Uniqueness of renormalized solutions for a class of parabolic equations with unbounded nonlinearities

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Abstract: We prove uniqueness and a comparison principle of renormalized solutions for a class of doubly nonlinear parabolic equations 
\[
\frac{\partial b(x,u)}{\partial t} - \text{div}(A(t,x)Du + \Phi(u)) = f, \quad \text{where the right side belongs to } L^1((0,T) \times \Omega) \text{ and where } b(x,u) \text{ is unbounded function of } u \text{ and where } A(t,x) \text{ is a bounded symmetric and coercive matrix, and } \Phi \text{ is continuous function but without any growth assumption on } u.
\]

1 – Introduction

In the present paper we establish the uniqueness and comparison principle for a renormalized solutions for a class of doubly nonlinear parabolic equations of the type

\[
\begin{align*}
\frac{\partial b(x,u)}{\partial t} - \text{div}(A(t,x)Du + \Phi(u)) &= f \quad \text{in } \Omega \times (0,T), \\
 b(x,u)(t = 0) &= b(x,u_0) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega \times (0,T).
\end{align*}
\]

In Problem (1.1)-(1.3) the framework is the following: \( \Omega \) is a bounded domain of \( \mathbb{R}^N, \ (N \geq 1) \), \( T \) is a positive real number while the data \( f \) and \( b(x,u_0) \) in \( L^1(\Omega \times (0,T)) \) and \( L^1(\Omega) \). And where \( b \) is a Carathéodory function such that,

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b(x, s) is unbounded function of s. The matrix A(t, x) is a bounded symmetric
and coercive matrix. The function Φ is just assumed to be continuous on \[ \mathbb{R} \].

When Problem (1.1)-(1.3) is investigated one of the difficulties is due to the
facts that the data \( f \) and \( b(x, u_0) \) only belong to \( L^1 \) and the growths of \( b(x, u) \)
and \( \Phi(u) \) are not controlled with respect to \( u \) (the function \( \Phi(u) \) does not belong
to \( (L^1_{\text{loc}}((0, T) \times \Omega))^N \) in general), so that proving existence of a weak solution
(i.e. in the distribution meaning) seems to be an arduous task. To overcome
this difficulty we use the framework of renormalized solutions. The existence of
a renormalized solutions of (1.1)-(1.3) is proved in H. Redwane [15].

The notion of renormalized solution is introduced by Lions and Di Perna
[14] for the study of Boltzmann equation (see also P.-L. Lions [10] for a few
applications to fluid mechanics models). This notion was then adapted to elliptic
version of (1.1)-(1.3) in Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [8],
in P.-L. Lions and F. Murat [11] and F. Murat [12], [13] (see also [2], [3],
[4], [5], [6], [7]). At the same the equivalent notion of entropy solutions have
been developed independently by Bénilan and al. [1] for the study of nonlinear
elliptic problems.

The paper is organized as follows: Section 2 is devoted to specify the as-
sumptions on \( b, \Phi, f \) and \( u_0 \) needed in the present study and gives the definition
and the existence (Theorem 2.0.3) of a renormalized solution of (1.1)-(1.3). In
Section 3 we establish uniqueness and a comparison principle of such a solution
(Theorem 3.0.4)

2 – Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true:
\( \Omega \) is a bounded open set on \( \mathbb{R}^N \) \((N \geq 1)\), \( T > 0 \) is given and we set \( Q = \Omega \times (0, T) \).

\begin{equation}
(2.1) \quad b : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function such that;}
\end{equation}

for every \( x \in \Omega \) : \( b(x, s) \) is a strictly increasing \( C^1 \)-function, with \( b(x, 0) = 0 \).
For any \( K > 0 \), there exists \( \lambda_K > 0 \), a function \( A_K \) in \( L^1(\Omega) \) and a function \( B_K \)
in \( L^2(\Omega) \) such that

\begin{equation}
(2.2) \quad \lambda_K \leq \frac{\partial b(x, s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \right| \leq B_K(x)
\end{equation}

for almost every \( x \in \Omega \), for every \( s \) such that \( |s| \leq K \).

\begin{equation}
(2.3) \quad A(t, x) \text{ is a symmetric coercive matrix field with coefficients}
\end{equation}
lying in $L^\infty(Q)$ i.e. $A(t, x) = (a_{ij}(t, x))_{1 \leq i, j \leq N}$ with:

- $a_{ij}(t, x) \in L^\infty(Q)$ and $a_{ij}(t, x) = a_{ji}(t, x)$ a.e. in $Q$, $\forall i, j$
- $\exists \alpha > 0$ such that a.e. in $Q$, $\forall \xi \in \mathbb{R}^N$ $A(t, x)\xi\xi \geq \alpha\|\xi\|_{\mathbb{R}^N}^2$

(2.4) $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function
(2.5) $f$ is an element of $L^1(Q)$.
(2.6) $u_0$ is a measurable function defined on $\Omega$ such that $b(x, u_0) \in L^1(\Omega)$.

**Remark 2.0.1.** In (2.2), we denote by $\nabla_x\left(\frac{\partial b(x, s)}{\partial s}\right)$ the gradient of $\frac{\partial b(x, s)}{\partial s}$ defined in the sense of distributions.

As already mentioned in the introduction Problem (1.1), (1.2), (1.3) does not admit a weak solution under assumptions (2.1)-(2.6), since the growths of $b(x, u)$ and $\Phi(u)$ are not controlled with respect to $u$ (so that these fields are not in general defined as distributions, even when $u$ belongs $L^2(0, T; W^{1,2}_0(\Omega))$).

Throughout this paper and for any non negative real number $K$ we denote by $T_K(r) = \min(K, \max(r, -K))$ the truncation function at height $K$. The definition of a renormalized solution for Problem (1.1), (1.2), (1.3) can be stated as follows.

**Definition 2.0.2.** A measurable function $u$ defined on $Q$ is a renormalized solution of Problem (1.1), (1.2), (1.3) if

(2.7) $T_K(u) \in L^2(0, T; W^{1,2}_0(\Omega))$ for any $K \geq 0$ and $b(x, u) \in L^\infty(0, T; L^1(\Omega))$;

(2.8) $\int_{\{(t, x) \in Q; \ n \leq |u(x, t)| \leq n + 1\}} A(x, t) Du \, dx \, dt \to 0$ as $n \to +\infty$;

and if, for every increasing function $S$ in $W^{2,\infty}(\mathbb{R})$, which is piecewise $C^1$ and such that $S'$ has a compact support, we have

(2.9) $\frac{\partial b_S(x, u)}{\partial t} - \text{div}(S'(u)A(t, x)Du) + S''(u)A(t, x)DuDu$

$- \text{div}(S'(u)\Phi(u)) + S''(u)\Phi(u)Du = fS'(u)$ in $D'(Q)$;

(2.10) $b_S(x, u)(t = 0) = b_S(x, u_0)$ in $\Omega$;

where $b_S(x, r) = \int_0^r \frac{\partial b(x, s)}{\partial s}S'(s) \, ds$.

The existence theorem of renormalized solution of (1.1)-(1.3):

**Theorem 2.0.3.** Under assumptions (2.1)-(2.6) there exists at least a renormalized solution $u$ of Problem (1.1)-(1.3).
Proof of Theorem 3.0.3. The existence theorem of renormalized solution of (1.1)-(1.3) is proved in H. Redwane [15]

3 – Comparison principle and uniqueness result

This section is concerned with a comparison principle (and an uniqueness result) for renormalized solutions. We establish the following theorem.

Theorem 3.0.4. Assume that assumptions (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) hold true and moreover that.

For any $K > 0$, there exists a positive real number $\beta_K > 0$, such that

\[
\left| \frac{\partial b(x, z_1)}{\partial s} - \frac{\partial b(x, z_2)}{\partial s} \right| \leq \beta_K \left| z_1 - z_2 \right|
\]

for almost every $x$ in $\Omega$, and for every $z_1$ and every $z_2$ such that $|z_1| \leq K$ and $|z_2| \leq K$.

(3.2) $\Phi$ is a locally Lipschitz continuous function on $\mathbb{R}$.

Let then $u_1$ and $u_2$ be renormalized solutions corresponding to the data $(f_1, u_{10})$ and $(f_2, u_{20})$. If these data satisfying $f_1 \leq f_2$ and $u_{10} \leq u_{20}$ almost every where, we have

$u_1 \leq u_2$ almost every where.

Proof of Theorem 3.0.4. The proof is divided into two steps. In Step 1, we define a smooth approximation $S_n$ of $T_n$, and we consider two renormalized solutions $u_1$ and $u_2$ of (1.1), (1.2), (1.3) for the data $(f_1, u_{10})$ and $(f_2, u_{20})$ respectively. We plug the test function $\frac{1}{\sigma}T_{\sigma}^{+}\left( b_{S_n}(x, u_1) - b_{S_n}(x, u_2) \right)$ in the difference of equations (2.9) for $u_1$ and $u_2$ in which we have taken $S = S_n$.

In Step 2, we investigate the behaviour of the different terms in the estimate obtained in Step 1 (estimates (3.5)) as $\sigma$ tends to 0 and when $n$ tends to $+\infty$.

Step 1. Remark that when $\Phi$ is locally Lipschitz continuous on $\mathbb{R}$ the following derivation is licit for any function $S$ and $u$ satisfying the conditions mentioned in Definition 2.0.2.

\[
\text{div} \left( S'(u)\Phi(u) \right) - S''(u)\Phi(u)Du = S'(u)\Phi'(u)Du = \text{div}(\Phi_S(u)).
\]

Where $\Phi_S = (\Phi_{S,1}, \Phi_{S,2}, \cdots, \Phi_{S,N})$ with

$\Phi_{S,i}(r) = \int_0^r \Phi'_{S,i}(t)S'(t) dt$. 

Let us now introduce a specific choice of function $S$ in (2.9). For all $n > 0$, let $S_n \in C^1(\mathbb{R})$ be the function defined by $S_n(0) = 0; S'_n(r) = 1$ for $|r| \leq n; S'_n(r) = n + 1 - |r|$ for $n \leq |r| \leq n + 1$ and $S'_n(r) = 0$ for $|r| \geq n + 1$.

It yields, taking $S = S_n$ in (2.9)

\[
\frac{\partial b_{S_n}(x, u_i)}{\partial t} - \text{div} \left( S'(u_i)A(t, x)Du_i \right) + S''(u_i)A(t, x)Du_iDu_i + \\
- \text{div} \left( \Phi_{S_n}(u_i) \right) = f_iS'_n(u_i) \quad \text{in} \ D'(Q);
\]

for $i = 1, 2$ and where $b_{S_n}(x, r) = \int_0^r \frac{\partial b(x, s)}{\partial s}S'_n(s) ds$.

We use \( \frac{1}{\sigma}T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \) as a test function in the difference of equations (3.4) for $u_1$ and $u_2$.

\[
\frac{1}{\sigma} \int_0^T \int_0^t \left( \frac{\partial (b_{S_n}(x, u_1) - b_{S_n}(x, u_2))}{\partial t} \right) ; T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \right) ds \, dt + A^\sigma_n = \\
B^\sigma_n + C^\sigma_n + D^\sigma_n,
\]

for any $\sigma > 0$, $n > 0$, and where

\[
A^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega \left[ S'_n(u_1)A(t, x)Du_1 - S'_n(u_2)A(t, x)Du_2 \right]. \\
\cdot DT_\sigma^+ \left( b_{S_n}(x, u_1) - b_{S_n}(x, u_2) \right) dx \, ds \, dt
\]

\[
B^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega S''_n(u_1)A(t, x)Du_1Du_1T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, ds \, dt + \\
- \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega S''_n(u_2)A(t, x)Du_2Du_2T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, ds \, dt
\]

\[
C^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega [\Phi_{S_n}(u_1) - \Phi_{S_n}(u_2)]DT_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, ds \, dt
\]

\[
D^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega [f_1S'_n(u_1) - f_2S'_n(u_2)]T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, ds \, dt.
\]
In the sequel we pass to the limit in (3.5) when $\sigma$ tends to 0 and then $n$ tends to $+\infty$. Upon application of Lemma 2.4 of [9], the first term in the right hand side of (3.5) is derived as

\begin{equation}
\frac{1}{\sigma} \int_0^T \int_0^t \left\langle \frac{\partial (b_{S_n}(x, u_1) - b_{S_n}(x, u_2))}{\partial t}, T_\sigma^+ (b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \right\rangle \, ds \, dt = \\
\int_0^T \int_0^t \left( \frac{\partial}{\partial t} (b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \right) \, dx \, dt
\end{equation}

(3.10)

$$= \frac{1}{\sigma} \int_Q \tilde{T}_\sigma^+ (b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \, dx \, dt +$$

$$- \frac{T}{\sigma} \int_\Omega \tilde{T}_\sigma^+ (b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \, dx$$

where $\tilde{T}_\sigma^+(r) = \int_r^T T_\sigma^+(s) \, ds$.

Due to the assumption $u_1^0 \leq u_0^2$ a.e. in $\Omega$ and the monotone character of $b_{S_n}(x, .)$ and $T_\sigma(.)$, we have

\begin{equation}
\int_\Omega \tilde{T}_\sigma^+ (b_{S_n}(x, u_1^0) - b_{S_n}(x, u_2^0)) \, dx = 0.
\end{equation}

(3.11)

It follows from (3.5), (3.10) and (3.11) that

\begin{equation}
\frac{1}{\sigma} \int_Q \tilde{T}_\sigma^+ (b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \, dx \, dt + A_\sigma^n = B_\sigma^n + C_\sigma^n + D_\sigma^n
\end{equation}

(3.12)

for any $\sigma > 0$ and any $n > 0$.

**Step 2.** In this step, we study the behaviors of the terms $A_\sigma^n$, $B_\sigma^n$, $C_\sigma^n$ and $D_\sigma^n$ when $\sigma$ tends to 0 and $n \to +\infty$. More precisely, we prove the following estimates,

\begin{equation}
\lim_{n \to +\infty} \lim_{\sigma \to 0} A_\sigma^n \geq 0,
\end{equation}

(3.13)

\begin{equation}
\lim_{n \to +\infty} \lim_{\sigma \to 0} B_\sigma^n = 0,
\end{equation}

(3.14)

\begin{equation}
\lim_{\sigma \to 0} C_\sigma^n = 0 \quad \text{for all } n,
\end{equation}

(3.15)

\begin{equation}
\lim_{n \to +\infty} \lim_{\sigma \to 0} D_\sigma^n \leq 0.
\end{equation}

(3.16)

**Proof of (3.13)**

\[ A_\sigma^n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega \left[ S_n'(u_1) A(t, x) D u_1 - S_n'(u_2) A(t, x) D u_2 \right] \, dx \, ds \, dt \]
To establish (3.13) we first write $A^\sigma_n$, as follows
\begin{equation}
A^\sigma_n = \int_Q \frac{(T - t)}{\sigma} \left[ S'_n(u_1) \left( \frac{\partial b(x, u_1)}{\partial s} \right)^{1/2} A(t, x)^{1/2} Du_1 + \right.
\left. - S'_n(u_2) \left( \frac{\partial b(x, u_2)}{\partial s} \right)^{1/2} A(t, x)^{1/2} Du_2 \right] \left( (T^+_\sigma)'(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) \right) dx \, dt + \int_Q \frac{(T - t)}{\sigma} \left[ \left( \frac{\partial b(x, u_1)}{\partial s} \right)^{1/2} - \left( \frac{\partial b(x, u_2)}{\partial s} \right)^{1/2} \right]^2 A(t, x) DS_n(u_1) DS_n(u_2) \cdot (T^+_\sigma)'(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, dt + \int_Q \frac{(T - t)}{\sigma} \left[ S'_n(u_1) A(t, x) Du_1 - S'_n(u_2) A(t, x) Du_2 \right] \left[ \int_{u_2}^{u_1} S'_n(s) \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) ds \right] \cdot (T^+_\sigma)'(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)) dx \, dt.
\end{equation}

We denote by $C_n$ the compact subset $[-n - 1, n + 1]$ of $\mathbb{R}$, and remark that due to (2.2) and (3.1), there exist a positive real numbers $\lambda_n$ and $\beta_n$ such that
\begin{equation}
\left| \left( \frac{\partial b(x, z_1)}{\partial s} \right)^{1/2} \right| \leq \left( \frac{\beta_n}{2\sqrt{\lambda_n}} \right) |z_1 - z_2| \quad \text{for all } z_1, z_2 \text{ lying in } C_n,
\end{equation}
for almost every $x$ in $\Omega$.

Due to the definition of $b_{S_n}(x, r)$, we have
\begin{equation}
|b_{S_n}(x, s) - b_{S_n}(x, r)| = \left| \int_r^s S'_n(z) \frac{\partial b(x, z)}{\partial z} \, dz \right| \geq \lambda_n |S_n(s) - S_n(r)|
\end{equation}
for almost every $x$ in $\Omega$, and $\forall \ s, \ r \in \mathbb{R}$.

As a consequence it follows that for $\sigma < n$ and if $s$ and $r$ are real numbers such that $|S_n(s) - S_n(r)| \leq \sigma$, then both $S_n(s)$ and $S_n(r)$ belong to concave or to convex branch of $S_n$. For $\sigma < n$, we then have:
\begin{equation}
\min \left( S'_n(s), \ S'_n(r) \right) |r - s| \leq |S_n(s) - S_n(r)|
\end{equation}
for all real numbers such that $|S_n(s) - S_n(r)| \leq \sigma$.

From the above inequality and since $\|S'_n\|_{L^\infty(\mathbb{R})} = 1$ we deduce that
\begin{equation}
|S_n(s) - S_n(r)| \leq \sigma < n \implies S'_n(s)S'_n(r)|s - r| \leq |S_n(s) - S_n(r)|.
\end{equation}
Due to the definition of \( T^{+}_\sigma \), it follows that
\[
S_\sigma'(s)S_\sigma'(r)|s - r|(T^{+}_\sigma)'(S_\sigma(s) - S_\sigma(r)) \leq \sigma (T^{+}_\sigma)'(S_\sigma(s) - S_\sigma(r))
\]
for all numbers \( s \) and \( r \).

Recalling that \( \text{supp}(S_n) \subset [-n+1, n+1] \), inequalities (3.18) and (3.19) lead to:
\[
\left| \int_Q \frac{(T - t)}{\sigma} \left[ \left( \frac{\partial b(x, u_1)}{\partial s} \right)^{\frac{1}{2}} - \left( \frac{\partial b(x, u_2)}{\partial s} \right)^{\frac{1}{2}} \right]^{2} \right.
\]
\[
\cdot (T^{+}_\sigma)'(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))A(t, x)DS_n(u_1)DS_n(u_2) \ dx \ dt \right| \leq \frac{T\beta_n}{2\sqrt{\lambda_n}} \int_Q \left| \left( \frac{\partial b(x, u_1)}{\partial s} \right)^{\frac{1}{2}} - \left( \frac{\partial b(x, u_2)}{\partial s} \right)^{\frac{1}{2}} \right| \chi_{\{u_1 \neq u_2\}}
\]
\[
\cdot (T^{+}_\sigma)'(S_n(u_1) - S_n(u_2))A(t, x)DT_{n+1}(u_1)DT_{n+1}(u_2) \ dx \ dt.
\]

The term just above is easily shown to converge to zero as \( \sigma \) goes to zero since the function
\[
\left| \left( \frac{\partial b(x, u_1)}{\partial s} \right)^{\frac{1}{2}} - \left( \frac{\partial b(x, u_2)}{\partial s} \right)^{\frac{1}{2}} \right| \chi_{\{u_1 \neq u_2\}}
\]
converges to zero almost everywhere in \( Q \) as \( \sigma \) goes to zero and (due to (3.1)) is bounded by the \( L^1(Q) \)-function \( 2\| \frac{\partial b(x, s)}{\partial s} \|_{L^\infty(\Omega \times C_n)}|A(t, x)DT_{n+1}(u_1)DT_{n+1}(u_2)| \).

We remark that
\[
\left| \int_Q \frac{(T - t)}{\sigma} \left[ S_n'(u_1)A(t, x)DU_1 - S_n'(u_2)A(t, x)DU_2 \right](T^{+}_\sigma)'(b_{S_n}(x, u_1) +
\]
\[
- b_{S_n}(x, u_2)) \left[ \int_{u_2}^{u_1} S_n(s)\nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \ dx \ dt \right] \right| \leq \left| \int_Q \frac{(T - t)}{\sigma} \left| S_n'(u_1)A(t, x)DU_1 - S_n'(u_2)A(t, x)DU_2 \right| \chi_{\{u_1 \neq u_2\}} \cdot
\]
\[
\cdot (T^{+}_\sigma)'(S_n(u_1) - S_n(u_2)) \| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \|_{L^\infty(C_n)} \ dx \ dt \leq T \int_Q \left| S_n'(u_1)A(t, x)DT_{n+1}(u_1) - S_n'(u_2)A(t, x)DT_{n+1}(u_2) \right| \chi_{\{u_1 \neq u_2\}}
\]
\[
\cdot (T^{+}_\sigma)'(S_n(u_1) - S_n(u_2)) \| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \|_{L^\infty(C_n)} \ dx \ dt.
\]
The term just above is easily shown to converge to zero as $\sigma$ goes to zero since the function
\[
\left| S'(u_1)A(t, x)DT_{n+1}(u_1) - S'_n(u_2)A(t, x)DT_{n+1}(u_2) \right| \chi\{u_1 \neq u_2\}.
\]
\[
\cdot (T^+_{x_n})' \left( S_n(u_1) - S_n(u_2) \right) \left\| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \right\|_{L^\infty(C_n)}
\]
converges to zero almost everywhere in $Q$ as $\sigma$ goes to zero and is bounded by the $L^1(Q)$-function
\[
\left| S'(u_1)A(t, x)DT_{n+1}(u_1) - S'_n(u_2)A(t, x)DT_{n+1}(u_2) \right| B_n(x)
\]
since $\left\| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \right\|_{L^\infty(C_n)} \leq B_n(x) \in L^2(\Omega)$ (see (2.2)).

From the above analysis we conclude that (3.13) holds true.

**Proof of (3.14).** We have
\[
|B^\sigma_n| \leq T \int_{\{n \leq |u_1| \leq n+1\}} A(t, x)Du_1Du_1 \, dx \, dt + \int_{\{n \leq |u_2| \leq n+1\}} A(t, x)Du_2Du_2 \, dx \, dt.
\]
(3.23)

As a consequence of (2.8), letting $n$ go to infinity in the above estimates of $B^\sigma_n$ shows that (3.14) holds true.

**Proof of (3.15).**
\[
C^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega \left[ \Phi_{S_n}(u_1) - \Phi_{S_n}(u_2) \right] DT^+_{\sigma} \left( b_{S_n}(x, u_1) - b_{S_n}(x, u_2) \right) \, dx \, ds \, dt.
\]

To establish (3.15), let us remark that for all $s, r$ in $\mathbb{R}$, the following inequality holds true
\[
\left\| \Phi_{S_n}(s) - \Phi_{S_n}(r) \right\|_{\mathbb{R}^N} \leq \|\Phi'\|_{L^\infty(C_n)}^N \left| S_n(s) - S_n(r) \right|
\]
indeed, since $\text{supp}S'_n \subset [-n - 1, n + 1]$ and $\Phi'$ is assumed to be locally Lipschitz continuous, it follows that
\[
\left| \Phi_{S_n}(s) - \Phi_{S_n}(r) \right| \leq \int_r^s S'(z)\Phi'(z) \, dz \leq \|\Phi'\|_{L^\infty(C_n)}^N \left| S_n(s) - S_n(r) \right|.
\]
With the help of (3.24) the term $C^\sigma_n$ may be estimated as follows

$$|C^\sigma_n| \leq T\|\Phi\|_{L^\infty(C_n)}^N \int_{\{0 \leq (b_{S_n}(x,u_1) - b_{S_n}(x,u_2)) \leq \sigma\}} \frac{|S_n(u_1) - S_n(u_2)|}{\sigma} \, dx \, dt \leq T\|\Phi\|_{L^\infty(C_n)}^N \int_{\{0 \leq (S_n(u_1) - S_n(u_2)) \leq \frac{\sigma}{\lambda_n}\}} \frac{|S_n(u_1) - S_n(u_2)|}{\sigma} \, dx \, dt \leq T\|\Phi\|_{L^\infty(C_n)}^N \int_Q |DT^+_{\sigma}(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))| \, dx \, dt.$$ 

It yields

$$|C^\sigma_n| \leq \frac{T}{\lambda_n}\|\Phi\|_{L^\infty(C_n)}^N \int_Q |DT^+_{\sigma}(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))| \, dx \, dt,$$

which in turn implies (3.15) since $DT^+_{\sigma}(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))$ converges to zero in $L^1(Q)$ as $\sigma$ goes to zero.

**Proof of (3.16).**

$$D^\sigma_n = \frac{1}{\sigma} \int_0^T \int_0^t \int_\Omega [f_1 S'_n(u_1) - f_2 S'_n(u_2)] T^+_{\sigma}(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)) \, dx \, ds \, dt.$$ 

We have as $n$ tends to $+\infty$,

$$f_1 S'_n(u_1) - f_2 S'_n(u_2) \rightarrow f_1 - f_2 \text{ strongly in } L^1(Q).$$  

(3.25) 

Letting $\sigma$ tends to 0, we have $\frac{1}{\sigma}T^+_{\sigma}(t)$ goes to $sg^+_0(t)$, for all $t \in \mathbb{R}$. For $n > 0$ fixed, we have

$$\lim_{\sigma \rightarrow 0} D^\sigma_n = \int_0^T \int_0^t \int_\Omega \left( f_1 - f_2 \right) sg^+_0(b_{S_n}(x,u_1)) - b_{S_n}(x,u_2)) \, dx \, ds \, dt.$$ 

Since $f_1 \leq f_2$ a.e. in $Q$ and $sg^+_0(t) \geq 0$ for all $t$ in $\mathbb{R}$, then shows that (3.16) holds true. In view of the definition of $T^+_{\sigma}$ and $T^\sigma_n$, we have

$$\lim_{n \rightarrow +\infty} \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_0^T \int_\Omega \bar{T}^+_{\sigma}(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)) \, dx \, dt = \int_Q (b(x,u_1) - b(x,u_2))^+ \, dx \, dt.$$  

(3.26)
In view of estimates (3.11), (3.12), (3.13), (3.14), (3.15), (3.16) and (3.26) we have

$$\int_Q \left( b(x, u_1) - b(x, u_2) \right)^+ \, dx \, dt \leq 0,$$

so that \( b(x, u_1) \leq b(x, u_2) \) a.e. in \( Q \) which in turn implies that \( u_1 \leq u_2 \) a.e. in \( Q \), Theorem 3.0.4 is then established.

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**REFERENCES**


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