Rendiconti di Matematica, Serie VII
Volume 25, Roma (2005), 17-32

Laurent type expansions of $\bar{\partial}$-closed (0, n − 1)-forms in $\mathbb{C}^n$

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Abstract: *We characterize the multiple sequences* $\varpi_{k_1,\ldots,k_n}$ *of complex numbers for which there exist* $\bar{\partial}$*-closed* $(0, n-1)$*-forms* $\theta(\zeta)$*, defined for* $\zeta \in \mathbb{C}^n - \{|\zeta| \le R\}$*, so that* $\int_{|\zeta|=\rho} \zeta_1^{k_1} \cdots \zeta_n^{k_n} \theta(\zeta) \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n = \varpi_{k_1,\ldots,k_n}$ $(\rho > R)$*. We also derive Laurent type expansions of such* $\bar{\partial}$*-closed* $(0, n-1)$*-forms in terms of the derivatives of the Bochner-Martinelli kernel and we discuss Mittag-Leffler type constructions in this setting.*

1 – Introduction

Let us recall that given a sequence ϖ_k, $k = 0, 1, 2, \ldots$, of complex numbers, there exists a holomorphic function $g(\zeta)$ defined for $\zeta \in \mathbb{C} - \{|\zeta| \le R\}$ (where $R \ge 0$) so that

$$\int_{|\zeta|=\rho} \zeta^k g(\zeta) d\zeta = \varpi_k, \quad k = 0, 1, 2, \ldots \ (\rho > R),$$

if and only if

$$\limsup_{k\to\infty} \sqrt[k]{|\varpi_k|} \le R,$$

and that, moreover, such a function is of the form

$$g(\zeta) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \varpi_k \frac{1}{\zeta^{k+1}} + \ a\ holomorphic\ function\ in\ \mathbb{C}.$$

Key Words and Phrases: *Laurent type expansions –* $\bar{\partial}$*-closed* $(0, n-1)$*-forms – Fourier-Laplace transform – Derivatives of the Bochner-Martinelli kernel – Mittag-Leffler type constructions.*
A.M.S. Classification: 32A26

In $\mathbb{C}^n$, we may consider systems $(g_1, \dots, g_n)$ of C^∞ functions, which satisfy the differential equation

$$\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial g_j}{\partial \bar{\zeta}_j} = 0$$

(equivalently: the $(0, n-1)$-form

$$\theta = \sum_{j=1}^{n} g_j d\bar{\zeta}_1 \wedge \dots (j) \dots \wedge d\bar{\zeta}_n$$

is $\bar{\partial}$-closed), and pose an analogous question as follows: For which multiple sequences $\varpi_{k_1,\dots,k_n}$ of complex numbers, do there exist $\bar{\partial}$-closed $(0, n-1)$-forms $\theta(\zeta)$, defined for $\zeta \in \mathbb{C}^n - \{|\zeta| \le R\}$, so that

$$\int_{|\zeta|=\rho} \zeta_1^{k_1} \cdots \zeta_n^{k_n} \theta(\zeta) \wedge d\zeta_1 \wedge \dots \wedge d\zeta_n = \varpi_{k_1,\dots,k_n} \quad (\rho > R)?$$

It turns out that we can characterize such sequences (see Theorem 2) and furthermore we can give an analogous expansion for these $\bar{\partial}$-closed $(0, n-1)$-forms θ, in terms of appropriate derivatives of the Bochner-Martinelli kernel (see Theorem 3). For background material, we refer to [2], [3], [4], and [7].

Notation. If D is an open subset of $\mathbb{C}^n$, we will denote by $Z_{\bar{\partial}}^{(0,n-1)}(D)$ the set of $\bar{\partial}$-closed $(0, n-1)$-forms with C^∞ coefficients in D and $H_{\bar{\partial}}^{(0,n-1)}(D)$ will denote the set of the corresponding $\bar{\partial}$-cohomology classes in D:

$$H_{\bar{\partial}}^{(0,n-1)}(D) = \{[\theta] : \theta \in Z_{\bar{\partial}}^{(0,n-1)}(D)\},$$

where $[\theta] = \{\theta + \bar{\partial} - exact\ (0, n-1) - forms\ in\ D\}$.

Also $\mathcal{O}(D)$ will denote the set of holomorphic functions in D.

2 – Fourier-Laplace transforms of $\bar{\partial}$-closed $(0, n-1)$-forms

Let E be a compact convex set in $\mathbb{C}^n$ and let $\xi \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - E)$. The Fourier-Laplace transform of ξ is the entire holomorphic function F_ξ defined by the integral

$$F_\xi(w) = \int_{\zeta \in \partial U} e^{\langle \zeta, w \rangle} \xi(\zeta) \wedge \omega(\zeta), \quad w \in \mathbb{C}^n,$$

where $\langle \zeta, w \rangle = \sum \zeta_j w_j$, $\omega(\zeta) = d\zeta_1 \wedge \dots \wedge d\zeta_n$ and U is an open and bounded convex set with smooth boundary which contains E. Since the differential form

$e^{\langle\zeta,w\rangle}\xi(\zeta)\wedge\omega(\zeta)$ is $d-$closed, it follows from Stokes' theorem that the above integral is independent of the choice of U. Indeed, if V is a sufficiently large ball, then

$$\int\limits_{\zeta\in\partial V} e^{\langle\zeta,w\rangle}\xi(\zeta)\wedge\omega(\zeta)-\int\limits_{\zeta\in\partial U} e^{\langle\zeta,w\rangle}\xi(\zeta)\wedge\omega(\zeta)=\int\limits_{\zeta\in V-U} d[e^{\langle\zeta,w\rangle}\xi(\zeta)\wedge\omega(\zeta)]=0.$$

Notice also that this integral depends only on the cohomology class $[\xi]\in H^{(0,n-1)}_{\bar\partial}(\mathbb{C}^n-E)$. For, if $\xi-\theta=\bar\partial u$ (where $\theta\in Z^{(0,n-1)}_{\bar\partial}(\mathbb{C}^n-E)$ and u is a $(0,n-2)-$form in $\mathbb{C}^n-E$), then

$$\int\limits_{\zeta\in\partial U} e^{\langle\zeta,w\rangle}\xi(\zeta)\wedge\omega(\zeta)-\int\limits_{\zeta\in\partial U} e^{\langle\zeta,w\rangle}\theta(\zeta)\wedge\omega(\zeta)=\int\limits_{\zeta\in\partial U} d[e^{\langle\zeta,w\rangle}u(\zeta)\wedge\omega(\zeta)]=0.$$

Now it is easy to see that the function F_ξ is an entire function of exponential type. In fact,

$$|F_\xi(w)|\le\int\limits_{\zeta\in\partial U} e^{|\langle\zeta,w\rangle|}|\xi(\zeta)\wedge\omega(\zeta)|\le Ae^{R|w|}\quad for\quad w\in\mathbb{C}^n,$$

where A and R are positive constants.

Conversely, using the derivatives of the Bochner-Martinelli kernel, we will show that every entire function of exponential type is the Fourier-Laplace transform of a $\bar\partial$-closed $(0,n-1)$-form.

The derivatives of the Bochner-Martinelli kernel. For $\zeta\ne z$, set

$$M(\zeta,z)=\frac{\beta_n}{|\zeta-z|^{2n}}\sum_{j=1}^{n}(-1)^{j-1}(\bar\zeta_j-\bar z_j)d\bar\zeta_1\wedge\ldots(j)\ldots\wedge d\bar\zeta_n,$$

where $\beta_n=(-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$, and for each $k=(k_1,\ldots,k_n)$, where k_j are non-negative integers, let us define the $(0,n-1)$-forms

$$\eta_k(\zeta)=\left.\frac{\partial^{k_1+\cdots+k_n}M(\zeta,z)}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right|_{z=0}.$$

A simple computation shows that

$$\eta_k(\zeta)=\beta_n n(n+1)\cdots(n+k_1+\cdots+k_n-1)\frac{\bar\zeta_1^{k_1}\cdots\bar\zeta_n^{k_n}}{|\zeta|^{2(n+k_1+\cdots+k_n)}}\times$$
$$\times\sum_{j=1}^{n}(-1)^{j-1}\bar\zeta_j d\bar\zeta_1\wedge\ldots(j)\ldots\wedge d\bar\zeta_n.$$

Since $\bar\partial_\zeta[M(\zeta,z)]=0$, it follows that $\bar\partial\eta_k=0$. Thus $\eta_k\in Z^{(0,n-1)}_{\bar\partial}(\mathbb{C}^n-\{0\})$.

Now recall the Bochner-Martinelli formula: For $f \in \mathcal{O}(\mathbb{C}^n)$,

$$f(z) = \int\limits_{\zeta \in \mathbb{S}_\rho} f(\zeta) M(\zeta, z) \wedge \omega(\zeta), \quad \textit{when } |z| < \rho, \tag{1}$$

where $\mathbb{S}_\rho = \{\zeta \in \mathbb{C}^n : \quad |\zeta| = \rho\}$ and $\rho > 0$.

Applying to both sides of (1) the differentiation

$$\mathfrak{D}^k = \frac{\partial^{k_1 + \cdots + k_n}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_{z=0},$$

we obtain the formula

$$\mathfrak{D}^k f = \frac{\partial^{k_1 + \cdots + k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_{z=0} = \int\limits_{\zeta \in \mathbb{S}_\rho} f(\zeta) \eta_k(\zeta) \wedge \omega(\zeta). \tag{2}$$

Construction of $\bar{\partial}$-closed $(0, n-1)$-forms with prescribed Fourier-Laplace transform. Let F be an entire holomorphic function of the following exponential type:

$$|F(w)| \leq A e^{R|w|}, \text{ for every } w \in \mathbb{C}^n, \tag{$\mathfrak{F}_R$}$$

where A and R are positive constants.

Now we will estimate the derivatives of F at zero, using Cauchy's formula in the polydisk:

$$\mathfrak{D}^k f = \frac{k_1! \ldots k_n!}{(2\pi i)^n} \int\limits_{\zeta \in \mathbb{T}_r} \frac{f(\zeta_1, \ldots, \zeta_n)}{\zeta_1^{k_1+1} \cdots \zeta_n^{k_n+1}} \, d\zeta_1 \wedge \ldots \wedge d\zeta_n, \tag{3}$$

where $\mathbb{T}_r$ is the torus of multi-radius $r = (r_1, \ldots, r_n)$:

$$\mathbb{T}_r = \{\zeta \in \mathbb{C}^n : \ |\zeta_1| = r_1, \ldots, |\zeta_n| = r_n\}.$$

Since for $\zeta \in \mathbb{T}_r$, $|F(\zeta)| \leq A e^{R\sqrt{r_1^2 + \cdots + r_n^2}}$, (3) implies that the coefficient σ_k, in the expansion $F(w) = \sum_k \sigma_k w^k$, satisfies the inequality

$$|\sigma_k| = \frac{1}{k_1! \ldots k_n!} |\mathfrak{D}^k F| \leq A \frac{e^{R\sqrt{r_1^2 + \cdots + r_n^2}}}{r_1^{k_1} \cdots r_n^{k_n}} \quad \textit{for every } r_1, \ldots, r_n > 0.$$

Applying this inequality with

$$r_1 = \sqrt{k_1(k_1 + \cdots + k_n)}/R, \ldots, r_n = \sqrt{k_n(k_1 + \cdots + k_n)}/R,$$

we obtain

$$|\sigma_k| \le A\frac{(eR)^{k_1+\cdots+k_n}}{k_1^{k_1/2}\cdots k_n^{k_n/2}(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)/2}} \quad \textit{for every } k_1,\dots,k_n. \tag{4}$$

(Convention: ${k_j}^{k_j/2}=1$, when $k_j=0$.)

Next let us recall that if $F(w)$ is to be the Fourier-Laplace transform of an analytic functional $\mathcal{T}$, then the action of $\mathcal{T}$ on a function $f\in\mathcal{O}(\mathbb{C}^n)$ will be given by the formula:

$$\mathcal{T}(f)=\sum_k \sigma_k\mathfrak{D}^k f.$$

Now we take an arbitrary $f\in\mathcal{O}(\mathbb{C}^n)$ and substitute the values of its derivatives $\mathfrak{D}^k f$ in the sum $\sum_k \sigma_k\mathfrak{D}^k f$, using formula (2). Interchanging the order of summation and integration, we obtain – at least formally – that

$$\sum_k \sigma_k\mathfrak{D}^k f=\int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\left(\sum_k \sigma_k\eta_k(\zeta)\right)\wedge\omega(\zeta). \tag{5}$$

We will show that the series $\sum_k \sigma_k\eta_k(\zeta)$ converges for $\zeta\in\mathbb{C}^n-\mathbb{B}_R$ (where $\mathbb{B}_R=\{|\zeta|\le R\}$), and defines a $\bar\partial$-closed $(0,n-1)$-form whose Fourier-Laplace transform is the given function F. In fact we will see that the convergence is uniform and absolute on compact subsets of $\mathbb{C}^n-\mathbb{B}_R$, and therefore (5) holds when $\rho>R$. In proving this, (4) will play the important role in conjunction with the following lemma.

Lemma 1. *If $t_1,\dots,t_n>0$ and $t_1^2+\cdots+t_n^2<1$ then*

$$\sum_{k_1,\dots,k_n}\frac{e^{k_1+\cdots+k_n}(k_1+\cdots+k_n)!}{k_1^{k_1/2}\cdots k_n^{k_n/2}(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)/2}}t_1^{k_1}\dots t_n^{k_n}<\infty.$$

Proof. First let us keep in mind that the validity of the assertion is not affected if the general term of the sum is multiplied (or divided) by a quantity of the form $k_1^{s_1}\dots k_n^{s_n}$ (for some nonnegative constants $s_1,\dots,s_n$).

Now to prove the lemma, it suffices to show that the sum of the terms, whose indices $k_1,\dots,k_n$ are all even, is finite, i.e.,

$$\sum_{k_1,\dots,k_n}\frac{e^{2k_1+\cdots+2k_n}(2k_1+\cdots+2k_n)!}{(2k_1)^{k_1}\dots(2k_n)^{k_n}(2k_1+\cdots+2k_n)^{k_1+\cdots+k_n}}t_1^{2k_1}\dots t_n^{2k_n}<\infty \tag{6}$$

$$\text{for } t_1^2+\cdots+t_n^2<1.$$

To justify this reduction we split the sum according to the parity of the $k_1, ..., k_n$. More precisely, if we call $C(k_1, ..., k_n)$ the general term of the sum in the statement of the lemma then on the one hand it is clear that

$$\sum_{k_1,...,k_n} C(k_1, ..., k_n) = \sum_{(u_1,...,u_n)\in\{0,1\}^n} \sum_{k_1,...,k_n} C(2k_1 + u_1, ..., 2k_n + u_n),$$

and on the other hand it is easy to see (using the remark at the beginning of this proof) that (6) implies that each of the 2^n sums

$$\sum_{k_1,...,k_n} C(2k_1 + u_1, ..., 2k_n + u_n) \quad is\ finite$$

(i.e., when $u_1, ..., u_n \in \{0, 1\}$), and the reduction of the proof of the lemma to (6), follows.

Using the notation $|k| = k_1 + \cdots + k_n$, we have

$$\frac{e^{2k_1+\cdots+2k_n}(2k_1 + \cdots + 2k_n)!}{(2k_1)^{k_1}...(2k_n)^{k_n}(2k_1 + \cdots + 2k_n)^{k_1+\cdots+k_n}} =$$

$$= \left(\prod_{j=1}^{n} \frac{e^{k_j} k_j!}{k_j^{k_j}}\right) \frac{e^{|k|}|k|!}{|k|^{|k|}} \frac{(2|k|)!}{4^{|k|}(|k|!)^2} \frac{|k|!}{k_1!...k_n!}.$$

But from Stirling's formula, for $min\{s_j :\ 1 \le j \le n\}$ large enough,

$$\frac{e^{k_j} k_j!}{k_j^{k_j}} \approx \sqrt{2\pi k_j}, \quad \frac{e^{|k|}|k|!}{|k|^{|k|}} \approx \sqrt{2\pi|k|} \quad and \quad \frac{(2|k|)!}{4^{|k|}(|k|!)^2} \approx \frac{1}{2\sqrt{\pi|k|}}.$$

Therefore, using also the expansion

$$\sum_{k_1,...,k_n} \frac{(k_1 + \cdots + k_n)!}{k_1!...k_n!} t_1^{2k_1}...t_n^{2k_n} = \frac{1}{1 - (t_1^2 + \cdots + t_n^2)} \quad \text{valid for}\ t_1^2+\cdots+t_n^2 < 1,$$

we conclude that there is a positive integer N so that

$$\sum_{min\{k_1,...,k_n\}>N} \frac{e^{2k_1+\cdots+2k_n}(2k_1 + \cdots + 2k_n)!}{(2k_1)^{k_1}...(2k_n)^{k_n}(2k_1 + \cdots + 2k_n)^{k_1+\cdots+k_n}} t_1^{2k_1}...t_n^{2k_n} < \infty.$$

Now (6) can be proved by induction on n.

Some computations. For each $\psi = (\psi_1, ..., \psi_n) \in \mathbb{C}^n - \{0\}$, we define the region

$$\mathbb{G}_\psi = \left\{\zeta \in \mathbb{C}^n - \{0\} :\ \frac{|\zeta_j|}{|\zeta|^2} < \frac{|\psi_j|}{|\psi|^2}\ \ for\ j = 1, ..., n\right\}.$$

We will show that for $|\psi| > R$, the series $\sum_k |\sigma_k \mathfrak{C}_k(\zeta)|$ converges uniformly in $\zeta \in \mathbb{G}_\psi$, where $\mathfrak{C}_k(\zeta)$ is the main coefficient of $\eta_k(\zeta)$, i.e., the quantity

$$\mathfrak{C}_k(\zeta) = n(n+1)\cdots(n+k_1+\cdots+k_n-1)\frac{\bar{\zeta}_1^{k_1}\cdots\bar{\zeta}_n^{k_n}}{|\zeta|^{2(n+k_1+\cdots+k_n)}}.$$

First notice that $\mathbb{G}_\psi \subset \{\zeta \in \mathbb{C}^n : |\zeta| > |\psi|\}$, because

$$\frac{|\zeta_j|}{|\zeta|^2} < \frac{|\psi_j|}{|\psi|^2} \quad \Rightarrow \quad \sum_{j=1}^n \frac{|\zeta_j|^2}{|\zeta|^4} < \sum_{j=1}^n \frac{|\psi_j|^2}{|\psi|^4} \quad \Rightarrow \quad \frac{1}{|\zeta|^2} < \frac{1}{|\psi|^2}.$$

Also $u\psi \in \mathbb{G}_\psi$ for every $u > 1$ (as it is easy to check) and for a fixed $\rho > R$,

$$\mathbb{C}^n - \mathbb{B}_\rho = \bigcup_{\psi \in \mathbb{S}_\rho} \mathbb{G}_\psi. \tag{7}$$

Indeed, if $\zeta \in \mathbb{C}^n - \mathbb{B}_\rho$ then it is easy to see that $\zeta \in \mathbb{G}_\psi$, where $\psi = \rho\zeta/|\zeta|$, and of course $\rho\zeta/|\zeta| \in \mathbb{S}_\rho$.

To prove the uniform convergence of the series $\sum_k |\sigma_k \mathfrak{C}_k(\zeta)|$ for $\zeta \in \mathbb{G}_\psi$ (with $|\psi| > R$), it suffices to notice that, since,

$$|\mathfrak{C}_k(\zeta)| = \frac{\prod_{l=1}^{n-1}(k_1+\cdots+k_n+l)}{(n-1)!|\zeta|^{2n}}(k_1+\cdots+k_n)!\left(\frac{|\zeta_1|}{|\zeta|^2}\right)^{k_1}\cdots\left(\frac{|\zeta_n|}{|\zeta|^2}\right)^{k_n},$$

inequality (4) implies that the series $\sum_k \sup\{|\sigma_k \mathfrak{C}_k(\zeta)| : \zeta \in \mathbb{G}_\psi\}$ is dominated by the convergent series

$$\sum_{k_1,\dots,k_n}\left[\frac{\prod_{l=1}^{n-1}(k_1+\cdots+k_n+l)}{(n-1)!|\psi|^{2n}}\frac{e^{|k|}(|k|)!}{k_1^{k_1/2}\cdots k_n^{k_n/2}|k|^{|k|/2}}\left(\frac{R|\psi_1|}{|\psi|^2}\right)^{k_1}\cdots\left(\frac{R|\psi_n|}{|\psi|^2}\right)^{k_n}\right].$$

The convergence of the above series follows from Lemma 1, since

$$\sum_{j=1}^n\left(\frac{R|\psi_j|}{|\psi|^2}\right)^2 = \left(\frac{R}{|\psi|}\right)^2 < 1.$$

Now we can prove the following theorem which is a Paley-Wiener type theorem. As it is well-known such theorems deal with the question of representing entire functions of exponential type as Fourier-Laplace transforms of measures and the related literature is quite extensive. This particular theorem expresses such measures in terms of the Bochner-Martinelli kernel.

THEOREM 1. *If* $F(w) = \sum_k \sigma_k w^k$ *is an entire function, which satisfies* $(\mathfrak{F}_R)$ *for some* $R > 0$*, then the series* $\sum_k \sigma_k \eta_k(\zeta)$ *defines a* $\bar{\partial}$*-closed* $(0, n-1)$*-form* $\eta(\zeta)$*, with* C^∞ *coefficients in* $\zeta \in \mathbb{C}^n - \mathbb{B}_R$*, and*

$$F(w) = \int\limits_{\zeta \in \mathbb{S}_\rho} e^{\langle \zeta, w \rangle} \eta(\zeta) \wedge \omega(\zeta), \ \ for\ w \in \mathbb{C}^n \ and\ \rho > R.$$

Thus an analytic functional $\mathcal{T}$*, which is carried by the ball* $\mathbb{B}_R$*, is represented by the measure*

$$d\lambda(\zeta) = \sum_k \sigma_k \eta_k(\zeta) \wedge \omega(\zeta)\big|_{\zeta \in \mathbb{S}_\rho},$$

supported by the sphere $\mathbb{S}_\rho$ $(\rho > R)$*, where* $\sigma_k = \mathcal{T}(z^k)/k!$.

In particular, any measure $d\mu$ *(in* $\mathbb{C}^n$ *and with compact support) is analytically equivalent to* $d\lambda$ *(given by the above formula), where* $\sigma_k = \int z^k d\mu(z)/k!$ *and* $\rho > \sup\{|z| : z \in supp(\mu)\}$

PROOF. Notice that

$$\eta_k(\zeta) = \sum_{j=1}^n (-1)^{j-1} \bar{\zeta}_j \mathfrak{C}_k(\zeta) d\bar{\zeta}_1 \wedge ...(j)... \wedge d\bar{\zeta}_n.$$

But if $\mathcal{P}$ is any derivative (of any order), with respect to $\zeta_1, ..., \zeta_n, \bar{\zeta}_1, ..., \bar{\zeta}_n$, then

$$\sum_k \sup\left\{|\sigma_k \mathcal{P}[\bar{\zeta}_j \mathfrak{C}_k(\zeta)]| : \ \zeta \in \mathbb{G}_\psi\right\} < \infty, \tag{8}$$

provided that $|\psi| > R$. This follows from Lemma 1, which implies that

$$\sum_{k_1, \dots, k_n} k_1^{s_1} ... k_n^{s_n} \frac{e^{|k|}(|k|)!}{k_1^{k_1/2} \cdots k_n^{k_n/2} |k|^{|k|/2}} t_1^{k_1} ... t_n^{k_n} < \infty \quad (t_1, ..., t_n > 0, \ t_1^2 + \cdots + t_n^2 < 1),$$

for every nonnegative constants $s_1, ..., s_n$. (At this point we use the fact that, since the function F satisfies the condition $(\mathfrak{F}_R)$, the coefficients σ_k satisfy (4), and, therefore, we can carry out computations, similar to the ones that follow the proof of Lemma 1, which lead to (8).)

But (8) implies that $\eta = \sum_k \sigma_k \eta_k$ has C^∞ coefficients $\mathbb{C}^n - \mathbb{B}_R$ and that

$$\bar{\partial}\eta = \sum_k \sigma_k \bar{\partial}\eta_k = 0.$$

Furthermore, for $f \in \mathcal{O}(\mathbb{C}^n)$,

$$\int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\eta(\zeta)\wedge\omega(\zeta) = \int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\left(\sum_k \sigma_k\eta_k(\zeta)\right)\wedge\omega(\zeta) =$$
$$= \sum_k \sigma_k \int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\eta_k(\zeta)\wedge\omega(\zeta) = \sum_k \sigma_k \mathfrak{D}^k f,$$

where we also used (2). Applying the above formula with $f(\zeta) = e^{\langle\zeta,w\rangle}$ (for fixed w), we obtain

$$\int\limits_{\zeta\in\mathbb{S}_\rho} e^{\langle\zeta,w\rangle}\eta(\zeta)\wedge\omega(\zeta) = \sum_k \sigma_k w^k = F(w).$$

This completes the proof. □

3 – Laurent type expansions of $\bar\partial$-closed $(0, n-1)$-forms

The computations of the previous section lead also to the following theorem.

THEOREM 2. *Let $R \geq 0$. Suppose that for each $k = (k_1, ..., k_n)$, where k_j are nonnegative integers, we are given a complex number $\varpi_k = \varpi_{k_1,...,k_n}$. Then a necessary and sufficient condition that there exist $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$ so that*

$$(\mathfrak{P}) \qquad \int\limits_{\zeta\in\mathbb{S}_\rho} \zeta_1^{k_1}\cdots\zeta_n^{k_n}\theta(\zeta)\wedge\omega(\zeta) = \varpi_{k_1,...,k_n}, \quad \text{for every } k \ (\text{where } \rho > R),$$

is that the sequence $\varpi_k = \varpi_{k_1,...,k_n}$ satisfy the condition

$(\mathfrak{G}_R)$ *For every $\epsilon > 0$ there is a positive constant $A(\epsilon)$ so that*

$$|\varpi_k| \leq A(\epsilon)\frac{[e(R+\epsilon)]^{k_1+\cdots+k_n} k_1!...k_n!}{k_1^{k_1/2}\cdots k_n^{k_n/2}(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)/2}} \quad \text{for every } k_1, ..., k_n.$$

Proof. Set

$$c_{k_1,\dots,k_n} = \frac{\varpi_{k_1,\dots,k_n}}{k_1!\dots k_n!}.$$

To prove the one direction, let us assume that $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$ and satisfies $(\mathfrak{P})$. Then

$$c_{k_1,\dots,k_n} w_1^{k_1}\dots w_n^{k_n} = \int\limits_{\zeta\in\mathbb{S}_\rho} \frac{\zeta_1^{k_1}\cdots\zeta_n^{k_n} w_1^{k_1}\dots w_n^{k_n}}{k_1!\dots k_n!}\theta(\zeta)\wedge\omega(\zeta).$$

Since

$$\sum_{k_1,\dots,k_n} \frac{\zeta_1^{k_1}\cdots\zeta_n^{k_n} w_1^{k_1}\dots w_n^{k_n}}{k_1!\dots k_n!} = e^{\langle\zeta,w\rangle},$$

it follows that the series $F(w) = \sum_k c_k w^k$ converges, it defines an entire holomorphic function $F(w)$, and that this function is given by the integral:

$$F(w) = \int\limits_{\zeta\in\mathbb{S}_\rho} e^{\langle\zeta,w\rangle}\theta(\zeta)\wedge\omega(\zeta) \quad for\ \rho > R.$$

Applying this with $\rho = R+\epsilon$ (where $\epsilon > 0$), we see that

$$|F(w)| \le A(\epsilon)e^{(R+\epsilon)|w|},$$

where

$$A(\epsilon) = \int\limits_{|\zeta|=R+\epsilon} |\theta(\zeta)\wedge\omega(\zeta)|.$$

Now we can prove (in the same way we proved that $(\mathfrak{F}_R)$ implies (4)) that

$$|c_k| \le A(\epsilon)\frac{[e(R+\epsilon)]^{k_1+\dots+k_n}}{k_1^{k_1/2}\cdots k_n^{k_n/2}(k_1+\dots+k_n)^{(k_1+\dots+k_n)/2}},$$

and this proves $(\mathfrak{G}_R)$.

To prove the other direction, let us assume that the sequence ϖ_k satisfies $(\mathfrak{G}_R)$. Then, it follows from the proof of Theorem 1, that the series $\theta(\zeta) = \sum_k c_k\eta_k(\zeta)$ defines a $\bar\partial$-closed $(0,n-1)$-form with C^∞ coefficients in $\zeta\in\mathbb{C}^n - \mathbb{B}_{R+\epsilon}$, and this is true for every $\epsilon > 0$. Thus $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$. Moreover

$$\int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\theta(\zeta)\wedge\omega(\zeta) = \sum_k c_k \int\limits_{\zeta\in\mathbb{S}_\rho} f(\zeta)\eta_k(\zeta)\wedge\omega(\zeta) = \sum_k c_k\mathfrak{D}^k f,$$

for $f \in \mathcal{O}(\mathbb{C}^n)$ and $\rho > R$. Applying this formula with $f(\zeta) = \zeta_1^{l_1} \cdots \zeta_n^{l_n}$ (with nonnegative integers $l_1, ..., l_n$), we see that, indeed, θ satisfies the required period condition $(\mathfrak{P})$. This completes the proof of the theorem. □

The following theorem is a variation of Theorem 2. It gives Laurent type expansions for $\bar{\partial}$-closed $(0,n-1)$-forms in $\mathbb{C}^n - \mathbb{B}_R$. (The case $R = 0$ of it, is in [2].)

Theorem 3. *Every $\theta \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$ has an expansion of the form*

$$\theta = \sum_k \frac{\varpi_k}{k!} \eta_k + \bar{\partial} \upsilon,$$

where the numbers ϖ_k are given by $(\mathfrak{P})$ and υ is a $(0,n-2)$-form with C^∞ coefficients in $\mathbb{C}^n - \mathbb{B}_R$.

Proof. Given $\theta \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$, we define ϖ_k by $(\mathfrak{P})$ and we set

$$\eta = \sum_k \frac{\varpi_k}{k!} \eta_k.$$

It follows from the proof of Theorem 2 that $\eta \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}_R)$ and that, for $\rho > R$,

$$\int_{\zeta \in \mathbb{S}_\rho} \zeta_1^{k_1} \cdots \zeta_n^{k_n} \eta(\zeta) \wedge \omega(\zeta) = \varpi_{k_1,...,k_n}.$$

Therefore

$$\int_{\zeta \in \mathbb{S}_\rho} \zeta_1^{k_1} \cdots \zeta_n^{k_n} [\theta(\zeta) - \eta(\zeta)] \wedge \omega(\zeta) = 0, \quad \textit{for every } k_1, ..., k_n.$$

Now [1, Lemma 5] (see also Lemma 2, below) implies that there exists a $(0,n-2)$-form υ, with C^∞ coefficients in $\mathbb{C}^n - \mathbb{B}_R$, so that $\theta - \eta = \bar{\partial}\upsilon$. This gives the required expansion and completes the proof of the theorem. □

Remarks. **1.** Writing the quantity

$$\frac{e^{k_1+\cdots+k_n} k_1! ... k_n!}{k_1^{k_1/2} \cdots k_n^{k_n/2} (k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)/2}}$$

in the form

$$\left(\prod_{j=1}^n \frac{e^{k_j} k_j!}{k_j^{k_j}} \right) \frac{k_1^{k_1/2} \cdots k_n^{k_n/2}}{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)/2}},$$

and using Stirling's formula

$$\frac{e^{k_j} k_j!}{k_j^{k_j}} \approx \sqrt{2\pi k_j},$$

it is easy to see that a sequence ϖ_k satisfies the condition $(\mathfrak{G}_R)$ if and only if for every $\epsilon > 0$ there is a positive constant $\tilde{A}(\epsilon)$ so that

$$|\varpi_k| \le \tilde{A}(\epsilon) \frac{(R+\epsilon)^{k_1+\cdots+k_n} k_1^{k_1/2} \cdots k_n^{k_n/2}}{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)/2}} \quad \textit{for every } k_1, ..., k_n.$$

2. Let $\theta \in Z_{\bar\partial}^{(0,n-1)}(U - \mathbb{B}(a,R))$, where U is an open neighborhood of the closed ball $\mathbb{B}(a,R) = \{\zeta \in \mathbb{C}^n : |\zeta - a| \le R\}$. Taking a $\rho > R$ so that $\mathbb{B}(a,\rho) \subset U$, we define the coefficients c_k by the formula:

$$c_k = \frac{1}{k_1!...k_n!} \int\limits_{\zeta \in \mathbb{S}_\rho} (\zeta_1 - a_1)^{k_1} \cdots (\zeta_n - a_n)^{k_n} \theta(\zeta) \wedge \omega(\zeta).$$

Let us also consider the differential forms $\eta_k(\cdot, a)$ defined by the formula

$$\begin{aligned} \eta_k(\zeta,a) &= \left.\frac{\partial^{k_1+\cdots+k_n} M(\zeta,z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\right|_{z=a} = \\ &= \beta_n n(n+1)\cdots(n+k_1+\cdots+k_n-1) \frac{(\bar\zeta_1 - \bar a_1)^{k_1} \cdots (\bar\zeta_n - \bar a_n)^{k_n}}{|\zeta - a|^{2(n+k_1+\cdots+k_n)}} \times \\ &\quad \times \sum_{j=1}^n (-1)^{j-1} (\bar\zeta_j - \bar a_j) d\bar\zeta_1 \wedge \ldots (j) \ldots \wedge d\bar\zeta_n. \end{aligned}$$

Then $\eta_k(\cdot,a) \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{a\})$ and they have properties analogous to those of η_k. We notice that although the differential form θ is defined only in $U - \mathbb{B}(a,R)$, the series

$$\sum_k c_k \eta_k(\zeta, a)$$

converges for $\zeta \in \mathbb{C}^n - \mathbb{B}(a,R)$ and defines there a $\bar\partial$-closed $(0,n-1)$-form.

Expansions in more general domains. Suppose that D is a pseudoconvex domain in $\mathbb{C}^n$, $a^1, ..., a^N \in D$ and $R_1, ..., R_N \ge 0$ so that

$$\mathbb{B}(a^j, R_j) \subset D \ \ (j = 1, ...N) \quad \textit{and} \quad \mathbb{B}(a^j, R_j) \cap \mathbb{B}(a^m, R_m) = \emptyset \ \ (j \ne m).$$

Let $\theta \in Z_{\bar\partial}^{(0,n-1)}\big(D - [\mathbb{B}(a^1,R_1)\cup\cdots\cup\mathbb{B}(a^N,R_N)]\big)$. Taking $\rho_j > R_j$ so that the balls $\mathbb{B}(a^j,\rho_j)$ are pairwise disjoint, we define

$$c_k^j = \frac{1}{k_1!...k_n!}\int\limits_{\zeta\in\mathbb{S}_{\rho_j}} (\zeta_1 - a_1^j)^{k_1}\cdots(\zeta_n - a_n^j)^{k_n}\theta(\zeta)\wedge\omega(\zeta).$$

Then, in view of the previous remark, $\sum_k c_k^j\eta_k(\zeta,a^j)\in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \mathbb{B}(a^j,R_j))$, and therefore

$$\xi \overset{\text{def}}{=} \theta - \sum_{j=1}^N\sum_k c_k^j\eta_k(\zeta,a^j) \in Z_{\bar\partial}^{(0,n-1)}\left(D - [\mathbb{B}(a^1,R_1)\cup\cdots\cup\mathbb{B}(a^N,R_N)]\right).$$

Moreover

$$\int\limits_{\zeta\in\mathbb{S}_{\rho_j}} (\zeta_1 - a_1^j)^{k_1}\cdots(\zeta_n - a_n^j)^{k_n}\xi(\zeta)\wedge\omega(\zeta) = 0 \quad \textit{for all } k \textit{ and } j.$$

It follows from Lemma 2 below that ξ is $\bar\partial$-exact in $D - [\mathbb{B}(a^1,R_1)\cup\cdots\cup\mathbb{B}(a^N,R_N)]$. The conclusion is that θ has the following expansion

$$\theta = \sum_{j=1}^N\sum_k c_k^j\eta_k(\zeta,a^j) + \bar\partial\upsilon,$$

for some $(0,n-2)$-form υ with C^∞ coefficients in $D - [\mathbb{B}(a^1,R_1)\cup\cdots\cup\mathbb{B}(a^N,R_N)]$.

LEMMA 2. *Let us consider an open set $\Omega\subset\mathbb{C}^n$ of the form $\Omega = D - (G_1\cup\ldots\cup G_N)$ where D is a pseudoconvex set and $G_1,\ldots,G_N$ are compact convex sets in $\mathbb{C}^n$ so that $G_j\subset D$ and $G_j\cap G_m = \emptyset$ for $j\neq m$. Let us also consider simple closed surfaces S_j, each one around the set G_j and close to it.*

Then a differential form $\chi\in Z_{\bar\partial}^{(0,n-1)}(\Omega)$ is $\bar\partial$-exact (in Ω) if and only if

$$\text{(9)}\qquad \int\limits_{\zeta\in S_j} e^{\langle w,\zeta\rangle}\chi(\zeta)\wedge\omega(\zeta) = 0, \quad \textit{for every } j = 1,\ldots,N \textit{ and } w\in\mathbb{C}^n.$$

Notice that (9) is equivalent to

$$\int_{\zeta \in S_j} f(\zeta)\chi(\zeta) \wedge \omega(\zeta) = 0, \quad for\ f \in \mathcal{O}(\mathbb{C}^n)\ and\ j = 1, ..., N,$$

because the set of the functions $e^{\langle w, \zeta \rangle}$, $w \in \mathbb{C}^n$, is dense in the space of entire functions (with the topology of uniform convergence on compact sets. Also this is equivalent to

$$\int_{\zeta \in S_j} (\zeta_1 - a_1^j)^{k_1} \cdots (\zeta_n - a_n^j)^{k_n} \chi(\zeta) \wedge \omega(\zeta) = 0 \quad for\ all\ k\ and\ j,$$

where a_j are any preassigned points.

Proof of Lemma 2. The one direction follows from Stokes's formula. The other direction is a generalization of [1, Lemma 5] and its proof is similar in this case too, so we will outline it.

First we exhaust the set Ω with a sequence of compact sets of the form

$$K = \{\lambda \le 0\} - (\{\rho_1 < 0\} \cup \ldots \cup \{\rho_N < 0\}),$$

so that the set $\{\lambda < 0\}$ is a bounded strictly pseudoconvex set with smooth boundary and the sets $\{\rho_1 < 0\}, \ldots, \{\rho_N < 0\}$ are strictly convex neighborhoods of the convex sets $G_1, \ldots, G_N$. In other words, the sets $\{\lambda < 0\}$ should exhaust the pseudoconvex set D, while the set $\{\rho_j < 0\}$ should shrink down to the set G_j, for $j = 1, \ldots, N$.

Fixing such a set K, we consider the map $\gamma : (\partial K) \times \mathrm{int}(K) \to \mathbb{C}^n$ as follows: For $(\zeta, z) \in (\partial K) \times \mathrm{int}(K)$, $\{\gamma_l(\zeta, z)\}_{l=1}^n$ is defined to be a Henkin-Ramirez map of the strictly pseudoconvex set $\{\lambda < 0\}$, if $\zeta \in \{\lambda = 0\}$, and

$$\gamma_l(\zeta, z) = \frac{\partial \rho_j}{\partial \zeta_l}(z) \quad if\ \zeta \in \{\rho_j = 0\}.$$

(For exhaustions of pseudoconvex sets by strictly pseudoconvex domains and constructions of Henkin-Ramirez maps, see [5] and [6]).

Then

$$\sum_{l=1}^{n} (\zeta_l - z_l)\gamma_l(\zeta, z) \ne 0, \quad for\ \ (\zeta, z) \in (\partial K) \times \mathrm{int}(K),$$

and therefore we may write down the Cauchy-Leray formula:

(10) $u = \bar\partial_z(T_{q-1}u) + T_q(\bar\partial u) + L_q^\gamma(u)$, *for* $(0, q)$*-forms* u *in a neighborhood of* K

(notation is as in [1, p. 912]).

Now if $\chi \in Z^{(0,n-1)}_{\bar\partial}(\Omega)$ satisfies (9), it follows, as in the proof of [1, Lemma 5], that $L^{\gamma}_{n-1}(\chi) = 0$, and therefore (10) gives

$$\chi = \bar\partial_z(T_{n-2}\chi), \quad in \ \operatorname{int}(K).$$

Now the conclusion that χ is $\bar\partial$-exact in Ω, follows from [1, Lemma 4], and this completes the proof of the lemma.

4 – Mittag-Leffler type constructions of $\bar\partial$-closed $(0, n-1)$-forms

In Theorem 2, we saw when and how we can construct a $\bar\partial$-closed $(0,n-1)$-form, in the complement of a closed ball, with prescribed certain weighted periods. The following theorem deals with the analogous question, when the closed ball is replaced by the union of an infinite sequence of pair-wise disjoint closed balls. Given the previous constructions, its proof is similar to the proof of [3,Theorem 2].

Theorem 4. *Let D be an open subset of $\mathbb{C}^n$ and $\mathbb{B}(a^j, R_j)$, $j = 1, 2, 3, ...$, a sequence of pair-wise disjoint closed balls, contained in D, with $R_j \geq 0$. Let us also assume that the set $\{a^1, a^2, a^3, ...\}$ of the centers of these balls is discrete in D and set $\mathbb{M} = \bigcup_{j=1}^{\infty} \mathbb{B}(a^j, R_j)$. Suppose that for each j we are given a sequence $\varpi^j_k = \varpi^j_{k_1,..,k_n}$ of complex numbers which satisfies the condition $(\mathfrak{G}_{R_j})$. Then there exists $\theta \in Z^{(0,n-1)}_{\bar\partial}(\mathbb{C}^n - \mathbb{M})$ so that*

$$(\mathfrak{M}) \quad \int\limits_{\zeta\in\mathbb{S}_{\rho_j}} (\zeta_1 - a^j_1)^{k_1} \cdots (\zeta_n - a^j_n)^{k_n} \theta(\zeta) \wedge \omega(\zeta) = \varpi^j_{k_1,..,k_n}, \quad \textit{for all } k \textit{ and } j,$$

where $\rho_j > R_j$, with the balls $\mathbb{B}(a^j, \rho_j)$ being pair-wise disjoint.

If we assume, in addition, that the open set D and the balls $\mathbb{B}(a^j, R_j)$ satisfy the condition

$(*)$ *D can be exshausted by a sequence of pseudoconvex sets G_ν $(\nu = 1, 2, 3, \dots)$*

$$\textit{so that } (\partial G_\nu) \cap \mathbb{M} = \emptyset \ (\forall \nu),$$

then the differential form θ, which satisfies $(\mathfrak{M})$, is unique up to a $\bar\partial$-exact $(0,n-1)$-form in $\mathbb{C}^n - \mathbb{M}$.

Corollary. *With the notation and under the assumptions of the above theorem (including condition $(*)$), we have an isomorphism:*

$$H^{(0,n-1)}_{\bar\partial}(\mathbb{C}^n - \mathbb{M}) \cong \prod_{j=1}^{\infty} \mathcal{O}(B_j),$$

where $B_j = \{\zeta \in \mathbb{C}^n : |\zeta| < 1/R_j\}$.

Proof. To define this isomorphism, let us associate, to each cohomology class $[\theta] \in H_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \mathbb{M})$, a sequence of holomorphic functions $(h_j)_{j=1}^{\infty}$ defined by the power series:

$$h_j(\tau) = \sum_k c_k^j \tau^k, \quad for\ \tau \in B_j,$$

where

$$c_k^j = \frac{1}{k_1!...k_n!} \int\limits_{\zeta \in \mathbb{S}_{\rho_j}} (\zeta_1 - a_1^j)^{k_1} \cdots (\zeta_n - a_n^j)^{k_n} \theta(\zeta) \wedge \omega(\zeta),$$

with the $\rho_j > R_j$ chosen so that the balls $\mathbb{B}(a^j, \rho_j)$ are pairwise disjoint.

Then it is easy to check (in view of the previous computations) that $h_j \in \mathcal{O}(B_j)$ and that the map

$$[\theta] \to (h_j)_{j=1}^{\infty},$$

gives the required isomorphism.

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Lavoro pervenuto alla redazione il 22 dicembre 2003
ed accettato per la pubblicazione il 5 aprile 2004.
Bozze licenziate il 14 febbraio 2005

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