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A new class of bivariate refinable functions suitable for cardinal interpolation

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Dedicated to Prof. Laura Gori in occasion of her 70th birthday.

Abstract: *In this paper a new class of bivariate refinable functions is constructed by means of the directional convolution product between the tensor product of two univariate refinable functions belonging to a particular class, shortly recalled in this paper, with a third one in the same class. The new functions have many properties useful in applications, such as compact support, positivity, central symmetry, refinability and linear independence of the integer translates. Moreover, the so constructed functions turn out to be suitable for solving the cardinal interpolation problem.*

1 – Introduction

There has been a variety of recent results on the construction of multivariate refinable functions, that is functions $\phi : \mathbb{R}^s \to \mathbb{R}$ which are solutions of a *refinement equation* of type

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \phi(Mx - \alpha), \qquad x \in \mathbb{R}^s . \tag{1.1}$$

The integer matrix M is the so-called *dilation matrix* with all eigenvalues having modulus greater than 1. Equivalently, M satisfies $\lim_{k\to\infty} M^{-k} = 0$. The matrix sequence $\boldsymbol{a} = \{a_\alpha, \alpha \in \mathbb{Z}^s\}$ is a compactly supported sequence called the *refi-*

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nement mask (see, for instance, [1], [9] and references therein). In the bivariate case, there are several examples of nonseparable, *i.e.* non tensor product based, refinable functions known. Beside the box splines [2], [8], satisfying the refinement equation with $M = 2I$, there are refinable functions with $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ or $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ [3], [12]. The use of nonseparable refinable functions is more appropriate for certain applications in image processing. Though, in general, their construction and the analysis of their properties are more involved than the tensor product based functions. This is not the case with box splines whose construction uses directional convolution of B-splines and, thus, is simple and their properties easily follow from the properties of the B-splines.

We remark that the convolution product is a simple tool commonly used to improve the smoothness and the order of approximation of a function or a family of functions when approximating a curve or surface (see [1] or [9]). For instance, in [5] repeated convolutions are used to define new families of refinable function vectors with increasing smoothness, degree and approximation order.

The above considerations motivated us in constructing and investigating a new family of bivariate refinable functions which are obtained by directional convolution. A first approach was proposed in [6] where the concept of directional convolution product has been used with one univariate function being the characteristic function of the interval $[0, 1)$. Here, we consider a more general case: given a bivariate function which is a tensor product of compactly supported univariate refinable functions, we construct a new bivariate function by means of directional convolution of the given function and some other univariate refinable function. We so obtain a wide class of bivariate functions tuned by free parameters which permit to control and improve smoothness and polynomial reproducibility. We observe that the functions introduced in [6] are contained in the family here presented so that the results on polynomial reproducibility and cardinal interpolation of Section 3 and 4 applied also to those functions.

The univariate functions here used belong to a large class of compactly supported refinable functions introduced in [10], which we will call GP functions. The GP functions have many useful properties, such as total positivity and central simmetry, like the B-splines with integer knots which are a particular case of them. The differences between the B-splines and the GP functions are mainly due to the fact that their masks contain one or more extra parameters. These parameters are additional degrees of freedom allowing to use GP functions more effectively in several applications (see, for instance, [7], [11]). In particular, GP functions apply better than B-splines in the cardinal interpolation of non smooth functions [16]. This is why we here investigate the behaviour of the newly constructed bivariate refinable functions when solving the cardinal interpolation problem.

The outline of the paper is as follows. In Section 2 the definition and the main properties of the GP refinable functions are recalled. In Section 3 the new class of convolved bivariate refinable functions is characterized and their main properties are investigated. Section 4 is devoted to the discussion of the solution of cardinal interpolation problem. Finally, in Section 5 some examples are given.

2 – Preliminaries

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be two refinable functions, bivariate and univariate, respectively. Here, we consider the special case $M = 2I$.

The *convolution product* between Φ and φ along the direction $e \in \mathbb{Z}^2$ is defined as

$$\Psi(x) := (\Phi *_e \varphi)(x) := \int_{\mathbb{R}} \Phi(x - et)\varphi(t)\, dt, \qquad x \in \mathbb{R}^2 . \tag{2.1}$$

If $\boldsymbol{a}^\Phi$ and $\boldsymbol{a}^\varphi$ are the refinement mask of Φ and φ, respectively, it is easy to show that Ψ is refinable too, with mask

$$\boldsymbol{a}^\Psi = \frac{1}{2}(\boldsymbol{a}^\Phi *_e \boldsymbol{a}^\varphi), \quad \text{that is} \quad a^\Psi_\alpha = \frac{1}{2}\sum_{\beta\in\mathbb{Z}} a^\Phi_{\alpha-e\beta}\, a^\varphi_\beta, \quad \alpha \in \mathbb{Z}^2 . \tag{2.2}$$

In order to construct a new class of bivariate refinable functions we choose Φ as tensor product of two univariate refinable functions, so that

$$\Psi = (\varphi_1 \otimes \varphi_2) *_e \varphi_3 . \tag{2.3}$$

In particular, as convolution factors φ_i, $i = 1, 2, 3$, we choose univariate GP refinable functions which are compactly supported, centrally symmetric and totally positive functions with prescribed smoothness. Therefore, before proceeding, we recall the main properties of the GP functions (see [10] for more details).

Let us denote by $\varphi^{(n,h)}$, a GP refinable function of support $[0, n+1]$, $n \geq 2$. For any (n, h), $1 \geq h \geq [n/2]$, held fixed, $\varphi^{(n,h)}$ depends on h real parameters $b^{(n,h)}_{\alpha,0}$, $\alpha = 0, 1, \dots , h - 1$, satisfying

$$\begin{cases} b^{(n,h)}_{h,0} = 2^{2h-n} - 2\displaystyle\sum_{\alpha=0}^{h-1} b^{(n,h)}_{\alpha,0} , \\ \det\, (b^{(n,h)}_{2\alpha-\beta}; \alpha, \beta = 1, \dots , p) > 0, \quad p = 1, \dots , 2h , \end{cases} \tag{2.4}$$

with symmetric conditions

$$b^{(n,h)}_{2h-\alpha,0} = b^{(n,h)}_{\alpha,0}, \qquad \alpha = 0, 1, \dots , 2h . \tag{2.5}$$

The corresponding mask $\boldsymbol{a}^{(n,h)} = \{a_\alpha^{(n,h)}, 0 \le \alpha \le n+1\}$ turns out to be

$$\text{(2.6)} \qquad \begin{cases} a_\alpha^{(n,h)} = \displaystyle\sum_{\beta=0}^{h} b_{\beta,\beta}^{(n,h)} \binom{n+1-2\beta}{\alpha-\beta}, & \alpha = 0,\dots,n+1\,, \\ a_\alpha^{(n,h)} = 0, & \text{otherwise}\,, \end{cases}$$

where the coefficients $b_{\beta,\beta}^{(n,h)}$ are defined recursively as follows:

$$\text{(2.7)} \quad b_{\alpha,\beta}^{(n,h)} = b_{\alpha,\beta-1}^{(n,h)} - \binom{2h-2\beta+2}{\alpha-\beta+1} b_{\beta-1,\beta-1}^{(n,h)}, \quad \beta = 1,2,\dots,\alpha,\ \alpha = 1,\dots,h\,.$$

(We assume $\binom{\alpha}{\beta} = 0$ for $\beta \le 0$ or $\beta > \alpha$.)

The properties of $\varphi^{(n,h)}$ are related to the structure of the *symbol*

$$\text{(2.8)} \qquad a^{(n,h)}(z) := \sum_{\alpha\in\mathbb{Z}} a_\alpha^{(n,h)} z^\alpha = (1+z)^{n-2h+1}\, q_{2h}^{(n,h)}(z)\,,$$

where

$$\text{(2.9)} \qquad q_{2h}^{(n,h)}(z) = \sum_{\alpha=0}^{2h} b_{\alpha,0}^{(n,h)} z^\alpha, \quad q_{2h}^{(n,h)}(1) = 2^{-n+2h}\,.$$

Due to the factor $(1+z)^{n-2h+1}$ in (2.8), the function $\varphi^{(n,h)}$ belongs to $C^{n-2h}(\mathbb{R})$ and reproduces polynomials up to degree $n-2h$. Moreover, $\varphi^{(n,h)}$ is positive in $(0,n+1)$ and it is symmetric with respect to the center of its support, *i.e.* $\varphi^{(n,h)}(x) = \varphi^{(n,h)}(n+1-x)$.

The Fourier transform $\widehat{\varphi}^{(n,h)}(\omega)$ satisfies the refinement equation

$$\text{(2.10)} \qquad \widehat{\varphi}^{(n,h)}(\omega) = a^{(n,h)}\left(e^{-i\frac{\omega}{2}}\right)\widehat{\varphi}^{(n,h)}\left(\frac{\omega}{2}\right), \qquad \omega \in \mathbb{R}\,,$$

thus, since on the unit circle $a^{(n,h)}(z)$ vanishes if and only if $z=-1$, from (2.10) it follows that $\widehat{\varphi}^{(n,h)}(\omega)$ vanishes if and only if $\omega \in 2\pi\mathbb{Z}\setminus\{0\}$. In other words, the GP functions do not have real periodic zeros.

Concerning the integer translates of $\varphi^{(n,h)}$, it is known that the function system $\{\varphi^{(n,h)}(x-\alpha), \alpha\in\mathbb{Z}\}$ is linearly independent, stable, totally positive and forms a partition of unity so that any GP refinable function generates a multiresolution analysis on $L^2(\mathbb{R})$. We recall that linear independence of the integer translates of a functions is equivalent to the property that its Fourier transform has no periodic complex zeros [15].

As a final remark, we observe that classical B-splines on integer knots are a particular case of GP functions. In fact, by choosing

$$\text{(2.11)} \qquad b_{0,0}^{(n,h)} = \frac{1}{2^n}, \qquad b_{\alpha,0}^{(n,h)} = \binom{2h}{\alpha} b_{0,0}^{(n,h)}, \qquad \alpha = 1,2,\dots,h-1\,,$$

the mask (2.6) reduces to the mask of the B-spline of degree n.

3 – A new class of bivariate refinable functions

Using in (2.3) the GP refinable functions, we here construct a convolved function $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ where $\boldsymbol{n} = (n_1, n_2, n_3)$ and $\boldsymbol{h} = (h_1, h_2, h_3)$. Let us call *admissible* all the values $(\boldsymbol{n}, \boldsymbol{h})$ such that $n_i \geq 2$ and $1 \geq h_i \geq [n_i/2]$, $i = 1, 2, 3$. The bivariate function $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ turns out to have the following expression

$$\begin{aligned}\Psi^{(\boldsymbol{n},\boldsymbol{h})}(x_1, x_2) :&= (\varphi^{(n_1,h_1)} \otimes \varphi^{(n_2,h_2)}) \ *_e \ \varphi^{(n_3,h_3)}(x_1, x_2) = \\ &= \int_{\mathbb{R}} \varphi^{(n_1,h_1)}(x_1 - e_1 t)\varphi^{(n_2,h_2)}(x_2 - e_2 t)\varphi^{(n_3,h_3)}(t)\, dt\,,\end{aligned} \tag{3.1}$$

where $e = (e_1, e_2)$. From now on, we are going to consider $e \in \{-1, 0, 1\}^2$. From the properties of $\varphi^{(n_i,h_i)}$ it follows that $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ is compactly supported on $D \subset [0, n_1 + 1 + e_1 \cdot (n_3 + 1)] \times [0, n_2 + 1 + e_2 \cdot (n_3 + 1)]$, symmetric with respect to the center of D, and non negative. Moreover, its symbol is

$$\begin{aligned}a^{(\boldsymbol{n},\boldsymbol{h})}(z_1, z_2) = (1 + z_1)^{n_1-2h_1+1}(1 + z_2)^{n_2-2h_2+1}(1 + z_1^{e_1} z_2^{e_2})^{n_3-2h_3+1} \times \\ \times\, q_{2h_1}^{(n_1,h_1)}(z_1)\, q_{2h_2}^{(n_2,h_2)}(z_2)\, q_{2h_3}^{(n_3,h_3)}(z_1^{e_1} z_2^{e_2})\,.\end{aligned} \tag{3.2}$$

Proposition 3.1. *For any admissible $(\boldsymbol{n}, \boldsymbol{h})$, the function $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ has linearly independent integer translates and generates a MRA on $L_2(\mathbb{R}^2)$.*

Proof. To prove the linear independence, it is sufficient to show that the set of the *complex periodic zeros* of $\widehat{\Psi}^{(\boldsymbol{n},\boldsymbol{h})}$ is empty [15] that is

$$Z^C_{\Psi^{(\boldsymbol{n},\boldsymbol{h})}} = \{\theta \in \mathbb{C}^2 | \widehat{\Psi}^{(\boldsymbol{n},\boldsymbol{h})}(\theta + 2\pi\alpha) = 0, \alpha \in \mathbb{Z}^2\} = \{\emptyset\}\,.$$

This follows from the fact that

$$\widehat{\Psi}(\theta_1, \theta_2) = \widehat{\varphi}^{(n_1,h_1)}(\theta_1)\widehat{\varphi}^{(n_2,h_2)}(\theta_2)\widehat{\varphi}^{(n_3,h_3)}(e_1\theta_1 + e_2\theta_2)$$

and the univariate GP functions have no periodic complex zeros.

Concerning the stability, we have to prove that the set of the *real periodic zeros* of $\widehat{\Psi}^{(\boldsymbol{n},\boldsymbol{h})}$ is empty, that is

$$Z^R_{\Psi^{(\boldsymbol{n},\boldsymbol{h})}} = \{\omega \in \mathbb{R}^2 | \widehat{\Psi}^{(\boldsymbol{n},\boldsymbol{h})}(\omega + 2\pi\alpha) = 0, \alpha \in \mathbb{Z}^2\} = \{\emptyset\}\,.$$

Now, as the Fourier transform of a GP function vanishes if and only if $\omega + 2\pi\alpha \in 2\pi\mathbb{Z}^2$, and

$$\widehat{\Psi}^{(\boldsymbol{n},\boldsymbol{h})}(\omega + 2\pi\alpha) = \widehat{\Phi}(\omega + 2\pi\alpha)\widehat{\varphi}(e \cdot (\omega + 2\pi\alpha))\,,$$

where $\widehat{\Phi} = \widehat{\varphi}^{(n_1,h_1)} \otimes \widehat{\varphi}^{(n_2,h_2)}$ and $\widehat{\varphi} = \widehat{\varphi}^{(n_3,h_3)}$, it follows that if ω is not a multiple of 2π, then $\omega + 2\pi\alpha \notin 2\pi\mathbb{Z}^2$ and $\widehat{\Phi}(\omega + 2\pi\alpha) \neq 0$, $\widehat{\varphi}(e \cdot (\omega + 2\pi\alpha)) \neq 0$, so that ω is not a periodic zero. If ω is a multiple of 2π, then $\omega + 2\pi\alpha \in 2\pi\mathbb{Z}^2$ and $\widehat{\Phi}(\omega + 2\pi\alpha)$ vanishes for any value $\alpha \in \mathbb{Z}^2$ except $\alpha = \alpha_0 := -\frac{\omega}{2\pi}$. But, for $\alpha = \alpha_0$ one has $\widehat{\varphi}(e \cdot (\omega + 2\pi\alpha_0)) = \widehat{\varphi}(0) = 1$, so that ω is not a periodic zero and $Z^R_{\Psi^{(\boldsymbol{n},\boldsymbol{h})}}$ is empty. ∎

The approximation properties of a given refinable function can be derived from the reproduction of polynomial spaces by projection on the space of its integer translates. For the just constructed refinable functions the following proposition holds.

PROPOSITION 3.2. *Let* $\Pi_p := \mathrm{span}\{x^\mu := x_1^{\mu_1} x_2^{\mu_2} : |\mu| = \mu_1 + \mu_2 \le p\}$, *be the space of the bivariate polynomials of degree* p. *For any admissible* $(\boldsymbol{n}, \boldsymbol{h})$, *the refinable function* $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ *reproduces bivariate polynomials up to degree*

$$p = \min(\, n_1 - 2h_1 + |e_1| \cdot (n_3 - h_3 + 1)\,,\; n_2 - 2h_2 + |e_2| \cdot (n_3 - h_3 + 1)\,)\,. \tag{3.3}$$

In particular, $\sum_{\alpha \in \mathbb{Z}^2} \Psi^{(\boldsymbol{n},\boldsymbol{h})}(\cdot - \alpha) = 1$.

PROOF. From (3.2) it follows that $D^\mu(a_\Psi^{(\boldsymbol{n},\boldsymbol{h})}(z))|_{z=(-1,-1)^\gamma} = 0$ for all $\gamma \in \{(1,0), (0,1), (1,1)\}$ and $|\mu| \le p$, where

$$p = \min(n_1 - 2h_1 + |e_1| \cdot (n_3 - h_3 + 1), n_2 - 2h_2 + |e_2| \cdot (n_3 - h_3 + 1))\,.$$

Thus, the polynomial reproduction follows from Proposition 2.1 in [4]. ∎

4 – The cardinal interpolation problem

Aim of this section is to show that any function $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ with admissible $(\boldsymbol{n}, \boldsymbol{h})$ can be profitably used in applications as cardinal interpolation.

By cardinal interpolation problem (for short, CIP) with the translates of a compactly supported function F we mean the following.

Given a data sequence $\boldsymbol{d} = \{d_\alpha, \alpha \in \mathbb{Z}^2\}$ *and the system* $\{F(\cdot - \alpha),\ \alpha \in \mathbb{Z}^2\}$, *we seek a sequence* $\boldsymbol{c} = \{c_\alpha, \alpha \in \mathbb{Z}^2\}$ *such that the function*

$$s(x) := \sum_{\alpha \in \mathbb{Z}^2} c_\alpha F(x - \alpha), \qquad x \in \mathbb{R}^2\,, \tag{4.1}$$

satisfies

$$s(\beta) = d_\beta, \quad \beta \in \mathbb{Z}^2\,. \tag{4.2}$$

It is well known that if the trigonometric polynomial $\tilde{F}(\omega) := \sum_{\alpha \in \mathbb{Z}^2} F(\alpha) \times$ $\times e^{-i\alpha\omega}$ does not vanish for all ω in $\mathbb{R}^2$, then there exists a unique solution of the CIP (see, for instance, [13, Sect. 3]). In that case the CIP is said to be *solvable* and we have $\tilde{C}(\omega) = \tilde{D}(\omega)/\tilde{F}(\omega)$, where $\tilde{D}(\omega) = \sum_{\alpha \in \mathbb{Z}^2} d_\alpha e^{-i\alpha\omega}$ and $\tilde{C}(\omega) = \sum_{\alpha \in \mathbb{Z}^2} c_\alpha e^{-i\alpha\omega}$, so that the sequence $\boldsymbol{c}$ can be obtained by inverse discrete Fourier transform.

THEOREM 4.1. *Let us denote by* $\Psi_c^{(\boldsymbol{n},\boldsymbol{h})}$ *the shift of a given* $\Psi^{(\boldsymbol{n},\boldsymbol{h})}$ *centered at* 0. *For any* $\Psi_c^{(\boldsymbol{n},\boldsymbol{h})}$ *with admissible* $(\boldsymbol{n}, \boldsymbol{h})$, *the CIP is solvable.*

PROOF. From the Poisson summation formula we can write

$$\sum_{\alpha\in\mathbb{Z}^2} \Psi_c^{(\boldsymbol{n},\boldsymbol{h})}(\alpha)e^{-i\alpha\omega} = \sum_{\alpha\in\mathbb{Z}^2} \widehat{\Psi}_c^{(\boldsymbol{n},\boldsymbol{h})}(2\pi\alpha-\omega)\,,$$

and by the symmetry

$$\sum_{\alpha\in\mathbb{Z}^2} \widehat{\Psi}_c^{(\boldsymbol{n},\boldsymbol{h})}(2\pi\alpha-\omega) = \sum_{\alpha\in\mathbb{Z}^2} \widehat{\Psi}_c^{(\boldsymbol{n},\boldsymbol{h})}(2\pi\alpha+\omega)\,.$$

Since $\{\Psi_c^{(\boldsymbol{n},\boldsymbol{h})}(x-\alpha), \alpha\in\mathbb{Z}^2\}$ are linearly independent, it follows that $\{\widehat{\Psi}_c^{(\boldsymbol{n},\boldsymbol{h})}(\omega+2\pi\alpha),\ \alpha\in\mathbb{Z}^2\}$ are linearly independent as well [14, Theorem 5.1], so that $\sum_{\alpha\in\mathbb{Z}^2} \Psi_c^{(\boldsymbol{n},\boldsymbol{h})}(\alpha)e^{-i\alpha\omega} \neq 0$ and the CIP is solvable. □

Obviously, the solvability of the CIP for any $\Psi_c^{(\boldsymbol{n},\boldsymbol{h})}$ with admissible $(\boldsymbol{n},\boldsymbol{h})$ guarantees the existence of the *fundamental function* $L^{(\boldsymbol{n},\boldsymbol{h})}$ such that $L^{(\boldsymbol{n},\boldsymbol{h})}(\beta) = \delta_{\beta,0}$ for all $\beta\in\mathbb{Z}^2$. As showed in [2, Chapter 9], $L^{(\boldsymbol{n},\boldsymbol{h})}$ has the analytical expression

$$L^{(\boldsymbol{n},\boldsymbol{h})}(x) := \sum_{\alpha\in\mathbb{Z}^2} \ell_\alpha^{(\boldsymbol{n},\boldsymbol{h})}\, \Psi_c^{(\boldsymbol{n},\boldsymbol{h})}(x-\alpha), \quad x\in\mathbb{R}^2\,, \tag{4.3}$$

where the sequence $\boldsymbol{\ell}^{(\boldsymbol{n},\boldsymbol{h})} = \{\ell_\alpha^{(\boldsymbol{n},\boldsymbol{h})}, \alpha\in\mathbb{Z}^2\}$ decays exponentially as $\|\alpha\|$ goes to infinity. Thus, $L^{(\boldsymbol{n},\boldsymbol{h})}$ also decays exponentially and the interpolating function $s^{(\boldsymbol{n},\boldsymbol{h})}$ can be represented as

$$s^{(\boldsymbol{n},\boldsymbol{h})}(x) := \sum_{\alpha\in\mathbb{Z}^2} d_\alpha\, L^{(\boldsymbol{n},\boldsymbol{h})}(x-\alpha), \qquad x\in\mathbb{R}^2\,. \tag{4.4}$$

We remark that the exponential decay of $L^{(\boldsymbol{n},\boldsymbol{h})}$, implies the convergence of the above given series also in case the data sequence $\boldsymbol{d}$ has power growth.

5 – A few examples

In this section we shall give some examples of GP functions and we shall construct the corresponding bivariate refinable functions for solving the cardinal interpolation problem.

In case $h = 1$ the mask (2.6) depends on one parameter $b_{0,0}^{(n,1)}$ that for easy computation we set as $b_{0,0}^{(n,1)} = 2^{-k}$, where $k \geq 2$ is a real number. In the following we shall denote the corresponding mask and refinable function as $\boldsymbol{a}^{(n,k)}$ and $\varphi^{(n,k)}$, respectively. From (2.6) it follows

$$a_\alpha^{(n,k)} = \frac{1}{2^h}\left[\binom{n+1}{\alpha} + 4(2^{k-n}-1)\binom{n-1}{\alpha-1}\right], \quad \alpha = 0,\dots,n+1\,, \tag{5.1}$$

so that the symbol is

$$a^{(n,k)}(z) = \frac{1}{2^k}(1+z)^{n-1}(z^2 + (2^{k-n+2} - 2)z + 1)\,. \tag{5.2}$$

To have an idea of the behavior of the GP functions in fig. 1 the graphs of $\varphi^{(n,k)}$ for $n = 3$ and $k = 3, 4, 8$ are shown. The corresponding masks are $\boldsymbol{a}^{(3,3)} = \frac{1}{2^3}\{1, 4, 6, 4, 1\}$, $\boldsymbol{a}^{(3,4)} = \frac{1}{2^4}\{1, 8, 14, 8, 1\}$, $\boldsymbol{a}^{(3,8)} = \frac{1}{2^8}\{1, 128, 254, 128, 1\}$. Note that $\varphi^{(3,3)}$ is just the cubic B-spline with integer knots.

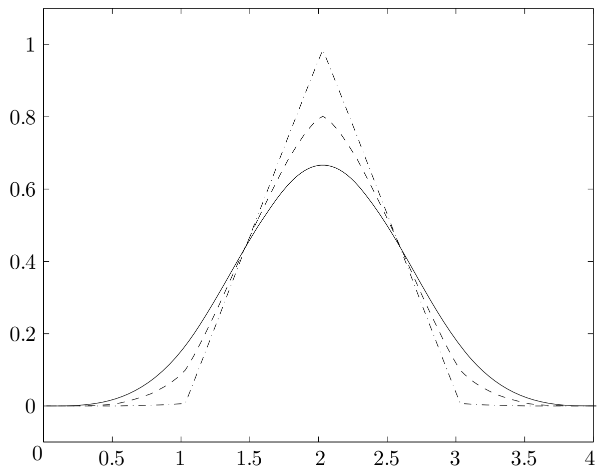

Fig. 1: Graphs of $\varphi^{(3,3)}(-), \varphi^{(3,4)}(--)$ and $\varphi^{(3,8)}(.-)$.

Using three GP functions $\varphi^{(n_i,k_i)}$, $i = 1, 2, 3$, in (3.1), we can construct the bivariate function

$$\Psi^{(\boldsymbol{n},\boldsymbol{k})} = (\varphi^{(n_1,k_1)} \otimes \varphi^{(n_2,k_2)}) *_e \varphi^{(n_3,k_3)}\,. \tag{5.3}$$

In particular, we consider here the case when $(n_1, k_1) = (n_2, k_2) = (n_3, k_3) = (n, k)$ and $e = (1, 1)$. For shortness in the following, we use the unique superscript (n, k). The mask of the convolved functions for $(n, k) = (3, 3)$ and $(n, k) = (3, 8)$ are

$$\boldsymbol{a}^{(3,3)} = \frac{1}{2^7}\begin{bmatrix} 0 & 1 & 4 & 6 & 4 & 1 \\ 1 & 8 & 22 & 28 & 17 & 4 \\ 4 & 22 & 48 & 52 & 28 & 6 \\ 6 & 28 & 52 & 48 & 22 & 4 \\ 4 & 17 & 28 & 22 & 8 & 1 \\ 1 & 4 & 6 & 4 & 1 & 0 \end{bmatrix}$$

and

$$\boldsymbol{a}^{(3,8)} = \frac{1}{2^{17}} \begin{bmatrix} 0 & 1 & 128 & 254 & 128 & 1 \\ 1 & 256 & 16638 & 32640 & 16385 & 128 \\ 128 & 16638 & 65024 & 80900 & 32640 & 254 \\ 254 & 32640 & 80900 & 65024 & 16638 & 128 \\ 128 & 16385 & 32640 & 16638 & 256 & 1 \\ 1 & 128 & 254 & 128 & 1 & 0 \end{bmatrix}.$$

The associated refinable functions are shown in fig. 2. Note that the function having mask $\boldsymbol{a}^{(3,3)}$ is a three directional box-splines.

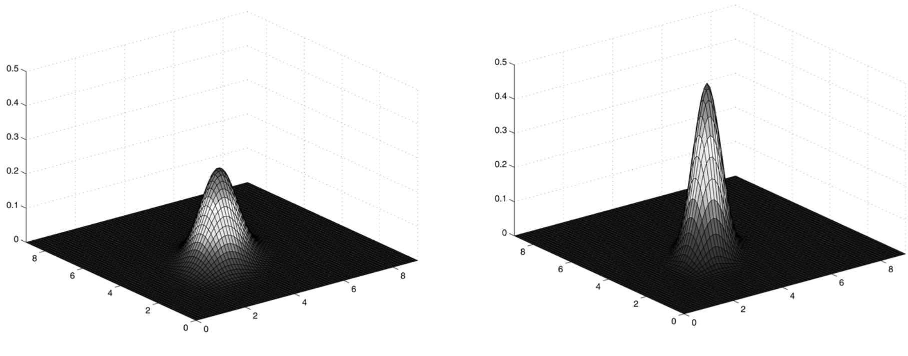

Fig. 2: Graph of the bivariate functions $\Psi^{(3,3)}$ (left) and $\Psi^{(3,8)}$ (right).

Finally, we consider the solution of the CIP first for data taken from the non continuous function $\chi_{[-4,4]^2}$. In particular the left picture displays the result we get by using the three directional box-spline $\Psi^{(3,3)}$ while the right picture concerns the use of $\Psi^{(3,8)}$. The reduction of the Gibbs effect when increasing k is evident: the overshoot of the interpolating function is 7% in the first case and 3% in the second one.

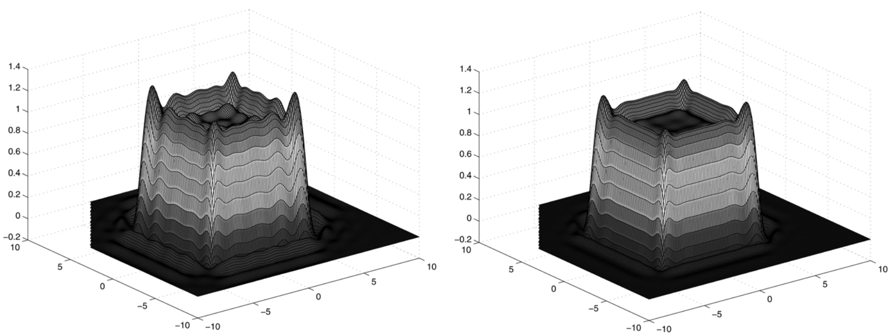

Fig. 3: Solution of the CIP with $\Psi^{(3,3)}$ (left) and $\Psi^{(3,8)}$ (right).

As a second example we consider the solution of the CIP for data taken from the pyramid having losang shape support with vertices $\{(2,0),\ (0,2),\ (-2,0),\ (0,-2)\}$; the pyramid is central symmetric and reaches the maximum value 2 at the origin. Again the left picture displays the result we get by using the three directional box-spline $\Psi^{(3,3)}$ while the right picture is the result with the convolved function $\Psi^{(3,8)}$. The picture shows that also in this case the Gibbs effect near the basis of the pyramid is reduced when k increases; moreover, while the interpolant constructed by means of $\Psi^{(3,3)}$ shows some smoothing of the edges of the pyramid, the interpolant constructed by means of $\Psi^{(3,8)}$ does not.

The examples show that the additional degree of freedom given by the extra parameter allows us to choose, among the functions in the family, the one which provides better performances in the application we are interested in.

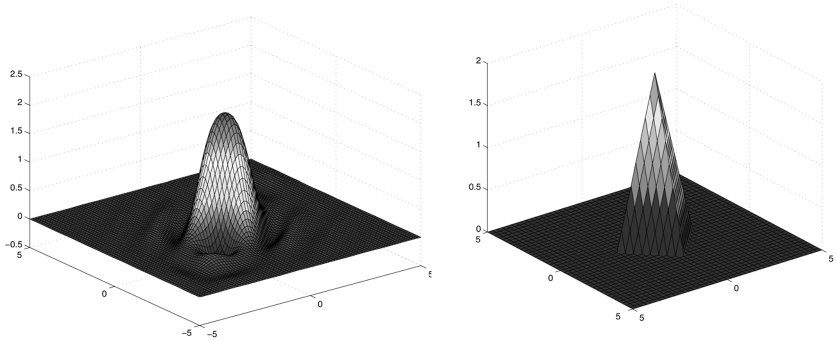

Fig. 4: Solution of the CIP with $\Psi^{(3,3)}$ (left) and $\Psi^{(3,8)}$ (right).

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