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# Analysis of the Laplacian of an incomplete manifold with almost polar boundary

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Abstract: *Motivated by recent interest in global analysis of singular manifolds, we establish the essential self-adjointness of a Laplacian, a Liouville property of sub-harmonic functions, conservativeness and parabolicity of an incomplete manifold. These results are applicable for manifolds with fractal Cauchy boundary.*

Let $M$ be a connected $C^{1,1}$-Riemann manifold without boundary. In our previous paper [13], we had studied that if the Cauchy boundary $\partial M = \overline{M} \setminus M$, where $\overline{M}$ is the completion of $M$, is almost polar (see Definition 3), then a Laplace-Beltrami operator (hereafter, Laplacian, in short) is essentially self-adjoint. We call such a manifold a *manifold with almost polar boundary*. The present paper is a continuation of this previous work. Here we will investigate the spectral theory of an incomplete manifold such as: The essential self-adjointness of the Laplacian, conservativeness and parabolicity of the manifold, a Liouville property of sub-harmonic functions.

The typical example $M = N \setminus \Sigma$ is a complete manifold $N$ deleted a closed manifold $\Sigma$ of co-dimension $\geq 2$. More crucial example is: $M$ itself is a manifold but the completion $\overline{M}$ is no more a manifold. For example, $\overline{M}$ may be an algebraic variety with singular set, a football, an orbifold, a $Met_1$-surface, so called *singular manifolds*. We allow $\partial M$ to be a fractal (see Section 5 for examples).

Key Words and Phrases: *Laplacian – Essential self-adjointness – Lioville property – Conservativeness – Parabolicity – Incomplete manifold – Singular manifold.*
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We consider a Laplacian $\Delta = \operatorname{div} \cdot \nabla$ with the following domain: the domain $D(\nabla)$ of $\nabla$ is the set of $C^{1,1}$-functions $f$ such that both $f$ and $\nabla f$ are square integrable. Similarly, the domain $D(\operatorname{div})$ of div is the set of $C^{1,1}$-vector fields $X$ such that $X$ and $\operatorname{div} X$ are square integrable. Then let the domain $D(\Delta)$ of $\Delta$ be the set of functions $f \in D(\nabla)$ such that $\nabla f \in D(\operatorname{div})$. $M$ is said to have *negligible boundary* if $-\operatorname{div}$ is a formal adjoint of $\nabla$. The following is our main result.

THEOREM 1. *Let $M$ be a connected $C^{1,1}$-Riemann manifold without boundary. If $\partial M$ is almost polar, then $M$ has negligible boundary. Moreover,*

(i) *If $M$ has negligible boundary, then the Laplacian $\Delta$ is essentially self-adjoint.*
(ii) *$\Delta$ is essentially self-adjoint if and only if two Sobolev spaces $W_0$ and $W$ coincide, and moreover, then the $L^2$-closure $\overline{\Delta}$ coincides with both Dirichlet and Neumann Laplacians.*
*In the sequel, assume $\Delta$ to be essentially self-adjoint.*
(iii) *If the volume $v(r)$ of the ball $B(x,r)$ of radius $r$ centered at an arbitrary but fixed point $x \in M$ satisfies*

$$\int^{\infty} \frac{r}{\log v(r)} = \infty, \tag{1}$$

*then $M$ is conservative. If $v(r)$ satisfies*

$$\int^{\infty} \frac{r}{v(r)} = \infty, \tag{2}$$

*then $M$ is parabolic.*
(iv) *Every sub-harmonic function $f$ belonging to $D(\overline{\Delta})$ or to $L^\infty \cap L^p$ for an arbitrary $p > 1$, is a constant.*

It is known that condition (1) (resp. (2)) of Theorem 1 implies the conservativeness (resp. parabolicity) of a complete manifold [8].

An immediate application of Theorem 1 is

COROLLARY 1. *Let us assume $M$ satisfies the condition of Theorem 1 and $\Delta$ is essentially self-adjoint. If $M$ is not bounded and the Ricci curvature is non-negative, then the following holds.*

(i) *Every harmonic 1-form $\alpha$ such that $|\alpha| \in D(\overline{\Delta})$ is 0.*
(ii) *Every harmonic map $f : M \to N$ such that the energy is in $D(\overline{\Delta})$ is constant, where $N$ is a complete smooth manifold whose sectional curvature is non-positive. The condition such that $M$ is not bounded and the Ricci curvature is non-negative may be replaced to that the Ricci curvature is positive at some point.*

We organize the paper in the following manner.

In Section 1, we establish notation.

In Section 2, we discuss Sobolev spaces $W_0^{1,2}$ and $W^{1,2}$. The nature of the research through the paper is the fact two Sobolev spaces $W_0^{1,2}$ and $W^{1,2}$ coincide if $\partial M$ is almost polar (Theorem 2). This result has been well known for an incomplete manifold $M = N \setminus \Sigma$, where $N$ is a complete manifold and $\Sigma$ is a closed almost polar set. Let us explain why our setting covers this case. Since $\Sigma$ is almost polar, it has volume 0, hence any ball centered at arbitrary point $x \in \Sigma$ has intersection with $N$. Therefore, there exists a sequence in $N$ that converges to $x$. This shows $N = \overline{M}$, because $N$ is complete. The contribution of our study is to generalize the previous result to an incomplete manifold such that whose completion is no more a manifold.

In Section 3, we study the essential self-adjointness of the Laplacian. In general, if a symmetric operator on a Hilbert space has a unique self-adjoint extensions, it is called *essentially self-adjoint*. This problem has been introduced to Riemannian geometry by M. P. Gaffney [6]. He established a sufficient condition for a manifold called *$M$ has negligible boundary* so that the Laplacian $\Delta$ on forms is essentially self-adjoint. We present two alternative proofs of Gaffney theorem. Subsequently, together with the main result in [5], Gaffney proved the essential self-adjointness of the Laplacian on forms of complete manifolds. We prove the Laplacian is essentially self-adjoint if and only if $W = W_0$.

In Section 4, we study the conservativeness, parabolicity, and a Liouville property. The idea of our study of conservativeness and parabolicity bases on the following fact. Consider again $M = N \setminus \Sigma$, where $N$ is a complete manifold and $\Sigma$ is a closed almost polar set. Then *the Brownian motion of $N$ does not hit $\Sigma$, accordingly, if $N$ is conservative or parabolic, then so is $M$.* On our setting, we discuss without asking if Brownian motion hits $\partial M$ or not, because we do not know if the Brownian motion could be extended to $\overline{M}$.

In order to establish the conservativeness, we decompose $M$ into $M_1$ and $M_2$, where $\partial M \subset \partial M_1$, $M_1$ has finite volume. We impose Neumann boundary condition to both manifolds. Then both manifolds are conservative (A. Grigor'yan [8] proved that (1) in Theorem 1 is a sufficient condition for the conservativeness of a complete manifold or a manifold with boundary of Neumann condition). As we had seen $M$ has no boundary condition (Theorem 2), we will obtain the conservativeness of $M$. One may prove the parabolicity in the same way, however, we present a different proof.

In Section 5, we provide examples. P. Li and G. Tian [11] proved the essential self-adjointness of the Laplacian, the conservativeness of an incomplete manifold $M \setminus \Sigma$, where $M \subset \mathbb{CP}^n$ is an algebraic variety deleted the singular set $\Sigma$ of co-dimension not less than 3. Since $M$ has finite volume and the singular set is almost polar, $M \setminus \Sigma$ is not only conservative but more strongly, parabolic. See more detailed proof for their result in [24]. The main reason why we study the manifold of class $C^{1,1}$ is because the most simple but non-trivial manifold with fractal boundary is merely $C^{1,1}$ (Example 7). Finally, let us introduce some related topics to our result. The Laplacian $\Delta_C$ on the set of forms with compact support of a complete manifold is essentially self-adjoint [20]. This result contains the Gaffney theorem on a complete manifold because $\Delta_C \subset \Delta$. However, $\Delta_C$ is not necessarily essentially self-adjoint on a manifold with polar boundary, while the corresponding Markov form $(\mathcal{E}, C_0^\infty)$ has unique Dirichlet extension. Indeed, $\Delta_C$ on $\mathbb{R}^3 \setminus \{0\}$, where $\{0\}$ is almost polar, has infinitely many self-adjoint extensions [9], [2]. The Laplacian $\Delta_C$ on $M \setminus N$, where $M$ is a complete and $N$ is a closed sub-manifold, is essentially self-adjoint if and only if the co-dimension of $N$ is greater than 3 [13].

S. Ozawa [16], studied the behavior of the first eigenvalue $\lambda_\epsilon$ of $M \setminus B_\epsilon$ where $M$ is a compact manifold as $\epsilon \to 0$. P. Li and G. Tian established an eigenvalue estimate of an algebraic variety deleted the singular set [11]. The author and W. Rossman proved the Weyl's asymptotic formula for an incomplete manifold [14]. G. C. Papanicolaou and S. R. S. Varadhan [17] analyzed the asymptotic behavior of the solution of the heat equation on a domain of an Euclidean space punched out small balls.

## 1 – Notation

We list up notation for convenience of reading. Most of them will be explained also in the main body of the paper.

$M$ - a connected Riemann manifold of class $C^{1,1}$ without boundary. $M$ admits an atlas such that every coordinate transformation is $C^{1,1}$ and Riemann metric is Lipschitz on every compact set.

- $\mu$ - the Riemann measure.
- $d$ - the intrinsic distance of $M$ - see Section 2.
- $r := d(\cdot, x)$ - the radius function from the point $x \in M$.
- $\overline{M}$ - the completion of $M$ with respect to $d$.
- $\partial M := \overline{M} \setminus M$.
- $B(\Sigma, r) := \{x \in \overline{M} | \, d(\Sigma, x) < r\}$ - the $r$-neighbourhood of the set $\Sigma \subset \overline{M}$.
- $\Delta := \operatorname{div} \nabla$ - the Laplace-Beltrami operator on $M$.
- $\overline{\Delta}$ - the $L^2$-closure of $\Delta$ - see Section 2.
- $\Delta_D$ (resp. $\Delta_N$) - Dirichlet (resp. Neumann) Laplacian - see Section 3.
- $p(t, x, y)$ - the heat kernel associated with $\frac{1}{2}\Delta$ - see Section 4.
- $\Omega$ - a bounded domain of $M$.
- $C^l$ - the set of real-valued functions of class $l$ on $M$.
- $C_0^l(\Omega)$ - the set of functions $f \in C^l$ with compact support in $\Omega$.
- $V^l$ - the set of real-valued vector fields of class $l$ on $M$.

– $V_0^l(\Omega)$ - the set of vector fields in $V^l$ with compact support in $\Omega$.
– $f|_\Omega$ - the function $f$ restricted to $\Omega$.
– $L^p := L^p(M,\mu)$ - the completion of $C_0^{1,1}$ with respect to the norm $\|f\|_p := (\int f^p)^{1/p} := (\int_M f^p(x)\, d\mu(x))^{1/p}$. Especially $\langle f, g\rangle := \int fg$ for $f, g \in L^2$.
– $e^{tT}$ - the semi-group generated by a non-positive self-adjoint operator $T$ on $L^2$.
– $W := W^{1,2}(M,\mu)$, $W_0 := W_0^{1,2}(M,\mu)$, and $H := H_2^1(M,\mu)$ - Sobolev spaces of order $(1,2)$ - see Section 2.
– $\mathcal{E}(f,g) := \langle \nabla f, \nabla g\rangle$ - the Dirichlet integral of $f, g \in W$.
– $\mathrm{Cap}(\Sigma)$ - the capacity of a Borel set $\Sigma \subset \overline{M}$ - see Section 2.

## 2 – Sobolev spaces

The main purpose of this section is to prove; if $\partial M$ is almost polar, then $W = W_0$. On a complete manifold, where the Cauchy boundary is empty, this goes back to Gaffney [5]. He cuts off the function $f \in W$ out side of a ball $B(r)$, and prove that the modified function $f_r$ belongs to $W_0$ and converges to $f$ as $r \to \infty$. If a manifold is incomplete, one should cut off $f$ also near the Cauchy boundary $\partial M$. We will prove that if $\partial M$ is almost polar, then this modified function $f_n$ belongs to $W_0$ and converges to $f$ (Theorem 2).

Definition 1. Denote by $W$ the completion of the set of real-valued $C^{1,1}$-functions $f$ on $M$ such that $\|f\|_{1,2} = \|f\|_2 + \|\nabla f\|_2 < \infty$, where $\|\cdot\|_2$ stands for the $L^2$-norm, with respect to the norm $\|\cdot\|_{1,2}$. The set $W_0$ is the completion of the set of functions in $C^{1,1}$ with compact support $C_0^{1,1}$ in $W$. Another Sobolev type space $H$ consists of measurable functions $f$ such that both $f$ and $\nabla f$ are square integrable.

The Riemann distance does not work on a $C^{1,1}$-manifold, so we work with the intrinsic distance [1].

Definition 2. The *intrinsic distance* $d$ is defined by

$$d(x,y) = \sup\{\psi(x) - \psi(y) |\, \psi \in C^{1,1},\ \|\nabla\psi\|_\infty \le 1\} \text{ for } x, y \in M.$$

We impose

Assumption 1. $d$ is non-degenerate and generates the original topology of $M$.

Remark 1. It is known that $d$ coincides with the Riemann distance, if the manifold is class $C^{2,1}$ [7].

Example 1. K. Th. Sturm [21] developed the conservativeness, parabolicity, and $L^p$-Liouville property of a local Dirichlet space utilizing the canonical intrinsic distance associated to the Dirichlet form. A local Dirichlet space is a generalization of a Riemann manifold, so his result covers complete manifolds, however, his assumption excludes incomplete manifolds.

We define the capacity for $\overline{M}$.

Definition 3. Let $\Sigma \subset \overline{M}$ be a Borel set. Denote by $\mathcal{O}$ the family of open sets $O$ of $\overline{M}$ such that $\Sigma \subset O$. Let $L(O)$ be the set of functions $f \in W_0$ such that

$$0 \leq f \leq 1 \text{ and } f|_O = 1.$$

The capacity $\mathrm{Cap}(\Sigma)$ of $\Sigma$ is

$$\mathrm{Cap}(\Sigma) = \inf_{O \in \mathcal{O}} \mathrm{Cap}(O),$$

where

$$\mathrm{Cap}(O) = \inf_{f \in L(O)} \|f\|_{1,2}.$$

We say $\Sigma$ is *almost polar* if $\mathrm{Cap}(\Sigma) = 0$.

Remark 2. The Brownian motion on $M$ hits $\Sigma \subset M$ if and only if $\mathrm{Cap}(\Sigma) > 0$, so the Brownian motion on $M$ and that of $M \setminus \Sigma$ (in order to make $M \setminus \Sigma$ a manifold, $\Sigma$ should be closed) are the same almost surely, if $\Sigma$ is almost polar. If $\Sigma$ is a manifold and co-dimension is not less than 2, or a fractal with Hausdorff co-dimension greater than 2, then it is almost polar.

Example 2. See Section 5 for examples of manifolds $M$ with almost polar boundary such that $\overline{M}$ is not a manifold.

Recall the definition of the *closure* of an operator.

Definition 4. An operator $S : H_1 \to H_2$, where $H_1$ and $H_2$ are Hilbert spaces, is called *closed* if the graph $G(S)$ is closed in $H_1 \times H_2$. $S$ is called *closable* if it has a closed extension. The operator $T$ whose graph $G(T)$ coincides with the the completion of $G(S)$ in $H_1 \times H_2$ is called the *$L^2$-closure* (hereafter, closure, for short) of $S$ and denoted by $\overline{S}$.

It is well known in functional analysis that

Proposition 1. *Every closable operator $S$ has its closure $\overline{S}$.*

We may state the main result of this section.

Theorem 2. *Let $M$ be a $C^{1,1}$-manifold without boundary. Then the following holds.*

(i) $W = H$.
(ii) *If* $\mathrm{Cap}(\partial M) = 0$, *then* $W_0 = W$.

Proof. We start to prove (i). For $f \in W$, let $f_n \in C^1$ be a sequence such that $f_n \to f$ in $W$ as $n \to \infty$. By Stokes theorem,

$$\langle f, \operatorname{div} X\rangle = \lim_{n\to\infty} \langle f_n, \operatorname{div} X\rangle = \lim_{n\to\infty} \langle \nabla f_n, X\rangle = \langle \overline{\nabla} f, X\rangle$$

for every $X \in V_0^1$. This shows $f \in H$.

Conversely, let $f$ be in $H$. Let $\{U_\alpha, \psi_\alpha\}_{\alpha>0}$ be a local chart, where each $U_\alpha$ is relative compact, and $\{\rho_\alpha\}_\alpha$ be an associated partition of unity such that $\rho_\alpha \in C_0^{1,1}(U_\alpha)$. We claim $f_\alpha := \rho_\alpha f \in H$. Let $X$ be the weak derivative of $f$. Then it holds

$$\langle \rho_\alpha X + f\nabla\rho_\alpha, Y\rangle = \langle f, -\operatorname{div}(\rho_\alpha Y)\rangle + \langle f\nabla\rho_\alpha, Y\rangle = \langle f_\alpha, \operatorname{div} Y\rangle$$

for every $Y \in V_0^1$. Hence $-(\rho_\alpha X + f\nabla\rho_\alpha) \in L^2$ is the weak derivative of $f_\alpha$. So $f_\alpha \in H$. Denote by $J_\epsilon$ the Friedrich mollifier. Define $J_\epsilon f_\alpha \in C^{1,1}$ by

$$J_\epsilon f_\alpha := \int J_\epsilon(\cdot, y) f_\alpha(y)\, \mu(dy).$$

Since $J_\epsilon(x,\cdot)$ has support in $B(x,\epsilon)$, for every $\alpha > 0$, there exists $\epsilon_\alpha > 0$ such that $J_\epsilon f_\alpha \in C_0^{1,1}(U_\alpha)$ for every $0 < \epsilon < \epsilon_\alpha$. Due to compactness argument,

$$\|J_\epsilon f_\alpha - f_\alpha\|_{1,2} \to 0$$

as $\epsilon \to 0$. For $\alpha > 0$ and $\delta > 0$, let $\epsilon_\alpha > 0$ be such that

$$\|J_{\epsilon_\alpha} f_\alpha - f_\alpha\|_{1,2} < 2^{-\alpha}\delta.$$

Then $f_\delta = \sum_\alpha J_{\epsilon_\alpha} f_\alpha \in C^{1,1}$ satisfies

$$\|f_\delta - f\|_{1,2} < \delta.$$

This shows $f \in W$. Now we have completed the proof of (i).

Next, we prove (ii). We would like to show that for every $f \in W$ there exists $f_n \in W_0$ such that $f_n \to f$ in $W$ as $n \to \infty$. First, we claim that we may assume $f$ to be bounded. Define

$$f \vee g := \max\{f, g\},$$
$$f \wedge g := \min\{f, g\}.$$

Then $f_n := (f \vee (-l)) \wedge l \to f$ in $W$ as $l \to \infty$ [11]. Hence, hereafter we assume $f$ is bounded.

Next we claim that we may assume that $f$ is 0 on some neighbourhood of $\partial M$. Since $\partial M$ is almost polar, there exists a sequence $e_n \in W$ such that

(a) $0 \leq e_n \leq 1$,
(b) there exists an open set $\partial M \subset O_n \subset \overline{M}$ such that $e_n|_{O_n}=1$,
(c) $\|e_n\|_{1,2} \to$ as $n \to \infty$.

From condition (c), we may assume that $e_n$ tends to 0 almost everywhere as $n \to \infty$. Set $f_n = (1 - e_n)f$. Because of condition (a), $f_n \in W$. Then

$$\|f - f_n\|_{1,2} \leq \|e_n f\|_2 + \|e_n \nabla f\|_2 + \|f \nabla e_n\|_2. \tag{3}$$

The first and second terms of R.H.S. of (3) tends to 0 as $n \to \infty$ by Lebesgue theorem. The third term of R.H.S. of (3) tends to 0 as $n \to \infty$ because $f$ is bounded. Due to (b), hereafter, we assume $f$ is 0 on some neighbourhood of $\partial M$.

Finally, we are going to cut off $f$ outside of a big ball. Define a function $\eta_n$ by

$$\eta_n(r) = ((2 - n^{-1}r) \vee 0) \wedge 1, \tag{4}$$

where $r$ is the radius function from an arbitrary but fixed point $x \in M$. Put $B(r) := B(x, r)$. We note that $B(r)$ has finite volume for every $r > 0$. Indeed, since $\partial M$ is almost polar, there exists an open set $\partial M \subset O$ with finite volume, and as $\overline{B(r)} \setminus O \subset M$ is compact because of Assumption 1, it has finite volume. Due to the definition of the intrinsic distance, $\|\nabla r\|_\infty \leq 1$, and thus $\eta_n(r) \in W$ for every $n > 0$. As $\eta_n$ is bounded,

$$f_n := f\eta_n \in W \text{ for every } n > 0.$$

Since $\|\nabla \eta_n\|_\infty \leq 1/n$, we have

$$\|f - f_n\|_{1,2} \leq \|(1 - \eta_n)f\|_2 + n^{-1}\|f\|_2 + \|(1 - \eta_n)\nabla f\|_2. \tag{5}$$

By Lebesgue theorem, R.H.S. of (5) tends to 0 as $n \to \infty$. Now we may assume $f$ has compact support. By the mollifier techniques as in the proof of (i) above, we obtain a sequence $f_n \in C_0^{1,1}$ such that $f_n \to f$ in $W$. This shows $W_0 = W$, and we have completed the proof. ∎

## 3 – Essential self-adjointness

A symmetric operator is called essentially self-adjoint if it has a unique self-adjoint extension. In a series of papers, M. P. Gaffney studied this problem for the Laplacian of a manifold. In [6] he established a criterion called *$M$ has negligible boundary* (see Definition 5) so that the Laplacian is essentially self-adjoint. Subsequently, in [5] he showed that a complete manifold has negligible boundary. In our previous paper [13] we had showed that if $W = W_0$, then $M$ has negligible boundary. In this section, we will prove that the converse is also true, namely

THEOREM 3. *The Laplacian is essentially self-adjoint if and only if* $W = W_0$.

First, we will present alternative two different proofs of Gaffney theorem (Theorem 4). Then Theorem 3 will follow immediately.

In order to make $\Delta$ symmetric, we need

DEFINITION 5. We say *$M$ has negligible boundary* if

$$\int \operatorname{div}(fX) = 0$$

for every $f \in D(\nabla)$ and $X \in D(\operatorname{div})$.

The following Gaffney theorem says the assumption such that $M$ has negligible boundary makes $\Delta$ not only symmetric but also essentially self-adjoint.

THEOREM 4. *$\Delta$ is essentially self-adjoint if and only if $M$ has negligible boundary.*

Before starting the proof, let us present a corollary which is the (i) of Theorem 1.

COROLLARY 2. *If $M$ has almost polar boundary, then $\Delta$ is essentially self-adjoint.*

PROOF. Let $f \in D(\nabla)$ and $X \in D(\operatorname{div})$. Due to Theorem 2, there exists $f_n \in C_0^{1,1}$ such that $f_n \to f$ in $W$. Then

$$\langle \nabla f, X \rangle = \lim_{n\to\infty} \langle \nabla f_n, X \rangle = -\langle f, \operatorname{div} X \rangle.$$

Hence $M$ has negligible boundary, and $\Delta$ is essentially self-adjoint by Theorem 4. ∎

The first proof is to combine Lemma 1 and 2. We prove that the closure $\overline{\Delta}$ is self-adjoint, because if $T$ is a self-adjoint extension of $\Delta$ then

$$\Delta \subset \overline{\Delta} \subset T = T^* \subset \overline{\Delta}^* = \Delta^*,$$

which shows $\overline{\Delta} = \overline{\Delta}^*$ implies $\overline{\Delta} = T$, the essential self-adjointness of $\Delta$.

LEMMA 1. *If $M$ has negligible boundary, then $\overline{\mathrm{div}} \cdot \overline{\nabla}$ is self-adjoint.*

PROOF. We would like to show $-\overline{\mathrm{div}} = \nabla^*$, where $\nabla^*$ stands for the adjoint-operator of $\nabla$, because then by Von Neumann theorem, $\overline{\mathrm{div}} \cdot \overline{\nabla} = -\nabla^* \cdot \overline{\nabla}$ is self-adjoint. One direction is obvious, because by the definition of negligible boundary, $-\mathrm{div} \subset \nabla^*$ and since $\nabla^*$ is closed, $-\overline{\mathrm{div}} \subset \nabla^*$.

Conversely, let $X \in D(\nabla^*)$. Because $D(\nabla_C^*) = H = W$ where $\nabla_C$ is the restriction of $\nabla$ to the set of functions with compact support, there exists a sequence $X_n \in D(\mathrm{div})$ such that $X_n \to X$ in $W$ as $n \to \infty$. This means $X \in D(\overline{\mathrm{div}})$. ∎

LEMMA 2. *If $M$ has negligible boundary, then $\overline{\Delta} = \overline{\mathrm{div}} \cdot \overline{\nabla}$.*

PROOF. As Lemma 1 says $\overline{\Delta} \subset \overline{\mathrm{div}} \cdot \overline{\nabla}$, we only prove the converse. Let $\{U_\alpha, \psi_\alpha\}_{\alpha>0}$ be a local chart, where each $U_\alpha$ is relative compact, and $\{\rho_\alpha\}_\alpha$ be an associated partition of unity such that $\rho_\alpha \in C_0^{1,1}(U_\alpha)$. Suppose $f \in D(\overline{\mathrm{div}} \cdot \overline{\nabla})$. Then $f_\alpha := \rho_\alpha f$ belongs to $D(\overline{\mathrm{div}} \cdot \overline{\nabla})$. By the definition of a closed operator, for every $\alpha > 0$ and $\epsilon > 0$, there exists a vector field $X_{\alpha,\epsilon} \in V_0^{1,1}(U_\alpha)$ such that

$$\|\overline{\nabla} f_\alpha - X_{\alpha,\epsilon}\|_{1,2} < \epsilon. \tag{6}$$

Due to Kodaira-Morrey-Eells decomposition, there exist $f_{\alpha,\epsilon} \in D(\nabla)$ and $Y_{\alpha,\epsilon} \in \mathrm{div}^{-1}(0)$ such that

$$X_{\alpha,\epsilon} = \nabla f_{\alpha,\epsilon} + Y_{\alpha,\epsilon}.$$

Since the $\mathrm{div}^{-1}(0)$ component of $\nabla f_{\alpha,\epsilon}$ is 0, we have

$$\|\overline{\nabla} f_\alpha - \nabla f_{\alpha,\epsilon}\|_2 \le \|\overline{\nabla} f_\alpha - X_{\alpha,\epsilon}\|_2 \tag{7}$$

and

$$\|\overline{\mathrm{div}} \cdot \overline{\nabla} f_\alpha - \mathrm{div} \cdot \nabla f_{\alpha,\epsilon}\|_2 = \|\overline{\mathrm{div}} \cdot \overline{\nabla} f_\alpha - \mathrm{div}\, X_{\alpha,\epsilon}\|_2. \tag{8}$$

By (6), (7), (8) and Poincaré inequality, for every $\alpha > 0$ and $\epsilon > 0$, there exists $h_{\alpha,\epsilon} \in D(\Delta)$ such that

$$\|f_\alpha - h_{\alpha,\epsilon}\|_2 + \|\overline{\mathrm{div}} \cdot \overline{\nabla}(f_\alpha - h_{\alpha,\epsilon})\|_2 < 2^{-\alpha}\epsilon.$$

Define a function $f_\epsilon \in D(\Delta)$ by

$$f_\epsilon = \sum_\alpha f_{\alpha,\epsilon}.$$

Then we have

$$\|f - f_\epsilon\|_2 + \|\overline{\mathrm{div}} \cdot \overline{\nabla} f - \Delta f_\epsilon\|_2 < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, now we have completed the proof. ∎

We proceed to the second proof. Let us recall two self-adjoint Laplacians. Both of them are an extension of the Laplacian on $C_0^{1,1}$. The Dirichlet Laplacian $\Delta_D$ is the self-adjoint operator defined on the set of functions $f \in W_0$ such that $\Delta f \in L^2$. The Neumann Laplacian $\Delta_N$ is the self-adjoint operator defined on the set of functions $f \in W$ such that $\langle \Delta f, \psi\rangle = -\langle \nabla f, \nabla\psi\rangle$ for every $\psi \in W$ [18]. Denote by $\Delta_{D,0}$ (resp. $\Delta_{N,0}$) the Laplacian defined on $C^{1,1} \cap D(\Delta_D)$ (resp. $C^{1,1} \cap D(\Delta_N)$).

Lemma 3. *$\Delta_{D,0}$ (resp. $\Delta_{D,0}$) is essentially self-adjoint and its closure is $\Delta_D$ (resp. $\Delta_N$).*

The proof is similar to that of main result of [15]. For the sake of completeness, we present the proof.

Proof. We would like to show $\overline{\Delta_{D,0}} = \Delta_D$. Let $f \in D(\Delta_D)$. Then by hypo-ellipticity of $\Delta$,

$$f_t = e^{t\Delta_D} f \in D(\Delta_{D,0}).$$

By definition of $e^{t\Delta_D}$,

$$f_t \to f \text{ in } L^2 \text{ as } t \to 0,$$

and

$$\Delta f_t = e^{t\Delta_D}\Delta_D f \to \Delta_D f \text{ in } L^2 \text{ as } t \to 0.$$

Hence $f_t \in D(\Delta_{D,0})$ is a Cauchy sequence with respect to the graph norm of $\Delta_{D,0}$, and thus, $f = \lim_{n\to\infty} f_t \in D(\overline{\Delta_{D,0}})$. Obviously, the same proof applies for $\Delta_{N,0}$. ∎

As $\Delta_{D,0} \subset \Delta$, if $\Delta$ is symmetric, then it is essentially self-adjoint. This completes the second proof of Theorem 4.

Now we prove Theorem 3.

Proof. In the proof above, we have observed that if $M$ has negligible boundary, then $\Delta$ is essentially self-adjoint and the closure $\overline{\Delta}$ is the Dirichlet Laplacian. Now we would like to see that the closure coincides also to the Neumann Laplacian, because if it is true, then the quadratic forms $\sqrt{-\Delta_D}$ and $\sqrt{-\Delta_N}$ coincide, where

$$D(\sqrt{-\Delta_D}) = W_0 \text{ and } D(\sqrt{-\Delta_N}) = W,$$

and accordingly, $W = W_0$ [18].

Suppose $f$ is in $D(\Delta_{N,0})$. Then

$$\langle \Delta f, \psi\rangle = -\langle \nabla f, \nabla\psi\rangle \text{ for every } \psi \in W$$

shows $\nabla f \in D(\nabla^*)$. Hence $\nabla f \in D(\overline{\mathrm{div}})$ by Lemma 1, and $f \in D(\overline{\Delta})$ by Lemma 2. Thus, $\Delta_D = \overline{\Delta} = \Delta_N$ and we have the proof. ∎

Remark 3. Consider the Laplacian $\Delta = \partial\delta + \delta\partial$ on forms with following domain [6]. The domain $D(\partial)$ of $\partial$ is the set of $C^1$-forms $\alpha$ such that both $\alpha$ and $\partial\alpha$ are square integrable. Similarly, the domain $D(\delta)$ of $\delta$ is the set of $C^1$-forms $\alpha$ such that both $\alpha$ and $\delta\alpha$ are square integrable. Then let the domain $D(\Delta)$ of the Laplacian $\Delta$ be the set of $C^1$-forms $\alpha \in D(\partial) \cap D(\delta)$ such that $\partial\alpha \in D(\delta)$ and $\delta\alpha \in D(\partial)$. One may prove the essential self-adjointness of $\Delta$ by a similar method of the second proof presented above. In fact, assume $\Delta$ is symmetric. Then, for the Friedrich self-adjoint extension $\Delta_F$ of $\Delta$,

$$\alpha_t = e^{-t\Delta_F}\alpha \in D(\Delta),$$

and both $\alpha_t$ and $\Delta\alpha_t$ converges to $\alpha$ and $\Delta\alpha$, respectively, as $t \to 0$.

## 4 – Conservative, parabolic, and Liouville property

In this section we prove conservativeness, parabolicity and a Liouville type property. Let us start from definitions. The *heat kernel* $p$ associated to $\frac{1}{2}\Delta$ is the smallest positive fundamental solution to the heat equation

$$\frac{1}{2}\Delta u_t = \frac{\partial}{\partial t}u_t.$$

J. Dodziuk [3] showed that every Riemann manifold (whether it is complete or incomplete) admits the heat kernel. Let us recall

Definition 6. A manifold $M$ is called *conservative* if the heat kernel $p$ satisfies

$$\int p(t,x,y)\,d\mu(y) = 1$$

for every $t > 0$ and $x \in M$.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2$ and $T$ be the generator. $(\mathcal{E}, D(\mathcal{E}))$ is called *conservative* if for every $f_n \in L^2$ such that $0 \le f_n \le 1$, $f_n \to 1$, it holds $e^{tT}f_n \to 1$ as $n \to \infty$ for every $t > 0$. For further study of a Dirichlet form, we refer [4]. Denote by $\Delta_D$ the Dirichlet Laplacian. Then, since

$$e^{t\Delta_D}f = \int p(t,\cdot,y)f(y)\,d\mu(y) \text{ for every } f \in L^2,$$

the conservativeness of $M$ is equivalent to that of the Dirichlet form $(\mathcal{E}, W_0)$. As $(\mathcal{E}, W_0)$ generates the Brownian motion, $M$ is conservative if and only if the Brownian motion $X_t$ starting from an arbitrary point of $M$ may be found on $M$ almost surely at every time $t > 0$. The terminology *conservative* originates on this fact.

A manifold with boundary is never conservative in the sense above, because the Brownian motion will be absorbed at the boundary. So instead of the heat kernel, we consider the Neumann heat kernel. In such case, the corresponding Dirichlet form is $(\mathcal{E}, W)$ and the Brownian motion is reflected at the boundary. A manifold with boundary of Neumann condition is conservative if it has volume growth condition (1) of Theorem 1 [8].

We start the proof of conservativeness.

Proof. Our argument bases on the following [4]. □

Theorem 5. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is conservative if and only if there exists a sequence $f_n \in D(\mathcal{E})$ such that*

$$0 \le f_n \le 1, \lim_{n\to\infty} f_n = 1, \text{ and } \lim_{n\to\infty} \mathcal{E}(f_n, \psi) = 0 \text{ for every} \psi \in D(\mathcal{E}) \cap L^1.$$

Let $\partial M \subset O_n$ be a decreasing family of open sets of $\overline{M}$ such that $\partial O_n \cap M$ is $C^{1,1}$ for every $n > 1$, and $\mathrm{Cap}(O_n) \to 0$ as $n \to \infty$. The manifold with boundary $M \setminus O_n$ is conservative with Neumann condition [8] for every $n > 1$. Hence, by Theorem 5, for every $n > 0$ there exists a sequence $f_{n,l} \in W(M \setminus O_n)$ such that $0 \le f_{n,l} \le 1$ on $M$,

$$\lim_{l\to\infty} f_{n,l} = 1, \text{ and } \lim_{l\to\infty} \mathcal{E}(f_{n,l}, \psi) = 0 \text{ for every } \psi \in W(M \setminus O_n) \cap L^1.$$

Set $h_{n,l} = e_n \vee f_{n,l}$, where $e_n$ is the equilibrium potential of $O_n \cap M$, that is the function $e_n \in W_0$ such that $0 \le e_n \le 1$ on $M$,

$$e_n|_{O_n \cap M} = 1, \text{ and } \mathrm{Cap}(O_n \cap M) = \|e_n\|_{1,2}.$$

Then $h_{n,l} \in W$, and

$$|\mathcal{E}(h_{n,l}, \psi)| \le |\langle \nabla e_n, \nabla \psi \rangle| + \Big| \int_{M \setminus O_n} \langle \nabla f_{n,l}, \nabla \psi \rangle \Big|. \tag{9}$$

The R.H.S. of (9) tends to 0 as $n, l \to \infty$. Since $h_{n,l} \in W_0$ for every $n, l > 1$, and $h_{n,l} \to 1$ as $n, l \to \infty$, $M$ is conservative by Theorem 5. □

A manifold is said to be *parabolic* if it does not admit a non-negative Green function $G$. By definition,

$$G(x, y) = \int_0^\infty p(t, x, y)\, dt, \text{ for every } x, y \in M.$$

This shows $M$ is parabolic if and only if the Brownian motion is recurrent. The concept of recurrence may be extended to general Dirichlet form, and it holds [4].

Theorem 6. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is recurrent if and only if there exists a sequence $f_n \in D(\mathcal{E})$ such that*

$$0 \le f_n \le 1, \lim_{n\to\infty} f_n = 1, \text{ and } \lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0.$$

We proceed to prove the parabolicity.

Proof. The volume growth condition (2) of Theorem 1 implies

$$\sum_{n>0} \frac{2^{n+1}}{v(2^{n+1}) - v(2^n)} = \infty,$$

where $v(r)$ stands for the volume of the ball $B(x, r)$ with arbitrary but fixed $x \in M$. Hence,

$$\lim_{n\to\infty} \frac{v(2^{n+1}) - v(2^n)}{2^{2n}} = 0. \tag{10}$$

Let $\eta_n \in W$ be the function defined by (4). Then

$$\mathcal{E}(\eta_n, \eta_n) = \frac{v(2^{n+1}) - v(2^n)}{2^{2n}}.$$

By (10) and Theorem 6, the Brownian motion is recurrent. □

Remark 4. The parabolicity may be proved by the same idea of the proof of conservativeness presented above. More precisely, decompose $M = M_1 \cup M_2$, where $\partial M \subset M_1$ and $M_1$ has finite volume. Imposing Neumann condition to both $M_1$ and $M_2$, by condition (2) of Theorem 1, both manifolds are parabolic. As $M$ does not have boundary condition, it is parabolic.

Finally, we prove a Liouville property.

Proof. Let $\eta_n \in W$ be the function defined by (4). Let $f$ be a non-negative sub-harmonic function. Assume $f \in D(\overline{\Delta})$. Since $\overline{\Delta} = \Delta_D$ ((ii) of Theorem 1),

$$0 \leq \langle \Delta f, \eta_n^2 f\rangle = -2\langle \nabla f, \eta_n f \nabla \eta_n\rangle - \langle \nabla f, \eta_n^2 \nabla f\rangle,$$

and hence

$$\|\eta_n \nabla f\|_2 \leq 2\|f \nabla \eta_n\|_2. \tag{11}$$

The R.H.S. of (11) tents to 0 as $n \to \infty$ by Lebesgue theorem. Therefore $\nabla f = 0$.

If one puts $h = f^{p/2}$, then $h$ is sub-harmonic and $f \in L^2$. Essentially the same proof described above applies to show $\nabla h = 0$, so we omit it. □

Remark 5. It is known that there exists non-constant non-negative sub-harmonic function such that $f \notin L^p$ for any $p > 1$ on a complete manifold. The next two examples show that we may not remove the additional assumption such that $f$ is bounded and $\partial M$ is almost polar.

Consider $M = \mathbb{R}^3 \setminus \{0\}$ and $f = (r^{-2} - 1) \vee 0$, where $r$ is the radius from the origin. Obviously, $M$ has almost polar, $f$ is sub-harmonic, and $f \in L^{5/4}$. Then, a standard smoothing technique of sub-harmonic functions on Euclidean space yields a non-negative, non-bounded, non-constant $C^2$-sub-harmonic function in $L^{5/4}$.

Consider $M = \mathbb{R} \setminus \{0\}$ and $f = ((r+1)^{-1} - 4) \vee 0$. Then $\{0\}$ is not almost polar, $f$ is bounded sub-harmonic and belongs to $L^p$ with every $p > 1$.

Remark 6. A similar estimate in the proof of parabolicity may be found in [10]. The conservativeness and parabolicity can be proved by the main result in [10]. The original proof of Liouville property of complete manifolds can be found in [22].

## 5 – Examples

In this section, we consider some examples. Let us recall a sufficient condition of $\partial M$ to be almost polar [13].

Definition 7. The lower Minkowski co-dimension of a Cauchy boundary $\partial M$ is

$$\liminf_{\epsilon \to 0} \frac{\log \operatorname{vol}(N_\epsilon)}{\log \epsilon}$$

where $N_\epsilon$ is the $\epsilon$-tubular neighbourhood of $\partial M$.

Theorem 7. *If the lower Minkowski co-dimension of $\partial M$ is greater than* 2, *then it is almost polar. In particular, if $\partial M$ is a manifold, and its lower Minkowski co-dimension is not less than* 2, *then the same conclusion holds.*

Example 3. Consider the incomplete manifold $M \setminus \Sigma$, where $M \subset \mathbb{CP}^n$ is an algebraic variety in complex projective space with singular set of co-dimension 2. P. Li and G. Tian [11] showed the essential self-adjointness of the Laplacian, conservativeness, and established an estimate of eigenvalues. As they proved that $\Sigma$ is almost polar (Theorem 4.1), and $M$ has finite volume, their manifold is not only conservative but more strongly, also parabolic. For a detailed proof, we refer [24].

Example 4. Let $M$ be a compact orbifold with singular locus $\Sigma$ of codimension $\geq 2$. In the same way as in [11], one may show that the Riemann manifold $M \setminus \Sigma$ has almost polar Cauchy boundary. The spectrum of the $L^2$-closure of the Laplacian is studied in [19].

Example 5. Consider an incomplete 2-dimensional manifold of finite volume having constant curvature $= 1$ with isolated conical singularities. Such a manifold is called $Met_1$-surface and is important for the study of minimal surfaces [12].

Example 6. Consider the so called "football". Set $M := \mathbb{C} \setminus \{0\}$

$$g = \frac{4(dx^2+dy^2)}{(1+r^2)^2}, \ f = \frac{\mu r^{\mu-1}(1+r^2)}{1+r^{2\mu}}, \ \mu \in \mathbb{R}_+, \ h = f^2 g,$$

where $r$ is the distance from the origin. Then $\{0\}$ is almost polar. In [14], we show that the Laplacian has pure point spectrum and it satisfies Weyl's asymptotic formula.

The next example has fractal Cauchy boundary.

Example 7. Let us recall the Cantor set $\Sigma$ in a real-line. Consider the union of $2^n$-segments

$$\Sigma_n = [0, 3^{-n}] \cup [2 \cdot 3^{-n}, 3^{-n+1}] \cup \ldots \cup [1-3^{-n}, 1] \subset \mathbb{R}.$$

Then $\Sigma$ is defined by $\Sigma = \lim_{n\to\infty} \cap_{1<l<n} \Sigma_l$. Let $(M, g) = (N \setminus \Sigma, f g_o)$ be an incomplete manifold defined as follows; $N$ is a 3-dimensional complete $C^2$-manifold with metric $g_o \in C^2$, $\Sigma$ is a Cantor set, by this we mean there exists a local chart $(U, \psi)$ of $N$ such that $C \subset U$ and $\psi(\Sigma)$ is a Cantor set in $\mathbb{R}$. The metric $g = f g_o$ is defined as follows. Denote by $r$ the distance from $\Sigma$ with respect to $g_o$, and set

$$f(x) = \begin{cases} r^{2\epsilon}, & \text{if } x \in B; \\ 1, & \text{otherwise,} \end{cases}$$

where $\epsilon > \frac{\log 2 - \log 3}{2\log 3 - \log 2}$ and $B = \{x \in N | \, r(x) < 1\}$.

Proposition 2. *$M$ is a $C^{1,1}$-manifold with almost polar boundary.*

Proof. Since $M$ is of class $C^2$ and $g$ is Lipschitz on every compact set, $M$ is of class $C^{1,1}$. Let us assume $\psi(\Sigma)$ lies in $x$-axis. We claim $\partial M = \Sigma$. Indeed, for every $a = (x, 0, 0) \in \Sigma$ (hereafter we identify $\Sigma$ with $\psi(\Sigma)$) and $a_n = (x, 0, 1/n) \in M$, there exists $C > 0$ such that

$$d(a - a_n) < C \int_0^{\frac{1}{n}} z^\epsilon \, dt \to 0 \text{ as } n \to \infty,$$

where $d$ is the Riemann distance with respect to $g$. As $N$ is complete, $\partial M = \Sigma$. Denote by $V_n$ the volume of $B(\Sigma_n, 3^{-1})$ with respect to $g$. By an explicit computation of $V_n$ and letting $n \to \infty$, Minkowski dimension of $\Sigma$ is

$$\dim(\Sigma) = 3 + \frac{\log V_n}{n \log 3} \leq 1 + \frac{\log 2}{\log 3} - \frac{2\epsilon + 1}{\epsilon + 1}.$$

Hence, co-dimension of $\Sigma \subset M$ is greater than 2 if $\epsilon > \frac{\log 2 - \log 3}{2\log 3 - \log 2}$. By Theorem 7, $\partial M$ is almost polar. ∎

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