Rendiconti di Matematica, Serie VII
Volume 25, Roma (2005), 85-108

On the surface tension for non local energy functionals

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ABSTRACT: *We consider the free energy functional* $F_\varepsilon(m)$, $\varepsilon > 0$ *a scaling parameter,* $m \in L^\infty(\mathcal{T}; [-1,1])$, $\mathcal{T}$ *the unit torus, which has been derived in a continuum limit from Ising spin systems with Kac interactions, see* [8]. *In* [1] *it is proved that* $F_\varepsilon(m)$ $\Gamma-$*converges to a perimeter functional* P. *We study here the free energy functional with an additional term describing the interaction with an external magnetic field* h. *We suppose that* h *takes only the two values* $\pm s$, $s > 0$. *Calling* E *the region of the torus where the external field is negative and* $F_{\varepsilon,s}(m; E)$ *the new functional, we then define* $G_{\varepsilon,s}(E) = \inf_m F_{\varepsilon,s}(m; E)$. *We prove that* $G_{\varepsilon,s}(\cdot)$ $\Gamma-$*converges to a perimeter functional which as a function of* s *converges pointwise as* $s \to 0$ *to* P.

1 – Introduction

In this paper we consider the non local, excess, free energy functional defined for all m on $L^\infty(\mathbb{R}^d; [-1,1])$, with values in $[0, +\infty]$, $+\infty$ included, by

$$\mathcal{F}_{\beta,h}(m) = \int_{\mathbb{R}^d} dr\, f_{\beta,h}\big(m(r)\big) + \frac{1}{4}\int_{\mathbb{R}^d} dr \int_{\mathbb{R}^d} dr'\, J(r,r')[m(r) - m(r')]^2 \tag{1.1}$$

where $h \in L^\infty(\mathbb{R}^d; \mathbb{R})$,

$$f_{\beta,h}(m) = \phi_{\beta,h}(m) - \min_{|m|\le 1} \phi_{\beta,h}(m) \tag{1.2}$$

$$\phi_{\beta,h}(m) = -\frac{m^2}{2} - hm - \frac{I(m)}{\beta}, \qquad m \in [-1,1] \tag{1.3}$$

KEY WORDS AND PHRASES: *Coexistence of Phases – Surface Tension –* Γ*- convergence.*
A.M.S. CLASSIFICATION: 47H – 82C

$$I(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}. \tag{1.4}$$

The interaction $J(r,r')$ is a translational invariant (i.e. $J(r,r') = J(0,r'-r)$), smooth, symmetric, probability kernel supported by $|r-r'| \le 1$. As $\mathcal{F}_{\beta,h}$ depends symmetrically on $J(r,r')$ there is no loss of generality in assuming

$$J(r,r') = J(r',r), \quad \text{and, equivalently,} \quad J(0,r) = J(0,-r).$$

The expression (1.1) arises in the study of Gibbs measures in Ising spin systems with Kac interactions, see [4] and in their time evolution with Glauber dynamics , where it is derived in a continuum limit, [6]; m is then interpreted as a magnetization density and $\beta^{-1} = \kappa T$, T the absolute temperature and κ the Boltzmann constant, h is an external magnetic field.

Due to the positivity of J, the second term is minimized by any constant function, while the first one is minimal when the constant is set equal to a minimizer, call it $m_{\beta,h}$, of $f_{\beta,h}(s)$, $s \in [-1,1]$.

Thus $\mathcal{F}_{\beta,h}(m^*) = 0$ when $m^*(r) = m_{\beta,h}$ for all $r \in \mathbb{R}^d$: $m^*(r)$ is therefore called an equilibrium phase and $\mathcal{F}_{\beta,h}(m)$ measures the increase of free energy in magnetization profiles m which deviate from equilibrium.

Phase transitions are related to the lack of uniqueness of the minimizers of the free energy functional, which, for $\mathcal{F}_{\beta,h}$, occurs at $h = 0$ and $\beta > 1$. In such cases in fact the equilibrium magnetization $m_{\beta,0}$ can take two values, $\pm m_\beta$, solutions of the mean field equation

$$m_\beta = \tanh\{\beta m_\beta\}. \tag{1.5}$$

We now turn to the main object of this paper, surface tension and more generally, coexistence of phases. Roughly speaking, the surface tension is the excess free energy per unit area needed to create a state with two coexisting phases. The area in the definition refers to the interface which separates the two phases and the surface tension may depend on its orientation when the interaction is anisotropic. Thus, in a macroscopic description, characterized by the assumption of local thermodynamic equilibrium, at all points the magnetization is either equal to m_β or to $-m_\beta$. Let us restrict, for simplicity, to a unit torus $\mathcal{T}$ of $\mathbb{R}^d$ (in macroscopic units). Then a macroscopic state is a magnetization profile $u(r) \in \{\pm m_\beta\}$ for any $r \in \mathcal{T}$. Call E the region in $\mathcal{T}$ where $u = m_\beta$ and E^c its complement, where $u = -m_\beta$, then, if the boundary ∂E of E is regular, the macroscopic free energy of u is

$$P(u) = \int_{\partial E} dH^{d-1}(r)\theta_\beta(\nu(r)) \tag{1.6}$$

where $dH^{d-1}(r)$ is the Hausdorff area measure and $\theta_\beta(n) = \theta_\beta(-n)$ is the surface tension of a planar surface with normal n, $\nu(r)$ the unit normal to ∂E at r.

Regularity of E is not really necessary and in fact the above expression keeps its validity for all u in $BV(\mathcal{T}, \{\pm m_\beta\}])$, as it will be discussed later on.

To relate the macroscopic theory to the functional (1.1) we interpret the latter as the result of a more accurate, microscopic, description of the system, where distances are magnified revealing deviations from the local equilibrium condition $|u(r)| = m_\beta$. Thus, calling ε^{-1} the magnifying factor of the blow up, a microscopic state is an element $m \in L^\infty(\varepsilon^{-1}\mathcal{T}, [-1,1])$ and its microscopic free energy is $\mathcal{F}^{\text{per}}_{\varepsilon^{-1}\mathcal{T}}(m)$, where the latter is the functional (1.1) restricted to $m \in L^\infty(\varepsilon^{-1}\mathcal{T}); [-1,1])$ with J replaced by its periodization on $\varepsilon^{-1}\mathcal{T}$.

To compare with (1.6) we first need to have objects on a same space. Let $V_\varepsilon : L^\infty(\mathcal{T};[-1,1]) \to L^\infty(\varepsilon^{-1}\mathcal{T};[-1,1])$ be defined by

$$V_\varepsilon m(r) = m(\varepsilon r) =: m^{(\varepsilon)}(r), \qquad r \in \varepsilon^{-1}\mathcal{T}. \tag{1.7}$$

Then $\mathcal{F}^{\text{per}}_{\varepsilon^{-1}\mathcal{T}} \circ V_\varepsilon$ becomes a functional on $L^\infty(\mathcal{T};[-1,1])$ which associates to any given $m \in L^\infty(\mathcal{T};[-1,1])$ a microscopic free energy, indexed by ε. Since we are interested in states with interface, their free energy must scale as an area, namely proportionally to ε^{-d+1}. We then define

$$\Phi_\varepsilon = \varepsilon^{d-1}\mathcal{F}^{\text{per}}_{\varepsilon^{-1}\mathcal{T}} \circ V_\varepsilon \tag{1.8}$$

Φ_ε is the "normalized, microscopic free energy functional" which we want to compare with the macroscopic functional P of (1.6).

Φ_ε and P are defined on different functional spaces, and to establish a relation between them we follow De Giorgi and his definition of Γ convergence. We start by arguing that a microscopic profile which describes the macroscopic state $u \in BV(\mathcal{T}, \{\pm m_\beta\})$ should look more and more like u as $\varepsilon \to 0$. To make it quantitative, we use the $L^1(\mathcal{T})$ norm, which weights both the volume of the region where two profiles differ and the amount of their discrepancy: this is therefore a natural candidate to quantify distances. In this language, the physical apparatus used to prepare a macroscopic state u is then schematized as a constraint which imposes the microscopic states m to be in a $L^1(\mathcal{T})$-ball of u. Thus a state u which looks sharp at the macroscopic level, becomes fuzzy after the microscopic blow up and it is better represented by a set of states, a ball in L^1 with center u, rather than by a single profile. The radius of the ball is related to the accuracy of the physical apparatus used in the preparation of the state and we imagine that it can be taken arbitrarily small, as $\varepsilon \to 0$.

To conclude, we only need to determine the free energy to associate to the L^1 ball which represents a macroscopic interface $u \in BV(\mathcal{T}, \{\pm m_\beta\})$ at the microscopic level. By invoking thermodynamic principles, the equilibrium free energy under a given constraint is the minimal free energy of the states satisfying the constraint, hence calling $\delta > 0$ the accuracy parameter identified to the radius of the L^1-ball, we set

$$\Phi_{\delta,\varepsilon}(u) = \inf_{\|m-u\|_{L^1(\mathcal{T})} \le \delta} \Phi_\varepsilon(m) \tag{1.9}$$

and call

$$\Phi_0'(u) = \lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \Phi_{\delta,\varepsilon}(u), \qquad \Phi_0''(u) = \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \Phi_{\delta,\varepsilon}(u). \tag{1.10}$$

The equality $P(u) = \Phi_0'(u) = \Phi_0''(u)$ is the De Giorgi definition that Φ_ε Γ-converges to P, in such a case we will write $\Phi_\varepsilon \xrightarrow{\Gamma} P$. Alberti and Bellettini, [1], have proved for a class of functionals which includes Φ_ε that

THEOREM. *For any u in $BV(\mathcal{T}, \{\pm m_\beta\})$ $\Phi_\varepsilon \xrightarrow{\Gamma} P$ and, if u is regular (i.e. its discontinuity set ∂E is a regular surface), then $P(u)$ is given by the expression* (1.6) *with $\theta_\beta(\nu)$ a continuous function on the unit ball of $\mathbb{R}^d$. The general theory of BV functions, see* [1] *defines for any $u \in BV(\mathcal{T}, \{\pm m_\beta\})$ a set $\partial^* E \subset \partial E$, a measure $d\mu$ on $\partial^* E$ and a unit vector function $\nu(r)$ on $\partial^* E$. In terms of these quantities,*

$$P(u) = \int_{\partial^* E} d\mu(r)\theta_\beta(\nu(r)). \tag{1.11}$$

There are also results about the value of the surface tension $\theta_\beta(\nu)$, expressed in terms of the one dimensional free energy of standing fronts, see [2], see also [5] for a related model, a uniqueness theorem for such one dimensional fronts, [7], and a proof of strict convexity and regularity of the surface tension as a function of the direction ν, [9].

The motivation of this paper is about the actual implementation of the previous definition of surface tension in a physical experiment. For that we would need a physical apparatus which forces the minus phase in the set E and the plus one in E^c. The natural way is to use an external magnetic field and, with a great deal of idealization, we will suppose to be able to set the magnetic field equal to $-s$ in a region B and equal to $+s$ in the complement, with the additional assumption that E and B are close in the symmetric difference distance, namely that $|B \vartriangle E| \le \delta$ (the same accuracy parameter as before). Under such a space dependent magnetic field

$$h_B(r) := s\mathbf{1}_{B^c}(r) - s\mathbf{1}_B(r) \tag{1.12}$$

equilibrium will be reached by minimizing over all m the functional $F_{\varepsilon,s}(m; B)$, defined in (2.2) below.

For any Borel subset B of the torus, we call $G_{\varepsilon,s}(B)$ the infimum of $F_{\varepsilon,s}(m; B)$ over all $m \in L^\infty(\varepsilon^{-1}\mathcal{T}; [-1, 1])$.

Our main result in this paper is a proof that $G_{\varepsilon,s}(\cdot)$ $\Gamma-$converges to a perimeter functional P_s and $P_s \to P$ as $s \to 0$, thus justifying from an operational point of view, the original definition of surface tension via $\Gamma-$convergence.

The paper is organized as follows: in the next section we give precise definitions and in Theorem 2.2.1 we state the main results. We divide the proof of Theorem 2.2.1 in two sections, the lower and the upper bound. To prove the lower bound we need some results about "contours",the argument is treated in Section 5. Finally, in the last section, we will prove the convergence to P of the surface free energy functional $P_s(E)$, see (2.4).

2 – Definitions and results

Let $\mathcal{T}$ be the unit torus in $\mathbb{R}^d$, $s > 0$ and $\varepsilon > 0$. Furthermore let $\mathcal{B}$ be the set of all Borel measurable subsets of the torus equipped with L^1-distance, which is the same as the volume of the symmetric difference:

$$|A \triangle B| := vol\,((A\backslash B) \cup (B\backslash A)) = \int |\mathbf{1}_A - \mathbf{1}_B| dr. \tag{2.1}$$

For all $m \in L^\infty(\varepsilon^{-1}\mathcal{T}; [-1; 1])$ and $B \in \mathcal{B}$ we define

$$\begin{aligned} F_{\varepsilon,s}(m(r); B) : = & \int_{\varepsilon^{-1}\mathcal{T}} f_{\beta,h_B}(m(r))dr + \\ & + \frac{1}{4}\int_{\varepsilon^{-1}\mathcal{T}}\int_{\varepsilon^{-1}\mathcal{T}} J(r,r')(m(r)-m(r'))^2 dr dr' \end{aligned} \tag{2.2}$$

where, by an abuse of notation, J is the periodization on $\varepsilon^{-1}\mathcal{T}$ of the probability kernel in (1.1), $h_B(r)$ as in (1.12), f_{β,h_B}, ϕ_{β,h_B} and $I(m)$ as in (1.2)-(1.4).
We next define, for any $B \in \mathcal{B}$,

$$G_{\varepsilon,s}(B) = \varepsilon^{d-1} \inf_{m\in L^\infty(\varepsilon^{-1}\mathcal{T};[-1,1])} F_{\varepsilon,s}(m; B). \tag{2.3}$$

Our main result is

Theorem 2.2.1. *For any s small enough, $G_{\varepsilon,s} \xrightarrow{\Gamma} P_s$ on $BV(\mathcal{T})$, where P_s is a perimeter functional in $BV(\mathcal{T})$. Namely for any $E \in BV(\mathcal{T})$, for any $m \in L^\infty(\varepsilon^{-1}\mathcal{T}; [-1, 1])$, and for any $\delta > 0$ there exists $s^\star > 0$ and a continuous function $\theta_{\beta,s}(\nu)$ on the unit ball of $\mathbb{R}^d$, such that for any $s \le s^\star$*

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} \inf_{B\in BV(\mathcal{T}):|B\triangle E|\le\delta\varepsilon^{-d}} G_{\varepsilon,s}(B) = \int_{\partial^* E}\theta_{\beta,s}(\nu)d\mu := P_s(E). \tag{2.4}$$

Moreover

$$\lim_{s\to 0}\theta_{\beta,s}(\nu) = \theta_\beta(\nu) \tag{2.5}$$

and $\lim_{s\to 0} P_s(E) = P(\chi_E)$, with P as in (1.11) *and $\chi_E = m_\beta\mathbf{1}_E - m_\beta\mathbf{1}_{E^c}$.*

In the next two sections we prove (2.4), while (2.5) will be proved in the last section.

3 – Lower bound

In this section we will prove that

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\inf_{B\in BV(\mathcal{T}):|B\triangle E|\le\delta\varepsilon^{-d}} G_{\varepsilon,s}(B)\ge\int_{\partial^* E} d\mu(r)\theta_{\beta,s}(\nu). \tag{3.1}$$

First of all, we need some basic notions and results for the theory of BV sets, which we state in the next subsection, for more details see, for example, [1].

3.1 – Geometric measure theory

We say that a function f on $\mathcal{T}$ has bounded variation, $f\in BV(\mathcal{T})$, if its gradient Df (in the sense of distributions) is a vector real valued Radon measure whose total variation measure has finite mass $\|\mu\|$:

$$\|\mu\|=\mu(\mathcal{T})=\sup_{\phi\in C^1(\mathcal{T},\mathbb{R}^d),\|\phi\|_\infty\le 1}\left|\int_{\mathcal{T}} dr f\,\mathrm{div}\,\phi\right|. \tag{3.2}$$

We say that E is a general BV set if $\mathbf{1}_E\in BV(\mathcal{T})$. If E is a C^1 set, the total variation $d\mu$ of $D\mathbf{1}_E$ is the usual Hausdorff measure $dH^{d-1}(r)$ on ∂E and for any $\phi\in C(\mathcal{T},\mathbb{R}^d)$

$$\int_{\mathcal{T}}\langle D\mathbf{1}_E,\phi\rangle=-\int_{\mathcal{T}} dr\mathbf{1}_E\,\mathrm{div}\,\phi=-\int_{\partial E} dH^{d-1}(r)\langle\nu(r),\phi\rangle \tag{3.3}$$

where $\nu(r)$ is the outward unit normal to ∂E at r.

If E is a general BV set, then there are a set $\partial^* E\subset\partial E$, called the reduced boundary of E, and a unit vector valued function $\nu(r)$ on $\partial^* E$ so that for any $\phi\in C(\mathcal{T},\mathbb{R}^d)$

$$\int_{\mathcal{T}}\langle D\mathbf{1}_E,\phi\rangle=-\int_{\partial^* E} dH^{d-1}(r)\langle\nu(r),\phi\rangle. \tag{3.4}$$

The following theorem states that BV sets can be regarded, measure theoretically, as C^1 sets:

THEOREM 3.3.1. *Let $E\in BV(\mathcal{T})$ and $D\mathbf{1}_E(r)=-d\mu\nu(r)$.Then for any $\varepsilon>0$ there are C^1 hyper-surfaces $S_1,\dots,S_m$ whose closures are disjoint from each other, and compact sets $K_1,\dots,K_m$ with $K_i\subset S_i\cap\partial^* E$, so that*

$$d\mu\big|_{K_i}=dH^{d-1}\big|_{K_i}\qquad\int_{\mathcal{T}} d\mu-\sum_{i=1}^m\int_{K_i} d\mu\le\varepsilon. \tag{3.5}$$

Moreover the normal to S_i ar $r \in K_i$, is the same as the unit vector $\nu(r)$ in (3.3) *and*

$$\max_{i=1\dots m}\max_{r,r'\in S_i}|\nu(r)-\nu(r')|\le\varepsilon. \tag{3.6}$$

The next Theorem states that a BV set E is with good approximation made of essentially flat parts plus a small remainder. We set

$$u=C(\mathbf{1}_{E^c}-\mathbf{1}_E). \tag{3.7}$$

THEOREM 3.3.2. *For any $\varepsilon>0$ there are $n\ge 1$ disjoint measurable sets Σ_i, each one contained in some $K_j^{(\varepsilon)}$, n cubes R_i, all of side h, and n unit vectors ν_i, ν_i orthogonal to a face of R_i, with the following proprieties so that*

$$\sup_{r\in\Sigma_i}|\nu(r)-\nu_i|<\varepsilon,\qquad \left|h^{d-1}-\int_{\Sigma_i}d\mu\right|<\varepsilon h^{d-1},\qquad \left|nh^{d-1}-\int_{\mathcal{T}}d\mu\right|<\varepsilon. \tag{3.8}$$

Moreover calling $\chi(r):=C(\mathbf{1}_{R_i^+}-\mathbf{1}_{R_i^-})$, with $R_i^{\pm}$ the upper and lower halves of R_i with to the direction ν_i,

$$\int_{R_i}dr|\chi_{R_i}-u|<\varepsilon h^d,\quad i=1,...,n. \tag{3.9}$$

3.2 – Proof of (3.1)

Let $R_n(L;C)$ be the cylinder in $\mathbb{R}^d$ whose axis is directed along n and whose cross section is LC, C an unit cube of $\mathbb{R}^{d-1}$ and $L>0$ a scaling parameter. We keep n and B fixed and to simplify notation we drop them, thus writing $R(L)$ and $R(L,k)$. We introduce coordinate axes with the origin the center of $R(L,k)$, x_d axis along n and the the others parallel to the side of C, so that C is a coordinate cube. Then

$$R(L,k)=\Big\{(x_1,..,x_d)\in\mathbb{R}^d: |x_d|\le k, |x_i|\le L, i=1,\dots,d-1\Big\}$$

and denote with $R^{\pm}_{L,k}$ the upper and lower halves of $R(L,k)$ with respect to the direction n. Calling

$$\chi(r)=m^+_{\beta,s}\mathbf{1}_{x_d\ge 0}+m^-_{\beta,s}\mathbf{1}_{x_d<0} \tag{3.10}$$

we denote by χ_Δ, $\Delta\subset\mathbb{R}^d$, the restriction of χ to Δ and we define

(3.11)
$$\theta_{\beta,s}(L,k):=\frac{1}{L^{d-1}}\inf_{\substack{m\in L^\infty(R(L,k);[-1;1])\\ B:|R^+_{L,k}\cap(R^+_{L,k}\triangle B^c)|\le\delta,\quad |R^-_{L,k}\cap(R^-_{L,k}\triangle B)|\le\delta}} F_s\left(m_{R(L,k)}|\chi_{R(L,k)^c};B\right).$$

We remember that the functional $F_{\varepsilon,s}(\cdot, B)$ is defined by making the interaction J periodic over each coordinate x_i, $i < d$, with period L, thus considering LB as a torus. We thus set

$$\theta_{\beta,s} := \liminf_{L\to\infty} \liminf_{k\to\infty} \theta_{\beta,s}(L,k). \tag{3.12}$$

In some cases, when the context is not clear, we indicate with $\theta_{\beta,s,\nu}$ the surface tension defined in the rectangle $R_\nu(L,k)$ directed along ν.

We consider the small parameter α and the cubes R_i as in Theorem 3.3.2 below, $i = 1,\dots,n$, all of side k so that if we call $\tilde{\chi}_i = s(\mathbf{1}_{\varepsilon^{-1}R_i^+} - \mathbf{1}_{\varepsilon^{-1}R_i^-})$, we have

$$\int_{\varepsilon^{-1}R_i} dr|h_B - \tilde{\chi}_i| \le \int_{\varepsilon^{-1}R_i} dr|h_B - h_E| + \int_{\varepsilon^{-1}R_i} dr|h_E - \tilde{\chi}_i|$$
$$\le \varepsilon^{-d}(\delta + \alpha k^d) \le 2\alpha k^d \varepsilon^{-d}.$$

Hence $|(B \vartriangle \varepsilon^{-1}R_i) \cap \varepsilon^{-1}R_i| \le 2\alpha k^d \varepsilon^{-d}$.

We next write

$$\Delta = \bigcup_{i=1}^n \varepsilon^{-1}R_i\,.$$

Then

$$F_{\varepsilon,s}(m;B) = F_{\varepsilon,s}(m_{\Delta^c};B) + \sum_{i=1}^n F_{\varepsilon,s}\left(m_{\varepsilon^{-1}R_i}|m_{\Delta^c};B\right)$$

$$\varepsilon^{d-1}F_{\varepsilon,s}(m;B) \ge \sum_{i=1}^n F_{\varepsilon,s}\left(m_{\varepsilon^{-1}R_i}|m_{\Delta^c};B\cap R_i\right) \tag{3.14}$$

where

$$F_{\varepsilon,s}(m_\Lambda|m_{\Lambda^c};B) := F_{\varepsilon,s}(m_\Lambda;B) + \frac{1}{2}\int_{\varepsilon^{-1}\Lambda}\int_{\varepsilon^{-1}\Lambda^c} J(r,r')(m_\Lambda(r) - m_{\Lambda^c}(r'))^2 dr dr' \tag{3.15}$$

$$F_{\varepsilon,s}(m_\Lambda;B) := \int_{\varepsilon^{-1}\Lambda} f_{\beta,h_B}(m_\Lambda(r))dr + \frac{1}{4}\int_{\varepsilon^{-1}\Lambda}\int_{\varepsilon^{-1}\Lambda} J(r,r')(m_\Lambda(r) - m_\Lambda(r'))^2 dr dr'.$$

At the end of section we are going to prove that

$$\inf_{\substack{m\in L^\infty(\mathbb{R}^d;[-1;1])\\ B\in BV(\mathcal{T}):|B\triangle E|\le\delta\varepsilon^{-d}}} F_{\varepsilon,s}(m;B) = \inf_{\substack{m\in L^\infty(\mathbb{R}^d;[-1;1]):F_{\varepsilon,s}(m;B)\le\delta\varepsilon^{-d}\\ B\in BV(\mathcal{T}):|B\triangle E|\le\delta\varepsilon^{-d}}} F_{\varepsilon,s}(m;B) \tag{3.16}$$

therefore, using (3.14) and (3.16), we obtain that

$$\text{l.h.s. of } (3.1) \geq$$

$$\liminf_{\alpha\to 0}\sum_i \left\{ \liminf_{\varepsilon\to 0}\varepsilon^{d-1} \inf_{\substack{F_{\varepsilon,s}(m;B)\leq 2\alpha k^d\varepsilon^{-d}\\ B\in BV(\mathcal{T}):|(B\triangle\varepsilon^{-1}R_i)\cap\varepsilon^{-1}R_i|\leq 2\alpha k^d\varepsilon^{-d}}} F_{\varepsilon,s}\left(m_{\varepsilon^{-1}R_i}|m_{\varepsilon^{-1}R_i^c};B\right)\right\}.$$

Now we state two results that we will prove later. The first one gives us a constraint on the function m, the second one gives us a lower bound on each rectangle $C(L) = R(L, L/2)$, where

$$R(L, L/2) = \left\{x \in \mathbb{R}^d : |x_i| \leq L, i = 1...d\text{-}1, |x_d| \leq L/2\right\}.$$

Notational remark: when we consider function on $L^\infty(\mathcal{T}; [-1, 1])$ we write $F_s(m; B)$ instead of $F_{\varepsilon,s}(m; B)$, $B \in BV(\mathcal{T})$.

Proposition 3.3.3. *Let $C(L)$ be the cylinders of the form $R(L; L/2)$. Then for any $m \in L^\infty(C(L); [-1; 1])$ such that $F_s\left(m_{C(L)}; B\right) \leq \delta L^{-d}$ we have*

$$\int_{C(L)} |m(r) - \chi(r)|dr \leq \delta' L^d \tag{3.17}$$

$\chi(r) = m^+_{\beta,s}\mathbf{1}_{x_d\geq 0} + m^-_{\beta,s}\mathbf{1}_{x_d<0}$, $r \in C(L)$.

Theorem 3.3.4. *There is a $c > 0$ and a continuous function $\theta_{\beta,s}(\nu)$ on the unit ball, so that for any $\varepsilon > 0$ there is $L_\varepsilon > 0$ and for any $L \geq L_\varepsilon$*

$$F_s\left(m_{C(L)}|m_{C(L)^c}; B\right) \geq L^{d-1}(\theta_{\beta,s}(\nu) - \varepsilon - c\sqrt{\delta}) \tag{3.18}$$

for any $\delta > 0$,for any m s.t. $\|m - \chi\|_{L^1(C(L))} \leq \delta' L^d$ and for any $B \in BV(\mathcal{T})$ such that

$$|(B^c \triangle C^-(L)) \cap C^-(L)| \leq \delta L^d \qquad |(B \triangle C^+(L)) \cap C^+(L)| \leq \delta L^d.$$

Then, using (3.18)

$$\text{l.h.s. of (3.1)} \geq \lim_{s\to 0} \liminf_{\alpha\to 0} \sum_i k^{d-1}(\theta_{\beta,s}(\nu_i) - cn\sqrt{\alpha}).$$

By (3.8) we have $nk^{d-1} \leq \mu(\mathcal{T}) + \alpha$ and for a suitable constant c',

$$|k^{d-1}\theta_{\beta,s}(\nu_i) - \int_{\Sigma_i} d\mu\theta_{\beta,s}(\nu)| \leq c' k^{d-1}\alpha \tag{3.19}$$

so, in conclusion $\exists \quad c'' > 0$ such that

$$\text{l.h.s. of (3.1)} \geq \lim_{s\to 0} \liminf_{\alpha\to 0} \sum_i \int_{\Sigma_i} d\mu\theta_{\beta,s}(\nu) - c''\sqrt{\alpha} \tag{3.20}$$

and, see the end of this section,

$$\lim_{\alpha\to 0} \sum_i \int_{\Sigma_i} d\mu\theta_{\beta,s}(\nu) \to \int_{\partial^* E} d\mu(r)\theta_{\beta,s}(\nu)$$

thus we obtain (3.1).

Proof of (3.16). It suffices to show that $\forall\delta > 0$ and for any $B \in BV(\mathcal{T})$ there exists $m \in L^\infty(\varepsilon^{-1}\mathcal{T}; [-1, 1])$ and $\delta' > 0$ such that

$$F_{\varepsilon,s}\,(m; B) \leq \delta'\varepsilon^{-d}.$$

It is enough to choose $\hat{m} = m^-_{\beta,s}\mathbf{1}_{B_n} + m^+_{\beta,s}\mathbf{1}_{B_n^c}$ where B_n are the polyedrical sets which approximate $B \in BV(\mathcal{T})$ in variation, namely $\mathbf{1}_{B_n}$ converges in variation to $\mathbf{1}_B$. Indeed, computing the functional $F_{\varepsilon,s}(\hat{m}; B)$

$$\begin{aligned} F_{\varepsilon,s}(\hat{m}; B) = &\int_{\varepsilon^{-1}\mathcal{T}} f_{\beta,h_B}(\hat{m})dr + \frac{1}{2}\int_{\varepsilon^{-1}B_n}\int_{\varepsilon^{-1}B_n^c} J(r,r')(m^+_{\beta,s} - m^-_{\beta,s})^2 dr dr' \leq \\ &\leq 2h\int_{\varepsilon^{-1}\mathcal{T}} dr|\mathbf{1}_{B_n} - \mathbf{1}_B| + \int_{\varepsilon^{-1}B_n}\int_{\varepsilon^{-1}B_n^c} J(r,r')dr dr' \leq \\ &\leq 2s\delta\varepsilon^{-d} + c_n\varepsilon^{-d+1} = \delta'\varepsilon^{-d} \end{aligned}$$

with $\delta' = 2s\delta + c_n\varepsilon$.

4 – Upper Bound

In this section we will prove that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\inf_{B\in BV(\mathcal{T}):|B\triangle E|\le\delta\varepsilon^{-d}} G_{\varepsilon,s}(B)\le\int_{\partial^* E} d\mu(r)\theta_{\beta,s}(\nu(r)). \tag{4.1}$$

Given $E\in BV(\mathcal{T})$ we can approximate in the sense of variations the function h_E by functions h_{E_k} equal to $\pm s$ outside and inside polyhedral sets E_k with boundary ∂E_k. For each k we will construct functions $m^{(\varepsilon,k,L,t)}$ so that

$$\limsup_{L\to\infty}\limsup_{t\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{d-1}F_{\varepsilon,s}(m^{(\varepsilon,k,L,t)};E_k)\le\int_{\partial^* E_k} d\mu_k(r)\theta_{\beta,s}(\nu(r)) \tag{4.2}$$

where $d\mu_k = d\mu|_{E_k}$ as in Theorem 3.3.1. Then by letting $k\to\infty$,

$$\begin{aligned}&\limsup_{k\to\infty}\limsup_{L\to\infty}\limsup_{t\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{d-1}F_{\varepsilon,s}(m^{(\varepsilon,k,L,t)};E_k)\le\\ &\qquad\le\limsup_{k\to\infty}\int_{\partial^* E_k} d\mu_k(r)\theta_{\beta,s}(\nu)\end{aligned} \tag{4.3}$$

and

$$\lim_{k\to\infty}\int_{\partial^* E_k} d\mu_k(r)\theta_{\beta,s}(\nu(r))=\int_{\partial^* E} d\mu(r)\theta_{\beta,s}(\nu(r)). \tag{4.4}$$

Then, by (4.2) and (4.3) there are $L(\varepsilon)$, $t(\varepsilon)$, and $k(\varepsilon)$ so that the family $m^{(\varepsilon,k(\varepsilon),L(\varepsilon),t(\varepsilon))}$ satisfies (4.1). Thus the proof of (4.1) follows from the existence of a family $m^{(\varepsilon,k,L,t,)}$ satisfying (4.2), which is proved in the rest of the subsection.

We fix k and we will drop it from the notation in the sequel. Thus we denote with E a polyhedral set and with $h_E = s(\mathbf{1}_{E^c}-\mathbf{1}_E)$. The faces of E are called σ_i, $i=1,..,n$, and their normal ν_i, directed toward the plus magnetization. On each hyperplane which contains $\varepsilon^{-1}\Sigma_i$, we introduce a partition into $d-1$ dimensional cubes of side L, the orientation of the cubes of the partition being the same for all ε. We first define $m^{(\varepsilon,L,t)}$ around $\varepsilon^{-1}\Sigma_1$: on each rectangle $R_{\nu_1}(L,t)$ of height $2t$ and mid cross section a cube entirely contained in $\varepsilon^{-1}\Sigma_1$, we choose $m^{(\varepsilon,L,t)}$ so that

$$\frac{1}{L^{d-1}}F_s(m^{(\varepsilon,L,t)}_{R_{\nu_1}(L,t)}|\chi_{R^c_{\nu_1}(L,t)};E_k)\le\theta_{\beta,s,\nu_1}(L,t)+\varepsilon. \tag{4.5}$$

When the mid cross section of $R_{\nu_1}(L,t)$ is not entirely contained in $\varepsilon^{-1}\Sigma_1$, we set $m^{(\varepsilon)} = m^{\pm}_{\beta,s}$ in the part of $R_{\nu_1}(L,t)$ which is above and below $\varepsilon^{-1}\Sigma_1\sqcap R_{\nu_1}(L,t)$. we follow the same rule in the other faces, except for the points where $m^{(\varepsilon)}$ has already been defined. On the remaining of the space we set $m^{(\varepsilon)}$ equal to $m^{\pm}_{\beta,s}$ outside and inside E respectively. If we fix t, if L is large enough, any rectangle $R_{\nu_i}(L,t)$ at distance $> L$ from the boundary of $\varepsilon^{-1}\Sigma_i$ has no intersection with any other rectangles, then, for a suitable constant c,

$$\varepsilon^{d-1}F_{\varepsilon,s}(m^{(\varepsilon)};E) \leq \sum_{i=1}^{n}\big([\theta_{\beta,s,\nu_i}(L,t)+\varepsilon]|\Sigma_i| + cLt\varepsilon\big). \tag{4.6}$$

Then (4.2) follows, and the proof of the upper bound is completed.

5 – Contours and dynamics

In this section we give a generalized definition of contours and we study some proprieties of the evolution. For this purpose we define three basic objects. The first one is the family of partitions of $\mathbb{R}^d$

$$\big\{\mathcal{D}^\ell,\ \ell = 2^n,\ n\in\mathbb{Z}\big\}$$

$\mathcal{D}^\ell$ is a decreasing sequence of partitions into cubes C^ℓ of side ℓ. $C_r^{(\ell)}$ denotes the cube of $\mathcal{D}^\ell$ which contains r. Another basic object is the coarse-grained image of $m \in L^\infty(\mathbb{R}^d;[-1,1])$ with grain ℓ, $Av^{(\ell)}(m;r)$

$$Av^{(\ell)}(m;r) = \frac{1}{C^{(\ell)}}\int_{C_r^{(\ell)}} dr' m(r')\ , \quad |C^{(\ell)}| = \ell^d,\ m\in L^\infty(\mathbb{R}^d;[-1;1]). \tag{5.1}$$

The last basic object is the "block spin" function

$$\eta^{(\zeta,\ell)}(m;r) = \begin{cases} \pm 1 & \text{if } |Av^{(\ell)}(m;r) - m^{\pm}_{\beta,s}| \leq \zeta, \\ 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

where $\zeta > 0$ and $\ell < 1$. Using these quantities we define:

- *Outer and inner boundaries.*
  The $\mathcal{D}^\ell$−outer boundary of a $\mathcal{D}^\ell$−measurable region Λ, denoted by $\delta^\ell_{out}[\Lambda]$, is the union of all the cubes C of $\mathcal{D}^\ell$ not in Λ which are connected to Λ. The $\mathcal{D}^\ell$−inner boundary $\delta^\ell_{out}[\Lambda]$ is the $\mathcal{D}^\ell$−outer boundary of Λ^c.
- *Phase Indicator.*
  Denoted by $\Theta^{(\zeta,\ell_-,\ell_+,s)}(m,B;r)$, $\ell_- < 1$, $\ell_+ > 10$, $\zeta > 0$, it is defined as $\Theta^{(\zeta,\ell_-,\ell_+,s)}(m,B;r) = \pm 1$ if $\eta^{(\zeta,\ell_-)}(m;r') = \pm 1$ for all $r' \in C_r^{(\ell_+)} \cup \delta^\ell_{out}[C_r^{(\ell_+)}]$ and $|C_r^{(\ell_-)} \cap (C_r^{(\ell_-)} \vartriangle B^c[B])| \leq \zeta$.
  Elsewhere $\Theta_B^{(\zeta,\ell_-,\ell_+)}((m,h);r) = 0$

- *Correct points.*
  The $\pm$ correct points of m for a set B are the points r where, respectively $\{\Theta^{(\zeta,\ell_-,\ell_+,s)}(m,B;r)=\pm 1\}$.
  The set $\{\Theta^{(\zeta,\ell_-,\ell_+,s)}(m,B;r)=0\}$, is instead the union of the spatial support of all the contours of m.
- *Approximate, local equilibrium phase spaces.*
  These are the spaces with elements m for which all points of Λ are $\pm$ correct. Such spaces are denoted by $M_{\zeta,\ell,\ell_+,\pm,\Lambda}$ and we drop Λ when $\Lambda=\mathbb{R}^d$.

5.1 – Invariance under evolution

In this subsection we will prove that the local equilibrium ensembles $M_{\zeta,\ell,\ell_+,\pm,\Lambda}$ are invariant under the partial dynamics and that the minimizers of free energy in $M_{\zeta,\ell,\ell_+,\pm,\Lambda}$ is pointwise close to $m^+_{\beta,s}$ [or to $m^-_{\beta,s}$], the closeness being exponentially with the distance from the boundaries. By simmetry, it is sufficient to prove the statement for the + ensemble, to which in the sequel we restrict.

We consider the Cauchy problem obtained, after a suitable scaling limit, by the Glauber dynamics, applied to Ising systems with Kac potentials,

$$\begin{cases} \dfrac{dm(r,t)}{dt} = -m(r,t) + \tanh\{\beta[J\star m(r,t)+h_B]\}\,, & r\in\mathbb{R}^d,\ t>0;\\ m(r,0)=m(r) & r\in\mathbb{R}^d. \end{cases} \tag{5.3}$$

We also consider dynamics where, outside region Λ, the function is frozen and it acts as a boundary condition for the evolution inside Λ. Namely, we define a partial dynamics in Λ by setting

$$\begin{cases} \dfrac{dm^{(\Lambda)}(r,t)}{dt} = -m^{(\Lambda)}(r,t) + \tanh\left\{\beta[J\star m^{(\Lambda)}+h_B]\right\}, & (r,t)\in\Lambda\times\{t>0\};\\ m^{(\Lambda)}(r,t)=m(r), & (r,t)\in(\Lambda^c\times\{t>0\})\cup(\mathbb{R}^d\times\{t=0\}). \end{cases} \tag{5.4}$$

Definition 5.5.1. Let T_t^Λ be the semigroup on $L^\infty(\mathbb{R}^d,[-1,1])$ defined by setting

$$T_t^\Lambda(m) = \text{solution of (5.4)}. \tag{5.5}$$

With similar arguments as in [5] it is possible to prove that the orbits $T_t^\Lambda(m)$ converge by subsequences as $t\to\infty$ and that the limits points satisfy the mean field equation

$$m^{(\Lambda)}(r) = \tanh\left\{\beta[J\star m^{(\Lambda)}+h_B]\right\} \quad r\in\Lambda. \tag{5.6}$$

LEMMA 5.5.2. *There are ζ_0', κ_0 and $s^\star$ positive, so that if $\zeta < \zeta_0'$ and $\ell < \ell_0(\zeta) = \kappa_0\zeta$, then, for any $m \in M_{\zeta,\ell,\ell_+,+,\Lambda}$, with $s < s^\star$ and $r \in \Lambda$*

$$|J \star m(r) - m^+_{\beta,s}| \le 2\zeta \tag{5.7}$$

$$|\tanh\{\beta[J \star m(r) + h_B]\} - m^+_{\beta,s}| \le \zeta - \varepsilon_0(\zeta), \qquad \varepsilon_0(\zeta) = \kappa_0\zeta. \tag{5.8}$$

PROOF. Calling

$$J^{(\ell)}(r,r') = Av^{(\ell)}(J(r,\cdot);r')$$

the average of $J(r,\cdot)$ over its second variable, for ℓ small enough

$$|J(r,r') - J^{(\ell)}(r,r')| \le c\ell\mathbf{1}_{|r-r'|\le 2}, \qquad c := d\|\nabla J\|_\infty < \infty. \tag{5.9}$$

Then

$$|J \star m - J^{(\ell)} \star m| \le 2^d c\ell$$

and since

$$J^{(\ell)} \star m = J^{(\ell)} \star u, \qquad u(r) \equiv Av^{(\ell)}(m;r)$$

$$|J \star m - J^{(\ell)} \star u| \le 2^d c\ell.$$

On other hand, by assumption, $|u(r) - m^+_{\beta,s}| \le \zeta$ for all r at distance ≤ 2 from Λ, hence

$$|J^{(\ell)} \star u(r) - m^+_{\beta,s}| \le \zeta \quad r \in \Lambda$$

thus concluding

$$|J \star m(r) - m^+_{\beta,s}| \le \zeta + 2^d c\ell. \tag{5.10}$$

By choosing κ_0 so small that $\kappa_0 2^d c < 1$ we derive (5.7) from (5.8). Since

$$\frac{d}{dm}\tanh\{\beta m\}\big|_{m=m^+_{\beta,s}} \le a < 1$$

$$\begin{aligned}|\tanh\{\beta[m(r) + h_B]\} - m^+_{\beta,s}| &\le a|J \star m(r) - m^+_{\beta,s} + (h_B - s)|\\ &\le a|\zeta + 2^d c\ell + (h_B - s)|.\end{aligned}$$

Choosing $s^\star = 2^{d-1}c\kappa_0\zeta$ for any $s < s^\star$ and $\kappa_0 \le (1-a)/(1+2^d c)$

$$|\tanh\{\beta[m(r) + h_B]\} - m^+_{\beta,s}| \le \zeta\big(1 - [(1-a) - 2^d c\kappa_0]\big) \le \zeta - \kappa_0\zeta.$$

The lemma is proved. □

The next Lemma proves the invariance of $M_{\zeta,\ell,\ell_+,+,\Lambda}$ under the partial dynamics T_t^Λ. We omit the proof.

LEMMA 5.5.3. *If* ζ, ℓ, $s^\star$ *and* Λ *are as in Lemma* 5.5.2, T_t^Λ, $t > 0$, *maps* $M_{\zeta,\ell,\ell_+,+,\Lambda}$ *into itself.*

We call

$$\mathcal{X}_{\Lambda,m} = \left\{ u \in M_{\zeta,\ell,\ell_+,+,\Lambda} \ : \ u_{\Lambda^c} = m_{\Lambda^c} \right\} \tag{5.11}$$

ψ_Δ standing for the restriction of a function ψ to a set Δ.

THEOREM 5.5.4. *There are* $\zeta_0 < \zeta_0'$ *(*ζ_0', $\ell_0(\zeta)$ *and* $s^\star$ *as in Lemma* 5.5.2*),* ω *and* c_ω *all positive, such that for any* $B \in BV(\mathcal{T})$, $m \in M_{\zeta,\ell,\ell_+,+,\Lambda}$, *and for any* $s \leq s^\star$, *the following holds:*

$$\inf_{u \in M_{\zeta,\ell,\ell_+,+,\Lambda}} F_s\left(u_\Lambda | m_{\Lambda^c}; B\right) = F_s(\psi | m_{\Lambda^c}) \tag{5.12}$$

where $\psi(r)$ *is the unique solution of the mean field:*

$$\psi(r) = \tanh\{\beta[J \star \psi(r) + s]\} \tag{5.13}$$

and

$$\begin{aligned} \psi_\Lambda \in & C^\infty(\Lambda, [m^+_{\beta,s} - \zeta, m^+_{\beta,s} + \zeta]) \\ & |\psi_\Lambda(r) - m^+_{\beta,s}| \leq c_\omega e^{-\omega \operatorname{dist}(r,\Lambda^c_{\neq})} \end{aligned} \tag{5.14}$$

where $\Lambda^c_{\neq} = \{r \in \Lambda^c \ \operatorname{dist}(r,\Lambda) \leq 1;\ m_{\Lambda^c}(r) \neq m^+_{\beta,s}\}$.

In (5.12) $F_s(\cdot)$ means that the magnetic field is constantly equals to s on the whole space. Moreover when $s = 0$, we simply write $F(\cdot)$.

PROOF. By Lemma 5.5.3 T_t^Λ leaves $\mathcal{X}_{\Lambda,m}$ invariant and since $\mathcal{X}_{\Lambda,m}$ is closed under uniform convergence on the compacts, for any $u \in \mathcal{X}_{\Lambda,m}$, $T_t^\Lambda u$ converges by subsequences to an element ψ of $X^0_{\Lambda,m}$:

$$X^0_{\Lambda,m} = \{\psi \in \mathcal{X}_{\Lambda,m} : \ \psi \text{ solves (5.6) } \}$$

and $F_s(u_\Lambda | m_{\Lambda^c}; B) \geq F_s(\psi_\Lambda | \psi_{\Lambda^c}; B)$, the inequality being strict unless $u \in X^0_{\Lambda,m}$. Therefore

$$F_s(u_\Lambda | m_{\Lambda^c}; B) > \inf_{\psi \in X^0_{\Lambda,m}} F_s(\psi_\Lambda | m_{\Lambda^c}; B), \qquad \text{for any } u \in \mathcal{X}_{\Lambda,m} \backslash X^0_{\Lambda,m}.$$

By (5.8), any $\psi \in X^0_{\Lambda,m}$ satisfies the first condition in (5.14). We show that if ζ is small enough then $X^0_{\Lambda,m}$ consists of only one element, ψ, which is therefore the strict minimizer of $F_s(u_\Lambda|m_{\Lambda^c};B)$. Suppose ψ and ϕ are both in $\mathcal{X}_{\Lambda,m}$, then by (5.7), $J\star\psi(r)$ and $J\star\phi(r)$, $r\in\Lambda$, are in $[m^+_{\beta,s}-2\zeta, m^+_{\beta,s}+2\zeta]$ so that, recalling that $s\le s^\star$

$$|\tanh\{\beta J\star\psi(r)+\beta h_B(r)\}-\tanh\{\beta J\star\phi(r)+\beta h_B(r)\}|\le$$
$$\le \frac{\beta}{\cosh^2\{\beta(m^+_{\beta,s}-2\zeta')\}}\Big(\int_\Lambda dr' J(r,r')|\psi(r')-\phi(r')|\Big)$$

since $\beta\cosh^{-2}\{\beta(m^+_{\beta,s})\}<1$ we have for $r\in\Lambda$ and a suitable constant $c<1$,

$$|\tanh\{\beta J\star\psi(r)+\beta h_B(r)\}-\tanh\{\beta J\star\phi(r)+\beta h_B(r)\}|\le c\sup_{r'\in\Lambda}|\psi(r')-\phi(r')|$$

which implies that $\phi=\psi$ in Λ, hence everywhere. By (5.8) applied to ψ_Λ

$$F_s(\psi_\Lambda|m_{\Lambda^c};B)\ge F_s(\psi_\Lambda|m_{\Lambda^c}).$$

Now we can repeat the same arguments and with $h_B=s$ every where and we obtain

$$F_s(\psi_\Lambda|m_{\Lambda^c})\ge F_s(\bar\psi_\Lambda|m_{\Lambda^c}) \tag{5.15}$$

where $\bar\psi$ satisfies (5.13). To prove the last inequality in (5.14), let $\psi\in X^0_{\Lambda,m}$ and $\phi\in X^0_{\Lambda,n}$, then, for $r\in\Lambda$

$$|\phi(r)-\psi(r)|\le e^{-2\omega}\Big(\int_\Lambda dr' J(r,r')|\psi(r')-\phi(r')|+\int_\Lambda dr' J(r,r')|m(r')-n(r')|\Big) \tag{5.16}$$

where we have chosen ζ_0 so small that

$$e^{-2\omega}:=\frac{\beta}{\cosh^2\{\beta(m^+_{\beta,s}-2\zeta_0)\}}<1.$$

Calling n_0 the smallest integer larger or equal to $\mathrm{dist}(r,\Lambda^c_{\neq})$, by iterating (5.16) we get

$$|\phi(r)-\psi(r)|\le\sum_{n\ge n_0}e^{-2\omega n}2\le\Big(2\sum_{n\ge 0}e^{-\omega n}\Big)e^{-\omega n_0}$$

which yields (5.14) with $c_\omega:=2/(1-e^{-\omega})$ and $n(r)=m^+_{\beta,s}$. □

5.2 – Free energy of Contours

We call the triple (ζ, ℓ_-, ℓ_+) good if the following holds:

- The pair (ζ, ℓ_-) is good if $\zeta < \zeta_0/2$ and $\ell_- < \ell^\star(\zeta)$, $\ell^\star(\zeta) = \kappa_0\zeta$ with ζ_0 and κ_0 as in Theorem 5.5.4.
- The triple (ζ, ℓ_-, ℓ_+) is good if he pair (ζ, ℓ_-) is good, $\ell_+ > 100$ and
$$c\ell_-^d\zeta^2 \geq 2^{d+3}\ell_+^d[c_\omega e^{-\ell_+\omega/6}]^2$$
with c a suitable positive constant and ω and c_ω as in Theorem 5.5.4.

The contours of a profile m relative to the parameters (ζ, ℓ_-, ℓ_+), are the pairs $\Gamma = (sp(\Gamma), \eta_\Gamma)$, where $sp(\Gamma)$, the spatial support of Γ, is a maximal connected component of $\{r \in \mathbb{R}^d \ : \ \Theta^{(\zeta,\ell_-,\ell_+,s)}(m, B; \cdot) = 0\}$ and η_Γ is the restriction of $\eta^{(\zeta,\ell)}(m; r)$ to $sp(\Gamma)$. Γ is a bounded contour if $sp(\Gamma)$ is bounded. If Γ is bounded we set

$$K = \delta_{in}^{\ell_+}[sp(\Gamma)], \qquad A = \delta_{out}^{\ell_+}[sp(\Gamma)] \tag{5.17}$$

K is the "safety zone" of Γ.

A_0 is the maximal connected component of A contained in the unbounded component of $sp(\Gamma)^c$. K^0 the maximal connected component of K which is connected to A_0; $\eta_\Gamma \equiv 1$ or $\eta_\Gamma \equiv -1$ on K_0; in the former case Γ is a + contour, in the latter a $-$ contour. The othe maximal connected components of K, if they exist, are denoted by $K_i^\pm$ $i = 1, .., n_\pm$, labelled so that $\eta_\Gamma = 1$ on K_i^+ and $\eta_\Gamma = -1$ on K_i^-. The maximal connected component of A connected to $K_i^\pm$ is called $A_i^\pm$. The maximal connected component of $sp(\Gamma)$ which contains $A_i^\pm$ is called $int_i^\pm(\Gamma)$ and we write

$$\begin{aligned} int^\pm(\Gamma) &= \bigcup_{i=1}^{n_\pm} int_i^\pm(\Gamma), \\ int(\Gamma) &= int^+(\Gamma) \cup int^-(\Gamma), \\ C(\Gamma) &= int(\Gamma) \cup sp(\Gamma) \end{aligned} \tag{5.18}$$

in the sequel we will choose ℓ_- "very small" and ℓ_+ very large, so that a correct point r is always inside a "large" region, where $\eta^{(\zeta,\ell_-)}(m; \cdot)$ is constantly equal to 1 or -1. At the same time, the region of correct points and the red zone where the deviations from equilibrium are localized, are separated by the safety zone, where $\eta^{(\zeta,\ell)}(m; r)$ has a constant non zero value.

Theorem 5.5.5. *Let (ζ, ℓ_-, ℓ_+) be good, $m \in L^\infty(\mathbb{R}^d, [-1, 1])$, $s^\star$ as in Lemma 5.5.2, $B \in BV(\mathcal{T})$ and Γ a (ζ, ℓ_-, ℓ_+), + bounded contour for m, then for any $s < s^\star$ there is $\psi \in L^\infty(\mathbb{R}^d, [-1, 1]$ equal to m on $C(\Gamma)^c$, to $m_{\beta,s}^+$ on $C(\Gamma)\backslash K_0$ and with ψ with values in $[m_{\beta,s}^+ - \zeta + \varepsilon, m_{\beta,s}^+ + \zeta - \varepsilon]$ on K_0 such that*

$$F_s\left(m_{C(\Gamma)}|m_{C(\Gamma)^c}; B\right) \geq F_s\left(\psi_{C(\Gamma)}|\psi_{C(\Gamma)^c}\right) \tag{5.19}$$

Proof. We need to prove that

$$F_s\,(m;B) \geq F_s\,(\psi)\,. \tag{5.20}$$

Let Σ_0 be a $\mathcal{D}^{(\ell'_-)}$-measurable circuit contained in K_0 whose complement is made of two unconnected components at mutual distance ≥ 1, calling $ext(\Sigma_0)$ the one which contains A_0. We also suppose that Σ_0 has distance $\leq \ell'/3$ from $S_0 := \delta^1_{in}[K_0]$. By Theorem 5.5.4 applied to $K_0\backslash S_0$ with boundary conditions the restriction of m to S_0 there is ϕ equal to m outside $K_0\backslash S_0$, which, on $K_0\backslash S_0$ has values in $[m^+_{\beta,s}-\zeta+\varepsilon, m^+_{\beta,s}+\zeta-\varepsilon]$, $\varepsilon = \varepsilon_0(\zeta')$ and such that

$$F_s(\phi_{K_0\backslash S_0}|m_{S_0}) \leq F_s(m_{K_0\backslash S_0}|m_{S_0};B)$$

$$|\phi(r) - m^+_{\beta,s}| \leq c_\omega e^{-\omega\ell'/3} \quad \text{on } \Sigma_0 \tag{5.21}$$

setting $\Delta = \Sigma_0 \cup ext(\Sigma_0)$, we have

$$F_s(\phi) = F_s(\phi_{\Delta^c}|\phi_\Delta) + F_s(\phi_\Delta) \geq F_s(\phi_\Delta).$$

Set $\psi = \phi$ on Δ and equal to $m^+_{\beta,s}$ on Δ^c, we are going to prove that

$$F_s(\phi_{\Delta^c}|\phi_\Delta) \geq F_s(\psi_{\Delta^c}|\psi_\Delta). \tag{5.22}$$

Indeed, since $F_s(\psi_{\Delta^c}) = 0$, we have

$$\begin{aligned} F_s(\psi_{\Delta^c}|\psi_\Delta) &= \frac{1}{2}\int_{\Sigma_0} dr \int_{\Delta^C} dr' J(r,r')(\phi(r) - m^+_{\beta,s})^2 \leq \\ &\leq F_s(\phi_{\Delta^c}) + \frac{1}{2}\int_{\Delta} dr \int_{\Delta^C} dr' J(r,r')(\phi(r) - \phi(r'))^2. \end{aligned} \tag{5.23}$$

The last inequality follows from the fact that the interaction between ψ_Δ and ψ_{Δ^c} is very small, i.e.

$$\int_{\Sigma_0} dr \int_{\Delta^C} dr' J(r,r')(\phi(r) - m^+_{\beta,s})^2 \leq \frac{|\Sigma_0|}{2}[c_\omega e^{-\ell'_+/3}]^2.$$

Then, since $F_s(\phi_\Delta) = F_s(\psi_\Delta)$ and from (5.23)

$$F_s(\phi) = F_s(\phi_\Delta) + F_s(\phi_{\Delta^c}|\phi_\Delta) \geq F_s(\psi_\Delta) + F_s(\psi_{\Delta^c}|\psi_\Delta) = F_s(\psi)$$

and then the theorem is proved. □

In the proof of Theorem 3.3.4 we use the following Corollary, whose for brevity we omit the proof.

Corollary 5.5.6. *Let* (ζ, ℓ_-, ℓ_+) *be good,* $s^\star$ *as in Lemma* 5.5.2, Λ *and* $\Delta \subset \Lambda$ *two bounded,* $\mathcal{D}^{(\ell_+)}$*-measurable regions;* $m \in L^\infty(\mathbb{R}^d, [-1,1])$ *with* $\eta^{(\zeta,\ell_-)}(m; r) = 1$, $r \in \delta^{\ell_+}_{out}[\Lambda] \cup \delta^{\ell_+}_{in}[\Lambda]$, $B \in BV(\mathcal{T})$ *with* $|C^{(\ell_-)}_r \cap (C^{(\ell_-)}_r \vartriangle B^c)| \le \zeta \ell^d_-$, $r \in \delta^{\ell_+}_{out}[\Lambda] \cup \delta^{\ell_+}_{in}[\Lambda]$. *Then there is a* $\phi \in L^\infty(\mathbb{R}^d, [-1,1])$ *so that* $\phi = m$ *on* Λ^c, $\phi = m^+_{\beta,s}$ *on* Δ, $\eta^{(\zeta,\ell_-)}(\phi; r) = 1$ *on* Λ *and for any* $s < s^\star$, *calling*

$$\delta\Delta = \{r \in \Delta : dist(r, \Delta^c) \le 1\}, \quad \Lambda^c_{\neq} = \{r \in \Lambda^c, m(r) \neq m^+_{\beta,s}, \ \mathrm{dist}(r,\Lambda) \le 1\}$$

$$F_s\left(m_\Lambda | m_{\Lambda^c}; B\right) \ge F_s\left(\phi_\Lambda | \phi_{\Lambda^c}\right) - \left(2c\omega \exp^{\omega} |\delta\Delta|\right) \exp^{-\omega\, \mathrm{dist}(\Delta, \Lambda^c_{\neq})} \tag{5.24}$$

6 – The surface tension

In this section we prove (2.5), Proposition 3.3.3 and Theorem 3.3.4.

Now we prove that

$$\lim_{s\to 0} \theta_{\beta,s} = \liminf_{k\to\infty} \lim_{s\to 0} \liminf_{L\to\infty} \theta_{\beta,s}(L,k) = \theta_\beta(\nu) \tag{6.1}$$

which clearly implies (2.5). We observe that (6.1) shows also that it is possible to obtain the same value by taking limits in the reverse order. To simplify the notation we omit the dependence on β writing θ_s instead of $\theta_{\beta,s}$. First of all we want to prove that

$$\lim_{s\to 0} \theta_s(L,K) = \theta(L,K) := \inf_{m \in L^\infty(R(L,K);[-1;1])} F\left(m_{R(L,k)} | \chi_{R(L,k)^c}\right). \tag{6.2}$$

Let m and B be so that

$$L^{d-1}\theta_s(L,k) = F_s\left(m_{R(L,k)} | \chi_{R(L,k)^c}; B\right).$$

Then for s small enough there exists ε

$$F_s\left(m_{R(L,k)} | \chi_{R(L,k)^c}; B\right) \ge F\left(m_{R(L,k)} | \chi^0_{R(L,k)^c}\right) - \varepsilon \ge L^{d-1}\theta(L,K) - \varepsilon$$

where $\chi^0(r) = m_\beta \mathbf{1}_{x_d \ge 0} - m_\beta \mathbf{1}_{x_d \le 0}$.

On the other hand let $\tilde m$ such that

$$L^{d-1}\theta(L,k) = F\left(\tilde m_{R(L,k)} | \chi^0_{R(L,k)^c}\right).$$

By straigthforward computation it is possible to show that

$$|F\left(\tilde{m}_{R(L,k)}|\chi^0_{R(L,k)^c}\right) - F_s\left(\tilde{m}_{R(L,k)}|\chi^0_{R(L,k)^c};B\right)| \le sKL^{d-1} \tag{6.3}$$

and

$$F_s\left(\tilde{m}_{R(L,k)}|\chi^0_{R(L,k)^c};B\right) \ge F_s\left(\tilde{m}_{R(L,k)}|\chi_{R(L,k)^c};B\right) + s^2L^d. \tag{6.4}$$

Then, using (6.3) and (6.4)

$$\theta(L,K) \ge \lim_{s\to 0}\theta_s(L,K).$$

Hence (6.2) is proved.

The next step is

$$\lim_{s\to 0}\theta_s \le \liminf_{k\to\infty}\lim_{s\to 0}\liminf_{L\to\infty}\theta_s(L,k) \le \liminf_{k\to\infty}\liminf_{L\to\infty}\theta(L,k). \tag{6.5}$$

It is easy to check that $\theta_s(L,k)$ is a non increasing function of K, i.e.

$$\liminf_{k\to\infty}\theta_s(L,k) = \inf_K \theta_s(L,k) := \theta_s(L).$$

This implies that

$$\theta_s(L) \le \theta_s(L,k) \qquad \liminf_{L\to\infty}\theta_s(L) \le \liminf_{L\to\infty}\theta_s(L,k).$$

By letting first $s\to 0$ and then $k\to\infty$ we obtain the first inequality in (6.5). The last inequality follows from (6.2) and by letting the limits in the following order: first $L\to\infty$, $s\to 0$ and then $k\to\infty$. Using again (6.2) we can obtain

$$\liminf_{k\to\infty}\liminf_{L\to\infty}\theta(L,k) \le \lim_{s\to 0}\theta_s \tag{6.6}$$

that together with (6.5) completes the proof of (2.5).

6.1 – Proof of Proposition 3.3.3

By definition of χ,

$$\int_{C(L)}|m(r)-\chi(r)|dr = \int_{C^-(L)}|m(r)-m^-_{\beta,s}|dr + \int_{C^+(L)}|m(r)-m^+_{\beta,s}|dr.$$

We define

$$A_\zeta = \left\{r\in C^-(L) \text{ s.t. } |m(r)-m^-_{\beta,s}| \le \zeta\right\}$$

and A_η the analogous on $C^+(L)$. Then

$$\int_{C(L)} |m(r)-\chi)|dr \le (\zeta+\eta)L^d + \int_{A^c_\zeta} |m - m^-_{\beta,s}|dr + \int_{A^c_\eta} |m - m^+_{\beta,s}|dr.$$

In A^c_ζ we have that

$$\int_{A^c_\zeta} |m - m^-_{\beta,s}|dr \le \frac{c}{(\zeta^2\wedge h)}\int_{A^c_\zeta} f^-_h(m) \le \frac{c}{(\zeta^2\wedge s)}\delta L^d.$$

With same arguments in A^c_η we obtain

$$\int_{C(L)} |m(r)-\chi(r)|dr \le (\zeta+\eta)L^d + \frac{c}{(\zeta^2\wedge s)}\delta L^d + \frac{c}{(\eta^2\wedge s)}\delta L^d \le \delta' L^d.$$

6.2 – Proof of the Theorem 3.3.4

We define the i-th layer, $i \in \mathbb{Z}$,

$$S_i = \{x \in C(L) : (x_d - \ell_+ i) \in [-\ell_+/2, \ell_+/2)\}. \tag{6.7}$$

Let

$$N = \min\{n \in \mathbb{N} : 2n\ell_+ \ge \sqrt{\delta}L\}. \tag{6.8}$$

Supposing $\sqrt{\delta}$ small enough, we define, for any $1 \le n \le N$

$$\Sigma_n := S_{2n-1} \cup S_{2n} \cup S_{2n-1+2N} \cup S_{2n+2N}$$

and in the same way Σ_{-n}, observing that $|\Sigma_n \cup \Sigma_{-n}| = 8|S_0|$.

We will use the estimates of Section 5 in the boxes delimited by Σ_n and Σ_{-n} respectively, to conclude that in the center layer of the boxes we can replace m by $m^+_{\beta,s}$ and $m^-_{\beta,s}$, and h by $\pm s$. Let

$$a_n = \frac{1}{8|S_0|}\left\{\int_{\Sigma_n\cup\Sigma_{-n}} dr|m-\chi| + \int_{\Sigma_n\cup\Sigma_{-n}} dr|h_B - \tilde{\chi}|\right\}. \tag{6.9}$$

Where $\tilde{\chi}$ is defined as (3.13) with $C^\pm(L)$ instead of $R^\pm_i$. Then

$$a = \min_{n\le N} a_n \le C\sqrt{\delta}. \tag{6.10}$$

In fact, by assumption

$$
\begin{aligned}
3\delta L^d &\geq \int_{C(L)} dr|m-\chi| + \int_{C(L)} dr|h_B - \tilde{\chi}| \geq \\
&\geq \sum_{i=1}^{N} 8|S_0|a_n \geq 8|S_0|aN = 8L^{d-1}N\ell^+ a \geq 4\sqrt{\delta}L^d a
\end{aligned}
$$

which proves (6.10).

Call n the integer where the minimum in (6.10) is achieved. Now we are going to use the analysis of Section 5. We shorthand $\eta(\cdot;\cdot)$ for $\eta^{(\zeta,\ell_-)}(\cdot;\cdot)$ and we define:

- $C^0(L)$ is the union of all cubes $C \in \mathcal{D}^{(\ell_+)}$ such that both C and $\delta_{out}^{\ell_+}[C]$are in $C(L)$.
- $\mathcal{M}_n$ is the union of all cubes $C \in \mathcal{D}^{(\ell_-)}$ contained in Σ_n where $\eta(m;\cdot) < 1$ and $C \cap B \neq 0$, of those in Σ_{-n} where $\eta(m;\cdot) > -1$ and $C \cap B^c \neq 0$ and of the set
$$\delta_{out}^{\ell_+}[C^0(L)] \sqcap \{\sqcup_{|j|\leq 4N} S_j\}.$$

We want to estimate the free energy cost changing m and h_B into new functions ϕ and $\tilde{h}_B$ set respectively equal to χ and $\tilde{\chi}$ on $\mathcal{M}_n$ and unchanged everywhere else. We need an estimate on the volume $|\mathcal{M}_n|$. It's easy to prove that for a suitable constant c the following estimate holds:

$$|\mathcal{M}_n| \leq c\sqrt{\delta}L^{d-1}. \tag{6.11}$$

Then there is a constant $c_0 > 0$ so that

$$F_s(m_{C(L)}|m_{C(L)^c};B) \geq F_{\tilde{s}}(\phi_{C(L)}|\phi_{C(L)^c};B) - c_0|\mathcal{M}_n|. \tag{6.12}$$

Indeed the first term in the functional does not increase when replacing m by ϕ and the other changes are proportional to the volume where have been made.

Recalling the definition of $C^0(L)$ and since

$$F_{\tilde{s}}(\phi_{C(L)}|\phi_{C(L)^c};B) \geq F_{\tilde{s}}(\phi_{C^0(L)}|\phi_{C^0(L)^c};B).$$

We have

$$F_s(m_{C(L)}|m_{C(L)^c};B) \geq F_{\tilde{s}}(\phi_{C^0(L)}|\phi_{C^0(L)^c};B) - C_0c\sqrt{\delta}L^{d-1}. \tag{6.13}$$

Let then Λ_+ be the box in $C^0(L)$ union of all $S_j \sqcap C^0(L)$ with $2n < j \leq 2n+2N-1$ and let Λ_- be its reflection around $x_d = 0$. We are going to apply the Corollary 5.5.6 with $\Lambda = \Lambda_+$ and $\Delta = S_{2n+N} \sqcap C^0(L)$ and then with their images under reflecion around $x_d = 0$. By symmetry we only consider the former and drop the sub fix $+$. The hypotheses of Corollary are here met because:

$$\eta(\phi; r) = 1,\ r \in \delta^{\ell_+}_{out}[\Lambda] \qquad \Lambda^c_{\neq} \subset \{S_{2n-1} \cup S_{2n+2N}\} \tag{6.14}$$

thus

$$\mathrm{dist}(\Delta, \Lambda^c_{\neq}) \geq \ell_+ N/2. \tag{6.15}$$

There is ψ equal to ϕ outside $\Lambda_\pm$ and equal to χ on $S_{2n+N} \sqcap C^0(L)$ and $S_{-2n-N} \sqcap C^0(L)$ such that

$$F_{\tilde{s}}(\phi_{C^0(L)}|\phi_{C^0(L)^c}; B) \geq F_{\tilde{s}}(\psi_{C^0(L)}|\psi_{C^0(L)^c}; B) - (2c_\omega e^\omega |S_0|) e^{-\omega \ell_+ N/2}. \tag{6.16}$$

Setting

$$U := \bigcup_{|j|<2n+N} \{S_j \sqcap C^0(L)\}. \tag{6.17}$$

We get

$$F_{\tilde{s}}(\psi_{C^0(L)}|\psi_{C^0(L)^c}; B) \geq F_{\tilde{s}}(\psi_U|\psi_{U^c}; B) = F_{\tilde{s}}(\psi_U|\chi_{U^c}; B) \tag{6.18}$$

U is a rectangle whose basis is a cube of side b, $L \geq b \geq L - 2\ell_+$; denoting by k the height of U we then have, recalling (3.12),

$$F_{\tilde{s}}(\psi_U|\chi_{U^c}; B) \geq b^{d-1} \theta_{\beta,s}(b,k). \tag{6.19}$$

Given $\varepsilon > 0$, we may choose $L_\varepsilon > 0$ so large that $\theta_{\beta,s}(b,k) > \theta_{\beta,s}(k) - \varepsilon/2$, and by letting $k \to \infty$ and using (6.13) we obtain (3.18).

Acknowledgements

We are indebted to A. De Masi and E. Presutti for helpful discussions and comments.

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Lavoro pervenuto alla redazione il 29 agosto 2003
ed accettato per la pubblicazione il 8 aprile 2004.
Bozze licenziate il 14 febbraio 2005

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