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Ekeland sequences compact in L^∞

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Abstract: *In this paper we use Ekeland's ε-variational principle and we prove that, for some integral functionals, there exists a minimizing sequence compact in L^∞.*

1 – Introduction

Let us recall Ekeland's ε-variational principle (see [8, 9]).

Lemma 1.1. *Let (V,d) be a complete metric space, and let $\mathcal{F} : V \to (-\infty, +\infty]$ be a lower semicontinuous function such that $\inf_V \mathcal{F}$ is finite. Let $\varepsilon > 0$ and $u \in V$ be such that*

$$\mathcal{F}(u) \leq \inf_{v\in V} \mathcal{F}(v) + \varepsilon\,.$$

Then there exists $v \in V$ such that

(i) $d(u,v) \leq \sqrt{\varepsilon}$;
(ii) $\mathcal{F}(v) \leq \mathcal{F}(u)$;
(iii) *v minimizes the functional* $\mathcal{G}(w) = \mathcal{F}(w) + \sqrt{\varepsilon}\, d(v,w)$.

Many papers used the above ε-variational principle in many different frameworks: it is impossible to quote all of them; we only recall [1, 10, 11, 12, 17, 18, 20] (and the references therein) and the papers referred below.

In this paper we carry on the study of additional properties of the Ekeland sequences (begun in [5] and developed in [7]), in the case of functionals defined through multiple integrals.

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In [5], we considered some properties of the minimizing sequences for integral functionals J. Thanks to the Ekeland Lemma, we proved the existence of a minimizing sequence compact in $L^s(\Omega)$ or in $C^{0,\alpha}$ for functionals which do not need to have a minimum, without using the integral representation of the relaxed functional J^*.

In [7], we improved the study done in the paper [5], under the assumption that the functional J has a minimum belonging to $L^\infty(\Omega)$. Using again Ekeland's ε-variational principle, we proved that there exists a minimizing sequence uniformly converging to a minimum u.

In this paper, we prove similar results if the coefficients satisfy a control assumption (quite natural and introduced in [2]; see (2.3) below).

2 – Setting and statement of the result

Let Ω be an open, bounded subset of $\mathbb{R}^N$, $N \geq 2$, and let p be a real number, with $2 \leq p < N$. We will denote by p^* the Sobolev exponent of p: $p^* = \frac{Np}{N-p}$.

Let $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function (i.e., measurable with respect to x for every $\xi \in \mathbb{R}^N$, and continuous with respect to ξ for almost every $x \in \Omega$) convex with respect to ξ, and such that

$$\alpha\, |\xi|^p \leq j(x,\xi) \leq \beta\, |\xi|^p\,, \tag{2.1}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where α, β are positive real numbers. Let $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$J(v) = \int_\Omega j(x, \nabla v) + \frac{1}{r}\int_\Omega b(x)|v|^r - \int_\Omega f(x)\, v, \qquad v \in W_0^{1,p}(\Omega)\,,$$

where

$$r > 1, \tag{2.2}$$

and the coefficient $b(x)$ and the datum $f(x)$, belonging to $L^1(\Omega)$, satisfy the dominated assumption

$$\text{there exists } Q \in \mathbb{R}^+ \text{ such that } \ |f(x)| \leq \ Q\, b(x). \tag{2.3}$$

Note that (2.1) implies

$$J(v) \geq \alpha \int_\Omega |\nabla v|^p + \frac{1}{r}\int_\Omega b(x)|v|^r - \int_\Omega f(x)\, v(x).$$

First of all, we point out that the assumptions (2.3) and (2.2) imply

$$\left[\begin{array}{l} \displaystyle\frac{1}{r}\int_\Omega b(x)|v|^r - \int_\Omega f(x)\, v \geq \frac{1}{r}\int_\Omega b(x)|v|^r - \int_\Omega Q\, b(x)\, |v| \\ \displaystyle = \int_{|v|^{r-1}\geq Q} b(x)\big[|v|^{r-1} - Q\big]|v| + \int_{|v|^{r-1}<Q} b(x)\big[|v|^{r-1} - Q\big]|v|, \end{array}\right. \tag{2.4}$$

where $b(x)\big[|v|^{r-1}-Q\big]|v|$ is positive on the set $\{|v|^{r-1}\geq Q\}$ and $b(x)\big[|v|^{r-1}-Q\big]|v|$ belongs to L^1 on the set $\{|v|^{r-1}<Q\}$.

Thus, under the above assumptions J is well defined on $W_0^{1,p}(\Omega)$ even though f is only in $L^1(\Omega)$; possibly with value $+\infty$.

Since J is both weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$, there exists a minimum u of J.

Moreover the assumption (2.1) and the inequality (2.4) say that the functional J is strongly lower semicontinuous in $W_0^{1,1}(\Omega)$, so that it is possible to apply the Ekeland Lemma 1.1.

Also we have the following results on the summability of minima of J.

Proposition 2.1. *Let u be a minimum of J on $W_0^{1,p}(\Omega)$. Then*

(i) *in* [19] (*see also* [14]) *is proved that, if $b(x)=0$, $f\in L^m(\Omega)$, $m>\frac{N}{p}$, then u belongs to $L^\infty(\Omega)$;*

(ii) *in* [4] *is proved that, if $b(x)=0$, $f\in L^m(\Omega)$, $1<m<\frac{N}{p}$, then u belongs to $L^\sigma(\Omega)$, $\sigma=\frac{(pm)^*}{p'}$;*

(iii) *in* [2] *is proved that, under the assumptions* (2.2) *and* (2.3), *$u\in L^\infty(\Omega)$, with the explicit bound*

$$|u(x)|\leq Q^{\frac{1}{r-1}}. \tag{2.5}$$

Remark 2.2. Let $\{\bar u_n\}$ be a minimizing sequence. Recall the definitions ($k>0$)

$$G_k(s)=(|s|-k)^+\,\frac{s}{|s|},\qquad T_k(s)=s-G_k(s)\,.$$

Before proving Theorem 2.3, note that, under the assumptions (2.1), (2.2), (2.3), since we have estimate (2.5), the sequence $\{u_n\}$, with $u_n=T_M(\bar u_n)$, $M=Q^{\frac{1}{r-1}}$, $\varepsilon_n\to 0$, satisfies

$$\begin{aligned}&\int_\Omega j(x,\nabla T_M(\bar u_n))+\frac1r\int_\Omega b(x)|T_M(\bar u_n)|^r-\int_\Omega f(x)\,T_M(\bar u_n)\\ &\leq \inf_{v\in W_0^{1,p}(\Omega)}J(v)+\varepsilon_n+\frac1r\int_\Omega b(x)[|T_M(\bar u_n)|^r-|\bar u_n|^r]+\int_\Omega f(x)\,G_M(\bar u_n)\,,\\ &\leq \inf_{v\in W_0^{1,p}(\Omega)}J(v)+\varepsilon_n+\int_\Omega f(x)\,G_M(\bar u_n)\,,\end{aligned}$$

and since

$$\lim_{n\to+\infty}\int_\Omega f(x)\,G_M(\bar u_n)=\int_\Omega f(x)\,G_M(u)=0\,,$$

we have that

$$\int_\Omega j(x,\nabla u_n)+\frac1r\int_\Omega b(x)|u_n|^r-\int_\Omega f(x)\,u_n\leq \inf_{v\in W_0^{1,p}(\Omega)}J(v)+\bar\varepsilon_n\,.$$

That is, the sequence $\{u_n\}$ is a minimizing sequence for J, and it is bounded in $L^\infty(\Omega)$: $|u_n(x)| \le M$.

We assume that

$$\text{there exists } a(x,\xi) = j_\xi(x,\xi), \text{ which satisfies the Leray-Lions assumptions} \tag{2.6}$$

(see [16]); that is $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that the following holds for almost every $x \in \Omega$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x,\xi)\xi \ge \alpha\,|\xi|^p\,, \\ |a(x,\xi)| \le \beta|\xi|^{p-1}\,, \\ [a(x,\xi) - a(x,\eta)](\xi-\eta) > 0\,, \end{cases} \tag{2.7}$$

where α, β are positive constants. Then the minimum $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of the Euler-Lagrange equation

$$\int_\Omega a(x,\nabla u)\nabla\varphi + \int_\Omega b(x)\,u|u|^{r-2}\varphi = \int_\Omega f(x)\,\varphi, \quad \forall\,\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{2.8}$$

With respect to the above Remark 2.2, if we assume (2.6), in Theorem 2.3, we prove more (as in [7], even with different assumptions): thanks to the ε-variational principle, it is possible to build a minimizing sequence not only bounded in $L^\infty(\Omega)$, but also strongly convergent to u in the same space.

Our main result is the following.

THEOREM 2.3. *We assume (2.1), (2.2), (2.3). Let u be a minimum of J on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let $\{\bar u_n\}$ be any minimizing sequence for J,*

$$\|\bar u_n\| \le M = Q^{\frac{1}{r-1}}, \text{ thanks to Remark 2.2.} \tag{2.9}$$

Then the minimizing sequence $\{u_n\}$ built after $\{\bar u_n\}$ using the ε-variational principle satisfies

$$\lim_{n\to\infty} \|u_n - u\|_{W_0^{1,p}(\Omega)} = 0\,, \tag{2.10}$$

and

$$\text{the sequence } \{u_n\} \text{ is compact in } L^\infty(\Omega)\,. \tag{2.11}$$

PROOF.

STEP 1. Let ε_n be a sequence of positive real numbers, converging to zero, and let $\bar u_n$ be such that, for every $n \in \mathbb{N}$,

$$J(\bar u_n) \le \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n\,.$$

Let us now consider the complete metric space $W_0^{1,1}(\Omega)$, endowed with the distance

$$d_n(w,v) = \frac{1}{\sqrt{\varepsilon_n}} \int_\Omega |\nabla w - \nabla v| \,.$$

Thanks to Fatou lemma, to the fact that $j(x,\xi) \geq 0$ and to (2.4), we have that J is strongly lower semicontinuous on $W_0^{1,1}(\Omega)$. Thus, in view of Lemma 1.1, there exists a sequence $\{u_n\}$ in $W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |\nabla u_n - \nabla \bar{u}_n| \leq \sqrt{\varepsilon_n} \,,$$

and

$$J(u_n) \leq J(\bar{u}_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n \,, \tag{2.12}$$

$$J(u_n) \leq J(w) + \sqrt{\varepsilon_n} \int_\Omega |\nabla u_n - \nabla w| \,, \qquad \forall w \in W_0^{1,1}(\Omega) \,. \tag{2.13}$$

Using the growth properties of J we now prove that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By (2.1), we have (recall (2.4))

$$\begin{aligned} &\int_\Omega j(x,\nabla u_n) + \int_{\{|u_n|^{r-1} < Q\}} \left[\frac{1}{r} b(x)|u_n|^r - f(x)\, u_n \right] \\ &\leq \int_\Omega j(x,\nabla u_n) + \frac{1}{r} \int_\Omega b(x)|u_n|^r - \int_\Omega f(x)\, u_n. \end{aligned}$$

Thus

$$\alpha \int_\Omega |\nabla u_n|^p \leq J(u_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + \int_{\{|u_n|^{r-1} < Q\}} \left[f(x)\, u_n - \frac{1}{r} b(x)|u_n|^r \right].$$

and

$$\alpha \int_\Omega |\nabla u_n|^p \leq J(u_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + C_Q,$$

which implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus $\|u_n\|_{W_0^{1,p}(\Omega)} \leq R$ and, up to subsequences, still denoted by $\{u_n\}$, there exists a function u in $W_0^{1,p}(\Omega)$ such that

$$u_n \to u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and almost everywhere in } \Omega. \tag{2.14}$$

By the weak lower semicontinuity of J on $W_0^{1,p}(\Omega)$, and by (2.12), u is a minimum of J on this space.

Now we follow a classic method by I. Ekeland and we use (2.6). Choosing $w = u_n - t\,\psi$ in (2.13), where t is a positive real number and ψ is a function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we obtain

$$J(u_n - t\,\psi) - J(u_n) + \sqrt{\varepsilon_n}\, t \int_\Omega |\nabla \psi| \geq 0\,.$$

Dividing by t, and letting t tend to zero, we get, since J is differentiable,

$$-\langle J'(u_n), \psi\rangle + \sqrt{\varepsilon_n} \int_\Omega |\nabla \psi| \geq 0\,,$$

so that

$$\langle J'(u_n), \psi\rangle \leq \sqrt{\varepsilon_n} \int_\Omega |\nabla \psi|\,. \tag{2.15}$$

Recalling that $J'(u) = 0$ since u is a minimum, we have

$$\langle J'(u_n) - J'(u), \psi\rangle \leq \sqrt{\varepsilon_n} \int_\Omega |\nabla \psi|\,,$$

for every ψ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Observe that

$$\langle J'(u_n), \psi\rangle = \int_\Omega a(x, \nabla u_n)\nabla \psi - \int_\Omega f(x)\psi, \tag{2.16}$$

that is

$$\langle J'(u_n) - J'(u), \psi\rangle = \int_\Omega [a(x, \nabla u_n) - a(x, \nabla u)]\nabla \psi + \int_\Omega b(x)(u_n|u_n|^{r-2} - u|u|^{r-2})\psi.$$

Choosing $\psi = T_k(u_n - u)$, and using the fact that $s \mapsto s|s|^{r-2}$ is monotone, we have

$$\left[\begin{array}{l} \displaystyle\int_{|u_n - u| \leq k} [a(x, \nabla u_n) - a(x, \nabla u)]\nabla(u_n - u) \\ \displaystyle\leq \sqrt{\varepsilon_n} \int_\Omega |\nabla T_k(u_n - u)| \leq \sqrt{\varepsilon_n} \int_\Omega |\nabla(u_n - u)| \leq C_R\sqrt{\varepsilon_n}\,. \end{array}\right.$$

Here we use the Fatou lemma, as $k \to \infty$, and we obtain

$$\int_\Omega [a(x, \nabla u_n) - a(x, \nabla u)]\nabla(u_n - u) \leq C_R\sqrt{\varepsilon_n}\,.$$

A result by J. Leray and J.-L. Lions says that the above limit implies that

$$\nabla u_n(x) \text{ converges a.e. to } \nabla u(x). \tag{2.17}$$

Then in [6] is proved that, under our assumption on the function $a(x,\xi)$, this a.e. convergence implies (2.10).

STEP 2. Let $0 \le \zeta \le 1$ be a smooth function and let $\varphi_k(t) = T_1[G_k(t)]$, with $k \ge M = Q^{\frac{1}{r-1}}$. Choose $\psi = \varphi_k(u_n)\,\zeta^p$ in (2.15); then we have

$$\left[\begin{array}{l} \displaystyle\int_\Omega a(x,\nabla u_n)\nabla[\varphi_k(u_n)\,\zeta^p] + \int_\Omega b(x)\,u_n|u_n|^{r-2}\varphi_k(u_n)\,\zeta^p - \int_\Omega f(x)\varphi_k(u_n)\,\zeta^p \\ \displaystyle\le \sqrt{\varepsilon_n}\int_\Omega |\nabla[\varphi_k(u_n)\,\zeta^p]| \end{array}\right.$$

and, recalling (2.3), (2.4), (2.9),

$$\left[\begin{array}{l} \displaystyle\int_\Omega a(x,\nabla u_n)\nabla[\varphi_k(u_n)\,\zeta^p] \\ \displaystyle\le \sqrt{\varepsilon_n}\int_\Omega |\nabla[\varphi_k(u_n)\,\zeta^p]| + \int_{\{|u_n|^{r-1}<Q\}} b(x)[Q-|u_n|^{r-1}]|\varphi_k(u_n)|\,\zeta^p \\ \displaystyle= \sqrt{\varepsilon_n}\int_\Omega |\nabla[\varphi_k(u_n)\,\zeta^p]|, \end{array}\right.$$

which implies (with $B_{n,k} = \{x\in\Omega : k \le |u_n(x)| < k+1\}$, $A_{n,k} = \{x \in \Omega : k \le |u_n(x)|\}$).

$$\left[\begin{array}{l} \displaystyle\alpha\int_{B_{n,k}} |\nabla u_n|^p\zeta^p \\ \displaystyle\le \sqrt{\varepsilon_n}\int_{B_{n,k}} |\nabla u_n|\,|\zeta^p| + p\int_\Omega |\nabla\zeta|\,|\varphi_k(u_n)|\,\zeta^{p-1} \\ \displaystyle+p\beta\int_\Omega (|\nabla u_n|\,\zeta)^{p-1}|\nabla\zeta||\varphi_k(u_n)| \end{array}\right.$$

and

$$\left[\begin{array}{l} \displaystyle\alpha\sum_{k=j}^\infty\int_{B_{n,k}} |\nabla u_n|^p\zeta^p \\ \displaystyle\le \sqrt{\varepsilon_n}\sum_{k=j}^\infty\int_{B_{n,k}} |\nabla u_n|\,\zeta^p + p\sum_{k=j}^\infty\int_\Omega |\nabla\zeta|\,|\varphi_k(u_n)|\,\zeta^{p-1} \\ \displaystyle+p\beta\sum_{k=j}^\infty\int_\Omega (|\nabla u_n|\,\zeta)^{p-1}|\nabla\zeta||\varphi_k(u_n)|\,. \end{array}\right.$$

Since

$$\sum_{k=j}^{+\infty}\varphi_k(t) = G_j(t)\,,$$

we have (note that $\nabla u_n\, G_j(u_n) = \nabla G_j(u_n)\, G_j(u_n)$ and use the Young inequality with $0 < B < \frac{\alpha}{2}$)

$$\begin{cases} \alpha \displaystyle\int_\Omega |\nabla G_j(u_n)|^p \zeta^p \\ \leq \sqrt{\varepsilon_n} \displaystyle\int_\Omega [|\nabla G_j(u_n)|\,\zeta]\zeta^{p-1} + p \int_\Omega |\nabla\zeta|\,|G_j(u_n)|\,\zeta^{p-1} \\ \quad + p\beta \displaystyle\int_\Omega (|\nabla G_j(u_n)|\,\zeta)^{p-1} |\nabla\zeta||G_j(u_n)|, \\ \leq B \displaystyle\int_\Omega |\nabla G_j(u_n)|^p\,\zeta^p + C_B(\sqrt{\varepsilon_n})^{p'} \int_{A_{n,j}} \zeta^p \\ \quad + C_1 \displaystyle\int_\Omega |\nabla\zeta|^p |G_j(u_n)|^p + C_2 \int_{A_{n,j}} \zeta^p \\ \quad + B \displaystyle\int_\Omega |\nabla G_j(u_n)|^p\,\zeta^p + C_B \int_\Omega |\nabla\zeta|^p |G_j(u_n)|^p. \end{cases}$$

Thus we have

$$(\alpha - 2B) \int_\Omega |\nabla G_j(u_n)|^p \zeta^p \leq (C_1 + C_B) \int_\Omega |\nabla\zeta|^p |G_j(u_n)|^p + (C_2 + C_B(\sqrt{\varepsilon_n})^{p'}) \int_{A_{n,j}} \zeta^p.$$

This estimate implies (see [13, 15]) that the sequence $\{u_n\}$ is bounded in the De Giorgi class $\mathcal{B}_2(\Omega, M)$; that is the functions u_n are equi-Hölder continuous in Ω and we proved the statement (2.11). □

REMARK 2.4. In [3] we presented a different method to prove the compactness in $W_0^{1,p}(\Omega)$ of minimizing sequences proved in (2.10).

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