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Representation formulas and Fatou-Kato theorems for heat operators on stratified groups

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Abstract: *In this note, we provide a characterization of non-negative $\mathcal{L}$-caloric functions on strips, where $\mathcal{L}$ is a sub-Laplacian on a stratified group. We prove representation results, Fatou-type and uniqueness theorems analogous to the classical Poisson-Stieltjes formula and to Kato's theorem concerning with positive solutions to the heat equation.*

1 – Introduction and main results

A stratified group is a connected and simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e., a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_r$ with $[\mathfrak{G}_1, \mathfrak{G}_i] = \mathfrak{G}_{i+1}$, $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$. Stratified groups (also known as Carnot groups) have been introduced by Folland [7] and afterwards deeply studied by various authors, see e.g., Rothschild and Stein [15], Folland and Stein [8], Varopoulos, Saloff-Coste and Coulhon [18]. In particular, Rothschild and Stein pointed out that any Hörmander operator, sum of squares of vector fields, can be locally approximated by a sub-Laplacian on a stratified group. Recently, analysis on such groups has received new and significant impulses in many directions. Indeed, Carnot groups appear as tangent groups of subriemannian manifolds (see e.g., [12]) and they find many applications in mechanics and in control theory. We also refer to the rich bibliography in the recent monographs [1], [9].

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In this paper, we give a contribution in the study of heat operators $\mathcal{H} = \mathcal{L} - \partial_t$, where $\mathcal{L}$ is a sub-Laplacian on $\mathbb{G}$. The study of parabolic-type operators on $\mathbb{G}$ has experienced an increasing interest, also in relation to some problems from image processing (see e.g., [5], [6], [13]) and from the geometric theory of several complex variables. In particular, the operator $\mathcal{H}$ (on a Carnot group $\mathbb{G}$) intervenes in the study of the linearizations of fully non-linear equations such as the Levi curvature equation [4], [11].

In this note, which is a natural sequel of a study started in [2] (and related to the above mentioned linearizations), we deal with a question left unanswered in [2], giving a characterization of non-negative $\mathcal{L}$-caloric functions and proving some representation formulas. We also prove some results analogous to the classical Fatou-type and uniqueness theorems of Kato [10] concerning with positive solutions to the heat equation. We point out that similar topics have been studied in [3], [14]: in [3], Fatou theory is generalized to the non-negative solutions of some sub-elliptic equations on non-tangentially accessible domain; in [14], Fatou-Kato results are obtained for a class of ultraparabolic Hörmander operators on different homogeneous Lie groups, making use of some Gaussian estimates of the fundamental solution analogous to the ones used here. Our main results are contained in Theorems 1.1, 1.3 and 1.4 below.

We point out that many results presented in this paper are valid in more general contexts. Indeed, the needed tools are mainly a local parabolic Harnack inequality and techniques related to Gaussian bounds (we refer to [18] for such results and related topics on general groups). However, our aim is only to answer to some questions arisen in the study of the above mentioned linearizations rather than to establish an axiomatic theory on the subject. Hence, we shall restrict to the setting of Carnot groups.

Theorem 1.1. *Let u be a real valued function defined on a strip. The following statements are equivalent*:

(i) *u is a non-negative $\mathcal{L}$-caloric function in some strip $\mathbb{R}^N \times (0, \delta_1)$.*
(ii) *For some Radon measure σ on $\mathbb{R}^N$, u has the representation*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\mathrm{d}\sigma(\xi), \tag{1}$$

for every (x,t) in some strip $\mathbb{R}^N \times (0, \delta_2)$.

Moreover, if (i)-(ii) *hold, then*

$$u(\cdot,t) \longrightarrow \sigma, \qquad \text{as} \quad t \to 0^+, \quad \text{in the weak sense of measures,} \tag{2}$$

$$u(x,t) \longrightarrow \varphi(x), \qquad \text{as} \quad t \to 0^+, \quad \text{for almost every } x \in \mathbb{R}^N, \tag{3}$$

where $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ is the density of the absolutely continuous part of σ w.r.t. the Lebesgue measure.

Throughout the paper, we call $\mathcal{L}$-*caloric* any solution u of the equation $\mathcal{H}u = 0$, where $\mathcal{H}$ is the heat operator defined in (7) below. Moreover, a Radon measure is understood to be a positive (regular) Borel measure on $\mathbb{R}^N$, finite on compact sets. Finally, by (2) we mean $\int \psi(x)\, u(x,t)\,\mathrm{d}x \longrightarrow \int \psi(x)\,\mathrm{d}\sigma(x)$, as $t \to 0^+$, for every continuous function ψ with compact support. The other notations are explained below. The following remark shows how the strips in (i) and (ii) of the above theorem are related.

Remark 1.2. If (i) holds then we have the representation (1) in the whole strip $\mathbb{R}^N \times (0, \delta_1)$ for a Radon measure σ on $\mathbb{R}^N$ satisfying the growth condition

$$\int_{\mathbb{R}^N} \exp(-\nu\, d^2(\xi))\,\mathrm{d}\sigma(\xi) < \infty, \tag{4}$$

where $\nu = \mathbf{c}/\delta_1$; vice-versa, if (ii) holds, then the measure σ satisfies (4) with $\nu = \mathbf{c}/\delta_2$ and (i) follows with $\delta_1 = \delta_2/\mathbf{c}^2$. Here $\mathbf{c} > 0$ is a structural constant only depending on $\mathcal{L}$.

The following result is a step in the proof of Theorem 1.1, beside being of its own interest.

Theorem 1.3. *Let u be a non-negative $\mathcal{L}$-caloric function in $\mathbb{R}^N \times (0,T)$. Then, for every $\varepsilon > 0$, we have the following Poisson-Stieltjes type representation formula*

$$u(x, t+\varepsilon) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\, u(\xi,\varepsilon)\,\mathrm{d}\xi, \qquad (x,t) \in \mathbb{R}^N \times (0, T-\varepsilon). \tag{5}$$

From Theorem 1.1 and using as a main step Lemma 2.7 in the next section, we can also derive the following Kato-type uniqueness result.

Theorem 1.4. *Let u be a non-negative $\mathcal{L}$-caloric function in $\mathbb{R}^N \times (0,T)$. If*

$$\begin{aligned} &\lim_{t\to 0^+} u(x,t) = 0 && \textit{for almost every } x \in \mathbb{R}^N, \\ &\limsup_{t\to 0^+} u(x,t) < \infty && \textit{for every } x \in \mathbb{R}^N, \end{aligned}$$

then u vanishes identically.

We explicitly remark that the lim sup-condition in Theorem 1.4 cannot be weakened, as one can easily realize taking $u = \Gamma$.

We now explain all the notation. First of all, we give an operative definition of Carnot group. Our definition is equivalent to the one of Folland, up to isomorphism. Let $\circ$ be an assigned Lie group law on $\mathbb{R}^N$. Suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by a given family of Lie group automorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called dilations) of the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}).$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \dots, r$ and $N_1 + \dots + N_r = N$. We denote by $\mathfrak{g}$ the Lie algebra of $(\mathbb{R}^N, \circ)$. For $i = 1, \dots, N_1$, let X_i be the vector field in $\mathfrak{g}$ that agrees at the origin with $\partial/\partial x_i^{(1)}$. We make the following assumption: the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$. With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a Carnot group. If $Y_1, \dots, Y_{N_1}$ is any basis for $\operatorname{span}\{X_1, \dots, X_{N_1}\}$, the second order differential operator

$$\mathcal{L} = \sum_{i=1}^{N_1} Y_i^2$$

is called a sub-Laplacian on $\mathbb{G}$. Since $X_1, \dots, X_{N_1}$ generate the whole $\mathfrak{g}$, which has rank N at every point, any sub-Laplacian $\mathcal{L}$ satisfies Hörmander's hypoellipticity condition. We denote by $Q = \sum_{j=1}^r jN_j$ the homogeneous dimension of $\mathbb{G}$. Then $|\delta_\lambda(E)| = \lambda^Q\,|E|$ for any measurable set E. Here and in the sequel, we denote by $|\cdot|$ the Lebesgue measure on $\mathbb{R}^N$. This measure is invariant w.r.t. the left and right translations on $\mathbb{G}$.

The simplest example of Carnot group is the additive Euclidean group $(\mathbb{R}^Q, +)$; in this case, the sub-Laplacians are exactly the constant coefficient elliptic operators. The most significant (and simple) non-abelian example of Carnot group is the Heisenberg group; in this case, a remarkable sub-Laplacian is the real part of the Kohn-Spencer Laplacian.

Throughout the paper, d will denote a fixed homogeneous norm on $\mathbb{G}$. For instance, we choose $d = \gamma^{1/(2-Q)}$, where γ denotes the fundamental solution of the sub-Laplacian $\sum_{i=1}^{N_1} X_i^2$. We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $d : \mathbb{R}^N \to [0, \infty)$, smooth away from the origin, such that $d(\delta_\lambda(x)) = \lambda\, d(x)$, $d(x^{-1}) = d(x)$, and $d(x) = 0$ iff $x = 0$. Hereafter, we also denote $d(y^{-1} \circ x)$ by $d(x, y)$ and use the notation $B_d(x, r)$ for the d-ball of center x and radius r. The following quasi-triangle inequality holds

$$d(x, y) \le \beta\,(d(x, z) + d(z, y)), \qquad x,\ y,\ z \in \mathbb{G}, \tag{6}$$

for a suitable constant β. Throughout the sequel, $\mathcal{L}$ will always denote a fixed sub-Laplacian on $\mathbb{G}$ and

$$\mathcal{H} = \mathcal{L} - \partial_t \tag{7}$$

the related heat operator on $\mathbb{G} \times \mathbb{R} \equiv \mathbb{R}^{N+1}$. Here $z = (x, t)$ is the point of $\mathbb{R}^{N+1}$ ($x \in \mathbb{G}$, $t \in \mathbb{R}$). The operator $\mathcal{H}$ is hypoelliptic by Hörmander theorem.

It is known that $\mathcal{H}$ possesses a fundamental solution with the properties recalled below (see [8], [18]; see also [2]). There exists a smooth function Γ on $\mathbb{R}^{N+1} \setminus \{0\}$ such that the fundamental solution for $\mathcal{H}$ is given by

$$\Gamma(x, t; \xi, \tau) = \Gamma(\xi^{-1} \circ x, t - \tau).$$

We have $\Gamma(x, t) \geq 0$ and $\Gamma(x, t) = 0$ iff $t \leq 0$; moreover

$$\Gamma(x, t) = \Gamma(x^{-1}, t), \qquad \Gamma(\delta_\lambda(x), \lambda^2 t) = \lambda^{-Q} \Gamma(x, t). \tag{8}$$

For every $\zeta \in \mathbb{R}^{N+1}$, $\Gamma(\cdot; \zeta)$ is locally integrable and $\mathcal{H}\Gamma(\cdot, \zeta) = -\delta_\zeta$ (the Dirac measure supported at $\{\zeta\}$). For every $x \in \mathbb{R}^N$, t, $\tau > 0$, we have

$$\int_{\mathbb{R}^N} \Gamma(\xi, t)\, \mathrm{d}\xi = 1, \qquad \Gamma(x, t + \tau) = \int_{\mathbb{R}^N} \Gamma(\xi^{-1} \circ x, t)\, \Gamma(\xi, \tau)\, \mathrm{d}\xi. \tag{9}$$

The main tool we shall employ in the proofs of our results is the following Gaussian estimate of Γ: there exists a positive constant $\mathbf{c}_0$ such that

$$\mathbf{c}_0^{-1}\, t^{-Q/2} \exp\Big(- \frac{\mathbf{c}_0\, d^2(x)}{t} \Big) \leq \Gamma(x, t) \leq \mathbf{c}_0\, t^{-Q/2} \exp\Big(- \frac{d^2(x)}{\mathbf{c}_0\, t} \Big), \tag{10}$$

for every $x \in \mathbb{R}^N$, $t > 0$. We finally recall the following result, related to the Cauchy problem for $\mathcal{H}$ (for the proof we refer to the results in [18] and to the classical method of Aronson; see also [2]).

Theorem 1.5. (i) *Let f be a continuous function on $\mathbb{R}^N$ satisfying the growth condition $|f(x)| \leq c \exp(\nu\, d^2(x))$, for some constants c, $\nu \geq 0$. Then the function*

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; \xi, 0)\, f(\xi)\, \mathrm{d}\xi, \qquad x \in \mathbb{R}^N,\ t \in (0, (\mathbf{c}\nu)^{-1})$$

is well posed and is a classical solution to the Cauchy problem

$$\mathcal{H}u = 0 \ \text{ in } \mathbb{R}^N \times (0, (\mathbf{c}\nu)^{-1}), \qquad u(\cdot, 0) = f.$$

Here $\mathbf{c}$ is a positive constant only depending on $\mathcal{L}$ and the structure of $\mathbb{G}$.

(ii) *Let u be a classical solution to the Cauchy problem*

$$\mathcal{H}u = 0 \ \text{ in } \mathbb{R}^N \times (0, r), \quad u(\cdot, 0) = 0.$$

Suppose that one of the following conditions holds: either u is non-negative or there exists $\nu > 0$ such that

$$\int_0^r \int_{\mathbb{R}^N} \exp\big(- \nu\, d^2(x) \big)\, |u(x, t)|\, \mathrm{d}x\, \mathrm{d}t < \infty.$$

Then u vanishes identically.

2 – Fatou-Kato theorems

For the reader convenience, we first recall the following weak maximum principle on strips, whose proof is standard and will be omitted.

PROPOSITION 2.1. *Let* $u \in C^2(\mathbb{R}^N \times (0,T))$. *If* $\mathcal{H}u \geq 0$, $\limsup u \leq 0$ *both in* $\mathbb{R}^N \times \{0\}$ *and at infinity, then* $u \leq 0$ *in the whole strip.*

In the sequel, we shall need the following Harnack theorem for $\mathcal{H}$, whose proof easily follows from the Harnack inequality in [18].

THEOREM 2.2. *Let us fix* $T > 0$ *and set* $S_T = \mathbb{R}^N \times (0,T)$.

(i) *For every* $z_0 = (x_0, t_0) \in S_T$ *and for every compact set* $K \subset S_{t_0}$, *there exists a positive constant* $\mathbf{c}$ *such that*

$$\sup_K u \leq \mathbf{c}\, u(z_0),$$

for every non-negative function u, $\mathcal{L}$*-caloric in* S_T.

(ii) *Let* $u_n \leq u_{n+1}$ *be a monotone sequence of* $\mathcal{L}$*-caloric functions in* S_T. *If there exists* $z_0 = (x_0,t_0) \in S_T$ *such that* $u_n(z_0)$ *is bounded, then* u_n *converges uniformly on the compact subsets of* S_{t_0} *to a function* u, $\mathcal{L}$*-caloric in* S_{t_0}.

We are now able to prove the Poisson-Stieltjes type representation formula.

PROOF OF THEOREM 1.3. For every $n \in \mathbb{N}$, we set

$$v_n(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\, \psi\Big(\frac{d(\xi)}{n}\Big)\, u(\xi,\varepsilon)\, \mathrm{d}\xi,$$

where $\psi \in C^\infty(\mathbb{R})$ is a fixed non-increasing cut-off function such that $\psi(r) = 1$ if $r \leq 1$, $\psi(r) = 0$ if $r \geq 2$. By Theorem 1.5, we know that v_n is a solution to the Cauchy problem

$$\begin{cases} \mathcal{H}v_n = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ v_n(\cdot,0) = \psi\Big(\dfrac{d(\cdot)}{n}\Big)\, u(\cdot,\varepsilon). \end{cases}$$

Moreover, for every $t \in (0,T)$, we have (by the estimates in (10))

$$\begin{aligned} 0 &\leq v_n(x,t) \leq \mathbf{c}_0\, t^{-Q/2} \int_{\mathbb{R}^N} \exp\Big(-\frac{d^2(x,\xi)}{\mathbf{c}_0\, t}\Big)\, \psi\Big(\frac{d(\xi)}{n}\Big)\, u(\xi,\varepsilon)\, \mathrm{d}\xi \leq \\ &\leq \mathbf{c}_0 \int_{d(x\circ\delta_{\sqrt{t}}\eta)\leq 2n} \exp\Big(-\frac{d^2(\eta)}{\mathbf{c}_0}\Big)\, u(x\circ\delta_{\sqrt{t}}\eta,\varepsilon)\, \mathrm{d}\eta \leq \\ &\leq \mathbf{c}_0 \max_{B_d(0,2n)} u(\cdot,\varepsilon) \int_{d(\eta)\geq(\beta^{-1}d(x)-2n)/\sqrt{T}} \exp(-d(\eta)^2/\mathbf{c}_0)\mathrm{d}\eta \to 0, \text{ as } d(x)\to\infty. \end{aligned}$$

We now apply the weak maximum principle for $\mathcal{H}$ to the $\mathcal{L}$-caloric function $w_n(x,t) = u(x,t+\varepsilon) - v_n(x,t)$ in the strip $\mathbb{R}^N \times (0, T-\varepsilon)$. Since $\psi \le 1$, we have $w_n(\cdot, 0) \ge 0$. Moreover, we have proved that v_n vanishes at infinity in the strip. Hence, recalling that u is non-negative, we get $\liminf w_n \ge 0$ at infinity in the strip. The maximum principle of Proposition 2.1 then yields $w_n \ge 0$ in $\mathbb{R}^N \times (0, T-\varepsilon)$. Recalling the definition of w_n and letting n go to infinity, from the above inequality, we finally obtain

$$u(x,t+\varepsilon) \ge \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\, u(\xi,\varepsilon)\, \mathrm{d}\xi =: v_\infty(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,T-\varepsilon),$$

since $v_n \nearrow v_\infty$ by monotone convergence. This proves in particular that v_∞ is finite in $\mathbb{R}^N \times (0,T-\varepsilon)$. Now, from the Harnack Theorem 2.2-(ii), it follows that v_∞ is $\mathcal{L}$-caloric in $\mathbb{R}^N \times (0,T-\varepsilon)$. Moreover, from the inequalities

$$v_n(x,t) \le v_\infty(x,t) \le u(x,t+\varepsilon), \quad (x,t) \in \mathbb{R}^N \times (0,T-\varepsilon),$$

and recalling that $v_n(x,0) = u(x,\varepsilon)$ if $d(x) \le n$, it follows that v_∞ is continuous in $\mathbb{R}^N \times [0, T-\varepsilon)$ and $v_\infty(\cdot,0) = u(\cdot,\varepsilon)$. As a consequence, setting $w_\infty(x,t) = u(x,t+\varepsilon) - v_\infty(x,t)$, w_∞ is a classical solution to $\mathcal{H}w_\infty = 0$ in $\mathbb{R}^N \times (0,T-\varepsilon)$, $w_\infty(\cdot,0) = 0$. Since moreover w_∞ is non-negative, it must vanish identically, by the uniqueness result in Theorem 1.5. This proves (5). ∎

We now turn to the proof of Theorem 1.1 which is split in various steps, starting with Lemma 2.3 below.

Lemma 2.3. *Let u be a non-negative $\mathcal{L}$-caloric function in the strip $\mathbb{R}^N \times (0,T)$. Then, there exists a Radon measure σ on $\mathbb{R}^N$ such that*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\, \mathrm{d}\sigma(\xi), \qquad (x,t) \in \mathbb{R}^N \times (0,T), \tag{11}$$

$$\int_{\mathbb{R}^N} \exp\Big(-\frac{2\mathbf{c}_0}{T}\, d^2(\xi)\Big)\, \mathrm{d}\sigma(\xi) < \infty, \tag{12}$$

where $\mathbf{c}_0 > 0$ is the constant in (10).

Proof. Let us fix $t_0 \in (0,T)$ and choose $j_0 \in \mathbb{N}$ such that $t_0 < T - 1/j_0$. From Theorem 1.3, it follows that

$$u\Big(x, t+\frac{1}{j}\Big) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\, u\Big(\xi,\frac{1}{j}\Big)\, \mathrm{d}\xi, \qquad (x,t) \in \mathbb{R}^N \times \Big(0, T-\frac{1}{j_0}\Big), \quad j > j_0.$$

Since $u(0, t_0 + \frac{1}{j}) \longrightarrow u(0,t_0)$ as $j \to \infty$, the sequence of Radon measures on $\mathbb{R}^N$

$$\mathrm{d}\mu_j(\xi) = \Gamma(0,t_0;\xi,0)\, u\Big(\xi,\frac{1}{j}\Big)\, \mathrm{d}\xi, \qquad j > j_0,$$

is bounded and hence weakly converges (up to a subsequence) to a certain Radon measure μ (with $\mu(\mathbb{R}^N) < \infty$) in the sense that

$$\int_{\mathbb{R}^N} f(\xi)\,\mathrm{d}\mu_j(\xi) \longrightarrow \int_{\mathbb{R}^N} f(\xi)\,\mathrm{d}\mu(\xi), \quad \text{for every } f \in C_{(0)}(\mathbb{R}^N) \tag{13}$$

(we have denoted by $C_{(0)}(\mathbb{R}^N)$ the space of continuous functions in $\mathbb{R}^N$, vanishing at infinity). We now set $M = 2\,\beta^2\,\mathbf{c}_0^2$, where β is defined by (6) and $\mathbf{c}_0$ is the constant in (10). For every $(x,t) \in \mathbb{R}^N \times (0, t_0/M)$, we have

$$u(x,t) = \lim_{j\to\infty} u\Big(x, t+\frac{1}{j}\Big) = \lim_{j\to\infty} \int_{\mathbb{R}^N} \frac{\Gamma(x,t;\xi,0)}{\Gamma(0,t_0;\xi,0)}\,\mathrm{d}\mu_j(\xi) = \int_{\mathbb{R}^N} \frac{\Gamma(x,t;\xi,0)}{\Gamma(0,t_0;\xi,0)}\,\mathrm{d}\mu(\xi),$$

by (13), observing that $\Gamma(x,t;\cdot,0)/\Gamma(0,t_0;\cdot,0) \in C_{(0)}(\mathbb{R}^N)$, since the estimates in (10) give, for $d(\xi) \ge 4\beta\, d(x)$,

$$\begin{aligned}
0 < \frac{\Gamma(x,t;\xi,0)}{\Gamma(0,t_0;\xi,0)} &\le \mathbf{c}(t,t_0)\,\exp\Big(\frac{\mathbf{c}_0\, d^2(\xi)}{t_0} - \frac{d^2(x,\xi)}{\mathbf{c}_0\, t}\Big) \le \\
&\le \mathbf{c}(t,t_0)\,\exp\Big(\frac{\mathbf{c}_0\, d^2(\xi)}{t_0} - \frac{1}{\mathbf{c}_0\, t}\Big(\frac{d^2(\xi)}{\beta^2} + d^2(x) - \frac{2\,d(x)\,d(\xi)}{\beta}\Big)\Big) \le \\
&\le \mathbf{c}(x,t,t_0)\,\exp\Big(-d^2(\xi)\Big(\frac{1}{2\,\beta^2\,\mathbf{c}_0\, t} - \frac{\mathbf{c}_0}{t_0}\Big)\Big) \longrightarrow 0,
\end{aligned}$$

as $d(\xi) \to \infty$, if $t < t_0/M$. Choosing

$$\mathrm{d}\sigma(\xi) = \frac{\mathrm{d}\mu(\xi)}{\Gamma(0,t_0;\xi,0)}, \tag{14}$$

we get (11) in the strip $\mathbb{R}^N \times (0, t_0/M)$. In order to extend the representation formula to the whole strip $\mathbb{R}^N \times (0,T)$, we shall exploit (9). For fixed $T > t \ge t_0/M > \varepsilon > 0$, by Theorem 1.3 we have

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^N} \Gamma(x,t-\varepsilon;\xi,0)\,u(\xi,\varepsilon)\,\mathrm{d}\xi = \\
&= \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \Gamma(x,t-\varepsilon;\xi,0)\,\Gamma(\xi,\varepsilon;y,0)\,\mathrm{d}\xi\Big)\mathrm{d}\sigma(y) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)\,\mathrm{d}\sigma(y).
\end{aligned}$$

We explicitly remark that σ is finite on the compact sets by the estimates in (10) and recalling that $\mu(\mathbb{R}^N) < \infty$. Moreover, again using (10) and from (11), it follows that

$$u\Big(0, \frac{T}{2}\Big) \ge \mathbf{c}_0^{-1}\Big(\frac{T}{2}\Big)^{-\frac{Q}{2}} \int_{\mathbb{R}^N} \exp\Big(-\frac{2\,\mathbf{c}_0\, d^2(\xi)}{T}\Big)\,\mathrm{d}\sigma(\xi),$$

which gives (12). $\square$

Theorem 2.4. *Let u be a non-negative $\mathcal{L}$-caloric function in the strip $\mathbb{R}^N \times (0,T)$. Then, there exists a non-negative function $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that*

$$u(x,t) \longrightarrow \varphi(x), \quad \text{as } t \to 0^+, \quad \text{for almost every } x \in \mathbb{R}^N.$$

Proof. Let σ be the Radon measure found in Lemma 2.3. By the Lebesgue decomposition theorem, there exists a non-negative function $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a singular Radon measure s on $\mathbb{R}^N$ such that

$$\mathrm{d}\sigma(\xi) = \varphi(\xi)\,\mathrm{d}\xi + \mathrm{d}s(\xi). \tag{15}$$

Moreover, for a.e. $x \in \mathbb{R}^N$ (w.r.t. the Lebesgue measure), we have

$$\frac{1}{|B_d(x,\rho)|}\int_{B_d(x,\rho)} |\varphi(\xi) - \varphi(x)|\,\mathrm{d}\xi \longrightarrow 0, \qquad \frac{s(B_d(x,\rho))}{|B_d(x,\rho)|} \longrightarrow 0, \text{ as } \rho \to 0^+. \tag{16}$$

The proof of (16) will be omitted. It follows e.g. adapting the arguments in [16, Chapter 8], replacing the Euclidean metric by the quasi-distance d. The doubling property of the d-balls ensures, for instance, a suitable d-version of the Vitali covering lemma (see e.g. [17]).

Let us now fix an $x \in \mathbb{R}^N$ where (16) holds and set, for brevity, $\mathrm{d}\alpha(\xi) = |\varphi(\xi) - \varphi(x)|\,\mathrm{d}\xi + \mathrm{d}s(\xi)$. Also fix $\varepsilon > 0$. Then there exists $\rho_0 \in (0, \sqrt{T})$ such that

$$\frac{1}{|B_d(x,\rho)|}\int_{B_d(x,\rho)} \mathrm{d}\alpha(\xi) < \varepsilon, \qquad \text{for every } \rho \in (0, 2\,\rho_0]. \tag{17}$$

Let now $t \in (0, \rho_0^2)$ and let $N(t) \in \mathbb{N}$ be such that $2^{N(t)-1} \le \rho_o/\sqrt{t} < 2^{N(t)}$. From Lemma 2.3, (9) and (15), we obtain

$$\begin{aligned}
|u(x,t) - \varphi(x)| &\le \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\mathrm{d}\alpha(\xi) \le \\
&\le \Big(\int_{B_d(x,\sqrt{t})} + \sum_{j=1}^{N(t)} \int_{2^{j-1}\sqrt{t} \le d(x,\xi) < 2^j\sqrt{t}} + \int_{d(x,\xi) > \rho_0}\Big)\Gamma(x,t;\xi,0)\,\mathrm{d}\alpha(\xi) = \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Using (17) and the estimates in (10), we get

$$\mathrm{I}_1 \le \frac{\mathbf{c}}{|B_d(x,\sqrt{t})|}\int_{B_d(x,\sqrt{t})} \exp\Big(-\frac{d^2(x,\xi)}{\mathbf{c}_0\, t}\Big)\,\mathrm{d}\alpha(\xi) \le \mathbf{c}\,\varepsilon,$$

recalling that $\sqrt{t} < \rho_0$. In the same way, we can prove the estimate

$$\begin{aligned}\mathrm{I}_2 &\le \mathbf{c} \sum_{j=1}^{N(t)} \frac{\exp(-4^{j-1}\mathbf{c}_0^{-1})\, 2^{j\,Q}}{|B_d(x, 2^j\sqrt{t})|} \int_{B_d(x,2^j\sqrt{t})} \mathrm{d}\alpha(\xi) \le \\ &\le \mathbf{c}\,\varepsilon \sum_{j=1}^{\infty} \exp(-4^{j-1}\mathbf{c}_0^{-1})\, 2^{j\,Q} = \mathbf{c}'\,\varepsilon,\end{aligned}$$

recalling that $2^j\sqrt{t} \le 2\,\rho_0$ for every $j \le N(t)$. Finally, using again (10) and recalling the definition (14) of σ, we have

$$\begin{aligned}\mathrm{I}_3 &\le \int_{d(x,\xi)>\rho_0} \Gamma(x,t;\xi,0)\,\mathrm{d}\sigma(\xi) + \varphi(x) \int_{d(x,\xi)>\rho_0} \Gamma(x,t;\xi,0)\,\mathrm{d}\xi \le \\ &\le \mathbf{c}(t_0) \int_{d(x,\xi)>\rho_0} t^{-\frac{Q}{2}} \exp\Big(- \frac{d^2(x,\xi)}{\mathbf{c}_0\, t} + \frac{\mathbf{c}_0\, d^2(\xi)}{t_0}\Big)\,\mathrm{d}\mu(\xi) + \\ &\quad + \mathbf{c}\,\varphi(x) \int_{d(\eta)>\frac{\rho_0}{\sqrt{t}}} \exp\big(- \mathbf{c}_0^{-1}\, d^2(\eta)\big)\,\mathrm{d}\eta,\end{aligned}$$

and then it is easy to see that I_3 vanishes as $t \to 0^+$. This concludes the proof. □

In order to complete the proof of Theorem 1.1, we are only left to prove Lemma 2.5 and Lemma 2.6 below.

LEMMA 2.5. *Let σ be a Radon measure on $\mathbb{R}^N$ satisfying the growth condition*

$$\int_{\mathbb{R}^N} e^{-\nu\, d^2(\xi)}\,\mathrm{d}\sigma(\xi) < \infty, \tag{18}$$

for some constant $\nu > 0$. Then the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\mathrm{d}\sigma(\xi) \tag{19}$$

is $\mathcal{L}$-caloric in the strip $\mathbb{R}^N \times (0, (\mathbf{c}^*\nu)^{-1})$, where $\mathbf{c}^*$ is a positive constant only depending on $\mathcal{L}$ and the structure of $\mathbb{G}$.

Proof. From the estimate (10) and recalling that σ is finite on compact sets, it follows

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\mathrm{d}\sigma(\xi) \le \mathbf{c}_0 t^{-Q/2} \int_{\mathbb{R}^N} \exp\Big(-\frac{d^2(x,\xi)}{\mathbf{c}_0 t}\Big)\mathrm{d}\sigma(\xi) \le$$
$$\le \mathbf{c}(x,t) + \mathbf{c}'(t) \int_{d(\xi)>4\beta d(x)} \exp\Big(-\frac{d^2(\xi)}{4\mathbf{c}_0\beta^2 t}\Big)\,\mathrm{d}\sigma(\xi) < \infty,$$

if $0 < t < (4\mathbf{c}_0\beta^2\nu)^{-1} =: (\mathbf{c}^*\nu)^{-1}$. Moreover, by dominated convergence, it is easy to see that u is continuous on the strip $\mathbb{R}^N \times (0, (\mathbf{c}^*\nu)^{-1})$. In order to prove that u is $\mathcal{L}$-caloric, one can differentiate under the integral sign, making use of the estimates of the derivatives of Γ along the vector fields $X_1, \dots, X_{N_1}$ (see e.g., [18]; see also [2]). Alternatively, one can use the Harnack Theorem 2.2, following the lines of the proof of Theorem 1.3: the function

$$v_{n,\varepsilon}(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t-\varepsilon;\xi,0)\,\psi\Big(\frac{d(\xi)}{n}\Big)\,u(\xi,\varepsilon)\,\mathrm{d}\xi,$$

is a solution to $\mathcal{H}v_{n,\varepsilon} = 0$ in $\mathbb{R}^N \times (\varepsilon,\infty)$, $v_{n,\varepsilon}(x,\varepsilon) = \psi(d(x)/n)\,u(x,\varepsilon)$; moreover, recalling that $0 \le \psi \le 1$, (9) and the definition (19) of u, we have

$$v_{n,\varepsilon}(x,t) \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Gamma(x,t-\varepsilon;\xi,0)\,\Gamma(\xi,\varepsilon;y,0)\,\mathrm{d}\xi\mathrm{d}\sigma(y) =$$
$$= \int_{\mathbb{R}^N} \Gamma(x,t;y,0)\,\mathrm{d}\sigma(y) = u(x,t) < \infty, \qquad \text{if } t < 1/(\mathbf{c}^*\nu);$$

hence, by Theorem 2.2, $v_{\infty,\varepsilon} = \lim_{n\to\infty} v_{n,\varepsilon}$ is $\mathcal{L}$-caloric in $\mathbb{R}^N \times (\varepsilon, 1/(\mathbf{c}^*\nu))$; finally, using again (9), we see that $v_{\infty,\varepsilon}(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t-\varepsilon;\xi,0)\,u(\xi,\varepsilon)\,\mathrm{d}\xi = u(x,t)$ in $\mathbb{R}^N \times (\varepsilon, 1/(\mathbf{c}^*\nu))$; since ε is arbitrary, this ends the proof. $\square$

Lemma 2.6. *Under the hypotheses of Lemma* 2.5 *above, we have*

$$u(\cdot,t) \longrightarrow \sigma, \quad \text{as } t \to 0^+, \text{ in the weak sense of measures.}$$

Proof. Let $f \in C_0(\mathbb{R}^N)$. We have to prove that

$$\int_{\mathbb{R}^N} f(x)\,u(x,t)\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} f(x)\,\mathrm{d}\sigma(x), \qquad \text{as } t \to 0^+.$$

For small $t > 0$, we have (see (8))

$$\int_{\mathbb{R}^N} f(x)\,u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,f(x)\,\mathrm{d}x\,\mathrm{d}\sigma(\xi) =$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Gamma(\xi,t;x,0)\,f(x)\,\mathrm{d}x\,\mathrm{d}\sigma(\xi).$$

Moreover, $\int_{\mathbb{R}^N} \Gamma(\xi, t; x, 0)\, f(x)\, \mathrm{d}x \longrightarrow f(\xi)$, as $t \to 0^+$, by Theorem 1.5. Hence it is sufficient to prove that

$$\Big| \int_{\mathbb{R}^N} \Gamma(\xi, t; x, 0)\, f(x)\, \mathrm{d}x \Big| \le \mathbf{c}(f)\, e^{-\nu\, d^2(\xi)} \tag{20}$$

holds for every $\xi \in \mathbb{R}^N$ and for every small $t > 0$, and then to use the dominated convergence (we recall that (18) holds). Let us set $k_0 = 4\beta \max_{\mathrm{supp}(f)} d$. Since (9) holds, the integral in the left-hand side of (20) is clearly uniformly bounded for $d(\xi) \le k_0$. On the other hand, if $d(\xi) > k_0$, the estimate (10) gives

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} \Gamma(\xi, t; x, 0)\, f(x)\, \mathrm{d}x \Big| \le \\
&\le \mathbf{c}(f) \exp\Big(- \frac{d^2(\xi)}{\mathbf{c}_0 \beta^2 t} + \frac{k_0\, d(\xi)}{2\mathbf{c}_0 \beta^2 t} \Big) \int_{\mathrm{supp}(f)} t^{-Q/2} \exp\Big(- \frac{d^2(x)}{\mathbf{c}_0 t} \Big)\, \mathrm{d}x \le \\
&\le \mathbf{c}(f)\, \exp\Big(- \frac{d^2(\xi)}{2\mathbf{c}_0 \beta^2 t} \Big) \int_{\mathbb{R}^N} e^{-d^2(\eta)/\mathbf{c}_0}\, \mathrm{d}\eta = \mathbf{c}'(f) \exp\Big(- \frac{d^2(\xi)}{2\mathbf{c}_0 \beta^2 t} \Big),
\end{aligned}$$

which finally yields (20) for sufficiently small t. □

Proof of Theorem 1.1. It directly follows collecting Lemma 2.3, Theorem 2.4, Lemma 2.5 and Lemma 2.6. □

Finally, we have to prove Theorem 1.4; our main tool will be Lemma 2.7 below. First, we fix a notation. Given a Radon measure σ on $\mathbb{R}^N$, we define the *upper d-symmetric derivative* of σ at $x \in \mathbb{R}^N$,

$$d\text{-}\overline{D}_{\mathrm{sym}}\sigma(x) = \limsup_{\rho \to 0^+} \frac{\sigma(B_d(x, \rho))}{|B_d(x, \rho)|}. \tag{21}$$

The following result generalizes [10, Lemma 1].

Lemma 2.7. *Let σ be a Radon measure on $\mathbb{R}^N$ such that $d\text{-}\overline{D}_{\mathrm{sym}}\sigma(x) < \infty$ for every $x \in \mathbb{R}^N$. Then σ is absolutely continuous w.r.t. the Lebesgue measure.*

Proof. We assume by contradiction that there exists a Borel set $E \subseteq \mathbb{R}^N$ such that $|E| = 0$ and $\sigma(E) > 0$. From the hypotheses, we infer that $E = \cup_{n \in \mathbb{N}} E_n$, where $E_n = \{x \in E \mid d\text{-}\overline{D}_{\mathrm{sym}}\sigma(x) < n\}$. Hence there exists $n_0 \in \mathbb{N}$ such that $\sigma(E_{n_o}) > 0$. Moreover, by the definition of $d\text{-}\overline{D}_{\mathrm{sym}}$, we have $E_{n_0} = \cup_{j \in \mathbb{N}} A_j$, where we have set

$$A_j = \{x \in E_{n_0} \mid \sup_{0 < \rho < 1/j} \sigma(B_d(x, \rho))/|B(x, \rho)| < n_0\}.$$

Thus, there exists $j_0 \in \mathbb{N}$ such that $\sigma(A_{j_o}) > 0$. From the regularity of σ, it follows that there exists a compact set K such that $K \subseteq A_{j_0}$ $(\subseteq E_{n_0} \subseteq E)$, $\sigma(K) > 0$. Clearly we have

$$\sigma(B_d(x,\rho)) < n_0\,|B_d(x,\rho)|, \qquad \text{for every } x \in K,\ 0 < \rho < 1/j_0. \tag{22}$$

We now fix $\varepsilon > 0$. Since $|E| = 0$ gives $|K| = 0$, there exists an open set V such that $K \subset V$, $|V| < \varepsilon$. We claim that there exists a disjoint family of d-balls $\{B_b(x_i,\delta)\}_{i=1}^p$ with the following properties: $x_i \in K$, $0 < \delta < (4\beta^2 j_0)^{-1}$, $K \subseteq \bigcup_{i=1}^p B_d(x_i, 4\beta^2\delta) \subseteq V$. As a consequence, by (22), we obtain

$$\begin{aligned} 0 < \sigma(K) &\le \sum_{i=1}^p \sigma(B_d(x_i,4\beta^2\delta)) \le n_0 \sum_{i=1}^p |B_d(x_i,4\beta^2\delta)| = \\ &= n_0\,(4\beta^2)^Q \sum_{i=1}^p |B_d(x_i,\delta)| = \\ &= n_0\,(4\beta^2)^Q \Big|\bigcup_{i=1}^p B_d(x_i,\delta)\Big| \le n_0\,(4\beta^2)^Q\,|V| < n_0\,(4\beta^2)^Q\,\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives a contradiction. Thus, in order to complete the proof, we only have to prove the claim. Let $\{\xi_n\}_n$ be a countable dense subset of K and let us choose a positive δ not exceeding $(4\beta^2 j_0)^{-1}$, such that $4\beta^2\delta < \min\{d(x,y) \mid x \in K,\ y \in \mathbb{R}^N \setminus V\}$ so that $\overline{B_d(x,4\beta^2\delta)} \subseteq V$ for every $x \in K$. We set $x_1 = \xi_1$. If $\{\xi_n\}_n \subset \overline{B_d(x_1,2\beta\delta)}$, then $K = \overline{\{\xi_n\}_n} \subseteq \overline{B_d(x_1,2\beta\delta)} \subset B_d(x_1,4\beta^2\delta)$. Otherwise, let $n_2 \in \mathbb{N}$ be such that $\xi_1,\dots,\xi_{n_2-1} \in \overline{B_d(x_1,2\beta\delta)}$, $\xi_{n_2} \notin \overline{B_d(x_1,2\beta\delta)}$. Setting $x_2 = \xi_{n_2}$, we clearly have $B_d(x_1,\delta) \cap B_d(x_2,\delta) = \emptyset$ (we recall that β is defined by (6)). Iterating this procedure, we obtain a (possibly finite) subsequence $\{x_i = \xi_{n_i}\}_i$ of $\{\xi_n\}_n$ and a sequence of disjoint d-balls $\{B_d(x_i,\delta)\}_i$ such that $\{\xi_n\}_n \subset \cup_i \overline{B_d(x_i,2\beta\delta)}$. This gives

$$K = \overline{\{\xi_n\}_n} \subseteq \overline{\bigcup_i B_d(x_i,2\beta\delta)} \subset \bigcup_i B_d(x_i,4\beta^2\delta)$$

(the radius of $B_d(x_i,2\beta\delta)$ has been chosen not depending on i, in order to allow this last inclusion). The claim is proved by taking a finite sub-covering. ∎

With Lemma 2.7 at hands, we are able to prove our uniqueness result.

Proof of Theorem 1.4. By Theorem 1.5, it is sufficient to prove that $u = 0$ in $\mathbb{R}^N \times (0,\delta)$ for some small $\delta > 0$. Let σ be the Radon measure introduced in Lemma 2.3. Let us prove that $d\text{-}\overline{D}_{\mathrm{sym}}\sigma(x) < \infty$ for every $x \in \mathbb{R}^N$. Assuming by contradiction that for some $x \in \mathbb{R}^N$ one has $d\text{-}\overline{D}_{\mathrm{sym}}\sigma(x) = \infty$, there exists a sequence of radii $\rho_j \to 0^+$ such that $\sigma(B_d(x,\rho_j))/|B_d(x,\rho_j)| \longrightarrow \infty$, as $j \to \infty$.

From the representation formula (11) and the estimates of Γ in (10), it follows that

$$u(x,\rho_j^2) = \int_{\mathbb{R}^N} \Gamma(x,\rho_j^2;\xi,0)\,\mathrm{d}\sigma(\xi) \geq \mathbf{c}\,\rho_j^{-Q} \int_{B_d(x,\rho_j)} \exp\Big(-\mathbf{c}_0 \frac{d^2(x,\xi)}{\rho_j^2}\Big)\,\mathrm{d}\sigma(\xi) \geq$$
$$\geq \mathbf{c}'\,\sigma(B_d(x,\rho_j))/|B_d(x,\rho_j)| \longrightarrow \infty, \quad \text{as } j \to \infty.$$

This contradicts the hypothesis $\limsup_{t\to 0^+} u(x,t) < \infty$. Hence we can apply Lemma 2.7 and obtain that σ is absolutely continuous w.r.t. the Lebesgue measure. From the Lebesgue decomposition (15) $\mathrm{d}\sigma(\xi) = \varphi(\xi)\,\mathrm{d}\xi + \mathrm{d}s(\xi)$, it immediately follows that $s = 0$. Moreover, Theorem 2.4 gives $\varphi(x) = \lim_{t\to 0^+} u(x,t) = 0$ for almost every $x \in \mathbb{R}^N$, by hypothesis. Therefore, we obtain $\sigma = 0$. In order to complete the proof, it is now sufficient to recall the representation formula (11). $\square$

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