A lower bound for the $b$-adic diaphony

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Abstract: The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a point set in the $s$-dimensional unit cube. In this note we show that the $b$-adic diaphony (for prime $b$) of a point set consisting of $N$ points in the $s$-dimensional unit cube is always at least of order $(\log N)^{(s-1)/2}/N$. This lower bound is best possible.

1 – Introduction

As the (classical) diaphony (see [25] or [8, Definition 1.29] or [16, Exercise 5.27, p. 162]) the $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the $s$-dimensional unit cube. This notion was introduced by Hellekalek and Leeb [15] for $b = 2$ and later generalized by Grozdanov and Stoilova [11] for general integers $b \geq 2$. The main difference to the classical diaphony is that the trigonometric functions are replaced by $b$-adic Walsh functions. Before we give the exact definition of the $b$-adic diaphony we recall the definition of Walsh functions.

Let $b \geq 2$ be an integer. For a non-negative integer $k$ with base $b$ representation $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1b + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function $b \operatorname{wal}_k : [0,1) \to \mathbb{C}$ by

$$b \operatorname{wal}_k(x) := e^{2\pi i (x_1\kappa_0 + \cdots + x_a\kappa_{a-1})/b},$$

for $x \in [0,1)$ with base $b$ representation $x = x_1b + x_2b^2 + \cdots$ (unique in the sense that infinitely many of the $x_i$ must be different from $b-1$).

Key Words and Phrases: $b$-adic diaphony – $L_2$ discrepancy – Uniform distribution of sequences.

A.M.S. Classification: 11K06 – 11K38
For dimension $s \geq 2$, $x_1, \ldots, x_s \in [0, 1)$ and $k_1, \ldots, k_s \in \mathbb{N}_0$ we define $b \text{wal}_{k_1, \ldots, k_s} : [0, 1)^s \to \mathbb{C}$ by

$$b \text{wal}_{k_1, \ldots, k_s}(x_1, \ldots, x_s) := \prod_{j=1}^s b \text{wal}_{k_j}(x_j).$$

For vectors $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in [0, 1)^s$ we write

$$b \text{wal}_k(x) := b \text{wal}_{k_1, \ldots, k_s}(x_1, \ldots, x_s).$$

If it is clear which base we mean we simply write $\text{wal}_k(x)$. It is clear from the definitions that Walsh functions are piecewise constant. It can be shown that for any integer $s \geq 1$ the system $\{b \text{wal}_k : k \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L^2([0, 1)^s)$, see for example [1], [17] or [20, Satz 1]. For more information on Walsh functions we refer to [1], [20], [24].

Now we give the definition of the $b$-adic diaphony (see [11] or [15]).

**Definition 1.** Let $b \geq 2$ be an integer. The $b$-adic diaphony of a point set $P_{N,s} = \{x_0, \ldots, x_{N-1}\} \subset [0, 1)^s$ is defined as

$$F_{b,N}(P_{N,s}) := \left( \frac{1}{(1+b)^s - 1} \sum_{k \in \mathbb{N}_0^s \atop k \neq 0} r_b(k) \left| \frac{1}{N} \sum_{h=0}^{N-1} b \text{wal}_k(x_h) \right|^2 \right)^{1/2},$$

where for $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $r_b(k) := \prod_{j=1}^s r_b(k_j)$ and for $k \in \mathbb{Z}$,

$$r_b(k) := \begin{cases} 
1 & \text{if } k = 0, \\
b^{-2a} & \text{if } b^a \leq k < b^{a+1} \text{ where } a \in \mathbb{N}_0.
\end{cases}$$

Note that the $b$-adic diaphony is scaled such that $0 \leq F_{b,N}(P_{N,s}) \leq 1$ for all $N \in \mathbb{N}$, in particular we have $F_{b,1}(P_{1,s}) = 1$. If $b = 2$ we also speak of dyadic diaphony.

The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence $\omega$ in the $s$-dimensional unit cube is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_{b,N}(\omega_N) = 0$, where $\omega_N$ is the point set consisting of the first $N$ points of $\omega$. This was shown in [15] for the case $b = 2$ and in [11] for the general case. Further it is shown in [5] that the $b$-adic diaphony is, up to a factor depending on $b$ and $s$, the worst-case error for quasi-Monte Carlo integration of functions from a certain Hilbert space of functions.

More general notions of diaphony can be found in [10], [13], [14].
Stoilova [22] proved that the $b$-adic diaphony of a $(t, m, s)$-net in base $b$ is bounded by

$$F_{b,N}(P) \leq c(b, s)b^t \frac{(m-t)^{\frac{s-1}{2}}}{b^m},$$

where $c(b, s) > 0$ only depends on $b$ and $s$. For the definition of $(t, m, s)$-nets in base $b$ we refer to Niederreiter [18], [19]. These are point sets consisting of $N = b^m$ points in the $s$-dimensional unit cube with outstanding distribution properties if the parameter $t \in \{0, \ldots , m\}$ is small. However, the optimal value $t = 0$ is not possible for all parameters $s \geq 1$ and $b \geq 2$. Niederreiter [18] proved that if a $(0, m, s)$-net in base $b$ exists, then we have $s - 1 \leq b$. Faure [9] provided a construction of $(0, m, s)$-nets in prime base $p \geq s - 1$ and Niederreiter [18] extended Faure’s construction to prime power bases $p^r \geq s - 1$. Hence if $b \geq s - 1$ is a prime power we obtain for any $m \in \mathbb{N}$ the existence of $N = b^m$ points in $[0,1)^s$ whose $b$-adic diaphony is bounded by

$$F_{b,N}(P) \leq c'(b, s)\frac{(\log N)^{\frac{s-1}{2}}}{N},$$

with $c'(b, s) > 0$. See also [6] where a similar bound on the dyadic diaphony of digital $(t, m, s)$-nets in base 2 (a subclass of $(t, m, s)$-nets) is shown.

The question for a general lower bound for the $b$-adic diaphony was pointed out in [22], see also [12]. In the following section we show that for prime $b$, the $b$-adic diaphony of an $N$-element point set in $[0,1)^s$ is always at least of order $\frac{\left(\log N\right)^{\frac{s-1}{2}}}{N}$, which shows that the above given upper bounds are best possible.

2 – A general lower bound for the $b$-adic diaphony

In the following we prove a lower bound on the $b$-adic diaphony for prime $b$. This is done using Roth’s lower bound on the $L_2$ discrepancy, which is another measure for the distribution properties of a point set.

**Theorem 1.** Let $b$ be a prime. For any dimension $s \geq 1$ there exists a constant $\tau(s, b) > 0$, depending only on the dimension $s$ and $b$, such that the $b$-adic diaphony of any point set $P_{N,s}$ consisting of $N$ points in $[0,1)^s$ satisfies

$$F_{b,N}(P_{N,s}) \geq \tau(s, b)\frac{(\log N)^{\frac{s-1}{2}}}{N}.$$
In the proof of our theorem below we use the generalized notion of weighted $L_2$ discrepancy, which was introduced in [23]. In the following let $D$ denote the index set $D = \{1, 2, \ldots, s\}$ and let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a sequence of non-negative real numbers. For $u \subseteq D$ let $|u|$ be the cardinality of $u$ and for a vector $x \in [0, 1)^s$ let $x_u$ denote the vector from $[0, 1)^{|u|}$ containing all components of $x$ whose indices are in $u$. Further let $\gamma_u = \prod_{j \in u} \gamma_j$, $\text{d}x_u = \prod_{j \in u} \text{d}x_j$, and let $(x_u, 1)$ be the vector $x$ from $[0, 1)^s$ with all components whose indices are not in $u$ replaced by $1$. Then the weighted $L_2$ discrepancy of a point set $P_{N,s} = \{x_0, \ldots, x_{N-1}\}$ is defined as

$$L_{2,\gamma}(P_{N,s}) = \left( \sum_{u \subseteq D} \gamma_u \int_{[0,1]^{|u|}} \Delta((x_u, 1))^2 \text{d}x_u \right)^{1/2},$$

where

$$\Delta(t_1, \ldots, t_s) = \frac{A_N([0, t_1) \times \ldots \times [0, t_s))}{N} - t_1 \ldots t_s,$$

where $0 \leq t_j \leq 1$ and $A_N([0, t_1) \times \ldots \times [0, t_s))$ denotes the number of indices $n$ with $x_n \in [0, t_1) \times \ldots \times [0, t_s)$. We can see from the definition of the weighted $L_2$ discrepancy that the weights $\gamma_u = \prod_{j \in u} \gamma_j$ modify the importance of different projections (see [7], [23] for more information on weights).

In [3] the authors considered point sets which are randomized in the following sense: for $b \geq 2$ let $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ and $\sigma = \frac{\sigma_1}{b} + \frac{\sigma_2}{b^2} + \cdots$ be the base $b$ representation of $x$ and $\sigma$. Then the digitally shifted point $y = x \oplus_b \sigma$ is given by $y = \frac{y_1}{b} + \frac{y_2}{b^2} + \cdots$, where $y_i = x_i + \sigma_i \in \mathbb{Z}_b$. For vectors $x$ and $\sigma$ we define the digitally shifted point $x \oplus_b \sigma$ component wise. Obviously, the shift depends on the base $b$. Now for $P_{N,s} = \{x_0, \ldots, x_{N-1}\} \subseteq [0, 1)^s$ and $\sigma \in [0, 1)^s$ we define the point set $P_{N,s,\sigma} = \{x_0 \oplus_b \sigma, \ldots, x_{N-1} \oplus_b \sigma\}$.

**Proof.** In [3] it was shown that if one chooses $\sigma$ uniformly from $[0, 1)^s$, then the expected value of the weighted $L_2$ discrepancy of a point set $P_{N,s,\sigma}$ is given by

$$\mathbb{E}(L_{2,\gamma}^2(P_{N,s,\sigma})) = \sum_{\mathbf{k} \in \mathbb{N}_0^s, \mathbf{k} \neq \mathbf{0}} \rho_b(\gamma, \mathbf{k}) \left| \frac{1}{N} \sum_{h=0}^{N-1} \text{wal}_k(x_h) \right|^2,$$

where $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{N}_0^s$, $\rho_b(\gamma, \mathbf{k}) = \prod_{j=1}^s \rho_b(\gamma_j, k_j)$, and

$$\rho_b(\gamma, k) = \begin{cases} 1 + \frac{\gamma}{3}, & \text{if } k = 0, \\ \frac{\gamma}{2b^{2(a+1)}} \left( \frac{1}{\sin^2 \left( \frac{\pi \kappa_a}{b} \right)} - \frac{1}{3} \right), & \text{if } b^a \leq k < b^{a+1} \text{ and } \\ \kappa_a = \left\lfloor \frac{k}{b^a} \right\rfloor, & \text{where } a \in \mathbb{N}_0. \end{cases}$$
If we take $\gamma_j = 3b^2$, for $j = 1, \ldots, s$ we have, $\rho_b(\gamma_j, 0) = (1 + b^2) = (1 + b^2)r_b(0)$ and for $k \geq 1$ we have $\rho_b(\gamma_j, k) = \frac{3}{2}r_b(k) \left( \frac{1}{\sin^2\left(\frac{\pi k}{b}\right)} - \frac{1}{3} \right)$. Let us denote $d_b := \max_{1 \leq \kappa \leq b - 1} \left( \frac{1}{\sin^2\left(\frac{\pi \kappa}{b}\right)} - \frac{1}{3} \right)$ and $c_b := \max\{1 + b^2, \frac{3}{2}d_b\}$.

For the above choice of the weights we have

$$\rho_b((3b^2), k) = \prod_{i=1}^{s} \rho_b(3b^2, k_i) \leq c_b^s \prod_{i=1}^{s} r_b(k_i) = c_b^sr_b(k).$$

Hence from the definition of $b$-adic diaphony we obtain the inequality

$$\mathbb{E}(\mathcal{L}_{2,(3b^2)}^2(P_{N,s}, \sigma)) \leq c_b^s((1 + b)^s - 1)F_{b,N}^2(P_{N,s}).$$

Roth [21] proved that for any dimension $s \geq 1$ there exists a constant $\tilde{c}(s) > 0$ such that for any point set consisting of $N$ points in the $s$-dimensional unit cube $[0,1)^s$ the classical $\mathcal{L}_2$ discrepancy of a point set satisfies

$$\mathcal{L}_2^2(P_{N,s}) \geq \tilde{c}(s) \frac{(\log N)^{s-1}}{N^2}.$$

Here we just note that the weights only change the constant $\tilde{c}(s)$, but do not change the convergence rate of the bound (see [2], [4], [23] for more information). Hence, for any point set $P_{N,s}$ consisting of $N$ points in the $s$-dimensional unit cube there is a constant $\tilde{c}(s,b)$, depending only on the dimension $s$, such that

$$\mathcal{L}_{2,(3b^2)}^2(P_{N,s}) \geq \tilde{c}(s,b) \frac{(\log N)^{s-1}}{N^2}.$$

From (2) it follows that there is a constant $\overline{c}(s,b)$, depending only on the dimension and the prime number $b$, such that

$$F_{b,N}^2(P_{N,s}) \geq \overline{c}^2(s,b) \frac{(\log N)^{s-1}}{N^2},$$

which completes the proof. $\square$
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Lavoro pervenuto alla redazione il 1 luglio 2006 ed accettato per la pubblicazione il 13 novembre 2006. 
Bozze licenziate il 14 maggio 2007

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The authors are supported by the Austrian Research Foundation (FWF), Project S9609, that is part of the Austrian National Research Network “Analytic Combinatorics and Probabilistic Number Theory”.