exists $T > 0$ such that $I(\tau) \le 0$ for every $\tau > T$. It follows that:

$$\begin{aligned} 0 &\le \int_{\mathbb{R}} v^{-2} e^{-y^2/2} \left(\frac{\partial v}{\partial y} + yv\right)^2 dy \le \\ &\le - \int_{\mathbb{R}} v^{-2} e^{-y^2/2} \left(\frac{\partial v}{\partial y} + yv\right) \frac{1}{e^{(p-2)\tau}} v^p dy. \end{aligned} \tag{60}$$

Thus, on integrating by parts in the last integral of (60), we have

$$\lim_{\tau\to+\infty} \int_{\mathbb{R}} v^{-2} e^{-y^2/2} \left(\frac{\partial v}{\partial y} + yv\right)^2 dy = 0.$$

3) The function $I(\tau)$ changes the sign as $\tau \in [0,\infty)$. Let $(\tau_i)_{i\in I} \in [0,\infty)$ such that $I(\tau_i) = 0$. Suppose $I(\tau) > 0$ as $\tau \in [\tau_{i-1}, \tau_i)$ and $I(\tau) \le 0$

as $\tau \in [\tau_i, \tau_{i+1}]$. Then $L(\tau_{i-1}) \geq L(\tau) \geq 0$ for $\tau \in [\tau_{i-1}, \tau_i)$ and $L(\tau_{i+1}) \geq L(\tau) \geq 0$ as $\tau \in [\tau_i, \tau_{i+1}]$.

As a consequence of $I(\tau_i) = 0$, we have that $L(\tau_i) \to 0$ as $\tau_i \to +\infty$. Thanks to the previous inequalities, $L(\tau) \to 0$ as $\tau \to +\infty$.

Step 2. Thanks to the inequality proved in Lemma 4.1, we obtain the result of Theorem 1.2 in a similar way as in the case where $p = 2$ if the intial value u_0 satisfies the assumption of Lemma 2.2.

By density argument the result can be proved in the general case. The solution to the Cauchy problem (1) satisfies indeed the following $L^1(\mathbb{R})$-contraction property proved in [5]:

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_1 \leq \|u_0 - \bar{u}_0\|_1,$$

for every $t \geq 0$. Consider now a nonnegative initial value $u_0 \in L^1(\mathbb{R})$ and approximate u_0 in $L^1(\mathbb{R})$ by a sequence of functions $(u_{0,n})_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ such that $u_{0,n}(x) \geq M_n \exp(-\frac{x^2}{2})$, a.e. $x \in \mathbb{R}$, where M_n are positive constants. Let u_n be the solution to (1) with initial value $u_{0,n}$. Thanks to the result on the long time behaviour of u_n and the $L^1(\mathbb{R})$-contraction property, we get the conclusion. □

5 – Concluding remarks

Consider the following class of convection-diffusion equations in $\mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u - a \cdot \nabla(u^p) & x \in \mathbb{R}^N, \qquad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{61}$$

where $p > 1 + \frac{1}{N}$, $N > 1$; $a \in \mathbb{R}^N$ and u_0 is a nonnegative function from $L^1(\mathbb{R}^N)$.

In the present section we will prove that our procedure can be used in the study of the long time behaviour of the solution to Cauchy problem (61).

Given $u_0 \in L^1(\mathbb{R}^N)$, there exists a unique classical solution $u \in C([0,\infty);\ L^1(\mathbb{R}^N))$ to (61), which satisfies the following properties (see [5]):

(i) for every $q\in(1,\infty)$, $u\in C((0,\infty);W^{2,q}(\mathbb{R}^N))\bigcap C^1((0,\infty);L^q(\mathbb{R}^N))$;

(ii) for every $q \in [1,\infty)$, there exists a constant $C_q = C(q, \|u_0\|_1)$ such that for every $t > 0$:

$$\begin{cases} \|u(t)\|_q \leq C_q t^{-N/2(1-1/q)}, \\ \|u(t)\|_1 \leq \|u_0\|_1. \end{cases} \tag{62}$$

(iii) Let t_0 be a nonnegative real number. Then there exists a positive constant C_∞ such that for every $t \geq t_0$:

$$\|u(t)\|_\infty \leq C_\infty t^{-N/2}. \tag{63}$$

By studying the problem in a similar way as the case where $p > 2$ and $N = 1$, we can prove the following result.

THEOREM 5.1. *Assume that $p > 1 + \frac{1}{N}$ and*

$$\begin{aligned} &u_0 \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} u_0(x)dx = 1; \\ &u_0(x) \geq 0 \ a.e.\ x \in \mathbb{R}^N. \end{aligned} \tag{64}$$

Let u be the solution to the Cauchy problem (61). *Then for every $q \in [1,\infty)$*

$$\lim_{t\to+\infty} (2t+1)^{N/2(1-1/q)} \|u(\cdot,t) - u_\infty(\cdot,2t+1)\|_{L^q(\mathbb{R}^N)} = 0,$$

where

$$u_\infty(x,2t+1) = \frac{1}{(2t+1)^{N/2}}\gamma \exp\left(-\frac{x^2}{2(2t+1)}\right),$$

and $\gamma = (\int_{\mathbb{R}^N} \exp(-\frac{x^2}{2})dx)^{-1}$.

We just give an outline of the proof.

PROOF. Let us divide the proof in the following four steps.

STEP 1. Following the same procedure of the proof of Lemma 2.2, we prove that if $u_0(x) \geq M \exp(-\frac{|x|^2}{2})$ for some positive constant M and for a.e. $x \in \mathbb{R}^N$, then there exist positive constants B and C such that

$$u(x,t) \geq \frac{Be^{-C}}{t^{N/2}} \exp\left(\frac{C}{t^{\frac{N(p-1)-1}{2}}}\right) \exp\left(-\frac{|x|^2}{4t}\right), \tag{65}$$

for every $t > 1$.

STEP 2. After performing a time-dependent scaling, we study the long time behaviour of the solution to the following problem:

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \nabla \cdot [\nabla v + yv - ae^{-(pN-N-1)\tau} v^p] & y \in \mathbb{R}^N, \quad \tau > 0, \\ v(y,0) = v_0(y) & y \in \mathbb{R}^N. \end{cases} \tag{66}$$

Notice that the solutions u and v to problems (61) and (66) respectively, are related by

$$\begin{aligned} u(x,t) &= \frac{1}{(2t+1)^{N/2}} v\left(\frac{x}{(2t+1)^{1/2}}, \log{(2t+1)^{1/2}}\right), \\ v(y,\tau) &= e^{N\tau} u\left(ye^{\tau}, \frac{e^{2\tau}-1}{2}\right). \end{aligned} \tag{67}$$

The large time behaviour of the solution to (66) is determined by the following equation:

$$\frac{\partial v}{\partial \tau} = \triangle v + \nabla \cdot (yv). \tag{68}$$

A stationary solution is given by the function:

$$\bar{v}_\infty(y) = \gamma \exp\left(-\frac{|y|^2}{2}\right), \tag{69}$$

where γ is a constant. We fix γ in such a way that

$$\int_{\mathbb{R}^N} \gamma \exp\left(-\frac{|y|^2}{2}\right) dy = 1. \tag{70}$$

STEP 3. Similarly as the case of the viscous Burgers equation, we define a nonnegative functional. Let v be the solution to (66). Let L be the function from $\mathbb{R}^+$ into $\mathbb{R}$ given by

$$L(\tau) = \int_{\mathbb{R}^N} \left(\frac{v(y,\tau)}{\bar{v}_\infty(y)} - 1 - \log \frac{v(y,\tau)}{\bar{v}_\infty(y)} \right) e^{-|y|^2/2} dy. \tag{71}$$

One can prove that if v is a solution to (66), then $L(\tau) < +\infty$.

Moreover, if v is the solution to (66), then

$$\lim_{|y_i| \to +\infty} \left(\frac{1}{\bar{v}_\infty} - \frac{1}{v} \right) \left(\frac{\partial v}{\partial y_i} + y_i v - a_i e^{-(Np-N-1)\tau} v^p \right) e^{-|y|^2/2} = 0, \tag{72}$$

for every $\tau > 0$ and $i = 1, ..., N$.

Consider L defined above. On integrating by parts, we prove that

$$\begin{aligned} \frac{dL}{d\tau} = &- \int_{\mathbb{R}^N} \sum_{i=1}^N \left(v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right)^2 + \right. \\ &\left. - v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right) a_i e^{-(Np-N-1)\tau} v^p \right) dy, \end{aligned} \tag{73}$$

for $\tau > 0$.

Set $v(y, \tau_j) = v_j(y)$, where $j \in \mathbb{N}$. Let v be the solution to (66). Then there exists a sequence $(v_k)_{k \in \mathbb{N}}$ such that:

$$\lim_{k \to +\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y_i} + y_i v_k \right)^2 dy = 0. \tag{74}$$

Let us suppose by contradiction that it does not exist a sequence $(\tau_k)_k$ in such a way that

$$\lim_{k \to +\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} {v_k}^{-2} e^{-|y|^2/2} \left(\frac{\partial v_k}{\partial y_i} + y_i v_k \right)^2 dy = 0.$$

Let I be the function $I : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\begin{aligned} I(\tau) = &\sum_{i=1}^N \int_{\mathbb{R}^N} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right)^2 dy + \\ &+ \sum_{i=1}^N \int_{\mathbb{R}^N} v^{-2} e^{-|y|^2/2} \left(\frac{\partial v}{\partial y_i} + y_i v \right) a_i e^{-(Np-N-1)\tau} v^p dy. \end{aligned} \tag{75}$$

On integrating by parts and making use of the L^∞-norm estimates for the function v, we deduce that the second integral in (75) tends to 0 as $\tau \to +\infty$. Therefore, there exists $T > 0$ such that for every $\tau > T$

$$I(\tau) > 0.$$

Moreover, $\dfrac{dL}{d\tau} \leq 0$ as $\tau > T$; then

$$\int_T^\infty I ds = L(T) - L(\infty) < \infty.$$

Thus we can find a sequence (τ_j) such that $I(\tau_j) \to 0$ as $j \to +\infty$ and we get a contradiction.

Similarly as the cases studied in Section 3, we can prove that there exists a real constant C in such a way that if $\bar{w}_\infty$ is the function defined by $\bar{w}_\infty = C\bar{v}_\infty$, then the sequence of functions $(v_k)_{k\in N}$ converges a.e. in $\mathbb{R}^N$ to $\bar{w}_\infty$.

Step 4. The inequality proved in Lemma 4.1 holds true even in the case where the functions v and w_∞ are defined in $\mathbb{R}^N$. Moreover, the conclusion of the proof of Theorem 5.1 is achieved by following the same procedure as for the proof of Theorem 1.2 in Section 4. $\square$

We have tried to apply the method to study the long time behaviour of the solution to (61) in the case where $p = 1 + \dfrac{1}{N}$ and $N > 1$. Unfortunately we are not able to conclude the proof because of technical difficulties due to the lack of informations about the qualitative properties of the self-similar solution to (61) (see [1]).

REFERENCES

[1] J. Aguirre – M. Escobedo – E. Zuazua: *Self-similar solutions of a convection diffusion equation and related semilinear elliptic problems*, Comm. Partial Differential Equations, **15** (1990), 139-157.

[2] J. A. Carrillo – G. Toscani: *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J., **49** (2000), 113-142.

[3] C. Cercignani – R. Illner – M. Pulvirenti: *The mathematical theory of diluite gases*, Springer-Verlag, 1994.

[4] I. L. Chern – T. P. Liu: *Convergence to diffusion waves of solutions for viscous conservation laws*, Comm. Math. Phys., **110** (1987), 503-517.

[5] M. Escobedo – E. Zuazua: *Large time behaviour for convection-diffusion equations in R^N*, J. Funct. Anal., **100** (1991), 119-161.

[6] A. Friedman – S. Kamin: *The asymptotic behaviour of gas in an n-dimensional porous medium*, Trans. Amer. Math. Soc., **262** (1985), 551-563.

[7] E. Hopf: *The partial differential equation $u_t + uu_x = \mu u_{xx}$*, Comm. Pure Appl. Math., **3** (1950), 201-230.

[8] S. Kamin: *Similar solutions and the asymptotics of filtration equations*, Arch. Rat. Mech. Anal., **60** (1985), 171-183.

[9] P. L. Lions – G. Toscani: *Diffusive limit for finite velocity Boltzmann kinetic models*, Revista Mat. Iberoamericana, **13** (1997), 473-513.

[10] M. H. Protter – H. F. Weinberger: *Maximum principles in differential equations*, Prentice-Hall, 1967.

[11] M. E. Schonbeck: *Decay of solutions to parabolic conservation laws*, Comm. Partial Differential Equations, **5** (1980), 449-473.

[12] J. L. Vazquez: *Asymptotic behaviour of nonlinear parabolic equations*, II Congreso on Applied Mathematics; Madrid (1993); Publicaciones de la Universidad Politecnica de Madrid.

[13] G. B. Whitham: *Linear and Nonlinear Waves*, Wiley Intersciance, 1974.

Lavoro pervenuto alla redazione il 9 aprile 2003
ed accettato per la pubblicazione il 9 giugno 2003.
Bozze licenziate il 4 settembre 2003

Indirizzo dell'Autore:

R. Cavazzoni – Dipartimento di Matematica e Appl. Architettura – Università di Firenze – Piazza Ghiberti, 27 – 50122 Firenze (Italia)
E-mail: cavazzon@interfree.it