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Optimal and approximate control of finite-difference approximation schemes for the 1D wave equation

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ABSTRACT: *We address the problem of control of numerical approximation schemes for the wave equation. More precisely, we analyze whether the controls of numerical approximation schemes converge to the control of the continuous wave equation as the mesh-size tends to zero.*

Recently, it has been shown that, in the context of exact control, i.e., when the control is required to drive the solution to a final target exactly, due to high frequency spurious numerical solutions, convergent numerical schemes may lead to unstable approximations of the control. In other words, the classical convergence property of numerical schemes does not guarantee a stable and convergent approximation of controls.

In this article we address the same problem in the context of optimal and approximate control in which the final requirement of achieving the target exactly is relaxed. We prove that, for those relaxed control problems, convergence (as the mesh-size tends to zero) holds. In particular, in the context of approximate control we show that, if the final condition is relaxed so that the final state is required to reach and ε-neighborhood of the final target with $\varepsilon > 0$, then the controls of numerical schemes (the so-called ε-controls) converge to the ε-controls of the wave equation. We also show that this result fails to be true in several space dimensions.

Although convergence is proved in the context of these relaxed control problems, the fact that instabilities occur at the level of exact control have to be considered as a serious warning in the sense that instabilities may ultimately arise if the control requirement is reinforced to exactly achieve the final target, i.e., as ε is taken smaller and smaller.

Contents:

1 – Introduction

In recent years important progress has been made on problems of observation and control of wave phenomena. Much less is known about numerical approximation schemes.

The problems of observability and controllability can be stated as follows:

- *Observability.* Assuming that waves propagate according to a given wave equation and with suitable boundary conditions, can one guarantee that their whole energy can be estimated in terms of the energy concentrated on a given subregion of the domain (or its boundary) where propagation occurs in a given time interval?
- *Controllability.* Can solutions be driven to a given state at a given final time by means of a control acting on the system on that subregion?

It is well known that the two problems are equivalent provided one chooses an appropriate functional setting, which depends on the equation (see for, instance, [53], [83]).

But several different variants are meaningful and possible. In particular, at the level of the controllability problem, one can consider several degrees of precision on the requirement of reaching the given target. For instance, one can require the control to drive the solution to the target exactly, this is the so-called *exact controllability* problem, or only in an approximate way, the so called *approximate controllability* one. One may also formulate the problem in the context of *optimal control*, minimizing a functional measuring the distance to the target in a suitable class of admissible controls.

Each of these control properties can be interpreted by duality as a suitable observability property. Obviously, stronger the control property under consideration is, stronger the corresponding observability property will be as well.

In this work we shall mainly focus on the issue of how these two properties behave under numerical approximation schemes for two particular control problems: *optimal* and *approximate control*. More precisely, we shall discuss the problem of whether, as the mesh-size tends to zero, the controls of numerical approximation schemes converge to the controls of the continuous wave equation.

This article is devoted to the wave equation as a simplified hyperbolic problem arising in many areas of Mechanics, Engineering and Technology. It is indeed, a model for describing the vibrations of structures, the propagation of acoustic or seismic waves, etc. Therefore, the control of the wave equation enters in a way or another in problems related with control mechanisms for structures, buildings in the presence of earthquakes, for noise reduction in cavities and vehicles, etc.

By now it is well known that, in the context of the exact controllability problem, the answer to the question is negative in the sense that exact controls of numerical approximation schemes may diverge as the mesh-size tends to zero. This is due to the classical numerical dispersion phenomena. Indeed, it is well known that the interaction of waves with a numerical mesh produces dispersion phenomena and spurious(1) high frequency oscillations [76], [74]. In particular, because of this nonphysical interaction of waves with the discrete medium, the velocity of propagation of numerical waves and, more precisely, the so called *group velocity*(2) may converge to zero when the wavelength of solutions is of the order of the size of the mesh and the latter tends to zero. As a consequence of this fact, the time needed to uniformly (with respect to the mesh size) observe (or control) the numerical waves *exactly* from the boundary or from a subset of the medium in which they propagate may tend to infinity

(1)The adjective spurious will be used to designate any component of the numerical solution that does not correspond to a solution of the underlying PDE. In the context of the wave equation, this happens at the high frequencies and, consequently, these spurious solutions weakly converge to zero as the mesh size tends to zero. Consequently, the existence of these spurious oscillations is compatible with the convergence (in the classical sense) of the numerical scheme, which does indeed hold for fixed initial data.
(2)At the numerical level it is important to distinguish the notions of phase and group velocity. Phase velocity refers to the velocity of propagation of individual monocromatic waves, while group velocity corresponds to the velocity of propagation of wave packets, that may significantly differ from the phase velocity when waves of similar frequencies are combined. See, for instance, [74].

as the mesh becomes finer. This is the reason for the unstable behavior of the control and observation properties of most numerical approximation schemes as the mesh-size tends to zero.

But that happens, as mentioned above, for the problems of exact observation and control. Exact observation means that the total energy of solutions is reconstructed from partial measurements uniformly, independently of the solution. Exact control means that one wishes to drive the solution exactly to a final target.

The main goal of this article is to show that when these requirements are relaxed, and one considers the problems of approximate and/or optimal control, then instabilities disappear and classical numerical schemes provide convergent approximations of controls.

In this paper we first briefly describe why numerical dispersion and spurious high frequency oscillations are an obstacle for the convergence of exact controls.

We then address the problems of approximate and optimal control. We prove, combining classical results on the convergence of numerical schemes and Γ-convergence arguments, that controls converge for the relaxed optimal and approximate control problems.

All we have said up to now concerning the wave equation can be applied with minor changes to several other models that are purely conservative like Schrödinger and beam equations (see the survey article [87] for a comparison between these models and their behavior in what concerns numerics and control).

However, many models from physics and mechanics have some damping mechanism built in. When the damping term is "mild" the qualitative properties are the same as those we discussed above. However, some other dissipative mechanisms may have much stronger effects. This is for instance the case for the thermal effects arising in the heat equation itself but also in some other more sophisticated systems, like the system of thermoelasticity. Roughly speaking one may say that the strong damping mechanisms help for the convergence of controls of numerical schemes. There is actually an extensive literature on optimal control of parabolic equations that confirms this fact [44], [68], [75], ... We also refer to E. Casas [9] for the analysis of finite-element approximations of elliptic optimal control problems and to [17] for an optimal shape design problem for the Laplace operator. But this has been done mainly in the context of optimal control and very little is known about the controllability issues that we address in this paper (we refer to [87] for a discussion of this topic and for a list of related open problems). For instance, as we shall see, in several space dimensions, the problem of analyzing the behavior of approximate controls for the heat equation is mainly open too.

Most of the analysis we shall present here has been also developed in the context of a more difficult problem, related to the behavior of the conservation/control properties in homogenization. There, the coefficients of the wave equation oscillate rapidly on a scale δ that tends to zero, so that the equation homogenizes to a constant coefficient one. In that framework the interaction of high frequency waves with the microstructure produces localized waves at high frequency. These localized waves are an impediment for the uniform observation/control properties to hold. But, once more, this impediments do not arise if the control requirement is relaxed [12] and [48]. This was already observed in the context of homogenization and approximate control of the heat equation in [84]. The analogies between both problems (homogenization and numerical approximation) are clear: the mesh size h in numerical approximation schemes plays the role of the parameter δ in homogenization (see [85] and [14] for a discussion of the connection between these problems). Although the analysis of the numerical problem is much easier from a technical point of view, it was only developed after the problem of homogenization was understood. This is due in part to the fact that, from a control theoretical point of view, there was a conceptual difficulty to match the existing finite-dimensional and infinite-dimensional theories. This article may also be viewed as a further step in that direction showing that although the instabilities do arise at the level of exact control, optimal and approximate control problems ate often well-behaved.

This paper is mainly concerned with finite-difference numerical approximation schemes for $1D$ wave equations but the results and techniques extend easily to most common numerical approximation schemes, like finite-element methods, and also to fully discrete approximations. As we shall see, however, interesting open problems arise in several space dimensions where the questions under investigation exhibit new and not completely understood geometrical aspects. The rest of this paper is organized as follows.

In Section 2 we recall the basic ingredients of the finite-dimensional theory we will need along the paper. In particular we shall introduce the Kalman rank condition.

Section 3 is devoted to presenting and discussing the problems of observability and controllability for the constant coefficient wave equation. In Section 4 we discuss the finite-difference space semi-discretization of the $1D$ wave equation and recall the main results on the lack of controllability and observability. We also comment on some remedies and cures that have been introduced in the literature to avoid these instabilities.

In Section 5 and 6 we show that numerical approximation schemes are well behaved if the control requirement is relaxed to an approximate or optimal control problem, respectively. In Section 7 we briefly discuss the problem of stabilization. Finally, in Section 8 we formulate an interesting open problem related with the extension of the results in this paper to several space dimensions.

The interested reader is referred to the survey articles [81] and [83] for a more complete discussion of the state of the art in the controllability of partial differential equations and to [87] for what concerns numerical issues.

2 – Preliminaries on finite-dimensional systems

Most of this article is devoted to analyze the wave equation and its numerical approximations. Numerical approximation schemes and more precisely those that are semi-discrete (discrete in space and continuous in time) yield finite-dimensional systems of ODE's. There is by now an extensive literature on the control of finite-dimensional systems and the problem is completely understood for linear ones [50]. The problem of convergence of controls as the mesh-size in the numerical approximation tends to zero is very closely related to passing to the limit as the dimension of finite-dimensional systems tends to infinity. The later topic is widely open and this article may be considered as a contribution in this direction.

In this section we briefly summarize the most basic material on finite-dimensional systems that will be used along this article (we refer to [59] for more details).

Consider the finite-dimensional system of dimension N:

$$x' + Ax = Bv, \quad 0 \le t \le T; \quad x(0) = x_0, \tag{2.1}$$

where x is the N-dimensional state and v is the M-dimensional control, with $M \le N$.

Here A is an $N \times N$ matrix with constant real coefficients and B is an $M \times N$ matrix. The matrix A determines the dynamics of the system and the matrix B models the way controls act on it.

Obviously, in practice, it would be desirable to control the N components of the system with a low number of controls. The best would be to do it by means of a scalar control, in which case $M = 1$. This is typically the situation when dealing with the boundary control of numerical approximation schemes of the $1D$ wave equation.

System (2.1) is said to be controllable in time T when every initial datum $x_0 \in \mathbb{R}^N$ can be driven to any final datum x_1 in $\mathbb{R}^N$ in time T and, more precisely, if for any $x_0, x_1 \in \mathbb{R}^N$ there exists $v \in L^2(0,T; \mathbb{R}^M)$ such that the solution of (2.1) satisfies

$$x(T) = x_1. \tag{2.2}$$

It turns out that for finite-dimensional systems there is a necessary and sufficient condition for controllability which is of purely algebraic nature. It is the so called *Kalman condition: System* (2.1) *is controllable in some time* $T > 0$ *iff*

$$\operatorname{rank}[B, AB, \ldots, A^{N-1}B] = N. \tag{2.3}$$

According to this, in particular, system (2.1) is controllable in some time T if and only if it is controllable for all time.

There is a direct proof of this result which uses the representation of solutions of (2.1) by means of the variation of constants formula. However, the methods we shall develop along this article rely more on the dual (but completely equivalent!) problem of observability of the adjoint system.

Consider the *adjoint system*

$$-\varphi' + A^*\varphi = 0, \quad 0 \le t \le T;\ \varphi(T) = \varphi_0. \tag{2.4}$$

It is not difficult to see that system (2.1) is controllable in time T if and only if the adjoint system (2.4) is *observable* in time T, i.e. if there exists a constant $C > 0$ such that, for all solution φ of (2.4),

$$\mid \varphi_0 \mid^2 \le C \int_0^T \mid B^*\varphi \mid^2 dt. \tag{2.5}$$

Before analyzing (2.5) in more detail let us see that this observability inequality does indeed imply controllability of the state equation.

Assume the observability inequality (2.5) holds and consider the following quadratic functional $J : \mathbb{R}^N \to \mathbb{R}$:

$$J(\varphi_0) = \frac{1}{2}\int_0^T \mid B^*\varphi(t) \mid^2 dt - < x_1, \varphi_0 > + < x_0, \varphi(0) > . \tag{2.6}$$

It is easy to see that, if $\tilde{\varphi}_0$ is a minimizer for J, then the control $v = B^*\tilde{\varphi}$, where $\tilde{\varphi}$ is the solution of the adjoint system (2.4) with that datum at time $t = T$, is such that the solution $x = x(t)$ of the state equation satisfies the control requirement (2.2). Indeed, it is sufficient to write down explicitly the fact that the differential of J at the minimizer vanishes.

Thus, the controllability problem is reduced to minimizing the functional J. Applying the Direct Method of the Calculus of Variations it can be shown that J achieves its minimum since the functional J is continuous and convex and it is also coercive according to the observability inequality (2.5). Indeed, note that when (2.5) holds the following variant holds as well, with possibly a different constant $C > 0$:

$$\mid \varphi_0 \mid^2 + \mid \varphi(0) \mid^2 \le C \int_0^T \mid B^*\varphi \mid^2 dt. \tag{2.7}$$

This gives a constructive way of building the controls, as a minimum of J.

The coercivity of J requires the Kalman condition (2.3) to be satisfied. The rank condition (2.3) turns out to be equivalent to the adjoint one

$$\text{rank}[B^*,\, B^*A^*, \dots ,\, B^*[A^*]^{N-1}] = N. \tag{2.8}$$

To see the equivalence between (2.7) and (2.8) let us note that, since we are in finite-dimension, using that all norms are equivalent[3], the observability inequality (2.7) is equivalent to a uniqueness property:

(2.9) (UP) Does the fact that $B^*\varphi \equiv 0$ for all $\leq t \leq T$ imply that $\varphi \equiv 0$?

And, as we shall see, this uniqueness property is precisely equivalent to the adjoint Kalman condition (2.8).

Remark 2.1. Before proving this statement we note that $B^*\varphi$ is only an M-dimensional projection of the solution φ who has N components. Therefore, in order for this property (UP) to be true the operator B^* has to be chosen in a strategic way, depending of the state matrix A. The Kalman condition is the right test to check whether the choice of B^* (or B) is appropriate.

Let us finally prove that the uniqueness property (UP) holds when the adjoint rank condition (2.8) is fulfilled. In fact, taking into account that solutions φ are analytic in time, the fact that $B^*\varphi$ vanishes is equivalent to the fact that all the derivatives of $B^*\varphi$ of any order at time $t = T$ vanish. But the solution φ admits the representation $\varphi(t) = e^{A^*(t-T)}\varphi_0$ and therefore all the derivatives of $B^*\varphi$ at time $t = T$ vanish if and only if $B^*[A^*]^k\varphi_0 \equiv 0$ for all $k \geq 0$. According to the Cayley-Hamilton's theorem this is equivalent to the fact that $B^*[A^*]^k\varphi_0 \equiv 0$ for all $k = 0, \ldots, N-1$. Finally, the latter is equivalent to $\varphi_0 \equiv 0$ (i.e. $\varphi \equiv 0$) if and only if the adjoint Kalman rank condition (2.8) is fulfilled.

Remark 2.2. It is important to note that in this finite-dimensional context, the time T of control plays no role. In particular, whether a system is controllable (or its adjoint observable) is independent of the time T of control.

Remark 2.3. In the finite-dimensional context of this section we have only considered the problem of exact controllability. This is so since, in this case, approximate and exact controllability are equivalent properties. Approximate controllability refers to the situation in which the set of reachable states is dense in the space where solutions live. In this case, since we are in $\mathbb{R}^N$, this is equivalent to the fact that the set of reachable states in the whole $\mathbb{R}^N$ and this is precisely when exact controllability holds. The dual version of this equivalence property reads as follows: in finite-dimensions, the observability inequality (2.5) holds if and only if the uniqueness property (UP) is satisfied. None of these equivalences hold in general for infinite-dimensional dynamical systems.

The main task to be undertaken in order to pass to the limit in numerical approximations of control problems for wave equations as the mesh-size tends to zero is to explain why, even though at the finite-dimensional level the value

[3] This is the key point where finite and infinite dimensional systems behave so differently in what concerns controllability problems.

of the control time T is irrelevant, it may play a key role for the controllability/observability of the continuous PDE, as it is for instance the case in the context of the wave equation due to the finite speed of propagation.

3 – The constant coefficient wave equation

3.1 – Problem formulation: Observability

Let us consider the constant coefficient $1D$ wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{3.1}$$

In (3.1) $u = u(x,t)$ describes the displacement of a vibrating string occupying the interval $(0,1)$.

The energy of solutions of (3.1) is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx = E(0),\ \ \forall 0 \le t \le T. \tag{3.2}$$

The problem of continuous boundary observability of (3.1) can be formulated, roughly, as follows: *to give sufficient conditions on the length of the time interval* T *such that there exists a constant* $C(T) > 0$ *so that the following inequality holds for all solutions of* (3.1):

$$E(0) \le C(T) \int_0^T |u_x(1,t)|^2\, dt. \tag{3.3}$$

This corresponds to the exact controllability property of the wave equation with control on $x = 1$ we shall discuss in the next subsection.

Inequality (3.3), when it holds, guarantees that the total energy of a solution can be "observed" or estimated from the energy concentrated or measured on the extreme $x = 1$ of the string during the time interval $(0,T)$ uniformly in the whole class of solutions of (3.1).

Here and in the sequel, the best constant $C(T)$ in inequality (3.3) will be referred to as the *observability constant.*

Of course, one can formulate a weakened version of this observability property which consists simply on the following uniqueness problem:

(3.4) If the solution u of (3.1) is such that $u_x(1,\ t) \equiv 0$ for $0 \le t \le T$, then $u \equiv 0$?

When this uniqueness property holds, we say that the system (3.1) is *weakly observable.* Of course, since we are now dealing with a PDE and therefore we are necessarily in the context of an infinite dimensional dynamical system, unlike in the previous section, the fact that this uniquenes property holds does not automatically guarantee that the observability inequality (3.3) holds as well.

Remark 3.1. This is just an example of a variety of similar observability problems. Among its possible variants, the following are worth mentioning: (a) one could observe the energy concentrated on the extreme $x = 0$ or in the two extremes $x = 0$ and 1 simultaneously; (b) the $L^2(0,T)$-norm of $u_x(1,t)$ could be replaced by some other norm, (c) one could also observe the energy concentrated in a subinterval (α, β) of the space interval $(0,1)$ occupied by the string, etc.

3.2 – Exact controllability

As we mentioned above, the observability problem above is equivalent to a boundary controllability one[(4)]. More precisely, the observability inequality (3.3) holds, if and only if, for any $\big(y^0, y^1\big) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $v \in L^2(0,T)$ such that the solution of the controlled wave equation

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ y(0,t) = 0;\ y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases} \tag{3.5}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1. \tag{3.6}$$

Remark 3.2. Needless to say, in this control problem the goal is to drive solutions to equilibrium at the time $t = T$. Once the configuration is reached at time $t = T$, the solution remains at rest for all $t \geq T$, by taking null control for $t \geq T$, i.e. $v \equiv 0$ for $t \geq T$.

Remark 3.3. It is convenient to note that (3.1) is not, strictly speaking, the adjoint of (3.5). The initial data for the adjoint system should be given at time $t = T$. But, in view of the time-irreversibility of the wave equations under consideration this is irrelevant. Obviously, one has to be more careful about this when dealing with time irreversible systems as the heat equation.

Let us check first that observability implies controllability since the proof is of a constructive nature and allows to build the control of minimal norm ($L^2(0,T)$-norm in the present situation) by minimizing a convex, continuous and coercive functional in a Hilbert space. In the present case, given $\big(y^0, y^1\big) \in$

(4) We refer to J. L. Lions [53] for a systematic analysis of the equivalence between controllability and observability through the so called Hilbert Uniqueness Method (HUM).

$L^2(0,1) \times H^{-1}(0,1)$ the control $v \in L^2(0,T)$ of minimal norm for which (3.6) holds is of the form

$$v(t) = u_x^*(1,t), \tag{3.7}$$

where u^* is the solution of the adjoint system (3.1) corresponding to initial data $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional,

$$J((u^0,u^1)) = \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1}\times H_0^1}, \tag{3.8}$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that the solutions of (3.1) satisfy the extra regularity property that $u_x(1,t) \in L^2(0,T)$ (a fact that holds also for the Dirichlet problem for the wave equation in several space dimensions, see [45], [53], [54]). More, precisely, for all $T > 0$ there exists a constant $C_*(T) > 0$ such that

$$\int_0^T \left[|u_x(0,t)|^2 + |u_x(1,t)|^2 \right] dt \le C_*(T) E(0), \tag{3.9}$$

for all solution of (3.1).

Thus, in order to guarantee that the functional J achieves its minimum, it is sufficient to prove that it is coercive. This is guaranteed by the observability inequality (3.3).

Once coercivity is known to hold the Direct Method of the Calculus of Variations (DMCV) allows showing that the minimum of J over $H_0^1(0,1) \times L^2(0,1)$ is achieved. By the strict convexity of J the minimum is unique and we denote it, as above, by $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$, the corresponding solution of the adjoint system (3.1) being u^*.

The functional J is of class C^1. Consequently, the gradient of J at the minimizer vanishes and this is equivalent to

$$\int_0^1 y(T) w_t(T) dx - < y_t(T), w(T) >_{H^{-1}\times H_0^1} = 0, \tag{3.10}$$

for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, w being the corresponding solution of (3.1). Obviously, this condition is equivalent to the exact controllability one ($y(T) \equiv y_t(T) \equiv 0$) since, whenever (w^0, w^1) covers the whole space $H_0^1(0,1) \times L^2(0,1)$, $(w(T), w_t(T))$ does it as well.

This argument shows that *continuous observability implies controllability.* The reverse is also true.

The main difference with respect to finite-dimensional systems is that the unique continuation property (3.4) does not imply the observability inequality to hold.

3.3 – Approximate controllability

Let us now discuss the control theoretical consequences of the weak observability or unique continuation property (3.4), a property that holds when $T \geq 2$ too. When this property holds the system is *approximately controllable* which means that, for all $\varepsilon > 0$ there is a control v_ε in $L^2(0, T)$ such that the solution satisfies

$$[\| y_\varepsilon(x, T) \|^2_{L^2(0,1)} + \| y_t(x, T) \|^2_{H^{-1}(0,1)}]^{1/2} \leq \varepsilon. \tag{3.11}$$

The control satisfying (3.11) can be built as above but this time the functional to be minimized has to be slightly perturbed(5):

$$\begin{aligned} J_\varepsilon((u^0, u^1)) = \frac{1}{2} \int_0^T \mid u_x(1,t) \mid^2 dt + \varepsilon \parallel (u^0, u^1) \parallel_{H_0^1(0,1)\times L^2(0,1)} + \\ + \int_0^1 y^0 u^1 dx - \int_0^1 y^1 u_0 dx. \end{aligned} \tag{3.12}$$

In [21] it was proved, in the context of the approximate controllability of the heat equation, that adding the ε-term in the functional J_ε guarantess its coercivity as a direct consequence of the weak observability property, without requiring the observability inequality to hold.

The same is true in the present case: if weak observability holds then the functional J_ε satisfies the coercivity property

$$\underline{\lim}_{\|(u^0,u^1)\|_{H_0^1(0,1)\times L^2(0,1)}\to\infty} \frac{J_\varepsilon(u^0, u^1)}{\parallel (u^0, u^1) \parallel_{H_0^1(0,1)\times L^2(0,1)}} \geq \varepsilon. \tag{3.13}$$

Moreover the functional J_ε achieves its minimum at a single point $(u^{0,*}, u^{1,*})$ of $H_0^1(0, 1) \times L^2(0, 1)$. The control $v = u_x^*(1, t)$ is then such that (3.11) is satisfied.

Remark 3.4. In the present $1D$ case both the unique continuation and observability inequality hold if and only if $T \geq 2$. But, in several space dimensions, the observability inequality requires of further geometric constrainst. More precisely, it is required that the so-called Geometric Control Condition (GCC) is satisfied by the subset of the boundary where observation is being made (see [4]). Recall that, roughly speaking, GCC consists on requiring that all rays of Geometric Optics enter the control region in a time which is less than the control time.

(5) Here and in the sequel $-\int_0^1 y^1 u_0 dx$ denotes the duality pairing between $u^0 \in H_0^1(0,1)$ and $y^1 \in H^{-1}(0,1)$.

3.4 – Observability

The following holds:

PROPOSITION 3.1. *For any* $T \geq 2$, *system* (3.1) *is observable. In other words, for any* $T \geq 2$ *there exists* $C(T) > 0$ *such that* (3.3) *holds for any solution of* (3.1). *Conversely, if* $T < 2$, (3.1) *is not observable, or, equivalently,*

$$\sup_{u \text{ solution of } (3.1)} \left[\frac{E(0)}{\int_0^T \mid u_x(1,t) \mid^2 dt} \right] = \infty. \tag{3.14}$$

The proof of observability for $T \geq 2$ can be carried out in several ways. The simplest one uses the Fourier representation of solutions [87] but it is insufficient to deal with multidimensional problems. In several space dimensions one may use multipliers (KOMORNIK, [45]; LIONS, [53]), Carleman inequalities (ZHANG, [79]), and microlocal tools (BARDOS et al., [4]; BURQ and GÉRARD, [7]).

On the other hand, for $T < 2$ the observability inequality does not hold, due to the finite speed of propagation (= 1 in the model under consideration).

Summarizing, Proposition 3.1 states that, in one space dimension, a necessary and sufficient condition for the observability (both in its strong and weak version) to hold is that $T \geq 2$.

4 – 1*D* Finite-difference semi-discretizations

In this section we discuss the observability/controllability properties of a semi-discrete finite-difference approximation of the wave equation. This problem arises naturally in the numerical approximation of controls.

We describe the following results, of negative nature:

- The observability constant for the semi-discrete model tends to infinity for any T as the mesh-size h tends to zero.
- There are initial data for the wave equation for which the exact controls of the semi-discrete models diverge as $h \to 0$, This proves that one can not simply rely on the classical convergence (consistency + stability) analysis of the underlying numerical schemes to design stable algorithms for computing the controls.

We also briefly recall some of the basic cures that have been developed in the literature to avoid this high frequency numerical pathologies.

4.1 – Finite-difference approximations

Let us now formulate these problems and state the corresponding results in a more precise way.

Given $N \in \mathbf{N}$ we define $h = 1/(N+1) > 0$. We consider the mesh

$$x_0 = 0;\ x_j = jh,\ j = 1, \dots, N;\ x_{N+1} = 1, \tag{4.1}$$

which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, \dots, N$.

Consider the following finite difference approximation of the wave equation (3.1):

$$\begin{cases} u_j'' - \dfrac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ u_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \dots, N. \end{cases} \tag{4.2}$$

Observe that (4.2) is a coupled system of N linear differential equations of second order. The function $u_j(t)$ provides an approximation of $u(x_j, t)$ for all $j = 1, \dots, N$, u being the solution of the continuous wave equation (3.1). The conditions $u_0 = u_{N+1} = 0$ reproduce the homogeneous Dirichlet boundary conditions, and the second order differentiation with respect to x has been replaced by the three-point finite difference.

We shall use a vector notation to simplify the expressions. Then, system (4.2) reads as follows

$$\begin{cases} \vec{u}\,''(t) + A_h \vec{u}(t) = 0, & 0 < t < T \\ \vec{u}(0) = \vec{u}^{\,0},\ \vec{u}\,'(0) = \vec{u}^{\,1} \end{cases} \tag{4.3}$$

where the matrix A is given by:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{4.4}$$

and the column vector

$$\overrightarrow{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \tag{4.5}$$

represents the whole set of unknowns of the system.

The solution $\overrightarrow{u}$ of (4.3) depends also on h but often this will not be made explicit in the notation.

The energy of the solutions of (4.2),

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[| u_j' |^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right], \tag{4.6}$$

is constant in time. It is a natural discretization of the continuous energy (3.2).

The problem of observability of system (4.2) can be formulated as follows: *to find $T > 0$ and $C_h(T) > 0$ such that*

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{4.7}$$

holds for all solutions of (4.2).

Observe that $| u_N/h |^2$ is a natural approximation of $| u_x(1,t) |^2$ for the solution of the continuous system (3.1). Indeed $u_x(1,t) \sim [u_{N+1}(t) - u_N(t)]/h$ and, taking into account that $u_{N+1} = 0$, it follows that $u_x(1,t) \sim -u_N(t)/h$.

System (4.2) is finite-dimensional. Therefore, if observability holds for some $T > 0$, then it holds for all $T > 0$ as we have seen in Section 2.

Inequality (4.7) does indeed hold for all $T > 0$ and $h > 0$. This can be seen analyzing the Kalman rank condition.

4.2 – Non uniform observability

But the observability constant $C_h(T)$ diverges as $h \to 0$. To see this let us consider the eigenvalue problem

$$-\left[w_{j+1} + w_{j-1} - 2w_j\right]/h^2 = \lambda w_j,\ j = 1, \ldots, N; \quad w_0 = w_{N+1} = 0. \tag{4.8}$$

The spectrum can be computed explicitly in this case (Isaacson and Keller [4.2]), the eigenvalues and eigenvectors being

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right) \tag{4.9}$$

and

$$\vec{w}_k^{\,h} = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi j h),\ k, j = 1, \ldots, N. \tag{4.10}$$

Obviously,

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0 \tag{4.11}$$

for each $k \geq 1$, $\lambda_k = k^2\pi^2$ being the k-th eigenvalue of the continuous wave equation (3.1). On the other hand we see that the eigenvectors $\vec{w}_k^{\,h}$ of the discrete system (4.8) coincide with the restriction to the mesh-points of the eigenfunctions $w_k(x) = \sin(k\pi x)$ of the continuous wave equation (3.1).

The main negative result on the lack of uniform (as $h \to 0$) observability inequality is as follows [39], [40]:

THEOREM 4.1. *For any $T > 0$ it follows that, as $h \to 0$,*

$$\sup_{u \text{ solution of } (4.2)} \left[\frac{E_h(0)}{\displaystyle\int_0^T \mid u_N/h \mid^2 dt} \right] \to \infty. \tag{4.12}$$

This negative result is a consequence of the following identity

$$h \sum_{j=0}^{N} \left| \frac{w_{j+1} - w_j}{h} \right|^2 = \frac{2}{4 - \lambda h^2} \left| \frac{w_N}{h} \right|^2 \tag{4.13}$$

and the fact that

$$\lambda_N^h h^2 \to 4 \text{ as } h \to 0. \tag{4.14}$$

But, the fact that isolated eigenvectors are badly observed on the boundary is not the only obstacle for the boundary observability property to be uniform as the mesh-size tends to zero. Indeed, let us consider the following solution of the semi-discrete system (4.2), constituted by the last two eigenvectors:

$$\vec{u} = \frac{1}{\sqrt{\lambda_N}} \left[\exp(i\sqrt{\lambda_N} t)\vec{w}_N - \exp(i\sqrt{\lambda_{N-1}} t)\vec{w}_{N-1} \right]. \tag{4.15}$$

This solution is a wave packet obtained as superposition of two monochromatic semi-discrete waves corresponding to the last two eigenfrequencies of the system. The total energy of this solution is of the order 1 (because each of both components has been normalized in the energy norm and the eigenvectors are orthogonal one to each other). However, the trace of its discrete normal derivative tend to zero in $L^2(0, T)$ as $h \to 0$. This is due to two facts.

- First, the trace of the discrete normal derivative of each eigenvector is of order h compared to its total energy.
- Second and more important, the gap between $\sqrt{\lambda_N}$ and $\sqrt{\lambda_{N-1}}$ is of the order of h.

Thus, by Taylor expansion, the difference between the two time-dependent complex exponentials $\exp(i\sqrt{\lambda_N}t)$ and $\exp(i\sqrt{\lambda_{N-1}}t)$ is of the order Th.

This construction makes it possible to show that, whatever the time T is, the observability constant $C_h(T)$ in the semi-discrete system is at least of order $1/h$. In fact, this idea but combining an increasing number of high eigenfrequencies, can be used to show that the observability constant has to blow-up at infinite order. We refer to [58] for a precise analysis of the exponential blow-up of the observability constant.

The careful analysis of this negative example is extremely useful when designing possible remedies, i.e., to determine how one could modify the numerical scheme in order to reestablish the uniform observability inequality, since we have only found two obstacles and both happen at high frequencies. The first remedy is very natural: to cut off the high frequencies or, in other words, to ignore the high frequency components of the numerical solutions. This Fourier filtering method will be discussed later in some more detail. But let us first state the main consequences of the negative results above on the lack of uniform controllability.

4.3 – On the lack of uniform controllability

We have shown that the uniform observability property of the finite difference approximations (4.2) fails for any $T > 0$. In this subsection we explain the consequences of this result in the context of controllability.

The corresponding control system is:

$$(4.16)\qquad \begin{cases} y_j'' - \dfrac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ y_0(0,t) = 0;\ y_{N+1}(1,t) = v(t), & 0 < t < T \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 1, \dots, N, \end{cases}$$

and the question we consider is whether, for a given $T > 0$ and given initial data $(\vec{y}^{\,0}, \vec{y}^{\,1})$, there exists a control $v_h \in L^2(0,T)$ such that

$$(4.17)\qquad \vec{y}(T) = \vec{y}\,'(T) = 0.$$

System (4.2) being observable for all $h > 0$ and $T > 0$, system (4.16) is controllable for all $h > 0$ and $T > 0$, too.

However, this does not mean that the controls will be bounded as h tends to zero. In fact they diverge, even if $T \geq 2$. More precisely, we have the following main results:

- Taking into account that for all $h > 0$ the Kalman rank condition is satisfied, for all $T > 0$ and all $h > 0$ the semi-discrete system (4.16) is controllable. In other words, for all $T > 0$, $h > 0$ and initial data $(\vec{y}^{\,0}, \vec{y}^{\,1})$, there exists $v \in L^2(0,T)$ such that the solution $\vec{y}$ of (4.16) satisfies (4.17). Moreover, the control v of minimal $L^2(0,T)$-norm can be built as in Section 3. It suffices to minimize the functional

$$J_h((\vec{u}^{\,0},\vec{u}^{\,1})) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \tag{4.18}$$

over the space of all initial data $(\vec{u}^{\,0},\vec{u}^{\,1})$ for the adjoint semi-discrete system (4.2).

Of course, in view of the observability inequality (4.7), this strictly convex and continuous functional is coercive and, consequently, has a unique minimizer.

Once we know that the minimum of J_h is achieved, the control is easy to compute. It suffices to take

$$v_h(t) = u_N^*(t)/h,\ 0 < t < T, \tag{4.19}$$

as control to guarantee that (4.17) holds, where $\vec{u}^{\,*}$ is the solution of the semi-discrete adjoint system (4.2), corresponding to the initial data $(\vec{u}^{\,0,*},\vec{u}^{\,1,*})$ that minimize the functional J_h.

The control we obtain in this way is optimal in the sense that it is the one of minimal $L^2(0,T)$-norm. We can also get an upper bound on its size. Indeed, using the fact that $J_h \le 0$ at the minimum (which is a trivial fact since $J_h((0,0)) \le 0$), and the observability inequality (4.7), we deduce that

$$||v_h||_{L^2(0,T)} \le 4C_h(T)||(y^0,y^1)||_{*,h}, \tag{4.20}$$

where $||\cdot||_{*,h}$ denotes the norm

$$||(y^0,y^1)||_{*,h} = \sup_{(u_j^0,u_j^1)_{j=1,\dots,N}} \left[\left|h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0\right| \Big/ E_h^{1/2}(u^0,u^1)\right]. \tag{4.21}$$

It is easy to see that this norm converges as $h \to 0$ to the norm in $L^2(0,1) \times H^{-1}(0,1)$. This norm can also be written in terms of the Fourier coefficients. It becomes a weighted euclidean norm whose weights are uniformly (with respect to h) equivalent to those of the continuous $L^2 \times H^{-1}$-norm.

The estimate (4.20) is sharp and the constant $C_h(T)$ blows-up as h tends to zero. This has important consequences on the limit behavior of the control problem.

Indeed, according to Theorem 4.1, for all $T > 0$ the constant $C_h(T)$ diverges as $h \to 0$. This shows, by the Banach-Steinhaus theorem, that there are initial data for the wave equation in $L^2(0,1) \times H^{-1}(0,1)$ such that the controls of the semi-discrete systems $v_h = v_h(t)$ diverge as $h \to 0$. There are different ways of making this result precise. For instance, given initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system, we can consider in the semi-discrete control system (4.16) the initial data that take the same Fourier coefficients as (y^0, y^1) for the indices $j = 1, \dots, N$. It then follows that, because of the divergence of the observability constant $C_h(T)$, there are necessarily some initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system such that the corresponding controls v_h for the semi-discrete system diverge in $L^2(0,T)$ as $h \to 0$. Indeed, assume that for any initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, the controls v_h remain uniformly bounded in $L^2(0,T)$ as $h \to 0$. Then, according to the uniform boundedness principle, we would deduce that the maps that associate the controls v_h to the initial data are also uniformly bounded. But this implies the uniform boundedness of the observability constant $C_h(T)$.

This lack of convergence is in fact easy to understand. As we have shown above, the semi-discrete system generates a lot of spurious high frequency oscillations. The control of the semi-discrete system has to take them into account. When doing this it gets further and further away from the true control of the continuous wave equation.

4.4 – Some remedies

Several remedies and cures have been proposed in the literature to avoid the unstabilities that high frequency numerical spurious solutions introduce both at the level of observation and control.

- **Fourier filtering**
  Filtering consists on considering subclasses of solutions of the adjoint system (4.2) constituted by the Fourier components corresponding to the eigenvalues $\lambda \leq \gamma h^{-2}$ with $0 < \gamma < 4$. This is equivalent to considering solutions whose only nontrivial components are those corresponding to the indices $0 < j < \delta h^{-1}$ with $0 < \delta < 1$. In these subclasses of solutions the observability inequality becomes uniform, i.e. the observability constant does not blow-up as h tends to zero. But for this to be true the time T of observability needs to be taken large enough and the value of the optimal observability time depends on the filtering parameters γ or δ. Note that these classes of solutions correspond to taking projections of the complete solutions by cutting off all frequencies with $\gamma h^{-2} < \lambda 4 h^{-2}$.
  More precisely, solutions of (4.2) can be developed in Fourier series as follows:

$$\vec{u} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) \vec{w}_k^h \tag{4.22}$$

where a_k, b_k are the Fourier coefficients of the initial data, i.e.,

$$\vec{u}^{\,0} = \sum_{k=1}^{N} a_k \vec{w}^{\,h}_{k}, \quad \vec{u}^{\,1} = \sum_{k=1}^{N} b_k \vec{w}^{\,h}_{k}.$$

Given $0 < \delta < 1$, the classes of filtered solutions are of the form:

$$\mathcal{C}_\delta(h) = \left\{ \vec{u} \text{ sol. of (4.2) s.t. } \vec{u} = \sum_{k=1}^{[\delta/h]} \left(a_k \cos\left(\sqrt{\lambda^h_k} t\right) + \frac{b_k}{\sqrt{\lambda^h_k}} \sin\left(\sqrt{\lambda^h_k} t\right)\right) \vec{w}^{\,h}_{k} \right\}. \tag{4.23}$$

The Fourier filtering is natural since the numerical scheme, which converges in the classical sense, reproduces, at low frequencies, as $h \to 0$, the whole dynamics of the continuous wave equation. But, it also introduces a lot of high frequency spurious solutions. The scheme then becomes more accurate if we ignore the high frequency components and this makes the observability inequality uniform provided the time is taken to be large enough.

To prove the uniform (as $h \to 0$) observability result for filtered solutions of system (4.2), it is sufficient to combine a sharp analysis of the spectrum of the semi-discrete system under consideration and the classical *Ingham inequality* in the theory of nonharmonic Fourier series (see Ingham [41] and Young [77]). This analysis gives an explicit estimate of the optimal observability time in the class $\mathcal{C}_\delta(h) : T(\delta) = 2/\cos(\pi\delta/2)$. The minimal time $T(\delta)$ of uniform observability in this subclasses of filtered solutions is such that $T(\delta) \to 2$ as $\delta \to 0$ and $T(\delta) \to \infty$ as $\delta \to 1$, as one could expect.

At the level of control, these results imply the uniform controllability of the projections of solutions of (4.16) over the subspace of the low eigenmodes that have not been cutted-off. One can then pass to the limit and prove the convergence towards the control of the continuous wave equation (3.5). This is so because, as h tends to zero, regardless of the value of the filtering parameter, one ends up recovering all the Fourier components of the state on the controlled projection.

We refer to [87] for a more details on the algorithm of control based on Fourier filtering.

In the context of the numerical computation of the boundary control for the wave equation the need of an appropriate filtering of the high frequencies was observed by R. Glowinski [29]. This issue was further investigated numerically by M. Asch and G. Lebeau in [1]. There, finite difference schemes were used to test the Geometric Control Condition in various geometrical situations and to analyze the cost of the control as a function of time.

However, this method, which is natural from a theoretical point of view, can be hard to implement in numerical simulations. Indeed, solving the semi-discrete system provides the nodal values of the solution. One then needs to compute its Fourier coefficients and, once this is done, to recalculate the nodal values of the filtered/truncated solution. Therefore, it is convenient to explore other ways of avoiding these high frequency pathologies that do not require going back and forth from the physical space to the frequency one. Several other possibilities have been introduced and analyzed in the literature. We mention them below.

- **Tychonoff regularization**
  Glowinski et al. in [32] proposed a Tychonoff regularization technique that allows one to recover the uniform (with respect to the mesh size) coercivity of the functional that one needs to minimize to get the controls in the HUM approach. The method was tested to be efficient in numerical experiments. The convergence of the argument has been discussed in [87].

- **A two-grid algorithm**
  Glowinski and Li in [31] introduced a two-grid algorithm that also makes it possible to compute efficiently the control of the continuous model. The method was further developed by Glowinski in [29].
  The relevance and impact of using two grids can be easily understood in view of the analysis above of the $1D$ semi-discrete model. In view of the explicit expression of the eigenvalues of the semi-discrete system (4.9), all of them satisfy $\sqrt{\lambda} \leq 2/h$. We have also seen that the observability inequality becomes uniform when one considers solutions involving eigenvectors corresponding to eigenvalues $\sqrt{\lambda} \leq 2\gamma/h$, with $\gamma < 1$. Glowinski's 2-grid algorithm is based on the idea of using two grids: one with step size h and a coarser one of size $2h$. In the coarser mesh the eigenvalues obey the sharp bound $\lambda \leq 1/h^2$. Thus, the oscillations in the coarse mesh that correspond to the largest eigenvalues $\sqrt{\lambda} \sim 1/h$, in the finer mesh are associated to eigenvalues in the class of filtered solutions with parameter $\gamma = 1/2$. Then, this corresponds to a situation where the observability inequality is uniform for T large enough.
  The convergence of this method has recently been proved rigorously in [64] where the time of control was found to be $T > 4$, twice the control time for the continuous wave equation.

- **Mixed finite elements**
  An alternative approach consists in using mixed finite element methods rather than finite differences or standard finite elements, which require some filtering, Tychonoff regularization or multigrid techniques, as we have shown. First of all, it is important to underline that the analysis we have developed for the finite difference space semi-discretization of the $1D$ wave equation can be carried out with minor changes for finite element semi-discretizations as well. In particular, due to the high frequency spurious oscillations, uniform observability does not hold [40]. It is thus natural to consider mixed finite

element (m.f.e.) methods. This idea was introduced by Banks et al. [2] in the context of boundary stabilization of the wave equation.

This method has been succesfully adapted in [11] for control purposes. It provides a good approximation of the wave equation and converges in classical terms. For this scheme the gap between the square roots of consecutive eigenvalues of its spectrum is uniformly bounded from below, and in fact tends to infinity for the highest frequencies as $h \to 0$. According to this and applying Ingham's inequality, the uniform observability property holds (see [11]).

The idea of correcting the dispersion diagram by modifying the numerical scheme has been previously explored in S. Krenk [46], for instance, where this was done by adding higher order terms in the approximation of the scheme. This approach has been also pursued by A. Munch [60] to enrich the class of schemes introduced in [11].

5 – Robustness of approximate controllability

In the previous sections we have shown that the exact controllability property behaves badly under most classical finite difference approximations. It is natural to analyze to what extent the high frequency spurious pathologies do affect other control problems and properties. In this section we focus on the problem of approximate controllability.

The approximate controllability problem is a relaxed version of the exact controllability one. The goal this time is to drive the solution of the controlled wave equation (3.5) not exactly to the equilibrium as in (3.6) but rather to an ε-state such that

$$\left[||y(T)||^2_{L^2(0,1)} + ||y_t(T)||^2_{H^{-1}(0,1)} \right]^{1/2} \leq \varepsilon. \tag{5.1}$$

When for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ and for all ε there is a control v such that (5.1) holds, we say that the system (3.5) is approximately controllable. Obviously, approximate controllability is a weaker notion than exact controllability and whenever the wave equation is exactly controllable, it is approximately controllable too.

As we have seen in Section 3.3, although exact controllability requires an observability inequality of the form of (3.3) to hold, for approximate controllability one only requires the uniqueness property (3.4).

This uniqueness property holds for $T \geq 2$ as well and can be easily proved using Fourier series or d'Alembert's formula. Its multidimensional version holds as well, as an immediate consequence of Holmgren's Uniqueness theorem (see [53]) for general wave equations with analytic coefficients and without geometric conditions, other than the time being large enough. In $1D$, because of the trivial geometry, both the uniqueness property and observability inequality hold simultaneously for $T \geq 2$ but this is not longer true in several space dimensions.

Of course, the approximate controllability property by itself, as seen in Section 3.3, does not provide any information of what the cost of controlling to an ε-state as in (5.1) is, i.e. on what is the norm of the control v_ε needed to achieve the approximate control condition (5.1)[(6)]. But this issue will not be addressed here.

In what follows we fix some $\varepsilon > 0$. As mentioned above and seen in Section 3.3, once ε is fixed, we know that when $T \geq 2$, for all initial data (y^0, y^1) in $L^2(0, 1) \times H^{-1}(0, 1)$, there exists a control $v_\varepsilon \in L^2(0, T)$ such that (5.1) holds. Moreover, the control can be obtained minimizing a functional of the form (3.12).

The question we are interested in is the behavior of this property under numerical discretization.

Thus, let us consider the semi-discrete controlled version of the wave equation (4.16). We also fix the initial data in (4.16) "independently of h". This can be done in several ways:

a) When the data (y^0, y^1) of the continuous wave equation are smooth enough, for instance continuous, we may take the initial data for (4.16) as being the restriction of (y^0, y^1) to the mesh-points.
b) One may also take as initial for (4.16) the projection of the Fourier coefficients of (y^0, y^1) over the first N modes that can be represented on the discrete model.

Of course, (4.16) is also approximately controllable[(7)]. The question we address is as follows: *given initial data which are "independent of h", as above, with ε fixed, and given also the control time $T \geq 2$, is the control v_h of the semi-discrete system* (4.16) (*such that the discrete version of* (5.1) *holds*) *uniformly bounded as $h \to 0$?*

In the previous sections we have shown that the answer to this question in the context of exact controllability (which corresponds to taking $\varepsilon = 0$) is negative. However, we have also seen that relaxing the final requirement of reaching the target exactly may help. The following result shows that this is the case in the context of approximate control too.

Theorem 5.1. *Assume that the initial data in* (4.16) *are essentially independent of h.*

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(6) Roughly speaking, when exact controllability does not hold (for instance, in several space dimensions, when the GCC is not fulfilled), the cost of controlling blows up exponentially as ε tends to zero (see [66]). This type of result has been also proved in the context of the heat equation in [24]. But there the difficulty does not come from the geometry but rather from the regularizing effect of the heat equation.

(7) In fact, in finite dimensions, exact and approximate controllability are equivalent notions and, as we have seen, the Kalman condition is satisfied for system (4.16).

Assume that $T \geq 2$.

Then, for $\varepsilon > 0$ *fixed, the controls* v_h *such that the solution of satisfies*

$$\| (\vec{y_h}(T), \vec{y}'_h(T)) \|_{*,h} \leq \varepsilon, \tag{5.2}$$

are uniformly bounded in $L^2(0, T)$ *as* $h \to 0$.

Moreover, the controls v_h *can be chosen such that they converge in* $L^2(0, T)$ *to a limit control* v *for which* (5.1) *is realized for the continuous wave equation* (3.5).

This positive result on the uniformity of the approximate controllability property under numerical approximation when $\varepsilon > 0$ does not contradict the fact that the controls blow up for exact controllability (i.e. when $\varepsilon = 0$). These are in fact two complementary and compatible facts. For approximate controllability, one is allowed to concentrate an ε amount of energy on the solution at the final time $t = T$. For the semi-discrete problem this is done precisely in the high frequency components that are badly controllable as $h \to 0$, and this makes it possible to keep the control fulfilling (5.1) bounded as $h \to 0$.

The approximate control of the semi-discrete system can be obtained by minimizing the functional

$$J_h^*(\vec{u}^0, \vec{u}^1) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + \varepsilon ||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1\times\ell^2} + h\sum_{j=1}^{N} y_j^0 u_j^1 - h\sum_{j=1}^{N} y_j^1 u_j^0 \tag{5.3}$$

over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semi-discrete system (4.2). In J_h^*, $||\cdot||_{\mathcal{H}^1\times\ell^2}$ stands for the discrete energy norm, i.e. $||\cdot|| = \sqrt{2E_h}$. Note that there is an extra term $\varepsilon||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1\times\ell^2}$ in this new functional compared with the one we used to obtain the exact control (see (4.18)). On the other hand, the functional in (5.3) is a discrete version of the functional (3.12) one needs to minimize to get the approximate control for the continuous wave equation. In both cases, the controls one finds that way are those of minimal $L^2(0, T)$-norm.

Theorem 5.1 states the convergence of controls which are closely related to the minimizers of these functionals. Indeed, while the control v of the continuous wave equation (3.5) is defined as

$$v(t) = u_x^*(1, t), \tag{5.4}$$

u^* being the solution of the adjoint equation (3.1) with the initial data being the minimizer of the functional in (3.12), the control v_h of the semi-discrete equation (4.16) is defined as

$$v_h(t) = u_N^*(t)/h, \tag{5.5}$$

where u^* is the solution of the semi-discrete adjoint equation (4.2) with the minimizer of the functionals $J^*_{h,\,\varepsilon}$ in (5.3) as initial data.

Therefore, roughly speaking, Theorem 5.1 can be viewed as a Γ-convergence result [19] of the functional $J^*_{h,\,\varepsilon}$ towards J^*_{ε}.

Similar results have been proved in several different but related problems:

a) The approximate control of parabolic equations with rapidly oscillating coefficients and perforated domains in any space dimension (see [84] and [20], respectively) and the null control in $1D$ [55].
b) The exact controllability of the space semi-discretizations of the beam equation [51].

The key ingredient in the proof of Theorem 5.1 is the uniform (with respect to h) coercivity of the functionals $J^*_{h,\,\varepsilon}$. The following holds;

$$\underline{\lim}_{||(\vec{u}^{\,0},\vec{u}^{\,1})||_{\mathcal{H}^1\times\ell^2}\to\infty} \frac{J^*_h(\vec{u}^{\,0},\vec{u}^{\,1})}{||(\vec{u}^{\,0},\vec{u}^{\,1})||_{\mathcal{H}^1\times\ell^2}} \geq \varepsilon, \tag{5.6}$$

uniformly in h, provided $T \geq 2$.

Once the uniform observability property (5.6) holds, the minimizers are immediately uniformly bounded and the controls as well. Once this is done one can proceed in two steps:

a) First one shows that the weak limit (in $L^2(0,\,T)$) of the controls is a control for the limit system;
b) one then shows by Γ-convergence arguments that the limit control is precisely the one associated with the minimization of the limit functional J^*_ε;
c) finally one proves, using convexity and weak lower semicontinuity arguments, that $J^*_{h,\,\varepsilon}(\vec{u}^{\,*,0}_h,\vec{u}^{\,*,1}_h)$ tends to $J^*_\varepsilon(\vec{u}^{\,*,0},\vec{u}^{\,*,1})$ as h tends to zero. This, together with the fact that the initial data to be controlled are essentially independent of h allows concluding that the $L^2(0,\,T)$-norms of the controls converge to the $L^2(0,\,T)$-norm of the limit controls. This guarantees that convergence holds in the strong topology.

We refer to [51] for the details of the proof in the closely related problem of the control of the beam equation.

Consequently, let us focus on the proof of the uniform coercivity property (5.6). At this level, the fact that $T \geq 2$ is essential. In order to show that the coercivity property above is uniform in $0 < h < 1$ we have to argue as in [84]. Mainly, we have to consider the case where $h \to 0$ and solutions of the adjoint semi-discrete system (4.2) converge to a solution of the continuous adjoint wave equation (3.1) such that $u_x(1,t) \equiv 0$ in $(0,T)$. Of course, if this happens with $T \geq 2$ we can immediately deduce that $u \equiv 0$ by the well known uniqueness property of the solutions of the wave equation discussed in Section 3.3. This suffices to conclude the uniform coercivity property.

This shows that the approximate controllability property is well-behaved under the semi-discrete finite-difference discretization of the wave equation. But the argument is in fact much more general and can be applied for other numerical approximation schemes. The two assumptions that are needed on the numerical scheme for this to hold are:

a) The scheme is convergent in the classical sense;
b) for all h the numerical scheme is controllable.

However, as we shall see, although these properties hold for most numerical schemes in $1D$, the second property may fail in several space dimensions unless some filtering is introduced or some extra geometric assumptions are imposed on the subset where the control is suppported.

6 – Robustness of optimal control

Finite horizon optimal control problems can also be viewed as relaxed versions of the exact controllability one.

Let us consider the following example in which the goal is to drive the solution of the wave equation (3.5) at time $t = T$ as closely as possible to the desired equilibrium state but penalizing the use of the control. In the continuous context the problem can be simply formulated as that of minimizing the functional

$$L^k(v) = \frac{k}{2}||(y(T), y_t(T))||^2_{L^2(0,1)\times H^{-1}(0,1)} + \frac{1}{2}||v||^2_{L^2(0,T)} \tag{6.1}$$

over $v \in L^2(0, T)$. This functional is continuous, convex and coercive in the Hilbert space $L^2(0, T)$. Thus it admits a unique minimizer that we denote by v_k. The corresponding optimal state is denoted by y_k. The penalization parameter establishes a balance between reaching the distance to the target and the use of the control. As k increases, the need of getting close to the target (the $(0, 0)$ state) is emphasized and the penalization on the use of control is relaxed.

When exact (resp. approximate) controllability holds, i.e. when $T \geq 2$, it is not hard to see that the control one obtains by minimizing L^k converges, as $k \to \infty$, to an exact (resp. approximate) control for the wave equation (see [23]).

When the value of the parameter $k > 0$ is fixed, the optimal control v_k does not guarantee that the target ((0, 0) in this case) is achieved in an exact way. One can then measure the rate of convergence of the optimal solution $(y_k(T), y_{k,t}(T))$ towards $(0, 0)$ as $k \to \infty$. When approximate controllability holds but exact controllability does not (a typical situation in several space dimensions when the GCC is not satisfied), the convergence of $(y_k(T), y_{k,t}(T))$ to $(0, 0)$ in $L^2(0, 1) \times H^{-1}(0, 1)$ as $k \to \infty$ is very slow (roughly spaking, of logarithmic nature).

But here, once again, we fix any $k > 0$ and we discuss the behavior of the optimal control problem for the semi-discrete equation as $h \to 0$.

It is easy to write the semi-discrete version of the problem of minimizing the functional L^k. Indeed, it suffices to introduce the corresponding semi-discrete functional L^k_h replacing the $L^2 \times H^{-1}$-norm in the definition of L^k by the discrete norm introduced in (4.21). It is also easy to prove by the arguments we have developed in the context of approximate controllability, that, *as $h \to 0$, the control v^k_h that minimizes L^k_h in $L^2(0,T)$ converges to the minimizer of the functional L^k and the optimal solutions y^k_h of the semi-discrete system converge to the optimal solution y^k of the continuous wave equation in the appropriate topology*(8) *as $h \to 0$ too.*

In this case the proof of the uniform boundedness of the control is much easier since the uniform coercivity of the functionals L^k_h is obvious as soon as $k > 0$.

This shows that the optimal control problem is also well-behaved with respect to numerical approximation schemes, like the approximate control problem.

The reason for this is basically the same: in the optimal control problem the target is not required to be achieved exactly and, therefore, the pathological high frequency spurious numerical components are not required to be controlled.

In view of this discussion it becomes clear that the source of divergence in the limit process as $h \to 0$ in the exact controllability problem is the requirement of driving the high frequency components of the numerical solution exactly to zero. As we mentioned in the introduction, taking into account that optimal and approximate controllability problems are relaxed versions of the exact controllability one, even though they are theoretically well behaved under the numerical approximation process as our results above show, this negative result should be considered as a warning about the limit process as $h \to 0$ in general control problems.

7 – Stabilization

The problem of controllability has been addressed along this paper. The connections between the problems of controllability and stabilization are well known (see for instance [69], [78]) and similar developments could be carried out in the context of stabilization.

In the context of the wave equation, it is well known that the GCC suffices for stabilization and more precisely to guarantee the uniform exponential decay of solutions when a damping term, supported in the control region, is added

(8)Roughly, in $L^p([0,T]; L^2(0,1)) \cap W^{1,p}[0,T]; H^{-1}(0,1))$ for all $1 \leq p < \infty$, once the solution of the semi-discrete problem has been extended to the interior conveniently (as a piecewise linear and continuous function, for instance).

to the system. More precisely, when the subdomain ω of the domain Ω where the wave equation holds satisfies the GCC the solutions of the damped wave equation

$$y_{tt} - \Delta y + 1_\omega y_t = 0$$

with homogeneous Dirichlet boundary conditions are known to decay exponentially in the energy space. In other words, there exist constants $C > 0$ and $\gamma > 0$ such that

$$E(t) \le Ce^{-\gamma t} E(0)$$

holds for every finite energy solution of the Dirichlet problem for this damped wave equation.

It is then natural to analyze whether the decay rate is uniform with respect to the mesh size for numerical discretizations. The answer is in general negative. Indeed, due to spurious high frequency oscillations, the decay rate fails to be uniform, for instance, for the classical finite difference semi-discrete approximation of the wave equation. This was established rigorously by F. Macià [56], [57] using Wigner measures. This negative result also has important consequences in many other issues related with control theory like infinite horizon control problems, Riccati equations for the optimal stabilizing feedback ([65]), etc.

We shall simply mention here that, even if the most natural semi-discretization schemes fail to be uniformly exponentially stable, the uniformity of the exponential decay rate can be reestablished if we add an internal viscous damping term to the equation (see [72], [73] and [61]).

In [72] we analyzed finite difference semi-discretizations of the damped wave equation

$$u_{tt} - u_{xx} + \chi_\omega u_t = 0, \tag{7.1}$$

where χ_ω denotes the characteristic function of the set ω where the damping term is effective. In particular we analyzed the following semi-discrete approximation in which an extra numerical viscous damping term is present:

$$\begin{cases} u_j'' - \dfrac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] - \left[u_{j+1}' + u_{j-1}' - 2u_j'\right] - u_j'\chi_\omega = 0, & \\ & 0 < t < T,\ j = 1, \dots, N \\ u_j(t) = 0, & 0 < t < T,\ j = 0,\ N+1 \\ u_j(0) = u_j^0, \quad u_j^1(0) = u_j^1, & j = 1, \dots, N. \end{cases} \tag{7.2}$$

It was proved that this type of scheme preserves the uniform stabilization properties of the wave equation (7.1). To be more precise we recall that solutions of the $1D$ wave equation (7.1) in a bounded interval with Dirichlet boundary conditions decay exponentially uniformly as $t \to \infty$ when a damping term as above is added, ω being an open non-empty subinterval (see [80]). Using the numerical scheme above, this exponential decay property is kept with a uniform rate as h tends to zero. The extra numerical damping that this scheme introduces adding the term $[u'_{j+1} + u'_{j-1} - 2u'_j]$ damps out the high frequency spurious oscillations that the classical finite difference discretization scheme introduces and that produce a lack of uniform exponential decay in the presence of damping.

The problem of whether this numerical scheme is uniformly observable or controllable as h tends to zero is an interesting open problem.

Note that the system above, in the absence of the damping term lozalized in ω, can be written in the vector form

$$\vec{u}'' + A_h\vec{u} + h^2 A_h\vec{u}' = 0. \tag{7.3}$$

Here $\vec{u}$ stands, as usual, for the vector unknown $(u_1, \ldots, u_N)^T$ and A_h for the tridiagonal matrix associated with the finite difference approximation of the Laplacian (4.4). In this form it is clear that the scheme above corresponds to a viscous approximation of the wave equation. Indeed, taking into account that A_h provides an approximation of $-\partial_x^2$, the presence of the extra multiplicative factor h^2 in the numerical damping term guarantees that it vanishes asymptotically as h tends to zero.

In [61] these results were extended to general domains in $2-d$. The subdomain ω was assumed to be a neighborhood of a subset of the boundary satisfying the classical multiplier condition, which constitutes a particular class of subdomains satisfying the GCC [80]. Then, adding a numerical viscosity term the uniform exponential decay was proved.

In the absence of geometric conditions on the subset ω, by only assuming that it is an open non-empty subset of Ω, using La Salle's invariance principle with the energy of the system as Lyapunov function, one can show that all solutions of the damped wave equation tend to zero as t goes to infinity without uniform exponential decay rate. This is true even in several space dimensions. This results turns out to be false at the semi-discrete level in the multi-dimensional case. Indeed, the property of decay relies on a unique continuation property similar to those we discussed in the context of approximate controllability. In the case of the continuous wave equation this property requires that whenever the solution u of the wave equation vanishes in $\omega \times (0, \infty)$, then it vanishes everywhere. This holds as a consequence of Holmgren's uniqueness theorem if $T > 0$ is large enough. But it fails to be true for the semi-discrete equation without further restrictions on the subdomain ω as we shall see in open problem #2 below.

If one adds a numerical viscosity term, obviously, these difficulties dissapear and one recovers the decay of solutions of the semi-discrete system. But uniform (with respect to h) exponential decay rates can only be expected under geometric restrictions in ω as in [72] and [61]. Similar devepments have been carried out in [73] in the context of boundary damping in one-space dimension. Very likely similar results are true for boundary damping in several dimensions too. But a complete analysis of this issue using the techniques in [61] and [73] is still to be done.

8 – Open problems

1. *Moment problems techniques.* We have considered finite difference space semi-discretizations of the wave equation. We have addressed the problem of boundary observability and, more precisely, the problem of whether the observability estimates are uniform when the mesh size tends to zero.

We have proved that the uniform observability property does not hold for any time T. We have also described some possible remedies.

The main consequences concerning controllability have been mentioned. In particular, we have shown that exact controls of numerical approximation schemes may diverge.

By the contrary, we have proved that the problems of approximate and optimal control are well-behaved and that the convergence of the semi-discrete controls holds as the mesh-size h tends to zero.

It would be interesting to see if the moment problems techniques and the sharp estimates in [58] on biorthogonal families allow giving an alternative proof of these positive results with some explicit estimates on the size of the controls.

2. *Discrete unique-continuation.* As we mentioned above, the extension of Theorem 5.1 to the multi-dimensional case is no completely obvious. In fact, the results one gets change significantly.

Let us for instance discuss the simplest case of the constant coefficient wave equation in a square of $\mathbb{R}^2$. In [82] the instability of the controls was proved for finite difference semi-discrete approximations in the context of exact controllability. But, in view of Theorem 5.1, one could expect this not to be the case at the level of approximate controllability. But a new phenomena, producing new instabilities, arises in several space dimensions that we describe now.

In several space dimensions, for the continuous wave equation, approximate controllability holds from any open subset of the boundary if the control time is large enough (twice the diameter of the square domain is enough although a sharper estimate needs to take into account the geometry of the subset where the control is located). This means that the support of the control can be taken to be in any open subset of the domain or its boundary. But this fails to be true for the semi-discrete equation. Indeed, in $2-d$ the unique continuation or uniqueness property that is needed for the controllability of the semi-discrete approximation to hold is not satisfied automatically. In fact it is not even sufficient to assume that $h>0$ is small enough to guarantee that this uniqueness property is satisfied.

The following example due to O. Kavian [43] shows that, at the discrete level, new phenomena arise in what concerns the uniqueness problem. It concerns the eigenvalue problem for the 5-point finite difference scheme for the Laplacian in the square. A grid function taking alternating values ± 1 along a diagonal and vanishing everywhere else is an eigenvector with eigenvalue $\lambda = 4/h^2$. According to this example, even at the level of the elliptic equation, the domain ω where the solution vanishes has to be assumed to be large enough to guarantee the unique continuation property. In [16] it was proved that when ω is a "neighborhood of one side of the boundary", then unique continuation holds for the discrete Dirichlet problem in any discrete domain. Here by a "neighborhood of one side of the boundary" we refer to the nodes of the mesh that are located immediately to one side of the boundary nodal points (left, right, top or bottom). Indeed, if one knows that the solution vanishes at the nodes immediately to one side of the boundary, taking into account that they vanish in the boundary too, the 5-point numerical scheme allows propagating the information and showing that the solution vanishes at all nodal points of the whole domain.

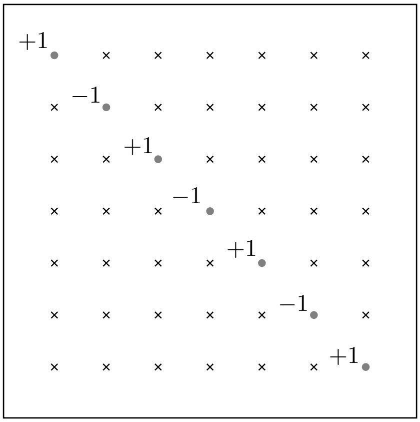

Fig. 1 The eigenvector for the 5-point finite difference scheme for the Laplacian in the square, with eigenvalue $\lambda = 4/h^2$, taking alternating values ± 1 along a diagonal and vanishing everywhere else in the domain.

Getting optimal geometric conditions on the set ω depending on the domain Ω where the equation holds, the discrete equation itself, the boundary conditions and, possibly, the frequency of oscillation of the solution for the unique continuation property to hold at the discrete level is an interesting and widely open subject of research.

One of the main tools for dealing with unique continuation properties of PDE are the so called *Carleman inqualities*. It would be interesting to develop the corresponding discrete theory.

Now, returning to the wave equation in the square domain and its semi-discrete approximations, we see that, in view of the explicit construction of the eigenvector above, one can build solutions of the semi-discrete system in separated variables that vanish everywhere in the domain except on the diagonal for all time. This example shows that the controllability property of the semi-discrete system fails for many open subsets of the boundary. Consequently, the $1D$ result in Theorem 5.1 showing that, whenever the wave equation is approximately controllable, its semi-discrete approximations are controllable as well and the convergence of controls is false in several space dimensions without further geometric restrictions on the support of the controls.

The same pathology is an obstacle for the approximate controllability of the semi-discrete approximations of other models like, for instance, the heat or the Schrödinger equations. It is interesting to note that this obstacle of lack of unique continuation does not arise in the context of the problem of homogenization we mentioned in the introduction. Although, in principle, the later is more difficult to deal with from a technical point of view it turns out that the problem of approximate controllability is well-behaved in that context in several space dimensions for parabolic equations too [84].

It would be interesting to analyze if a filtering mechanism allows reestablishing the uniformity of the approximate controllability property without imposing additional geometric restrictions on the supports of the controls.

Concerning the problem of decay of solutions of wave equations in the presence of damping discussed in the previous section we emphasize that the counterexample above to unique continuation allows showing that, at the semi-discrete level, in contrast with what happens in the continuous case, the decay of solutions may fail without further restrictions on the geometry of the subdomain ω where the damping is effective.

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