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Input-output techniques for the stability of evolution families on the half-line

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Abstract: *The aim of the paper is to obtain general input-output conditions for the uniform exponential stability of evolution families. Thus it is involved a very general class of function spaces which are translation invariant. The present approach includes as particular cases, many interesting situations among them we note the results obtained by Y. Latushkin, S. Montgomery-Smith, T. Randolph, N. van Minh, F. Rabiger and R. Schnaubelt.*

1 – Introduction

Linear evolution equations in Banach spaces have seen important developments in the recent decades. An important role in the early development of qualitative theory of differential systems, was played by the paper "Die Stabilitätsfrage bei Differentialgleichungen" [13], where Perron gave a characterization of exponential stability of the solutions to the linear differential equations

$$\frac{dx}{dt} = A(t)x, \quad t \in [0, +\infty), \quad x \in \mathbb{R}^n, \text{ where}$$

$A(t)$ is a matrix bounded continuous function, in terms of the existence of bounded solutions of the equations $\frac{dx}{dt} = A(t)x + f(t)$, where f is a continuous bounded function on $\mathbb{R}_+$. After these seminal researches of O. Perron,

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valuable contributions concerning the extension of Perron's problem in the more general framework of infinite-dimensional Banach spaces were obtained, in their pioneering monographs, by M. G. Krein and J. L. Daleckij [5], R. Bellman [1], J. L. Massera and J. J. Schäffer [11]. In last years, the input-ouput techniques led to several results about exponential stability and exponential dichotomy for the case of exponentially bounded and strongly continuous evolution families (see for instances the contributions of N. van Minh [10], [11], F. Räbiger [10], Y. Latushkin [2], [6], [7], [8], P. Preda [13], [14], [15], [16], T. Randolph [7], [8], R. Schnaubelt [8], [10], [17]. Generally, if the characterizations of certain asymptotic properties rely on the use of the theory of evolution semigroups, then the input space and the output space must be the same (see [2] or [10]). Here, for instance the proofs are direct being based on the use of appropriate input functions, so one obtains characterizations of diverse concepts such that the input and the output space may be distinct. In the spirit of Perron's idea and of all the above recent results, the aim of this paper is to propose a general, unitary and complete study for the uniform exponential stability of evolution families in terms of input-output conditions. Therefore, we involve a general class of translation invariant function spaces and accordingly to Massera and Schaffer (see [11, pag. 57]) we will call this class as the set of all $\mathcal{T}$-spaces. We want to emphasize that until now the most common classes of spaces used as input or output spaces were the L^p or M^p spaces. These are in particular $\mathcal{T}$-spaces, so our treatment include as particular cases almost all the situations required by the input-output techniques, among them we note (L^p, L^q)-admissibility, (M^p, M^q)-admissibility, (L^p, M^q)-admissibility. Thus, generalizations of the well-known results due to N. van Minh, F. Rabiger and R. Schnaubelt, are also obtained. Moreover, the large class of Orlicz spaces is also contained in this general context of $\mathcal{T}$-spaces and from here other interesting applications arise (as for instance Example 2.3 that present a $\mathcal{T}$-space contained in each $L^p, p \in [1,\infty)$ but different than any L^p-space, $p \in [1,\infty)$).

2 – Preliminaries

Let $B(X)$ be the Banach algebra of all linear and bounded operators acting on the Banach space X. Also, we denote by $\mathcal{M}(\mathbb{R}_+, X)$ the space of all Bochner measurable functions from $\mathbb{R}_+$ to X and by:

$$L^1_{\mathrm{loc}}(\mathbb{R}_+, X) = \{f \in \mathcal{M}(\mathbb{R}_+, X) : \int_K ||f(t))||dt < \infty, \text{ for each compact } K \text{ in } \mathbb{R}_+\}$$

$$L^p(\mathbb{R}_+, X) = \{f \in \mathcal{M}(\mathbb{R}_+, X) : \int_{\mathbb{R}_+} ||f(t)||^p dt < \infty\}, \text{ where } p \in [1,\infty)$$

$$L^\infty(\mathbb{R}_+, X) = \{f \in \mathcal{M}(\mathbb{R}_+, X) : \text{ess}\sup_{t\in\mathbb{R}_+} ||f(t)|| < \infty\}$$

$$M^p(\mathbb{R}_+, X) = \{f \in \mathcal{M}(\mathbb{R}_+, X) : \sup_{t\in\mathbb{R}_+} \int_t^{t+1} ||f(s)||^p ds < \infty\} \text{ where } p \in [1,\infty),$$

$T(\mathbb{R}_+, X)$ the space of all functions $f \in L^1_{\text{loc}}(I, X)$ with the property that there exist $(\tau_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ two sequences of positive real numbers such that

$$\sum_{n=0}^\infty a_n < \infty \quad \text{and} \quad ||f|| \le \sum_{n=0}^\infty a_n \chi_{[\tau_n, \tau_n+1]} .$$

We note that $L^p(\mathbb{R}_+, X), L^\infty(\mathbb{R}_+, X), M^p(\mathbb{R}_+, X), T(\mathbb{R}_+, X)$ are Banach spaces endowed with the respectively norms:

$$||f||_p = \left(\int_{\mathbb{R}_+} ||f(t)||^p dt\right)^{\frac{1}{p}}$$

$$||f||_\infty = \text{ess}\sup_{t\in\mathbb{R}_+} ||f(t)||$$

$$||f||_{M^p} = \sup_{t\in\mathbb{R}_+} \left(\int_t^{t+1} ||f(s)||^p ds\right)^{\frac{1}{p}} \quad \text{and}$$

$$||f||_T = \inf\left\{\sum_{n=0}^\infty a_n : \text{where} \quad (a_n)_{n\in\mathbb{N}} \quad \text{satisfy the above inequality}\right\}.$$

In order to simplify the notations we put $L^p := L^p(\mathbb{R}_+, \mathbb{R})$, $L^\infty := L^\infty(\mathbb{R}_+, \mathbb{R})$, $M^p := M^p(\mathbb{R}_+, \mathbb{R})$, for all $p \in [1,\infty)$ and $L^1_{\text{loc}} = L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}), T = T(\mathbb{R}_+, \mathbb{R})$. Next we recall the definition of $\mathcal{T}$-spaces.

Definition 2.1. A Banach space E is said to be a $\mathcal{T}$-space if the following conditions hold:

s_1) $E \subset L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ and for each compact $K \subset \mathbb{R}_+$ there is $\alpha_K > 0$ such that

$$\int_K |f(t)|dt \le \alpha_K ||f||_E, \text{ for all } f \in E$$

s_2) $\chi_{[0,t]} \in E$, for each $t \ge 0$, where $\chi_{[0,t]}$ denotes the characteristic function (indicator) of the interval $[0, t]$

s_3) If $f \in E$ and $h \in \mathcal{M}(\mathbb{R}_+, \mathbb{R})$ with $|h| \le |f|$, then $h \in E$ and $||h||_E \le ||f||_E$.

s_4) If $f \in E$, $t \ge 0$, $g_t : \mathbb{R}_+ \to \mathbb{R}$, $g_t(s) = \begin{cases} 0, & s \in [0,t) \\ f(s-t), & s \in [t,\infty) \end{cases}$ then $g_t \in E$ and $||g_t||_E = ||f||_E$.

EXAMPLE 2.1. It is a routine to verify that M^p, L^p, L^∞ and T, the spaces mentioned above, are $\mathcal{T}$-spaces. One can easy remark that $T \subset E \subset M^1$, for any $\mathcal{T}$-space E. (More results in this direction can be found in [9].)

EXAMPLE 2.2. It can be observed that the Orlicz spaces are $\mathcal{T}$-spaces, too. For more convenience we recall in the next the definition of Orlicz spaces. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function which is non-decreasing, left-continuous, $\varphi(t) > 0$, for all $t > 0$. Define

$$\Phi(t) = \int_0^t \varphi(s)ds\,.$$

A function Φ of this form is called a Young function. For $f : \mathbb{R}_+ \to \mathbb{R}$ a measurable function and Φ a Young function we define

$$M^\Phi(f) = \int_0^\infty \Phi(|f(s)|)ds\,.$$

The set L^Φ of all f for which there exists a $k > 0$ that $M^\Phi(kf) < \infty$ is easily checked to be a linear space. With the norm

$$\rho^\Phi(f) = \inf\Big\{k > 0 : M^\Phi\Big(\frac{1}{k}f\Big) \le 1\Big\}$$

the space (L^Φ, ρ^Φ) becomes a Banach space which is easy to check that verify the conditions $s_2), s_3), s_4)$. Inorder to verify the condition $s_1)$ consider $f \in L^\Phi, t > 0, k > 0$ such that $M^\Phi(\frac{1}{k}f) \le 1$. Then we have that

$$\Phi\Big(\frac{1}{kt}\int_0^t |f(s)|ds\Big) \le \frac{1}{t}\int_0^t \Phi\Big(\frac{1}{k}|f(s)|\Big)ds \le \frac{1}{t}\,,$$

and so

$$\int_0^t |f(s)|ds \le t\Phi^{-1}\Big(\frac{1}{t}\Big)k$$

which implies that

$$\int_0^t |f(s)|ds \le t\Phi^{-1}\Big(\frac{1}{t}\Big)\rho^\Phi(f)\,,$$

for all $f \in L^\Phi, t > 0$, and hence the condition $s_1)$ is also verified. The connection between Orlicz spaces and the L^p spaces is given by

REMARK 2.1. $L^\Phi = L^p$ if and only if $\Phi(t) = t^p$, for all $t \geq 0$.

The "only if" part is obvious. Conversely if $L^\Phi = L^p$ then $||\chi_{[0,t]}||_\Phi = ||\chi_{[0,t]}||_p$, for all $t > 0$, and so $\Phi^{-1}(s) = s^{\frac{1}{p}}$ for all $s > 0$, which implies that $\Phi(t) = t^p$, for all $t \geq 0$.

From here other useful situations can arise. For instance, in order to verify easier the admissibility condition it is important to find smaller input spaces. Thus, since this approach offers a very large degree of liberty in choosing the input-output spaces, we can construct easily an Orlicz space (and as it can be seen implicit a $\mathcal{T}$-space) which is contained in each L^p, for all $p \in [1,\infty)$, but different than any L^p-space, $p \in [1,\infty)$.

EXAMPLE 2.3. If we take $\Phi(t) = e^t - 1$ then $L^\Phi \subset L^p$, for all $p \in [1,\infty)$.

Indeed one can see that $t^m \leq m!\Phi(t)$ for all $t \geq 0$ and all $m \in \mathbb{N}^*$ which implies that $L^\Phi \subset L^m$, for all $m \in \mathbb{N}^*$. Having in mind that $L^m \bigcap L^{m+1} \subset L^p$ for all $p \in [m, m+1]$, and all $m \in \mathbb{N}^*$, it follows that $L^\Phi \subset L^p$, for all $p \in [m, m+1]$, and all $m \in \mathbb{N}^*$.

If E is a $\mathcal{T}$-spaces we denote by

$$E(X) = \{f \in \mathcal{M}(\mathbb{R}_+, X) : t \mapsto ||f(t)|| : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is in } E\}.$$

REMARK 2.2. $E(X)$ is a Banach space endowed with the norm

$$||f||_{E(X)} = ||\ ||f(\cdot)||\ ||_E .$$

REMARK 2.3. If $\{f_n\}_{n\in\mathbb{N}} \subset E(X)$, $f \in E(X)$, $f_n \to f$ in $E(X)$ when $n \to \infty$, then there exists $\{f_{n_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ such that

$$f_{n_k} \to f \ a.e.$$

For a $\mathcal{T}$-space E we denote by $\alpha_E, \beta_E : \mathbb{R}_+ \to \mathbb{R}_+$ the following functions:

$$\alpha_E(t) = \inf\left\{\alpha > 0 : \int_0^t |f(s)|ds \leq \alpha||f||_E, \text{ for all } (t,f) \in \mathbb{R}_+ \times E\right\},$$

$$\beta_E(t) = ||\chi_{[0,t]}||_E .$$

It is known (see for instance [9]) that α_E, β_E are nondecreasing functions and moreover

$$(*) \quad t \leq \alpha_E(t)\beta_E(t) \leq 2t, \quad \text{for all} \quad t \geq 0 .$$

EXAMPLE 2.4. It is easy to see that for L^p and M^p we have:

$$\alpha_{L^p}(t) = \begin{cases} t^{1-\frac{1}{p}}, & p \in [1,\infty),\ t \geq 0 \\ t, & p = \infty,\ t \geq 0 \end{cases}$$

$$\beta_{L^p}(t) = \begin{cases} t^{\frac{1}{p}}, & p \in [1,\infty),\ t \geq 0 \\ 1, & p = \infty,\ t \geq 0 \end{cases}$$

$t \leq \alpha_{M^p}(t) \leq [t] + \{t\}^{1-\frac{1}{p}}$, for each $(p,t) \in [1,\infty) \times \mathbb{R}_+$, where $[t]$ denotes the largest integer less or equal than t and $\{t\} = t - [t]$.

$$\beta_{M^p}(t) = \begin{cases} t^{\frac{1}{p}}, & t \in [0,1) \\ 1, & t \geq 1 \end{cases}$$

$$\alpha_{L^\Phi}(t) = t\Phi^{-1}\Big(\frac{1}{t}\Big) \quad \beta_{L^\Phi}(t) = \Big(\Phi^{-1}\Big(\frac{1}{t}\Big)\Big)^{-1} .$$

Definition 2.2. A $B(X)$-valued function $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is called an evolution family if:

- the identity on X can be obtained as $U(t,t)$, for each $t \geq 0$;
- the evolution property $U(t,s) = U(t,r)U(r,s)$ holds for all $t \geq r \geq s \geq 0$;
- $U(\cdot,s)x$ is continuous on $[s,\infty)$, for all $s \geq 0,\ x \in X$;
- $U(t,\cdot)x$ is continuous on $[0,t)$, for all $t \geq 0,\ x \in X$;
- $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ has an exponential growth (i.e. there are $M,\omega > 0$ such that
$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \text{ for all } t \geq s \geq 0 .$$

Definition 2.3. The evolution families $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is called uniformly exponentially stable (u.e.s) if there exist two strictly positive constants N, ν such that the following statement hold:

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)} .$$

Definition 2.4. The pair (E,F) is said to be admissible to $\mathcal{U}$ if for all $f \in E(X)$ the function $x_f : \mathbb{R}_+ \to X$ defined by $x_f(t) = \int_0^t U(t,s)f(s)ds$ lies in $F(X)$.

3 – The main result

Let (E,F) be a pair of $\mathcal{T}$-spaces.

Lemma 3.1. *If the pair (E,F) is admissible to $\mathcal{U}$ then there is $K > 0$ such that*

$$||x_f||_{F(X)} \leq K||f||_{E(X)} .$$

Proof. Let us define the linear operator $T : E(X) \to F(X)$, given by

$$(Tf)(t) = \int_0^t U(t,s)f(s)ds\,.$$

If $\{g_n\}_{n\in\mathbb{N}} \subset E(X)$, $g \in E(X)$, $h \in F(X)$ such that

$$g_n \overset{E(X)}{\longrightarrow} g, \quad Tg_n \overset{F(X)}{\longrightarrow} h$$

$$||(Tg_n)(t) - (Tg)(t)|| \le \int_0^t ||U(t,s)(g_n(s) - g(s))||ds \le$$

$$\le \int_0^t Me^{\omega t}||g_n(s) - g(s)||ds \le$$

$$\le Me^{\omega t}\alpha_E(t)||g_n - g||_{E(X)}\,,$$

for all $t \ge 0$ and all $n \in \mathbb{N}$.

It follows, using again the Remark 2.3, that $Tg = h$, and hence T is closed and so, by the Closed-Graph theorem it is also bounded. So we obtain that

$$||x_f||_{F(X)} = ||Tf||_{F(X)} \le ||T||\;||f||_{E(X)}, \text{ for all } f \in E(X) \quad \text{as required}\,.$$

Lemma 3.2. *If F is a $\mathcal{T}$-space, $h \in F$, $h \ge 0$ and if there are two constants $a, b > 0$ such that $h(r) \le ah(t) + b$, for all $r \ge t \ge 0$ with $r - t \le 1$, then $h \in L^\infty$.*

Proof. By the hypothesis we have that

$$h(n+1) \le ah(s) + b, \text{ for all } n \in \mathbb{N} \text{ and all } s \in [n, n+1]$$

and from here

$$h(n+1) \le a\int_n^{n+1} h(s)ds + b \le a\alpha_F(1)||h||_F + b, \text{ for all } n \in \mathbb{N}$$

which implies that

$$c = \sup_{n\in\mathbb{N}} h(n) < \infty\,.$$

Using again the hypothesis, we obtain that

$$h(t) \le ah(n) + b \le ac + b, \text{ for all } \ n \in \mathbb{N}, \text{ and all } t \in [n, n+1]\,.$$

We consider again E and F being two $\mathcal{T}$-spaces and we have:

Lemma 3.3. *If the pair (E, F) is admissible to $\mathcal{U}$ then the following statements hold*:

i) *for all $f \in E(X)$ there exist $a, b > 0$ such that*

$$||x_f(r)|| \le a||x_f(t)|| + b, \text{ for all } r \ge t \ge 0 \text{ with } r - t \le 1\,;$$

ii) *the pair (E, L^∞) is admissible to $\mathcal{U}$.*

PROOF. i) We have that:

$$x_f(r) = \int_0^r U(r,s)f(s)ds =$$

$$= \int_0^t U(r,t)U(t,s)f(s)ds + \int_t^r U(r,s)f(s)ds =$$

$$= U(r,t)x_f(t) + \int_t^r U(r,s)f(s)ds, \text{ for all } r \geq t \geq 0\,.$$

It results that

$$||x_f(r)|| \leq Me^{\omega(r-t)}||x_f(t)|| + \int_t^r Me^{\omega(r-s)}||f(s)||ds \leq$$

$$\leq Me^{\omega}||x_f(t)|| + Me^{\omega}\int_t^{t+1} ||f(s)||ds \leq$$

$$\leq Me^{\omega}||x_f(t)|| + Me^{\omega}\alpha_E(1)||f||_{E(X)}$$

for all $r \geq t \geq 0$ with $r - t \leq 1$.

The condition ii) follows directly from i) and Lemma 3.2.

LEMMA 3.4. *Let* $g : \{(t,t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\} \to \mathbb{R}_+$ *be a function such that the following properties hold*:

1) $g(t,t_0) \leq g(t,s)g(s,t_0)$, *for all* $t \geq s \geq t_0 \geq 0$;
2) *there exist* $M, a > 0$ *and* $b \in (0,1)$ *such that*

$$g(t,t_0) \leq M, \quad \textit{for all } \ t_0 \geq 0 \textit{ and all } t \in [t_0, t_0 + a]$$

$$g(t_0 + a, t_0) \leq b, \qquad \textit{for all } t_0 \geq 0\,.$$

Then there exist two constants $N, \nu > 0$ *such that*

$$g(t,t_0) \leq Ne^{-\nu(t-t_0)}, \textit{ for all } t \geq t_0 \geq 0\,.$$

PROOF. Let $t \geq t_0 \geq 0$ and $n = \left[\frac{t-t_0}{a}\right]$, the largest integer less than or equal with $\frac{t-t_0}{a}$.

Then we have that

$$g(t,t_0) \leq g(t,t_0+na)g(t_0+na,t_0) \leq$$

$$\leq g(t,t_0+na)b^n \leq Mb^n = Me^{-\nu na} \leq Ne^{-\nu(t-t_0)}$$

where $\nu = -\frac{1}{a}\ln b,\ N = Me^{\nu a}$, as required.

Now we can state the main result of this paper.

Theorem 3.1. *$\mathcal{U}$ is u.e.s. if and only if there exists a pair (E, F) of $\mathcal{T}$-spaces, admissible to $\mathcal{U}$ with $\lim_{t\to\infty} \alpha_E(t)\beta_F(t) = \infty$.*

Proof. *Necessity.* It follows easily that the pair (L^∞, L^∞) is admissible to $\mathcal{U}$.

Sufficiency. First observe that if the pair (E, F) is admissible to $\mathcal{U}$ then by Lemma 3.3 the pair (E, L^∞) is admissible to $\mathcal{U}$.

Let $x \in X,\ t_0 \geq 0$ and $f : \mathbb{R}_+ \to X$,

$$f(t) = \begin{cases} U(t, t_0)x, & t \in [t_0, t_0 + 1] \\ 0, & t \in \mathbb{R}_+ \setminus [t_0, t_0 + 1]\,. \end{cases}$$

It is easy to check that $f \in E(X)$ and $||f||_{E(X)} \leq Me^{\omega}\beta_E(1)||x||$ and

$$x_f(t) = \begin{cases} 0, & 0 \leq t \leq t_0 \\ \displaystyle\int_{t_0}^{t_0+1} U(t, s)f(s)ds, & t \geq t_0 + 1\,. \end{cases}$$

If $t \geq t_0 + 1$ then,

$$x_f(t) = \int_{t_0}^{t_0+1} U(t, s)U(s, t_0)xds = U(t, t_0)x$$

which implies that

$$||U(t, t_0)x|| = ||x_f(t)|| \leq ||x_f||_\infty \leq K||f||_{E(X)} \leq KMe^{\omega}\beta_E(1)||x||$$

for all $t \geq t_0 + 1,\ t_0 \geq 0$ and all $x \in X$.

Hence there exits $L > 0$ such that

$$||U(t, t_0)|| \leq L, \text{ for all } t \geq t_0 \geq 0\,.$$

Let $t_0 \geq 0,\ \delta > 0,\ x \in X$ and $g : \mathbb{R}_+ \to X$

$$g(t) = \begin{cases} U(t, t_0)x, & t \in [t_0, t_0 + \delta] \\ 0, & t \in \mathbb{R}_+ \setminus [t_0, t_0 + \delta]\,. \end{cases}$$

Then $g \in E(X)$, and $||g||_{E(X)} \leq L\beta_E(\delta)||x||$. It follows that

$$x_g(t) = \int_0^t U(t,s)f(s)ds = \begin{cases} 0, & t \in [0,t_0) \\ (t-t_0)U(t,t_0)x, & t \in [t_0, t_0+\delta) \\ \delta U(t,t_0)x, & t \in [t_0+\delta, \infty) \end{cases}$$

and so

$$\begin{aligned} \frac{\delta^2}{2}||U(t_0+\delta,t_0)x|| &= \int_{t_0}^{t_0+\delta}(s-t_0)||U(t_0+\delta,t_0)x||ds \leq \\ &\leq \int_{t_0}^{t_0+\delta}(s-t_0)L||U(s,t_0)x||ds = \\ &= L\int_{t_0}^{t_0+\delta}||x_g(s)||ds \leq L\alpha_F(\delta)||x_g||_{F(X)} \leq \\ &\leq KL\alpha_F(\delta)||g||_{E(X)} \leq \\ &\leq KL^2\alpha_F(\delta)\beta_E(\delta)||x|| \leq \frac{4KL^2\delta^2}{\alpha_E(\delta)\beta_F(\delta)}||x|| \end{aligned}$$

for all $t_0 \geq 0,\ \delta > 0,\ x \in X$.

We obtain that

$$||U(t_0+\delta,t_0)|| \leq \frac{8KL^2}{\alpha_E(\delta)\beta_F(\delta)},\ \text{for all } t_0 \geq 0,\ \delta > 0\,.$$

By Lemma 3.4 it results that there exist two constants $N, \nu > 0$ such that

$$||U(t,t_0)|| \leq Ne^{-\nu(t-t_0)},\ \text{for all } t \geq t_0 \geq 0\,.$$

Now we conclude with

THEOREM 3.2. *The following assertions are equivalent*

1) $\mathcal{U}$ *is u.e.s*;
2) *there exists* E *a* $\mathcal{T}$*-space, such that the pair* (E,E) *is admissible to* $\mathcal{U}$;
3) *there exist* $p,q \in [1,\infty],\ (p,q) \neq (1,\infty)$ *such that the pair* (L^p, L^q) *is admissible to* $\mathcal{U}$;
4) *there exists* $p,q \in [1,\infty)$ *such that the pair* (M^p, M^q) *is admissible to* $\mathcal{U}$;
5) *there exists* $p \in (1,\infty],\ q \in [1,\infty)$ *such that the pair* (L^p, M^q) *is admissible to* $\mathcal{U}$.

Proof. Follows easily from Theorem 3.1 and Example 2.2.

Remark 3.1. From the statement (2) of the Theorem 3.2 and Example 2.2 it follows also that $\mathcal{U}$ is u.e.s. if and only if there is an Orlicz space L^Φ such that (L^Φ, L^Φ) is admissible to $\mathcal{U}$.

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