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Finite geometries: classical problems and recent developments

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ABSTRACT: *In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory, cryptography and design of experiments have made the field even more attractive. Pioneering work has been done by B. Segre and each of the four topics of this paper is related to his work; two classical problems and two recent developments will be discussed. First I will mention a purely combinatorial characterization of Hermitian curves in* $PG(2, q^2)$*; here, from the beginning, the considered pointset is contained in* $PG(2, q^2)$*. A second approach is where the object is described as an incidence structure satisfying certain properties; here the geometry is not a priori embedded in a projective space. This will be illustrated by a characterization of the classical inversive plane in the odd case. A recent beautiful result in Galois geometry is the discovery of an infinite class of hemisystems of the Hermitian variety in* $PG(3, q^2)$*, leading to new interesting classes of incidence structures, graphs and codes; before this result, just one example for* $GF(9)$*, due to Segre, was known. An exemplary example of research combining combinatorics, incidence geometry, Galois geometry and group theory is the determination of embeddings of generalized polygons in finite projective spaces. As an illustration I will discuss the embedding of the generalized quadrangle of order (4,2), that is, the Hermitian variety* $H(3, 4)$*, in* $PG(3, K)$ *with* K *any commutative field.*

1 – Introduction

In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory,

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cryptography and design of experiments have made the field even more attractive. Basic works on the subject are: "Projective Geometries over Finite Fields", "Finite Projective Spaces of Three Dimensions" and "General Galois Geometries", the first two volumes being written by Hirschfeld (1979 with a second edition in 1998, 1985) and the third volume by Hirschfeld and Thas (1991); the set of three volumes was conceived as a single entity. I also mention the "Handbook of Incidence Geometry: Buildings and Foundations", edited in 1995 by Buekenhout, which covers an enormous amount of material.

In his investigations on graph theory, design theory and finite projective spaces, the statistician R. C. Bose mainly used purely combinatorial arguments in combination with some linear algebra. Another great pioneer in finite projective geometry was Beniamino Segre. His celebrated result of 1954 stating that in the projective plane $\mathrm{PG}(2,q)$ over the Galois field $\mathrm{GF}(q)$ with q odd, every set of $q+1$ points, no three of which are collinear, is a conic, stimulated the enthusiasm of many young geometers. The work of Segre and his followers has many links with error-correcting codes and with maximal distance separable codes, in particular.

Finally, the fundamental and deep work in the last four decades on polar spaces, generalized polygons, and, more generally, incidence geometry, in the first place by Tits, but also by Shult, Buekenhout, Kantor and others, gave a new dimension to finite geometry.

Here I will state some important and elegant results, all related to the work of B. Segre, say something about the used techniques and mention some open problems.

2 – The geometry of PG(2, q)

First I will consider the geometry of $\mathrm{PG}(2,q)$, that is, the projective plane over the finite field $\mathrm{GF}(q)$. It is the purpose to show how classical algebraic curves can be characterized in purely combinatorial terms. I will illustrate this with the famous theorem of Segre on conics and with a theorem on Hermitian curves.

A k-arc of $\mathrm{PG}(2,q)$ is a set of k points of $\mathrm{PG}(2,q)$ no three of which are collinear. Then clearly $k \leq q+2$. By Bose (1947), for q odd, $k \leq q+1$. Further, any nonsingular conic of $\mathrm{PG}(2,q)$ is a $(q+1)$-arc. It can be shown that each $(q+1)$-arc K of $\mathrm{PG}(2,q)$, q even, extends to a $(q+2)$-arc $K \cup \{x\}$ (see, e.g., Hirschfeld (1998), p. 177); the point x, which is uniquely defined by K, is called the *kernel* or *nucleus* of K. The $(q+1)$-arcs of $\mathrm{PG}(2,q)$ are called *ovals*. The following celebrated theorem is due to Segre (1954).

Theorem 1. *In* $\mathrm{PG}(2,q)$, *q odd, every oval is a nonsingular conic.*

For q even, Theorem 1 is valid if and only if $q \in \{2,4\}$; see e.g., THAS (1995).

Crucial for the proof of Theorem 1 is Segre's *Lemma of Tangents* (see, e.g., HIRSCHFELD (1998), p. 179) which we now explain. For any k-arc K with $3 \leq k \leq q+1$, choose three of its points as the triangle of reference $u_0u_1u_2$ of the coordinate system. The lines intersecting K in one point are called the *tangent lines* of K. A tangent line of K through one of u_0, u_1, u_2 has respective equation

$$X_1 - dX_2 = 0, X_2 - dX_0 = 0, X_0 - dX_1 = 0\,,$$

with $d \neq 0$. We call d the *coordinate* of such a line. Suppose the $t = q+2-k$ tangent lines at each of u_0, u_1, u_2 are

$$X_1 - a_iX_2 = 0, X_2 - b_iX_0 = 0, X_0 - c_iX_1 = 0\,,$$

$i \in \{1, 2, \cdots, t\}$. Then, relying on the fact that the product of the non-zero elements of $\mathrm{GF}(q)$ is -1, Segre obtains the following

LEMMA 2 (Lemma of Tangents). *The coordinates a_i, b_i, c_i of the tangent lines at u_0, u_1, u_2 of a k-arc K through these points satisfy*

$$\Pi_{i=1}^{t} a_ib_ic_i = -1\,.$$

For an oval K we have $t = 1$, and so the lemma becomes $abc = -1$.

Geometrically this means that for q odd the triangles formed by three points of an oval and the tangent lines at these points are in perspective, and for q even these tangent lines are concurrent.

A Hermitian arc or *unital* H of $\mathrm{PG}(2,q)$, with q a square, is a set of $q\sqrt{q}+1$ points of $\mathrm{PG}(2,q)$ such that any line of $\mathrm{PG}(2,q)$ intersects H in either 1 or $\sqrt{q}+1$ points. The lines intersecting H in one point are called the *tangent lines* of H. At each of its points H has a unique tangent line. Let S be a unitary polarity of $\mathrm{PG}(2,q), q$ a square. Then the absolute points of ζ, that is, the points x of $\mathrm{PG}(2,q)$ which lie on their image x^ζ, form a Hermitian arc. Such a Hermitian arc is called a *nonsingular Hermitian curve.* For any nonsingular Hermitian curve coordinates in $\mathrm{PG}(2,q)$ can always be chosen in such a way that it is represented by the equation

$$X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} = 0\,.$$

In 1992 the following theorem was obtained, solving a longstanding conjecture on Hermitian curves; see THAS (1992).

THEOREM 3. *In $\mathrm{PG}(2,q), q$ a square, a Hermitian arc H is a nonsingular Hermitian curve if and only if tangent lines of H at collinear points are concurrent.*

In the proof we combine an argument, similar to Segre's one in the Lemma of Tangents, with the same argument applied to the unital $\hat{H}$ of the dual plane consisting of the $q\sqrt{q}+1$ tangent lines of H. Finally, we rely on the following characterization due to Lefèvre-Percsy (1982), and independently to Faina and Korchmáros (1983).

Theorem 4. *In* $\mathrm{PG}(2,q), q$ *square and* $q > 4$*, a Hermitian arc* H *is a nonsingular Hermitian curve if and only if every line of* $\mathrm{PG}(2,q)$ *meeting* H *in more than one point meets it in a subline* $PG(1,\sqrt{q})$*.*

Theorems 1 and 3 are purely combinatorial characterizations of algebraic curves. Finally we give a characterization, due to Hirschfeld, Storme, Thas and Voloch (1991), in terms of algebraic curves, that is, we will assume from the beginning that our pointset is an algebraic curve.

Theorem 5. *In* $\mathrm{PG}(2,q), q$ *a square and* $q \neq 4$*, any algebraic curve of degree* $\sqrt{q}+1$*, without linear components, and with at least* $q\sqrt{q}+1$ *points in* $\mathrm{PG}(2,q)$*, must be a nonsingular Hermitian curve.*

In the last part of the proof of Theorem 3 we can also rely on Theorem 5 instead of on Theorem 4; see Thas, Cameron and Blokhuis (1992).

3 – Finite inversive planes

A second approach to characterize geometries is that where the object is described as an incidence structure satisfying certain properties; here the geometry is not a priori embedded in a projective space, even the finite field is in many cases a priori absent. Hence the finite projective space must be constructed.

We will give examples concerning circle geometries and designs. A $t-(v,k,\lambda)$ *design*, with $v > k > 1, k \geq t \geq 1, \lambda > 0$, is a set P with v elements called *points*, provided with subsets of size k called *blocks*, such that any t distinct points are contained in exactly λ blocks. A $3-(n^2+1, n+1, 1)$ design is usually called an *inversive plane* or *Möbius plane* of *order n*; here the blocks are mostly called *circles*. An *ovoid* O of $\mathrm{PG}(3,q)$, $q > 2$, is a set of q^2+1 points no three of which are collinear; an *ovoid* of $\mathrm{PG}(3,2)$ is a set of 5 points no four of which are coplanar. For properties on ovoids we refer to Hirschfeld (1985). If O is an ovoid, then O provided with all intersections $\pi \cap O$, where π is any plane containing at least 2 (and then automatically $q+1$) points of O, is an inversive plane $\mathcal{I}(O)$ of order n. An inversive plane arising from an ovoid is called *egglike*. The following famous theorem is due to Dembowski (1964).

Theorem 6. *Each (finite) inversive plane of even order is egglike.*

If the ovoid O is an elliptic quadric, then the inversive plane $\mathcal{I}(O)$ is called *classical* or *Miquelian.* Barlotti (1955), and, independenly, Panella (1955) proved that for q odd any ovoid is an elliptic quadric. Hence for q odd any egglike inversive plane is Miquelian. For odd order no other inversive planes are known. To the contrary, in the even case Tits (1962) showed that for any $q = 2^{2e+1}$, with $e \geq 1$, there exists an ovoid which is not an elliptic quadric; these ovoids are called *Tits ovoids* and are related to the simple Suzuki groups $Sz(q)$. In fact, for $q = 8$ Segre (1959) discovered an ovoid which is not an elliptic quadric, and which was shown to be a Tits ovoid by Fellegara (1962). For even order no other nonclassical inversive planes than the ones associated to the Tits ovoids are known.

For the even case, the following beautiful theorem is due to M. R. Brown (2000).

Theorem 7. *If an ovoid O of* $\mathrm{PG}(3, q)$*, q even, contains a conic section, then O is an elliptic quadric.*

Let $\mathcal{I}$ be an inversive plane of order n. For any point x of $\mathcal{I}$, the points of $\mathcal{I}$ different from x, together with the circles containing x (minus x), form a $2-(n^2, n, 1)$ design, that is, an *affine plane* of order n. That affine plane is denoted by $\mathcal{I}_x$, and is called the *internal* or *derived plane* of $\mathcal{I}$ at x. For an egglike inversive plane $\mathcal{I}(O)$ of order q, each internal plane is Desarguesian, that is, is the affine plane $\mathrm{AG}(2, q)$. The following theorem, due to Thas (1994), solves a longstanding conjecture on circle geometries.

Theorem 8. *Let $\mathcal{I}$ be an inversive plane of odd order n. If for at least one point x of $\mathcal{I}$ the internal plane $\mathcal{I}_x$ is Desarguesian, then $\mathcal{I}$ is Miquelian.*

In the proof of Theorem 8 we first represent $\mathcal{I}$ in the plane $\mathcal{I}_x$, where the circles of $\mathcal{I}$ not containing x become conics, by Segre's famous theorem. Then the key idea is to use a fundamental result on Minkowski planes (another type of circle geometries), which in turn depends on the classification of a particular class of quasifields. As a corollary of Theorem 8 we obtain the first computer-free proof of the uniqueness (up to isomorphism) of the inverse plane of order 7.

4 – Hemisystems of Hermitian varieties

In this section I will discuss a problem posed by Segre (1965), but only solved very recently.

Let us consider a *nonsingular Hermitian variety* $H(3, q^2)$ in $\mathrm{PG}(3, q^2)$. Coordinates in $\mathrm{PG}(3, q^2)$ can always be chosen in such a way that $H(3, q^2)$ is represented by the equation

$$X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0\,.$$

Each line of $H(3,q^2)$ contains q^2+1 points, and each point of $H(3,q^2)$ is on $q+1$ lines of $H(3,q^2)$.

Regular systems of $H(3,q^2)$ were introduced by Segre (1965), as sets of lines of $H(3,q^2)$ with the property that every point is on a constant number m of lines of the set, with $0 < m < q+1$. Segre (1965) shows that, if K exists, then $m = (q+1)/2$. In the latter case K consists of $(q+1)(q^3+1)/2$ lines and is called a *hemisystem.* So, for q even, $H(3,q^2)$ admits no regular system. Another corollary is that $H(3,q^3)$ admits no spread (since $m \neq 1$), that is, $H(3,q^2)$ cannot be partitioned by lines. In fact, the proof of Segre is restricted to q odd, but Bruen and Hirschfeld (1978) remark that, with their definition of a quadric permutable with a Hermitian variety, it also holds for q even.

A very short proof of Segre's result is given by Thas (1981). It goes as follows. First he shows that for any regular system K, the graph G with as vertices the lines of $H(3,q^2)$ in K and as adjacency being concurrent, is strongly regular with parameters $v = (q^3+1)m, k = (q^2+1)(m-1), \lambda = m-2$ and $\mu = mq - 2q + m$. Since any strongly regular graph satisfies $(v-k-1)\mu = k(k-\lambda-1)$, it follows that $m = (q+1)/2$.

A (finite) *partial quadrangle* is an incidence structure $\mathcal{S} = (P, B, \mathrm{I}\,)$ in which P and B are disjoint nonempty sets of objects called points and lines, respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
(iii) if x is a point and L is a line not incident with x, then there is at most one pair $(y, M) \in P \times B$ for which $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.
(iv) the point graph of $\mathcal{S}$, that is, the graph with vertex set P, two distinct vertices being adjacent if and only if they are incident with a common line, is strongly regular.

Partial quadrangles were introduced by Cameron (1975). A (finite) *generalized quadrangle* is an incidence structure satisfying (i), (ii) and

(iii)$'$ if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

For generalized quadrangles (iv) is automatically satisfied. Generalized quadrangles were introduced by Tits (1959), and for more details on this topic we refer to the monographs by Payne and Thas (1984), and Van Maldeghem (1998).

The points and lines of $H(3,q^2)$ form a generalized quadrangle, also denoted $H(3,q^2)$, with $s = q^2$ and $t = q$. The points and lines of a nonsingular elliptic quadric $Q(5,q)$ of $\mathrm{PG}(5,q)$ form a generalized quadrangle, also denoted $Q(5,q)$, with parameters $s = q$ and $t = q^2$. It is well-known that the generalized quadrangle $H(3, q^2)$ is the dual of the generalized quadrangle $Q(5, q)$; see, e.g. 3.2.3 of Payne and Thas (1984). Let θ be an anti-isomorphism of $H(3, q^2)$ onto $Q(5, q)$.

Consider a hemisystem K of $H(3, q^2)$ and its image K^θ on $Q(5, q)$. Let $\mathcal{S}$ be the incidence structure with as points the points of K^θ, as lines the lines of $Q(5, q)$, and as incidence relation that of $Q(5, q)$. Then, as the graph G introduced in a foregoing section is strongly regular, the incidence structure $\mathcal{S}$ is a partial quadrangle with $s = (q-1)/2$ and $t = q^2$; the parameters of the point graph G of $\mathcal{S}$ are $v = (q^3+1)(q+1)/2$, $k = (q^2+1)(q-1)/2$, $\lambda = (q-3)/2$ and $\mu = (q-1)^2/2$.

Next, we intersect K^θ with a hyperplane Π of $\mathrm{PG}(5, q)$. Then $|K^\theta \cap \pi| \in \{(q^2+1)(q+1)/2, 1 + (q^2+1)(q-1)/2\theta\}$. As K^θ has two intersection numbers with respect to hyperplanes, it defines a strongly regular graph $\hat{G}$ with $\hat{v} = q^6$ vertices and also a linear projective two-weight code; see Calderbank and Kantor (1986).

Segre (1965) discovered a hemisystem of $H(3, 9)$; independently, Hill (1973) discovered the corresponding set of 56 points on $Q(5, 3)$. The corresponding graph G is the well-known graph of Gewirtz (1969). Thas (1995) conjectured that there are no hemisystems for $q > 3$. It came as a great surprise when Cossidente and Penttila (2005) proved the following beautiful result.

Theorem 9. *For each odd prime power q the Hermitian variety $H(3, q^2)$ has a hemisystem.*

In fact, for several values of q they discovered more than one hemisystem.

As a corollary new strongly regular graphs, new partial quadrangles and new projective linear two-weight codes arise; these objects are not only new, but, for $q > 3$, graphs, codes, partial quadrangles with these parameters were previously unknown.

5 – Generalized quadrangles in projective spaces

In 1974 Buekenhout and Lefèvre (1974) published their beautiful theorem classifying all finite generalized quadrangles fully embedded in $\mathrm{PG}(d, q)$. Dienst (1980) proved the analogue of the Buekenhout-Lefèvre theorem for infinite generalized quadrangles. Weak (or polarized) and lax embeddings of generalized quadrangles in finite projective spaces were considered by Thas and Van Maldeghem (1998, 2001, 200*). In one of these papers Thas and Van Maldeghem (2001) overlooked a small, but interesting, case; in this Section 5 I will consider that case.

A *lax embedding* of a generalized quadrangle $\mathcal{S}$ (see Section 4) with pointset P in a projective space $\mathrm{PG}(d, K)$, $d \geq 2$ and K a commutative field, is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of $\mathrm{PG}(d,K)$ satisfying

(i) the set P^θ generates $\mathrm{PG}(d,K)$.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *laxly embedded* in $\mathrm{PG}(d,K)$.

A *weak* or *polarized embedding* in $\mathrm{PG}(d,K)$ is a lax embedding which also satisfies

(ii) for any point x of $\mathcal{S}$, the subspace generated by the set

$$X = \{y^\theta \parallel y \in P \text{ is collinear with } x\}$$

meets P^θ precisely in X, which is equivalent to $< X > \neq \mathrm{PG}(d,K)$ (here $< X >$ is the subspace of $\mathrm{PG}(d,K)$ generated by the set X).

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *weakly* or *polarizedly embedded* in $\mathrm{PG}(d,K)$.

A *full embedding* in $\mathrm{PG}(d,K)$ is a lax embedding with the following additional property

(iii) for every line L of $\mathcal{S}$, all points of $\mathrm{PG}(d,K)$ on the line L^θ have an inverse image under θ.

In such a case we say that the image $\mathcal{S}^\theta$ of $\mathcal{S}$ is *fully embedded* in $\mathrm{PG}(d,K)$.

Usually, we simply say that $\mathcal{S}$ is laxly, or weakly, or fully embedded in $\mathrm{PG}(d,K)$ without referring to θ, that is, we identify the points and lines of $\mathcal{S}$ with their images in $\mathrm{PG}(d,K)$.

Generalized quadrangles were defined in Section 4. If s and t are the parameters of the finite generalized quadrangle $\mathcal{S}$, then we say that $\mathcal{S}$ has *order* (s,t); if $s=t$ we say that $\mathcal{S}$ has *order* s. If $s>1$ and $t>1$ we say that the generalized quadrangle is *thick*.

The geometry of points and lines of a nonsingular quadric of projective index 1, that is, of Witt index 2, in $\mathrm{PG}(d,q)$ is a generalized quadrangle denoted by $Q(d,q)$. Here only the cases $d=3,4,5$ occur and $Q(d,q)$ has order (q,q^{d-3}). The geometry of all points of $\mathrm{PG}(3,q)$, together with all totally isotropic lines of a symplectic polarity in $\mathrm{PG}(3,q)$, is a generalized quadrangle of order (q,q) denoted by $W(q)$. The geometry of points and lines of a nonsingular Hermitian variety of projective index 1 in $\mathrm{PG}(d,q^2)$ is a generalized quadrangle $H(d,q^2)$ of order (q^2,q^{2d-5}); here either $d=3$ or $d=4$. Any generalized quadrangle isomorphic to one of these examples is called *classical*; the examples themselves are called the *natural embeddings* of the classical generalized quadrangles.

The following beautiful theorem is due to BUEKENHOUT and LEFÈVRE (1974).

THEOREM 10. *If $\mathcal{S}$ is a generalized quadrangle fully embedded in* $\mathrm{PG}(d,q)$, *then $\mathcal{S}$ is one of the natural embeddings of the classical generalized quadrangles.*

Thas and Van Maldeghem (1998) determined all thick generalized quadrangles weakly embedded in $\mathrm{PG}(d,q)$. Only clasical quadrangles show up. It should be mentioned that alle weak embeddings in $\mathrm{PG}(3,q)$ of thick generalized quadrangles were classified by Lefèvre-Percsy (1981) although she used a stronger definition for "weak embedding". For thick generalized quadrangles "being fully or weakly embedded" in $\mathrm{PG}(d,q)$ characterizes the finite classical generalized quadrangles amongst the others. This is no longer true for laxly embedded generalized quadrangles. To handle laxly embedded generalized quadrangles completely different combinatorial and geometric methods are needed than in the full and the weak case. Also, these last methods do not work in the case of laxly embedded generalized quadrangles in the plane. By projection every generalized quadrangle which admits an embedding in some projective space admits a lax embedding in a plane. This makes the classification in dimension 2 very hard and probably impossible. Hence we restrict our attention to the case $d \geq 3$. In Thas and Van Maldeghem (2001) there are two theorems on these lax embeddings with in total a proof of 35 pages. In the first theorem they prove that if a generalized quadrangle $\mathcal{S}$ of order (s,t), with $s > 1$, is laxly embedded in $\mathrm{PG}(d,q)$, then $d \leq 5$; for $d \in \{3,4,5\}$ and several infinite classes of (s,t) they prove that $\mathcal{S}$ is classical. In the second theorem they determine the generalized quadrangles $\mathcal{S}$ of order (s,t) which are laxly embedded in $\mathrm{PG}(d,q)$, with $d \geq 3$, and isomorphic to one of $Q(5,s), Q(4,s), H(4,s), H(3,s)$ or the dual of $H(4,t)$. In this theorem the authors overlooked a class of lax embeddings of $H(3,4)$, which is up to isomorphism the unique generalized quadrangle of order $(4,2)$, in $\mathrm{PG}(3,q)$. As the solution of this problem is particularly nice and elegant, and related to the work of Segre (1942, 1949, 1951), I will state it here; this is again joint work of Thas and Van Maldeghem (200*).

Consider a nonsingular cubic surface $\mathcal{F}$ in $\mathrm{PG}(3,K)$, K any commutative field, and assume that $\mathcal{S}$ has 27 lines. Then necessarily $K \neq \mathrm{GF}(q)$ with $q \in \{2,3,5\}$; see Chapter 20 of Hirschfeld (1985). Let $\mathcal{S}' = (P', B', \mathrm{I}')$ be the following incidence structure: the elements of P' are the 45 tritangent planes of $\mathcal{F}$ (that are the planes which intersect $\mathcal{F}$ in 3 lines), the elements of B' are the 27 lines of $\mathcal{F}$, and a point $\pi \in P'$ is incident with a line $L \in B'$ if and only if $L \subset \pi$. It is well-known that $\mathcal{S}'$ is the unique generalized quadrangle of order $(4,2)$. Let β be an anti-isomorphism of $\mathrm{PG}(3,K)$, let $(P')^\beta = P$ and let $(B')^\beta = B$. If I is containment, then $\mathcal{S} = (P, B, \mathrm{I})$ is again isomorphic to $H(3,4)$, and is contained in the dual surface $\hat{\mathcal{F}}$ of $\mathcal{F}$ which again contains exactly 27 lines. Clearly $\mathcal{S}$ is laxly embedded in $\mathrm{PG}(3,K)$. An *Eckardt point* y of $\mathcal{F}$ is a point contained in 3 lines of $\mathcal{F}$, which are then contained in the tangent plane of $\mathcal{F}$ at y. If $x \in P$, then the 3 lines of $\mathcal{S}$ incident with x are contained in a plane π if and only if $\pi^{\beta^{-1}}$ is an Eckhardt point of $\mathcal{F}$. Thas and Van Maldeghem (200*) show that every lax embeddding in $\mathrm{PG}(3,K)$ of the unique generalized quadrangle of order $(4,2)$ is of the type described above. Such a lax embedding is uniquely defined by 5 mutually skew lines $A_1, A_2, \cdots, A_5$ with a transversal B_6 such that each five of the six lines are linearly independent (in the sense that their Plücker (or line) coordinates define 5 independent points in $\mathrm{PG}(5, K)$). Such a configuration exists for every commutative field K except for $K = \mathrm{GF}(q)$ with $q = 2, 3$ or 5; see Chapter 20 of Hirschfeld (1985). The embedding is weak if and only if $\mathcal{F}$ has 45 Eckhardt points; in such a case GF(4) is a subfield of K, see Hirschfeld (1985). Finally, by Thas and Van Maldeghem (1998), in that case $\mathcal{S}$ is a full embedding of that generalized quadrangle in a subspace $\mathrm{PG}(3, 4)$ of $\mathrm{PG}(3, K)$, so by Buekenhout and Lefèvre (1974) is the natural embedding of $H(3, 4)$ in that subspace $\mathrm{PG}(3, 4)$. So we have the following theorem

Theorem 11. *Let K be any commutative field and let $\mathcal{S}$ be a lax embedding of the unique generalized quadrangle of order* $(4, 2)$ *in* $\mathrm{PG}(3, K)$. *Then* $|K| \neq 2, 3, 5$ *and $\mathcal{S}$ arises from a unique nonsingular cubic surface $\mathcal{F}$ as explained above. Also, the embedding is polarized if and only if $\mathcal{F}$ admits* 45 *Eckhardt points. In that case* GF(4) *is a subfield of K and $\mathcal{S}$ is a natural embedding of $H(3, 4)$ in a subspace* $\mathrm{PG}(3, 4)$ *of* $\mathrm{PG}(3, K)$.

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