Rendiconti di Matematica, Serie VII
Volume 27, Roma (2007), 1-16

The “form” of a triangle

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Abstract: *Heron's formula, maybe due to Archimedes, expresses the squared area of a triangle as a polynomial in the squared lengths of its sides. A true understanding of this formula comes from an adequate coding of the shape of a triangle, i.e. of the triangle up to translations, rotations and reflexions. One can then build a "space of triangle shapes" which, if orientation is added becomes the "shape sphere" and possesses very nice symmetries which were recently used in researches on the 3-body problem. The invariant interpretation of the shape sphere in turn sheds light on the nature of Heron's formula.*

This is the english version (with an appendix added) of a text in French which will appear in a book dedicated to the memory of Gilles Chatelet. The french title "la forme de n corps", has a pun, hard to render into English because it plays on the two meanings of the french word "forme": shape and quadratic form.

– Introduction: Heron's formula

The form is quadratic and the bodies are points, even if of celestial origin. At the beginning, there is the formula of Heron of Alexandria [H] for the area $|\Delta|$ of a triangle whose sides have lengths α, β, γ:

$$\text{(H)} \qquad 16|\Delta|^2 = (\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(\alpha-\beta+\gamma)(\alpha+\beta-\gamma)\,.$$

Key Words and Phrases: *Heron's formula – Volumes – Inertias – Dispositions – Invariants.*
A.M.S. Classification: 51K05 – 34C14 – 15A63 – 70E15

The last three factors are easily understood as they vanish when the triangle is flat (the second when the length of the longest side is α, the third if it is β, the fourth if it is γ). If one sets

$$a=\alpha^2,\ b=\beta^2,\ c=\gamma^2\,,$$

the formula becomes

$$16|\Delta|^2 = 2ab+2bc+2ca-a^2-b^2-c^2\,.$$

Surfaces and determinants have much in common and so it is not too astonishing to discover that this formula takes the form of the *Cayley determinant*

$$\text{(C)}\qquad -16|\Delta|^2=\det\begin{pmatrix}0&1&1&1\\1&0&c&b\\1&c&0&a\\1&b&a&0\end{pmatrix}.$$

Nevertheless, a 2×2 determinant would be less surprising[(1)] than a 4×4 determinant. What follows originates from work ([AC], see also [A3] and [C1]) done in collaboration with Alain Albouy on the *symmetries* of the *n-body Problem* and their *reduction*. That work generalizes Lagrange's fundamental memoir [L] to more than three bodies. In it one understands that the above determinant is simply a means of computing a subtler determinant, namely that of an endomorphism of a two-dimensional vector space which possesses no privileged basis. (Think of the plane of equation $x+y+z=0$ in $\mathbb{R}^3$. This plane is naturally equipped with the three lines of intersection with the coordinate planes $x=0,\ y=0$ and $z=0$ but certainly not with a *canonical* basis.) Choices or tricks are necessary to compute in such a plane.

1 – The "dispositions" and the reduction of translations

Let us consider a triangle in the plane $\mathbb{R}^2$ or the space $\mathbb{R}^3$. Its area is not affected by a rigid motion or by a symmetry. This is the fundamental property that we want to exploit. It explains why the area depends only on the lengths of the sides and not on the absolute positions of the vertices.

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(1) Indeed, if X is the matrix whose columns are the components of the two vectors $\vec{V},\vec{W}\in\mathbb{R}^2$, $\det X$ is the oriented area of the parallelogram generated by the two vectors and $\det{}^tXX$ is the squared area.

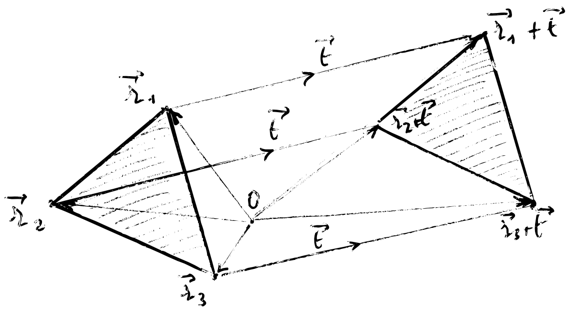

Fig. 1: Invariance of the area by translation.

The way to compute “up to translations” goes back to the works of Jacobi on the *n-body Problem* [J]. What follows is a formalization of his ideas. Giving n points $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ in the space $E \equiv \mathbb{R}^k, k = 1, 2, 3, \dots$ is the same as giving the linear mapping $X : \mathbb{R}^n \to E$ defined by $X(\xi_1, \xi_2, \dots, \xi_n) = \sum_{i=1}^n \xi_i \vec{r}_i$. In other words, one represents the “configuration” of n points (bodies) in E by the $k \times n$ matrix, whose i^{th} column consists of the coordinates of $\vec{r}_i$ in $E \equiv \mathbb{R}^k$. This representation was introduced by Alain Albouy in his PhD thesis [A1].

The source $\mathbb{R}^n$ of the mapping X represents the “side of the bodies”, the target E represents the “side of space”.

Now, let $\mathcal{D}^*$ be the subspace of $\mathbb{R}^n$ which consists of the n-tuples $(\xi_1, \xi_2, \dots, \xi_n)$ whose sum equals 0:

$$\mathcal{D}^* = \left\{ (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n,\ \sum_{i=1}^n \xi_i = 0 \right\} .$$

Whatever be $\vec{t} \in E$ and $(\xi_1, \xi_1, \dots, \xi_n) \in \mathcal{D}^*$, we have

$$\sum_{i=1}^n \xi_i (\vec{r}_i + \vec{t}) = \sum_{i=1}^n \xi_i \vec{r}_i \,.$$

Hence, the restriction x of X to $\mathcal{D}^*$ no longer distinguishes the two n-tuples $(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$ and $(\vec{r}_1 + \vec{t}, \vec{r}_2 + \vec{t}, \dots, \vec{r}_n + \vec{t})$. Yet, it gives the differences $\vec{r}_i - \vec{r}_j$ and hence the positions of the bodies once the position of one of them has been fixed. It follows that giving n points up to a translation in E is the same as giving the mapping $x : \mathcal{D}^* \to E$.

Such a mapping can be represented by the $k \times n$ matrix whose columns consist in the components of the vectors $\vec{r}_i \in \mathbb{R}^k$ but as well by the matrix whose columns consist in the components of the vectors $\vec{r}_i + \vec{t}$, where $\vec{t} \in \mathbb{R}^k$. It is only after we choose a basis of $\mathcal{D}^*$ that we can get a representation by a $(k-1) \times n$ matrix.

But where does $\mathcal{D}^*$ come from? One has to interpret the subspace $\mathcal{D}^*$ as the dual of the space $\mathcal{D}$ of n-tuples of points on the real line up to a translation. This latter space, called the *disposition space*, is the quotient of $\mathbb{R}^n$ by the line generated by the vector $(1,1,\dots,1)$: whatever be $t \in \mathbb{R}$, the n-tuples $(x_1,x_2,\dots,x_n)$ and $(x_1+t,x_2+t,\dots,x_n+t)$ represent the same element in $\mathcal{D}$. To $(\xi_1,\xi_2,\dots,\xi_n) \in \mathcal{D}^*$ and $(x_1,x_2,\dots,x_n)$ representing an element of $\mathcal{D}$, the duality associates the well defined real number $\sum \xi_i x_i = \sum \xi_i(x_i+t)$. A homomorphism $x : \mathcal{D}^* \to E$ may then be interpreted as an element of the tensor product $\mathcal{D} \otimes E$. The representation of x by an equivalence class of $k \times n$ matrices corresponds to the one of $\mathcal{D} \otimes E$ as the quotient of $\mathbb{R}^n \otimes E \equiv E^n$ by the diagonal action of the translations in E.

2 – From the "side of space" to the "side of the bodies": the reduction of rotations and symmetries

The invariance under translations of the area $|\Delta|$ means that it depends only on x; its invariance under linear isometries (i.e. rotations and symmetries with respect to a vector subspace) means that it depends in fact only on the Gram matrix, whose coefficients are the scalar products $\langle \vec{r}_i, \vec{r}_j \rangle_E$. In particular, if x is represented by the $k \times n$ matrix X whose columns are the components of the vectors $\vec{r}_i \in E \equiv \mathbb{R}^k$, $|\Delta|$ depends only on the $n \times n$ matrix ${}^tXX$.

The Gram matrix must of course be interpreted as the matrix of a quadratic form β on $\mathcal{D}^*$ (and not on $\mathbb{R}^n$). This quadratic form is a complete coding for the shape ("forme" in french) defined by the n points up to isometries; it may be written:

$$\beta(\xi,\eta) = \sum_{i,j} \langle \vec{r}_i, \vec{r}_j \rangle_E \, \xi_i \eta_j = \sum_{i,j} \left(-\frac{1}{2} r_{ij}^2\right) \xi_i \eta_j \, ,$$

where $r_{ij} = ||\vec{r}_i - \vec{r}_j||$ is the distance between i and j. Notice that the last equality does not hold on $\mathbb{R}^n$ but only on $\mathcal{D}^*$.

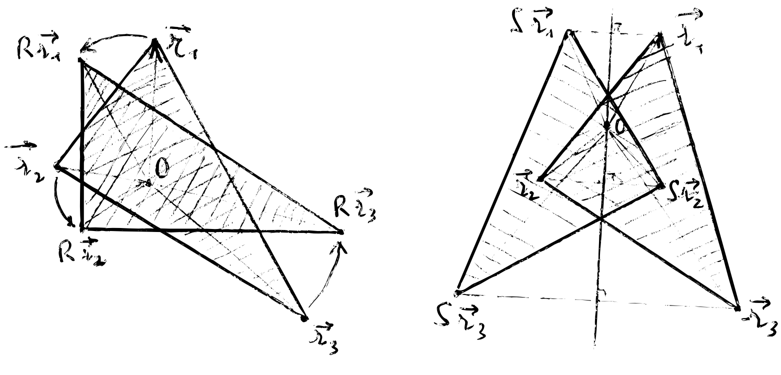

Fig. 2: Invariance of the area by rotation and symmetry.

Remark. One can show that the mutual distances r_{ij} are the coordinates of β in a natural basis of the vector space of quadratic forms on $\mathcal{D}^*$. Hence giving these is the same as giving β and no mention has to made of the ambient space E. We have passed from *“the side of space”* to *“the side of the bodies”.*

From the definition of β one deduces that $\beta(\xi,\xi) = ||\sum_i \xi_i \vec{r}_i||_E^2 \geq 0$. The following theorem states that the converse holds (see [Bo], [AC]):

Theorem (Borchart 1866). *The real numbers r_{ij} are the mutual distances of n points in an euclidean space E (whose dimension is not imposed) if and only if the quadratic form $\beta = \sum_{i,j}\left(-\frac{1}{2}r_{ij}^2\right)\xi_i\eta_j$ on $\mathcal{D}^*$ is non negative. Moreover, on can choose E of dimension k if and only if the rank of β is less than or equal to k.*

This theorem was rediscovered several times. See for example [S] and [Bl].

3 – Masses and volumes

Let us transform the points $\vec{r}_i$ into “bodies”, possibly “celestial bodies”, by assigning positive masses m_i to them. A classical way of reducing the translation symmetry is to fix at the origin of $\mathbb{R}^n$ the *center of mass* $x_G = (1/\sum m_i)\sum m_i x_i$ of an n-tuple $(x_1, x_2, \ldots, x_n)$. Doing so, one identifies $\mathcal{D}$ to the hyperplane of $\mathbb{R}^n$ with equation $\sum m_i x_i = 0$. In restricting to this hyperplane the *mass scalar product* (or *kinetic energy scalar product*) defined on $\mathbb{R}^n$ by the formula

$$(x_1, x_2, \ldots, x_n)\cdot(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} m_i x_i y_i\,,$$

one turns $\mathcal{D}$ into an *Euclidean space* whose scalar product may be written

$$(x_1, x_2, \ldots, x_n)\cdot(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} m_i (x_i - x_G)(y_i - y_G)\,.$$

To this euclidean structure we associate the isomorphism

$$\mu : \mathcal{D} \to \mathcal{D}^*, \quad \mu(x_1, x_2, \ldots, x_n) = \big(m_1(x_1 - x_G), m_2(x_2 - x_G), \ldots, m_n(x_n - x_G)\big)$$

which endows $\mathcal{D}^*$ with the euclidean scalar product

$$(\xi_1, \xi_2, \ldots, \xi_n)\cdot(\eta_1, \eta_2, \ldots, \eta_n) = \sum_{i=1}^{n} \frac{1}{m_i}\xi_i\eta_i\,.$$

Using this scalar product we can turn the quadratic form β into a symmetric endomorphism B of the euclidean space $\mathcal{D}^*$, defined by

$$\beta(\xi,\eta) = B(\xi)\cdot\eta = \xi\cdot B(\eta)\,.$$

We come back to the case of 3 bodies, where $\dim\mathcal{D}^* = 2$. We can give now an interpretation of the 4×4 determinant in formula (C) as an artefact in the computation of the determinant of an endomorphism of a two-dimensional space without privileged basis.

Proposition. *The formula* (C), *which expresses the squared area* $|\Delta|^2$ *of the triangle with vertices* $\vec r_1, \vec r_2, \vec r_3$ *as a Cayley determinant, is equivalent to the equation*

$$\det B = \frac{m_1 m_2 m_3}{m_1+m_2+m_3}(2|\Delta|)^2\,.$$

Proof. The proof is a sequence of exercises in elementary linear algebra, the tricks of which were announced in the introduction. After a possible translation, *one can assume that the center of mass of the* $\vec r_i$ *lies at the origin of* $\mathbb{R}^2$.

1) One defines the extension $\hat B$ of B to an endomorphism of $\mathbb{R}^3$ by the condition that $\hat B$ sends to 0 the vector (m_1,m_2,m_3) (this vector generates the orthogonal of $\mathcal{D}^*$ for the scalar product dual to the mass scalar product on $(\mathbb{R}^3)^*\equiv\mathbb{R}^3$. Show that the matrix of $\hat B$ in the canonical basis of $\mathbb{R}^3$ is

$$\hat B = (\hat b_{ij})_{1\le i,j\le 3},\quad \hat b_{ij} = m_i < \vec r_i, \vec r_j > \ .$$

2) Working in the basis of $\mathbb{R}^4 = \mathbb{R}\times\mathbb{R}^3$ formed by $(1,0,0,0)$, $(0,m_1,m_2,m_3)$, $(0,a_1,b_1,c_1)$, $(0,a_2,b_2,c_2)$, where the (a_i,b_i,c_i), $i=1,2$, generate $\mathcal{D}^*$, show that

$$\det B = -\frac{1}{M}\det\tilde B\,,$$

where $M=\sum_{i=1}^n m_i$ is the sum of the masses and $\tilde B$ is the 4×4 matrix obtained by adding on top of $\hat B$ the line $(\,0\quad 1\quad 1\quad 1\,)$ and on the left the column $(\,0\quad m_1\quad m_2\quad m_3\,)$.

3) Use the identities

$$< \vec r_i, \vec r_j > - < \vec r_1, \vec r_j > - < \vec r_{i1}, \vec r_1 > = < \vec r_{i1}, \vec r_{j1} >\,,$$
$$< \vec r_i, \vec r_j > -\frac12\|\vec r_i\|^2 - \frac12\|\vec r_j\|^2 = -\frac12 r_{ij}^2\,,$$

to conclude, with the help of line and column operations, that on the one hand

$$\det\tilde B = \det\begin{pmatrix} 0 & 1 & 1 & 1\\ m_1 & 0 & 0 & 0\\ m_2 & 0 & m_2 < \vec r_{21}, \vec r_{21} > & m_2 < \vec r_{21}, \vec r_{31} >\\ m_3 & 0 & m_3 < \vec r_{31}, \vec r_{21} > & m_3 < \vec r_{31}, \vec r_{31} >\end{pmatrix}$$

is the product by $-m_1m_2m_3$ of the squared area of the parallelogram generated by the vectors $\vec{r}_{21}, \vec{r}_{31}$, and that on the other hand

$$\det \tilde{B} = \det \begin{pmatrix} 0 & 1 & 1 & 1 \\ m_1 & 0 & -\frac{1}{2}m_1 r_{12}^2 & -\frac{1}{2}m_1 r_{13}^2 \\ m_2 & -\frac{1}{2}m_2 r_{21}^2 & 0 & -\frac{1}{2}m_2 r_{23}^2 \\ m_3 & -\frac{1}{2}m_3 r_{31}^2 & -\frac{1}{2}m_3 r_{32}^2 & 0 \end{pmatrix}$$

equals the product by $\frac{1}{4}m_1m_2m_3$ of the determinant which appears in formula (C).

4) Deduce the proposition by recalling that the area of any of the parallelograms generated by the $\vec{r}_i$ equals twice the area of the triangle they define.
5) More generally, show that

$$\det(\mathrm{Id}_{\mathcal{D}^*} - \lambda B) = 1 - I\lambda + \frac{m_1m_2m_3}{m_1+m_2+m_3} 4|\Delta|^2\lambda^2 ,$$

where $I = \mathrm{trace} B = \frac{1}{M}(m_2m_3r_{23}^2 + m_3m_1r_{31}^2 + m_1m_2r_{12}^2) = \frac{1}{M}(m_2m_3a + m_3m_1b + m_1m_2c)$ is the moment of inertia of the three masses with respect to their center of mass.

Remark. All this generalises with the same proof to the case of n bodies in a finite dimensional euclidean space (exercise). The characteristic polynomial of B is given by the formula

$$\det(\mathrm{Id}_{\mathcal{D}^*} - \lambda B) = 1 - \eta_1\lambda + \cdots + (-1)^{n-1}\eta_{n-1}\lambda^{n-1} ,$$
$$\eta_{k-1} = \frac{1}{M}\sum_{i_1<\cdots<i_k} m_{i_1}\cdots m_{i_k}[(k-1)!\mathrm{vol}_{i_1\cdots i_k}]^2, \quad M = \sum_{i=1}^{n} m_i ,$$

where $\mathrm{vol}_{i_1\cdots i_k}$ is the $(k-1)$-dimensional volume of the simplex with vertices the bodies $\vec{r}_{i_1}, \ldots, \vec{r}_{i_k}$. In particular, the squared $(n-1)$-dimensional volume V of the simplex whose vertices are the n points is given by the formula

$$\det B = \frac{m_1\cdots m_n}{M}\big((n-1)!V\big)^2, \text{ or}$$

$$(-1)^n 2^{n-1}(n-1)!^2V^2 = \det \begin{pmatrix} 0 & 1 & . & . & . & 1 \\ 1 & 0 & . & . & . & . \\ . & . & . & . & r_{ij}^2 & . \\ . & . & . & . & . & . \\ . & . & r_{ji}^2 & . & 0 & . \\ 1 & . & . & . & . & 0 \end{pmatrix} .$$

4 – Back to ambient space: inertias

The map x which represents the n-body configuration $(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)$ up to a translation is the restriction to the hyperplane $\mathcal{D}^*$ of the linear map $X : \mathbb{R}^n \to E$ defined by $X(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{i=1}^n \xi_i \vec{r}_i$. The transposed map sends the dual E^* of E into the dual of $\mathcal{D}^*$, which is canonically identified with $\mathcal{D}$. But the euclidean structures of $E = \mathbb{R}^k$ and $\mathcal{D}$ give an identification of each of these spaces with its dual. Finallly, the transposed ${}^t x : E \to \mathcal{D}^*$ of x may be defined by the formula:

$$\forall e \in E, \forall \xi \in \mathcal{D}^*, \quad e \cdot x(\xi) = {}^t x(e) \cdot \xi\,,$$

where the scalar products $\cdot$ are respectively taken in E and in $\mathcal{D}^*$.

The endomorphism $B : \mathcal{D}^* \to \mathcal{D}^*$ now becomes $B = {}^t x \circ x$. But the endomorphism $\mathcal{I} = x \circ {}^t x : E \to E$ is well-known by mechanicians, at least in dimension three where the bivectors can be identified with vectors once an orientation has been chosen. It is the *dual inertia form* of the rigid body defined by the n point masses (which one supposes to be held rigidly to each other by massless rods). Indeed, if $n = k = 3, \vec{r}_i = (x_i, y_i, z_i)$ and $\sum_i m_i \vec{r}_i = 0$, x and ${}^t x$ may be represented respectively by the matrices

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \quad \text{and} \quad {}^t X = \begin{pmatrix} m_1 x_1 & m_1 y_1 & m_1 z_1 \\ m_2 x_2 & m_2 y_2 & m_2 z_2 \\ m_3 x_3 & m_3 y_3 & m_3 z_3 \end{pmatrix},$$

and the extension $\hat{B}$ of B to $\mathbb{R}^3$ introduced in Paragraph 3 becomes $\hat{B} = {}^t X X$ while

$$\mathcal{I} = X {}^t X = \begin{pmatrix} \sum\limits_k m_k x_k^2 & \sum_k m_k x_k y_k & \sum_k m_k x_k z_k \\ \sum\limits_k m_k y_k x_k & \sum_k m_k y_k^2 & \sum_k m_k y_k z_k \\ \sum\limits_k m_k z_k x_k & \sum_k m_k z_k y_k & \sum_k m_k z_k^2 \end{pmatrix}.$$

Let us call $\mathcal{J} : \wedge^2 E^* \to \wedge^2 E$ the *inertia operator*, which turns the instantaneous rotation of a rigid body motion of the configuration x into its angular momentum. After identifying its source and target with $\mathbb{R}^3$, it is represented by the matrix:

$$\mathcal{J} = \begin{pmatrix} \sum\limits_k m_k (y_k^2 + z_k^2) & -\sum_k m_k x_k y_k & -\sum_k m_k x_k z_k \\ -\sum\limits_k m_k y_k x_k & \sum_k m_k (z_k^2 + x_k^2) & -\sum_k m_k y_k z_k \\ -\sum\limits_k m_k z_k x_k & -\sum_k m_k z_k y_k & \sum_k m_k (x_k^2 + y_k^2) \end{pmatrix} = (\mathrm{trace}\mathcal{I})\mathrm{Id} - \mathcal{I}\,.$$

In particular, the knowledge of the spectrum of any one of the three operators $B, \mathcal{I}, \mathcal{J}$ implies that of the other two. Fixing B is therefore equivalent to fixing up to rotation the inertia ellipsoid of the configuration. Hence B deserves the name of *intrinsic inertia*. Defined on the side of the bodies and no more on the side of ambient space, it is invariant under the group $O(\mathbb{R}^k)$ of linear isometries of $E = \mathbb{R}^k$ and covariant under the group $O(\mathcal{D})$ of linear isometries of $\mathcal{D}$ (the so-called *“democracy group”* of physicists); on the contrary, the inertia of mechanicians ($\mathcal{I}$ or $\mathcal{J}$, which define the inertia ellipsoid in $\mathbb{R}^k$) is invariant under $O(\mathcal{D})$ and covariant under $O(\mathbb{R}^k)$.

Remark. Once the masses are given, the newtonian potential function $U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$ depends only on the r_{ij}, i.e. on β. Let us call the *isospectral manifold* the set of all quadratic forms β on $\mathcal{D}^*$ such that the corresponding endomorphism B has a given spectrum; this manifold consists of the set of shapes whose inertia ellipsoid is fixed up to an isometry of E. One defines in [AC] the *balanced configurations (= configurations équilibrées*) of n positive masses to be the critical points of the restriction of U to an isospectral manifold. One shows that these are exactly the configurations which admit a *homographic motion* in some space E of arbitrary dimension (recall that a homographic motion is one along which the shape does not change up to similarity). This is another example of how ambient space is forgotten. These configurations generalise the classical *central configurations* which share the same property but only in a space E of dimension less or equal to 3. Indeed, the central configurations are the critical points of the restriction of U to the set of configurations whose moment of inertia $I = \operatorname{trace} B$ with respect to the center of mass is fixed.

5 – The 3-body problem in the plane: from area to oriented area and from Heron’s formula to the theory of invariants

In the quadrant $\mathbb{R}^3_+$ consisting of triplets (a, b, c) of non negative real numbers, the triplets which represent the squared side lengths of a triangle are those which belong to the cone of equation $2ab + 2bc + 2ca - a^2 - b^2 - c^2 \geq 0$. This follows from Borchart’s theorem above, as this inequality is equivalent to the non negativity of the quadratic form β defined on $\mathcal{D}^*$ by

$$\beta(\xi, \eta) = -\frac{1}{2}(a\xi_2\eta_3 + b\xi_3\eta_1 + c\xi_1\eta_2)\,.$$

Fixing arbitrarily positive masses, this amounts to the non negativity of the trace and the determinant of the corresponding endomorphism B. As $a, b, c \geq 0$, this is equivalent to the stated condition. One can say that *the positivity of the right hand side of Heron’s formula embodies the three triangle inequalities.*

We now fix the size of the triangles by imposing that their *moment of inertia with respect to the center of mass* be equal to 1:

$$I = \operatorname{trace} B = m_2m_3a + m_3m_1b + m_1m_2c = 1\,.$$

The intersection of this affine plane with the cone above is an elliptic domain $\mathcal{T}$ (a disc if the three masses are the same): it parametrizes the set of triangles with fixed inertia up to an isometry. The boundary of this domain corresponds to the flat triangles, whose area is zero. It contains three marked points, the *collision points* which label the degenerate triangles with two coincident vertices (fig. 3).

When the ambient space E is of dimension 2, one can take the quotient by the rotations (i.e. the linear isometries which preserve orientation). This leads to the space of *oriented* triangles with fixed inertia: it is a sphere $\mathcal{S}$ (*the shape sphere*), obtained by gluing along their boundaries two copies of $\mathcal{T}$ which correspond to the two possible orientations. We indicate in what follows a more conceptual way of getting this sphere and the structures which are naturally associated to it.

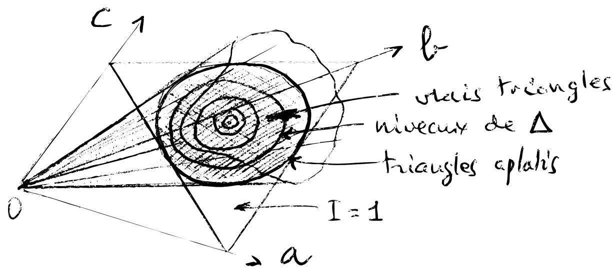

Fig. 3: In $\mathbb{R}^3$.

The first remark is that we can now enrich the notion of area by attaching to it a sign which depends on the orientation of the triangle. We shall denote by Δ this *oriented area.* Analytically, $\Delta = \frac{1}{2}\vec{u}_1 \wedge \vec{u}_2$, where $\vec{u}_1$ is the vector with origin at the first body and extremity at the second and $\vec{u}_2$ is any vector with origin on the segment between the two first bodies and extremity at the third body (if masses are attributed to the bodies and if the origin of $\vec{u}_2$ is the center of mass of the the first two bodies, $\vec{u}_1$ and $\vec{u}_2$ are the *Jacobi coordinates* in the space $\mathrm{Hom}(\mathcal{D}^*, \mathbb{R}^2) \equiv \mathcal{D} \otimes \mathbb{R}^2 \equiv \mathcal{D}^2$ of planar three-body configurations modulo translation; these coordinates can be obtained by choosing an orthogonal basis in $\mathcal{D}^*$).

The equation (C), which expresses Δ^2 as a function of the squared mutual distances $a = r_{23}^2$, $b = r_{31}^2$, $c = r_{12}^2$, defines a quadratic cone in $\mathbb{R}^4$ (coordinates a, b, c, Δ) or a half-cone $\mathcal{C}$ if one is interested only in the quadrant $a \geq 0, b \geq 0, c \geq 0$. The *shape sphere* $\mathcal{S}$ identifies with the set of generatrices of $\mathcal{C}$, i.e. with the quotient of $\mathcal{C} \setminus \{0\}$ by homotheties (fig. 4).

It follows that the cone $\mathcal{C}$ and the sphere $\mathcal{S}$ appear respectively as a realization of the quotient of $\mathcal{D}^2$ by the action of rotations or by the action of oriented similarities (rotations and homotheties). To make this point more precise, let us notice that the action of the group $SO(2)$ of rotations of $\mathbb{R}^2$ endows the space $\mathrm{Hom}(\mathcal{D}^*, \mathbb{R}^2) \equiv \mathcal{D} \otimes \mathbb{R}^2 \equiv \mathcal{D}^2$ of planar 3-body configurations modulo translation with the structure of a vector space on the field $\mathbb{C}$ of complex numbers, the multiplication by i corresponding to the rotation by $+\frac{\pi}{2}$ in $\mathbb{R}^2 \equiv \mathbb{C}$. In other words, if $(u, v) \in \mathcal{D}^2$, $i(u, v) = -(v, u)$.

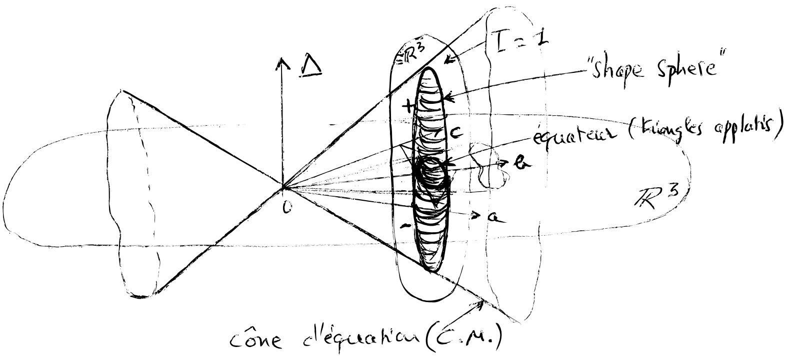

Fig. 4: In $\mathbb{R}^4$.

To go further on, one has to get rid of the choice of coordinates and, for this, give an intrinsic interpretation of the space $\mathbb{R}^4$ that we have introduced and of the cone $\mathcal{C}$ that it contains. The key remark is that the squared mutual distances a, b, c and the oriented area Δ are quadratic functions with real values on the vector space $\mathcal{D}^2$, i.e. functions which in *any* system of linear coordinates on $\mathcal{D}$, are expressed as second degree polynomials in these coordinates. Moreover, these functions are invariant under rotation: they are *quadratic invariants* under the action of the rotation group $SO(2)$. Now, it is classical in algebraic geometry to characterize a space by the space of functions that one can define on it. In the case we are interested in, the problem of understanding the quotient $\mathcal{D}^2/SO(2)$ of $\mathcal{D}^2$ by the action of the group $SO(2)$ of complex numbers with modulus 1 is equivalent to the problem of understanding the set of real polynomials $P : \mathcal{D}^2 \to \mathbb{R}$ invariant under this action, i.e. of polynomials P such that $P(\lambda v) = P(v)$ for all $\lambda \in \mathbb{C}$ with modulus 1. But any such polynomial can be expressed as a polynomial in the quadratic invariant polynomials. Indeed, if one chooses a basis of $\mathcal{D}^2$ on the field of complex numbers, i.e. if one identifies $\mathcal{D}^2$ with $\mathbb{C}^2$ (say by the choice of Jacobi coordinates associated with a choice of masses), the action of $SO(2)$ becomes $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$ and a polynomial $P(z_1, z_2) = \sum a_{ijkl} z_1^i \bar{z}_1^j z_2^k \bar{z}_2^l$ is invariant if and only if $a_{ijkl} \neq 0 \Rightarrow i - j + k - l = 0$. One deduces that P is a polynomial in the quadratic invariants $|z_1|^2, |z_2|^2, \mathrm{Re}\bar{z}_1 z_2, \mathrm{Im}\bar{z}_1 z_2$. It follows that we need only understand the quadratic invariants, i.e. the real four-dimensional vector space $\mathcal{Q}(\mathcal{D}^2)$ generated by the above functions. This space is of course independant of the choice of a basis in $\mathcal{D}^2$. As quadratic invariants are enough to construct all the polynomial invariants, the quotient $\mathcal{D}^2/SO(2)$ injects into the space $\mathcal{Q}(\mathcal{D}^2)$, more precisely into its dual $\mathcal{Q}(\mathcal{D}^2)^*$, by the *evaluation map* $e:\mathcal{D}^2\to\mathcal{Q}(\mathcal{D}^2)^*$ which to $v\in\mathcal{D}^2$ associates the linear form $q\mapsto q(v)$.

But the image of this map is easy to determine. Choose a complex basis of $\mathcal{D}^2$ whose coordinates are noted z_1,z_2. Then $\mathcal{Q}(\mathcal{D}^2)$ is generated by $|z_1|^2,|z_2|^2,\mathrm{Re}\bar z_1z_2,\mathrm{Im}\bar z_1z_2$, and the image $\mathcal{C}$ of $\mathcal{D}^2$ is defined by the sole quadratic relation $|\mathrm{Re}\bar z_1z_2|^2+|\mathrm{Im}\bar z_1z_2|^2=|z_1|^2|z_2|^2$, to which must be added the inequalities $|z_1|^2\geq 0,|z_2|^2\geq 0$. It is more pleasant to replace these last coordinates by

$$w_0=|z_1|^2+|z_2|^2, w_1=|z_1|^2-|z_2|^2, w_2=2\mathrm{Re}(\bar z_1z_2), w_3=2\mathrm{Im}(\bar z_1z_2)\,,$$

because the half-cone $\mathcal{C}$, defined by the equations $-w_0^2+w_1^2+w_2^2+w_3^2=0,\ w_0\geq 0$, appears as the *light cone* in Minkowski space. In order to identify it with the half-cone defined by (C), it remains to notice that a,b,c,Δ form another basis of $\mathcal{Q}(\mathcal{D}^2)$: Heron's formula expresses simply the quadratic relation satisfied by the elements of this basis.

REMARKS.

1) We have just seen that the image of the map $\mathcal{H}:\mathbb{C}^2\to\mathbb{R}^4$ defined by

$$(z_1,z_2)\mapsto(|z_1|^2+|z_2|^2,|z_1|^2-|z_2|^2,2\mathrm{Re}(\bar z_1z_2),2\mathrm{Im}(\bar z_1z_2))$$

is the half-cone defined by the equations $-w_0^2+w_1^2+w_2^2+w_3^2$, $w_0\geq 0$. Its composition $H=\pi_0\circ\mathcal{H}:\mathbb{C}^2\to\mathbb{R}^3$ with the projection π_0 parallel to w_0 on the subspace $\mathbb{R}^3$ generated by w_1,w_2,w_3 is the classical *Hopf map*, which sends the unit sphere of $\mathbb{C}^2$ onto the unit sphere of $\mathbb{R}\times\mathbb{C}$ by the even more classical *Hopf fibration.*

2) The space of generatrices of the half-cone above is the intrinsic definition (independent in particular of any choice of the masses) of the *shape sphere.* One shows (see [M]) that it inherits a well defined *conformal structure* by noticing that, independently of any choice of coordinates, one can define the notion of circle as a set of generatrices of the half-cone contained in a three-dimensional vector subspace (if one knows the circles, one knows in particular the infinitesimal circles and one deduces a notion of angle, i.e. a conformal structure). One obtains riemannian metrics in this conformal class (i.e. defining the same notion of angle) by considering the shape sphere as the set of oriented isometry classes classes of triangles whose moment of inertia with respect to their center of mass is equal to 1. To check this, it is enough, once the masses are given, to choose a basis on $\mathbb{C}$ of $\mathcal{D}^2$ coming from an orthogonal basis of $\mathcal{D}^*$ (*Jacobi type coordinates*). For example, for the metric corresponding to equal masses, one can take $z_1=\frac{1}{\sqrt2}(\vec r_2-\vec r_1)$ and $z_2=\sqrt{\frac23}(\vec r_3-\frac12(\vec r_1+\vec r_2))$. In any case, if one has chosen a basis which

is orthonormal, the metric on $\mathcal{D}^2 \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$ is the standard euclidean metric.

3) Topologically, the oriented situation is much richer than the non oriented one: the shape sphere possesses a symmetry group with twelve elements (the dihedral group D_6, which is also the symmetry group of the regular hexagon). Deprived of the three collision points, it acquires a fundamental group isomorphic to the free group on two generators. Recent works have shown that part of this richness shows up in the periodic solutions of the 3 body problem, but this is another story (see [C2]).

– Appendix. Darboux's interpretation of the quadratic form β

In [D], Gaston Darboux gives the following interpretation of β in the case when it is non degenerate i.e. when it describes a non degenerate n-simplex in $\mathbb{R}^{n-1}$ (he considers only the case $n = 4$ but this is immaterial; compare to Proposition 14 of [A2]). To each point $\vec{r}$ in $\mathbb{R}^{n-1}$ he attaches its barycentric homogeneous coordinates $(\xi_1, \cdots, \xi_n)$ with respect to the given simplex, defined (as an element of the projective space) by

$$\left(\sum_{i=1}^{n} \xi_i\right) \vec{r} = \sum_{i=1}^{n} \xi_i \vec{r}_i \,,$$

where $\vec{r}_1, \cdots, \vec{r}_n$ are the vertices of the simplex. This amounts to identifying $\mathbb{R}^{n-1}$ with the hyperplane $T = 1$ in $\mathbb{R}^n$(coordinates $X, Y, \ldots, T$) and calling $\xi_1, \ldots, \xi_n$ the coordinates of any point on the line generated by $(\vec{r}, 1) = (x, y, \ldots, 1)$ in the basis $\{(\vec{r}_1, 1), \ldots, (\vec{r}_{n-1}, 1)\}$ of $\mathbb{R}^n$. Hence

$$\vec{r} = \left(\frac{X}{T}, \frac{Y}{T}, \ldots\right) \in \mathbb{R}^{n-1}, \quad X = \sum_{i=1}^{n} \xi_i x_i,\ Y = \sum_{i=1}^{n} \xi_i y_i,\ \ldots,\ T = \sum_{i=1}^{n} \xi_i \,.$$

He then notices that in such coordinates the sphere circumscribed to the simplex (defined by $X^2+Y^2+\cdots-R^2T^2 = 0$ *if we suppose that the center* $\vec{s}$ *of this sphere is at the origin of* $\mathbb{R}^{n-1}$ *and call* R *its radius*) has equation $\sum_{i,j} r_{ij}^2 \xi_i \xi_j = 0$. A direct proof is easily found; one can also, as explained to me by Martin Celli, use Huyghens formula for the momentum of inertia:

$$\forall \vec{s} \in \mathbb{R}^{n-1},\ \sum_{i=1}^{n} \xi_i |\vec{s} - \vec{r}_i|^2 = \sum_{i=1}^{n} \xi_i |\vec{r} - \vec{r}_i|^2 + (\sum_{i=1}^{n} \xi_i)|\vec{s} - \vec{r}|^2 \,.$$

Choosing as $\vec{s}$ the center of the circumscribed sphere, the formula becomes

$$-\sum_{i=1}^{n} \xi_i |\vec{r} - \vec{r}_i|^2 = \left(\sum_{i=1}^{n} \xi_i\right) \left(\frac{X^2}{T^2} + \frac{Y^2}{T^2} + \cdots - R^2\right) ,$$

which implies that an equation of this sphere is $\sum_{i=1}^n \xi_i|\vec{r}-\vec{r}_i|^2 = 0$ (recall that for the points not at infinity, $\sum_{i=1}^n \xi_i \neq 0$). But, by the very definition of barycentric coordinates, $\sum_{i=1}^n \xi_i(\vec{r}-\vec{r}_i) = 0$ and it follows from a formula of Leibniz that

$$-\frac{1}{2}\sum_{i,j=1}^n \xi_i\xi_j r_{ij}^2 = -\left(\sum_{j=1}^n \xi_j\right)\sum_{i=1}^n \xi_i|\vec{r}-\vec{r}_i|^2 = \left(\sum_{i=1}^n \xi_i\right)^2\left(\frac{X^2}{T^2}+\frac{Y^2}{T^2}+\cdots-R^2\right),$$

that is

$$-\frac{1}{2}\sum_{i,j=1}^n \xi_i\xi_j r_{ij}^2 = X^2+Y^2+\cdots-R^2T^2\,,$$

so that the equation of the circumscribed sphere is $\sum_{i,j=1}^n \xi_i\xi_j r_{ij}^2 = 0$.

In particular, the quadratic form $\beta(\xi,\xi)$ appears as the restriction of the equation of this sphere to the hyperplane at infinity.

Let us see now how Darboux deduces the formula for the squared volume of an n-simplex (see end of Section 3) from the consideration of the "contravariant" of the triple formed by the equation of the circumscribed sphere $X^2+Y^2+\cdots-R^2T^2=0$ (in coordinates with the origin at the center of the sphere) and twice the linear form $T=0$ defining the "hyperplane at infinity". This means that he deduces the formula from the computation of the determinants of both sides of the following identity (the notations are those of Section 3 and the identity is obvious either directly or as a consequence of what is proved in 3):

$$\begin{pmatrix} 0 & 1 & 1 & . & . & 1\\ 1 & 0 & . & . & . & .\\ 1 & . & 0 & . & -\frac{1}{2}r_{ij}^2 & .\\ . & . & . & . & . & .\\ . & . & -\frac{1}{2}r_{ji}^2 & . & 0 & .\\ 1 & . & . & . & . & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & 0\\ 0 & x_1 & y_1 & . & 1\\ 0 & x_2 & y_2 & . & 1\\ . & . & . & . & .\\ 0 & x_n & y_n & . & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & . & . & 1\\ 0 & 1 & . & . & 0\\ 0 & 0 & . & . & .\\ 0 & . & . & 1 & 0\\ 1 & 0 & . & . & -R^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & . & . & 0\\ 0 & x_1 & x_2 & . & x_n\\ 0 & y_1 & y_2 & . & y_n\\ 0 & . & . & . & .\\ 0 & 1 & 1 & . & 1 \end{pmatrix}.$$

This identity expresses the transformation of the "contravariant" formed by the quadratic form and the two linear forms under the action of the linear group $GL(n,\mathbb{R})$ through its natural extension to $GL(n+1,\mathbb{R})$ in which it acts only on the last n coordinates. In the same way, one deduces immediately the formula

$$\det\begin{pmatrix} 0 & . & . & . & . \\ . & . & . & r_{ij}^2 & . \\ . & . & . & . & . \\ . & r_{ji}^2 & . & 0 & . \\ . & . & . & . & 0 \end{pmatrix} = (-1)^{n+1}2^n\big((n-1)!VR\big)^2$$

from the identity

$$\begin{pmatrix} 0 & . & . & . & . \\ . & . & . & -\frac{1}{2}r_{ij}^2 & . \\ . & . & . & . & . \\ . & -\frac{1}{2}r_{ji}^2 & . & 0 & . \\ . & . & . & . & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 & y_1 & . & 1 \\ x_2 & y_2 & . & 1 \\ . & . & . & . \\ x_n & y_n & . & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & . \\ 0 & . & . & 1 & 0 \\ 0 & 0 & . & . & -R^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & x_n \\ y_1 & y_2 & . & y_n \\ . & . & . & . \\ 1 & 1 & . & 1 \end{pmatrix}.$$

Acknowledgements

Thanks to Alain Albouy for references [Bo], [D], to Daniel Meyer for reference [S] and to Richard Montgomery for correcting my English. Thanks to Italo Capuzzo Dolcetta and Piero Negrini for inviting me to speak on this topic in the math colloquium of La Sapienza. Thanks to Charles Alunni, the editor of the volume dedicated to Gilles Chatelet, for the permission to publish this english version.

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Lavoro pervenuto alla redazione il 16 maggio 2006
ed accettato per la pubblicazione il 13 giugno 2006.
Bozze licenziate il 31 gennaio 2007

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