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Diagonal elliptic Bellman systems to stochastic differential games with discount control and noncompact coupling

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Dedicated to Umberto Mosco[1]

ABSTRACT: *We consider Bellman systems to stochastic differential games with quadratic cost functionals and a discount factor which may be influenced by the players. This leads to a diagonal elliptic system*

$$-\Delta u = H(x, u, \nabla u)$$

subject to boundary conditions where the Hamiltonian grows quadratically in grad u and contains a discount term $uF_0(u, \nabla u)$. *We mainly consider the two-dimensional case and 2 or 3 players. Under appropriate conditions we obtain the existence of regular solutions. Examples of cost functionals are presented, where no regularity theory is available up to now.*

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1 – Introduction

In the present paper we consider diagonal elliptic systems

$$-\Delta u_\nu + \alpha u_\nu = H_\nu(x,u,\nabla u), \qquad \nu = 1,2 \text{ or } 3 \tag{1.1}$$

in a bounded domain Ω of $\mathbb{R}^n$ where

$$H_\nu(x,u,\nabla u) = H_{\nu 0}(x,u,\nabla u) - u_\nu F(x,u,\nabla u) + f_\nu(x)\,. \tag{1.2}$$

The functions $H_{\nu 0}$, F may have quadratic growth in ∇u

$$|H_{\nu 0}(x,u,\eta)| + |F(x,u,\eta)| \le K|\eta|^2 + K\,, \qquad K = K(\|u\|_\infty) \tag{1.3}$$

and have to satisfy the Caratheodory conditions

$$H_{\nu 0}(x,u,\eta), F(x,u,\eta) \textit{ is measurable in } x \textit{ and continuous in } (\mu,\eta)\,. \tag{1.4}$$

We require

$$f \in L^\infty(\Omega) \tag{1.5}$$

$$F \ge 0\,, \tag{1.6}$$

and further structure conditions on $H_{\nu 0}$, see below.

We confine ourselves to the case of Dirichlet-zero-boundary conditions. The Laplace operator Δ in (1.1) may be replaced by an uniformly elliptic operator $\sum\limits_{i,k=1}^{n} D_i(a_{ik}(x)D_k)$.

It is well known (see [1]) that Bellman equations to stochastic games with infinite horizon, constant discount factor α and two players lead to diagonal systems (1.1) with $F = 0$. In a recent paper ([6]) the authors studied the case of discount control, i. e. the discount factor e^{-ct} in the cost functional depends on the controls $v, c = c(v)$.

In our paper [6] we treated Hamiltonians

$$H_\nu(x,u,\nabla u) = H_{\nu 0}(x,u,\nabla u_\nu) - u_\nu F_\nu(x,u,\nabla u) + f_\nu(x) \tag{1.7}$$

with $F_\nu \ge 0$ and $H_{\nu 0}, F_\nu$ having quadratic growth in ∇u. We obtained L^∞ and H^1-estimates for solutions u of (1.1), and corresponding approximations in arbitrary dimension, furthermore compactness and C^α-regularity of solutions (C^α at this moment), however only in the case of two space dimensions.

Note that in (1.7) $H_{\nu 0}$ depends only on u and ∇u_ν, not on the full gradient ∇u. This decreases the applicability of the results since it does not cover cases of cost functionals containing terms like $\theta v_1 \cdot v_2$, i. e. a non compact coupling of the controls. Up to now the coupling of the controls which has been admitted in the case of discount control is only via the state variables x which are controlled by v via the stochastic differential equation, cf. the example in section 5 of this paper. On this background it is important to generalize the structure of the term $H_{\nu 0}$ in (1.7), maintaining L^∞, H^1 and C^α estimates. This is the purpose of the present paper. Rather than (1.7) we assume the structure condition

$$H_\nu((x,u,\nabla u) = H_{\nu 0}(x,u,\nabla u) - u_\nu F(x,u,\nabla u) + f_\nu(x) \tag{1.8}$$

$$|H_{10}(x,u,\nabla u)| \le C|\nabla u_1|^2 + C|\nabla u_1||\nabla u_2| + C_1 \tag{1.9}$$

$$|H_{20}(x,u,\nabla u)| \le C|\nabla u|^2 + C_1 \tag{1.10}$$

with constantS $C = C(\|u\|_\infty), C_1$.

Note that the function F may not depend on ν - this is the prize to be paid for the more general structure of $H_{\nu 0}$. Conditions (1.9), (1.10) are exactly those used in our paper [1]. Condition (1.9) could be generalized by adding a term $\delta|\nabla u_2|^2$ to the right hand side of (1.9), with δ very small. This would allow the presence of a sub-quadratic term in ∇u_2 in the growth condition for H_{10}. The conditions (1.5), (1.6), (1.9), (1.10) imply H^1-bounds if an L^∞-estimate for u is available and, for $n = 2$, C^α-regularity and C^α-estimates for the solution and its approximations.

Once C^α-regularity is established it is well known how to obtain $H^{2,p}$-regularity, $1 \le p < \infty$.

Concerning L^∞-estimates – with reasonable application to stochastic games – we assume two conditions based on maximum principle arguments.

$$\begin{gathered}\textit{There exist constants } a_1, a_2, a_2 \neq 0 \\ \textit{such that } a_1 H_{10} + a_2 H_{20} \ge -K_1\end{gathered} \tag{1.11}$$

and

$$(\operatorname{sign} a_2) H_{20}(x,u,\eta)|_{\eta_2 = 0} \le K_3\,. \tag{1.12}$$

(A standard case would be $a_1 = a_2 = 1$.)

With these conditions, i. e. (1.3), (1.4), (1.6), (1.9), (1.10), (1.11), (1.12) - *not* using (1.7) - we obtain $H^{2,p} \cap H_0^{1,2}$-solutions of the system (1.1), cf. Theorem 1 of the next section. Concerning the bibliography of diagonal elliptic systems with lower order term growing quadratically in ∇u, cf. [6].

2 – The Theorem

By a weak solution to the Dirichlet problem of system (1.1) we mean a function $u \in L^\infty \cap H_0^{1,2}(\Omega;\mathbb{R}^n)$ such that

$$(2.1) \qquad (\nabla u_\nu, \nabla \varphi_\nu) + \alpha(u_\nu, \varphi_\nu) = \big(H_\nu(.,u,\nabla u), \varphi_\nu\big)\,, \qquad \nu = 1,2$$

for all $\varphi_\nu \in C_0^\infty(\Omega)$. Here $(v,w) = \int vw\,dx$.

Theorem 2.1. *Let the H_ν satisfy the Caratheodory conditions* (1.4), *the growth conditions* (1.3), *the structure conditions* (1.6), (1.8), (1.9), (1.10), *and the maximum principle type conditions* (1.11), (1.12) *and* (1.5), *and let $n = 2$. Then there exists a weak solution $u \in L^\infty \cap H_0^{1,2}$ which is contained in $H^{2,p}_{\text{loc}}$ for all $p < \infty$.*

Remark A cone condition or a uniform Wiener condition for the boundary would imply $u \in C^\alpha$ up the boundary of $\partial\Omega$.
$H^{2,\infty}$-boundary implies $u \in H^{2,p}$ up to the boundary.

Proof of the Theorem The proof proceeds similar as in [6] with adaptations to the new situation.

(i) We approximate (1.1) by replacing $H_{\nu 0}$ and F by $H_{\nu 0}^\delta = H_{\nu 0}(1+\delta|\nabla u|^2)^{-1}$, $F = F(1+\delta|\nabla u|^2)^{-1}$. Then there is a solution u^δ of the approximate systems and regularity theory tells us that the solution $u^\delta \in L^\infty \cap H^{2,p}_{\text{loc}}$ for all p.

(ii) We want to establish a uniform L^∞-bound for u^δ. Using the maximum principle type condition (1.11), (1.12) and $f \in L^\infty$, we conclude similarly as in [6] a uniform bound ($u = u^\delta$)

$$\alpha\|u_1\|_\infty \le C_1 + \|f\|_\infty\,,$$

thereafter we conclude from (1.11) via a maximum principle type argument that

$$\alpha(a_1 u_1 + a_2 u_2) \ge -K_1 - \|f\|_\infty$$

hence u_2 is uniformly bounded from below. A bound for u_2 from above finally follows from (1.12), again with the truncation techniques explained in [6]. Note that the arguments are simple and classical if a setting is arranged where $u \in C^2$ and $H, F, f \in C$.

(iii) From the basic inequality of the following chapter we conclude a uniform H^1-bound for u^δ in terms of $\|u^\delta\|_{L^\infty}$ which is bounded due to the consideration in (ii).

(iv) We select a subsequence still denoted by u^δ, $\delta \to 0$, such that

$$u^\delta \to u \text{ weakly in } H^{1,2} \qquad (\delta \to 0)\,.$$

We need strong convergence in $H^{1,2}$ in order to interchange the limit with the nonlinear function H. (Strong $H^{1,2}_{\mathrm{loc}}$-convergence is sufficient.)

(v) For the strong $H^{1,2}_{\mathrm{loc}}$-convergence of a subsequence (u^δ) we need a (locally) uniform C^α-estimate for the u^δ. Thereafter, one applies the usual monotonicity argument. At the present state of the research we are restricted to the case of *two* space dimensions. The steps (i)–(iv) work in n-dimensions.

(vi) Similarly as in our paper [6] we establish a weighted logarithmic estimate

$$\int_\Omega |\nabla u|^2 |\ln|x-x_0||^\theta \, dx \le K_\theta \tag{2.2}$$

with some $\theta \in (0,1)$, uniformly for $x_0 \in \Omega$ and $u = u^\delta$. This is one of the consequences of the basic inequality of the next chapter, see Lemma 3.2. In fact, we can prove (2.2) with θ arbitrarily near to 1.

[(vii)] From (2.2) we obtain a uniform smallness property

$$\int_{B_R\cap\Omega} |\nabla u|^2 \, dx \le \varepsilon\,, \qquad 0 < R \le R(\varepsilon)$$

for all balls $B_R = B_R(x_0)$, $x_0 \in \Omega$.

From this smallness condition in the case of two dimensions one derives a uniform C^α (or locally uniform C^α-estimate) via a global hole filling argument as it was done in [8], [6]. We do not repeat this argument and refer to these publications.

This proves the Theorem.

□

3 – Basic Inequalities

From the growth condition (1.8), (1.9), (1.10) we obtain the existence of bounded measurable functions $g_i, \sigma_{ik} : \Omega \to \mathbb{R}$, $i = 1,2$ and $\sigma_0 : \Omega \to \mathbb{R}^n$, such that

$$H_{10}(x,u,\nabla u) = \sigma_{11}|\nabla u_1|^2 + (|\nabla u_1| + |\nabla u_2|)\sigma_0 \nabla u_1 + g_1 \tag{3.1}$$

$$H_{20}(x,u,\nabla u) = \sigma_{22}|\nabla u_2|^2 + \sigma_{21}|\nabla u_1|^2 + g_2\,. \tag{3.2}$$

σ_{ik}, σ_0 will depend on ∇u in a terrible way.

Hint: First σ_0, then σ_{11} are constructed, thereafter look at

$$H_2(x,u,\nabla u) - (|\nabla u_1| + |\nabla u_2|)\sigma_0 \nabla u_2$$

which is bounded by $C|\nabla u_1|^2 + C|\nabla u_2|^2 + K$. One can arrange that, say,

$$|\sigma_{11}|, |\sigma_0|, \le C\,; \quad |\sigma_{22}|, |\sigma_{21}| \le 2C\,, \qquad g \in L^\infty\,. \tag{3.3}$$

For the sake of simplicity we consider only the case $g = 0$. The techniques can be extended easily to treat the case $g \neq 0$.

Now, choose $\lambda = 4C$, $C \neq 0$, and let $\beta = \beta(C, \|u\|_\infty)$ be a very large number chosen later. (Our construction is possible only for L^∞ solutions u of approximate or limiting problems.)

In the first equation we choose the test function

$$\varphi_1 = \beta\tau\big(e^{\lambda u_1} - e^{-\lambda u_1}\big)\exp$$

with

$$\exp = \exp\big[\gamma(\beta e^{\lambda u_1} + \beta e^{-\lambda u_1} + e^{\lambda u_2} + e^{-\lambda u_2})\big]$$

with a Lipschitz continuous non-negative function τ and a constant γ determined later. In the second equation we choose $\varphi_2 = \tau(e^{\lambda u_2} - e^{-\lambda u_2})\exp$.

At the left hand side of the equations, among others, we obtain the terms

$$\lambda\int_\Omega \beta\tau|\nabla u_1|^2(e^{\lambda u_1} + e^{-\lambda u_1})\exp\,dx \tag{3.4}$$

$$\lambda\int_\Omega \tau|\nabla u_2|^2(e^{\lambda u_2} + e^{-\lambda u_2})\exp\,dx \tag{3.5}$$

and on the right hand the terms

$$\int \beta\tau\sigma_{11}|\nabla u_1|^2(e^{\lambda u_1} - e^{-\lambda u_1})\exp\,dx$$
$$\int \tau\sigma_{22}|\nabla u_2|^2(e^{\lambda u_2} - e^{-\lambda u_2})\exp\,dx\,.$$

The latter terms are dominated by the terms (3.4), (3.5) since $\lambda \geq 4C$. Since $\xi(e^{\lambda\xi} - e^{-\lambda\xi}) \geq 0$ we may drop the terms coming from u_1F, u_2F while estimating.

Thus, from the first equation, we remain with the inequality

$$\begin{aligned}
&\frac{3}{4}\lambda\int_\Omega \beta\tau|\nabla u_1|^2(e^{\lambda u_1} + e^{-\lambda u_1})\exp\,dx + B_1 \leq\\
&\leq E_1 + \int_\Omega \beta\tau(f_1 + g_1)(e^{\lambda u_1} - e^{-\lambda u_1})\exp\,dx + \text{ pollution}_1\\
&B_1 = \lambda^{-1}\beta\big(\nabla(e^{\lambda u_1} + e^{-\lambda u_1})\,,\ \tau\nabla\exp\big)\\
&E_1 = \lambda^{-1}\beta\big(\tau(|\nabla u_1| + |\nabla u_2|)\,,\ \sigma_0\nabla(e^{\lambda u_1} + e^{-\lambda u_1})\exp\big)
\end{aligned}$$

pollution$_1$ = the term containing $\nabla\tau$ and g_1. Similarly, from the second equation,

$$\frac{3}{4}\lambda \int\limits_\Omega \tau|\nabla u_2|^2(e^{\lambda u_2}+e^{-\lambda u_2})\exp\, dx + B_2 \le$$

$$\le E_2 + D_2 + \int\limits_\Omega \tau(f_2+g_2)(e^{\lambda u_2}-e^{-\lambda u_2})\exp\, dx + \text{ pollution}_2$$

$$B_2 = \lambda^{-1}\big(\nabla(e^{\lambda u_2}+e^{-\lambda u_2}),\ \tau\nabla\exp\big)$$

$$E_2 = \lambda^{-1}\Big(\tau(|\nabla u_1|+|\nabla u_2|),\ \sigma_0\nabla(e^{\lambda u_2}+e^{-\lambda u_2})\exp\Big)$$

$$D_2 = \int\limits_\Omega \tau\sigma_{22}|\nabla u_1|^2(e^{\lambda u_2}-e^{-\lambda u_2})\exp\, dx \le$$

$$\le K_0\int\limits_\Omega \tau|\nabla u_1|^2\exp\, dx$$

$$K_0 \le 2K(1+e^{\lambda\|u_2\|_\infty})\,.$$

We add the inequalities just obtained and obtain, rewriting $B_i, E_i, i=1,2$,

$$\frac{3}{4}\lambda \int\limits_\Omega \tau\big[\beta|\nabla u_1|^2(e^{\lambda u_1}+e^{-\lambda u_1})+|\nabla u_2|^2(e^{\lambda u_2}+e^{-\lambda u_2})\big]\exp\, dx +$$

$$+\lambda^{-1}\gamma\int\limits_\Omega \tau|\nabla(\beta e^{\lambda u_1}+\beta e^{-\lambda u_1}+e^{\lambda u_2}+e^{-\lambda u_2})|^2\exp\, dx \le$$

$$\le \lambda^{-1}\Big(\tau(|\nabla u_1|+|\nabla u_2|),\ \sigma_0\nabla(\beta e^{\lambda u_1}+\beta e^{-\lambda u_1}+e^{\lambda u_2}+e^{-\lambda u_2})\exp\Big)+$$

$$+K_0\int\limits_\Omega \tau|\nabla u_1|^2\, dx + K\int\limits_\Omega \tau\, dx + \sum_{i=1}^{2}\ \text{pollution}_i\,.$$

$K=K(\beta,\|u\|_\infty)$. The term $K\int\limits_\Omega \tau\, dx$ arises on account of the terms containing f_i, g_i, after using Youngs's inequality.

The second integral in the left hand side of the last inequality can be used to dominate the integral on the right hand side containing the factor

$$\nabla(\beta e^{\lambda u_1}+\beta e^{-\lambda u_1}+e^{\lambda u_2}+e^{-\lambda u_2})$$

For this, we chose γ large enough. Using Young's inequality this yields

$$\int_\Omega \tau\big[\big(\frac{3}{4}\lambda\beta - \lambda^{-1}\gamma^{-1}K^2\big)|\nabla u_1|^2(e^{\lambda u_1}+e^{-\lambda u_1})\big]\exp\,dx +$$

$$+\int_\Omega \tau\big[\big(\frac{3}{4}\lambda - \lambda^{-1}\gamma^{-1}K^2\big)|\nabla u_2|^2(e^{\lambda u_2}+e^{-\lambda u_2})\big]\exp\,dx \leq$$

$$\leq K_0\int_\Omega \tau|\nabla u_1|^2\,dx + K\int_\Omega \tau\,dx + \sum_{i=1}^{2}\ \text{pollution}_i\,. \tag{3.6}$$

We now chose β so large such that

$$\frac{1}{4}\lambda\beta \geq K_0\,.$$

The inequality $(e^{\lambda u_1}+e^{-\lambda u_1})\exp \geq 1$ implies that the first summand in (3.6) can be used to dominate

$$K_0\int_\Omega \tau|\nabla u_1|^2\,dx\,.$$

With this we arrive at the estimate (let $\beta \geq 1$)

$$\frac{1}{2}\lambda\int_\Omega \tau(|\nabla u_1|^2+|\nabla u_2|^2)\,dx \leq K\int_\Omega \tau\,dx + \sum_{i=1}^{2}\ \text{pollution}_i\,. \tag{3.7}$$

The sum of the pollution terms reads

$$\sum_{i=1}^{2}\ \text{pollution}_i = -\lambda^{-1}\Big(\nabla(\beta e^{\lambda u_1}+\beta e^{-\lambda u_1}+e^{\lambda u_2}+e^{-\lambda u_2})\exp,\nabla\tau\Big)+$$

$$+\,k\int_\Omega \tau dx = -\lambda^{-1}\gamma^{-1}(\nabla\exp,\nabla\tau) + k\int_\Omega \tau dx\,. \tag{3.8}$$

Furthermore, it is clear that

$$\Big|\sum_{i=1}^{2}\ \text{pollution}_i\Big| \leq K_1\int_\Omega (|\nabla u_1|+|\nabla u_2|)|\nabla\tau|\,dx + k\int_\Omega \tau dx\,. \tag{3.9}$$

$K_1 = K_1(\|u\|_\infty)$.

We then can state

Lemma 3.1. *Let $u \in H_0^{1,2}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega)$ be a weak solution of the system* (1.1) *and assume the Caratheodory and growth conditions* (1.3), (1.5) *and the structure conditions* (1.6), (1.8), (1.9), (1.10)*. Then u satisfies*

$$\int_\Omega |\nabla u|^2 \tau\, dx + 2\lambda^{-2}\gamma^{-1}(\nabla \exp, \nabla\tau) \le K \int_\Omega \tau\, dx \tag{3.10}$$

with some constant $K = K(\|u\|_\infty)$ where $\tau \ge 0, \tau$ Lipschitz and λ, β, γ are chosen large enough, the choice depending on the growth constants and $\|u\|_\infty$.

Lemma 3.1 obviously yields an $H^{1,2}$-bound for u once an L^∞-bound is known.

A further important consequence is

Lemma 3.2. *Under the assumption of Lemma* 3.1 *for each $\theta \in (0,1)$ there is a uniform constant K_0 depending on $\|u\|_\infty$ and the growth constants such that*

$$\int_\Omega |\nabla u|^2 \big|ln|x - x_0|\big|^\theta\, dx \le K_0$$

uniformly for $x_0 \in \Omega$.

Proof. In Lemma 3.1 we chose $\tau = \big|ln(|x - x_0| + \delta_1)\big|^\theta$ and pass to the limit $\delta_1 \to +0$. We apply inequality (3.10). Obviously, the term $K \int \tau\, dx$ is bounded. We estimate

$$\begin{aligned} \big|(\nabla \exp, \nabla\tau)\big| &\le K \int |\nabla u| |ln|x - x_0||^{\theta-1} |x - x_0|^{-1}\, dx \le \\ &\le \varepsilon_0 \int |\nabla u|^2 \big|ln|x - x_0|\big|^\theta\, dx + K_{\varepsilon_0} \int |x - x_0|^{-2} \big|ln|x - x_0|\big|^{\theta-2}\, dx\,. \end{aligned}$$

The second summand is bounded by some constant K_0 since $2 - \theta > 1$. The term $\varepsilon_0 \int |\nabla u|^2 |ln|^\theta\, dx$ is absorbed by the corresponding term on the left hand side. This proves Lemma 3.2.

4 – The Case of Three Players

We present the structure conditions for the Hamiltonian H for a system of three equations where the analogue of the basic inequality for two equations can be derived. An analogous approach for the PDE-theory of stochastic games without discount controls has been presented in [7] and (for the parabolic case) in [5]. The conditions are now

$$H_\nu(x, u, \nabla u) = H_{\nu 0}(x, u, \nabla u) - u_\nu F(x, u, \nabla u) \tag{4.1}$$

where

$$0 \le F(x,u,\nabla u) \le C|\nabla u|^2 + C\,, \tag{4.2}$$

$$\big|H_{10}(x,u,\nabla u) - L(x,u,\nabla u)\nabla u_1\big| \le C|\nabla u_1|^2 + C|\nabla u_1||\nabla u_2| + K \tag{4.3}$$

$$\big|H_{20}(x,u,\nabla u) - L(x,u,\nabla u)\nabla u_2\big| \le C|\nabla u_1|^2 + C|\nabla u_2|^2 + K \tag{4.4}$$

$$\big|H_{30}(x,u,\nabla u)\big| \le C|\nabla u|^2 + K \tag{4.5}$$

with some Caratheodory function L such that

$$\big|L(x,u,\nabla u)\big| \le C|\nabla u| + C\,. \tag{4.6}$$

The conditions (4.3), (4.4), (4.5) have to be interpreted as follows:

(i) No condition on H_{30} (except quadratic growth on ∇u);

(ii) H_{20} may be a sum of terms which have quadratic growth in ∇u_1, ∇u_2, but the term containing ∇u_3 may be only of the form $L(\nabla u)\nabla u_2$ where L has linear growth in ∇u.

(iii) In the term H_{10} only quadratically growing terms in ∇u_1 are admitted, the other terms must be estimated by

$$|\nabla u_1||\nabla u_2| \quad \text{and} \quad |\nabla u_3||\nabla u_1|$$

(iv) the occurrence of terms of growth $|\nabla u_1||\nabla u_3|$ in H_{10} and $|\nabla u_2||\nabla u_3|$ in H_{20} is not arbitrary, but there is a coupling via the function L which is the same for H_{10} and H_{20}.

It seems to the authors that the more natural condition

$$\begin{aligned}
&|H_{10}| \le C|\nabla u_1|^2 + C\big(|\nabla u_2| + |\nabla u_3|\big)|\nabla u_1| + K\\
&|H_{20}| \le C|\nabla u_1|^2 + C|\nabla u_2|^2 + C|\nabla u_3||\nabla u_1| + K\\
&|H_{20}| \le C|\nabla u|^2 + K
\end{aligned}$$

has not been shown yet to be sufficient for the analogue of the basic inequality in section 3.

We proceed as in section 3 with more complicated test function $\varphi_1, \varphi_2, \varphi_3$.

$$\begin{aligned}
\varphi_1 &= \beta_1\tau(e^{\lambda u_1} - e^{-\lambda u_1})Exp_0Exp_1\\
\varphi_2 &= \beta_2\tau(e^{\lambda u_2} - e^{-\lambda u_2})Exp_0Exp_1\\
\varphi_3 &= \tau(e^{\lambda u_3} - e^{-\lambda u_3})Exp_1\,,
\end{aligned}$$

where

$$\begin{aligned}
Exp_0 &= exp\big[\gamma(\beta_1 e^{\lambda u_1} + \beta_1 e^{-\lambda u_1} + \beta_2 e^{\lambda u_2} + \beta_2 e^{-\lambda u_2})\big]\\
Exp_1 &= exp\big[\eta(\gamma^{-1}Exp_0 + e^{\lambda u_3} + e^{-\lambda u_3})\big]\,.
\end{aligned}$$

Here τ is a non-negative Lipschitz function. Using the above test function φ_ν in the ν-th equation we obtain at the left hand side of the equation, after summation $\nu = 1,2,3$, a sum of the type

$$A_{12} + A_3 + B_{12} + C_{12} + C_3 + T \tag{4.7}$$

with

$$
\begin{aligned}
A_{12} &= \sum_{\nu=1}^{2} \lambda\beta_\nu \int_\Omega \tau|\nabla u_\nu|^2 (e^{\lambda u_\nu} + e^{-\lambda u_\nu}) Exp_0 Exp_1 \, dx \\
A_3 &= \lambda \int_\Omega \tau|\nabla u_3|^2 (e^{\lambda u_3} + e^{-\lambda u_3}) Exp_1 \, dx \\
B_{12} &= \lambda^{-1} \int_\Omega \tau\nabla\big(\sum_{\nu=1}^{2}(\beta_\nu e^{\lambda u_\nu} + \beta_\nu e^{-\lambda u_\nu})\big)\nabla Exp_0 Exp_1 \, dx \\
C_{12} &= \lambda^{-1}\gamma^{-1} \int_\Omega \tau\nabla Exp_0 \nabla Exp_1 \, dx \\
C_3 &= \lambda^{-1} \int_\Omega \tau\nabla(e^{\lambda u_3} + e^{-\lambda u_3})\nabla Exp_1 \, dx \\
T &= \lambda^{-1} \int_\Omega \nabla\tau\big[\gamma^{-1}\nabla Exp_0 + \nabla(e^{\lambda u_3} + e^{-\lambda u_3})\big] Exp_1 \, dx \\
&= \eta^{-1}\lambda^{-1} \int \nabla\tau\nabla Exp_1 \, dx \, .
\end{aligned}
$$

We have

$$
C_{12} + C_3 = \eta\lambda^{-1} \int_\Omega \tau\big|\nabla[\gamma^{-1} Exp_0 + e^{\lambda u_3} + e^{-\lambda u_3}]\big|^2 Exp_1 \, dx \, . \tag{4.8}
$$

On the right hand side we use that

$$
u_\nu F\varphi_\nu \le 0
$$

due to the sign situation; so these terms are not considered any more. For the analysis of the remaining right hand side, the partial Hamiltonians are rewritten

$$
\begin{aligned}
H_{10}(x,u,\nabla u) &= \sigma_1|\nabla u_1|^2 + (|\nabla u_1| + |\nabla u_2|)\sigma_{12}\nabla u_1 \\
&\quad + L(x,u,\nabla u)\nabla u_1 + g_1 \\
H_{20}(x,u,\nabla u) &= \sigma_2|\nabla u_1|^2 + \sigma_2|\nabla u_2|^2 + (|\nabla u_1| + |\nabla u_2|)\sigma_{12}\nabla u_2 \\
&\quad + L(x,u,\nabla u)\nabla u_2 + g_2 \\
H_{30}(x,u,\nabla u) &= \sigma_3|\nabla u_1|^2 + \sigma_3|\nabla u_2|^2 + \sigma_3|\nabla u_3|^2 \\
&\quad + L(x,u,\nabla u)\nabla u_3 + g_3 \, .
\end{aligned}
$$

Here $\sigma_i, g_i, \in L^\infty(\Omega)$, $i = 1,2,3$, $\sigma_{12} \in L^\infty(\Omega;\mathbb{R}^n)$, the L^∞-bounds depending on the growth constants.

To obtain this representation, we derive from (4.3)

$$H_{10} - L\nabla u_1 = \sigma_1 |\nabla u_1|^2 + (|\nabla u_1| + |\nabla u_2|)\sigma_{12} \nabla u_1 + g_1 \,.$$

Thereafter we rewrite

$$\begin{aligned} H_{20} - L\nabla u_2 &= \sigma_2 |\nabla u_1|^2 + \sigma_2 |\nabla u_2|^2 + (|\nabla u_1| + |\nabla u_2|)\sigma_{12} \nabla u_2 + g_2 \\ H_{30} - L\nabla u_3 &= \sigma_3 |\nabla u|^2 + g_3 \,. \end{aligned}$$

We define $L_{12} = (|\nabla u_1| + |\nabla u_2|)\sigma_{12}$ and have

$$\begin{aligned} H_{10} &= \sigma_1 |\nabla u_1|^2 + L_{12} \nabla u_1 + L\nabla u_1 + g_1 \\ H_{20} &= \sigma_2 \big(|\nabla u_1|^2 + |\nabla u_2|^2\big) + L_{12} \nabla u_2 + L\nabla u_2 + g_2 \\ H_{30} &= \sigma_3 |\nabla u|^2 + L\nabla u_3 + g_3 \,. \end{aligned}$$

We have $\|g_i\|_\infty \le K$, $\|\sigma_i\|_\infty$, $\|\sigma_{ik}\|_\infty \le C' = C'(C)$, $i, k = 1, 2$.

We analyze the remaining right hand side

$$\sum_{\nu=1}^{3} (H_{\nu 0}, \varphi_\nu)$$

and try to dominate the summands by terms on the left hand side (4.7). Firstly, choosing $\lambda \ge 4C'$, the terms $(\sigma_\nu |\nabla u_\nu|^2, \varphi_\nu)$ are dominated by a fraction (say $\frac{1}{4}$) of A_{12} and A_3. Then we choose β_2 so large such that a fraction of $\beta_2 |\nabla u_2|^2 (e^{\lambda u_2} + e^{-\lambda u_2})$ dominates the term $\sigma_3 |\nabla u_2|^2 (e^{\lambda u_3} - e^{-\lambda u_3})$. This is possible since $u \in L^\infty$. Thereafter, we choose β_1 so large such that a fraction of $\beta_1 |\nabla u_1|^2 (e^{\lambda u_1} + e^{-\lambda u_1})$ dominates $\sigma_2 |\nabla u_1|^2 (e^{\lambda u_2} - e^{-\lambda u_2})$ and $\sigma_3 |\nabla u_1|^2 (e^{\lambda u_3} - e^{-\lambda u_2})$. Thus the inequality is simplified to

$$\begin{aligned} \frac{1}{4} A_{12} + \frac{3}{4} A_3 + B_{12} + C_{12} + C_3 + T &\le \sum_{\nu=1}^{2} (L_{12} \nabla u_\nu, \varphi_\nu) + \sum_{\mu=1}^{3} (L\nabla u_\mu, \varphi_\mu) + \\ &+ \text{ pollution } \textit{coming from } f \textit{ and } g_i \,. \end{aligned}$$

Now the term B_{12} is used to dominate the term

$$\sum_{\nu=1}^{2} (L_{12} \nabla u_\nu, \varphi_\nu) = \lambda^{-1} \Big(L_{12} \nabla \sum_{\nu=1}^{2} \big(\beta_\nu e^{\nabla u_\nu} + \beta_\nu e^{-\nabla u_\nu} \big), \tau Exp_0 Exp_1 \Big)$$

similarly as in the case of two players by choosing γ large.

We are left with the term

$$\sum_{\mu=1}^{3}(L\nabla u_\mu,\varphi_\mu)=\lambda^{-1}\big(L\nabla(e^{\lambda u_3}+$$

$$+e^{-\lambda u_3})+L\nabla[\sum_{\nu=1}^{2}(\beta_\nu e^{\lambda u_\nu}+\beta_\nu e^{-\lambda u_\nu})]Exp_0,\tau Exp_1\big)=$$

$$=\lambda^{-1}(L\nabla(e^{\lambda u_3}+e^{-\lambda u_3}+\gamma^{-1}Exp_0),Exp_1\tau)$$

The right hand side of the last equation can be estimated by

(4.9)

$$K'\lambda^{-1}\eta^{-1/2}\int_\Omega|\nabla u|^2\tau\,dx+\eta^{1/2}\int_\Omega|\nabla(e^{\lambda u_3}+e^{-\lambda u_3}+\gamma^{-1}Exp_0)|^2\tau\,dx\,.$$

Choosing $\eta=\eta(\|u\|_\infty)$ large the term $C_{12}+C_3$ in (4.8) and fractions of A_{12}, A_3 dominate (4.8).

Thus we have proved the basic inequality for three players.

Lemma 4.1. *Let $u\in H_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega)$ be weak solution of the system* (1.1) *with $\nu=1,2,3$ and assume the Caratheodory growth condition* (1.3), (1.5) *and the structure condition* (1.6), (4.1) *up to* (4.6). *Then u satisfies*

$$\int_\Omega|\nabla u|^2\tau\,dx+\lambda^{-1}\eta^{-1}(\nabla Exp_1,\nabla\tau)\le K\int_\Omega\tau\,dx$$

with some constant $K=K(\|u\|_\infty)$. Here $\tau\ge0,\tau$ Lipschitz and $\lambda,\beta_1,\beta_2,\gamma,\eta$ are chosen large enough, the choice depending on the growth constants and $\|u\|_\infty$.

The simplest condition for obtaining an L^∞-bound for the solution u are the structure conditions

$$|H_{\nu0}(x,u,\nabla u)|\le K|\nabla u||\nabla u_\nu|+K$$

however for functionals with non-compact control coupling one has to find analogues of our approach in [2], BF02a for two players.

5 – Simple Hamiltonians with Non-Compact Control Coupling

There is a standard formalism to construct the Hamiltonians from the Lagrange functions of a stochastic game (cf. [6], also for references).

To each player, there is the associated Lagrange function L,

$$L_i(x,\lambda_i,p_i,v) = l_i(x,v) + p_i g(x,v) - \lambda_i c_i(x,v)\,.$$

The function g comes from the stochastic differential equation

$$dy = g(y,v) + dw\,,$$

$y =$ state variables, v control variables and the l_i come from the cost functional of the i-th player, say

$$l_i(x,v) = \varphi_i(v) + f_i(x)\,.$$

The function c_i is the discount factor. The deterministic analog of the value function of the i-th player is

$$\int_0^{\tau} l_i\big(y(t),v(t)\big) exp\Big(-\int_o^t c_i(y(s),v(s))\,ds\Big)\,dt\,.$$

λ_i and p_i are parameters. The term $exp\big(-\int_o^t c_i(y(s),v(s))\big)$ is the discount factor.

For illustration we discuss the following simple examples for two players

$$\begin{aligned} l_i(x,v) &= \frac{1}{2}v_i^2 + \theta_i v_1 v_2 + f_i(x)\,, \qquad i=1,2\\ c_i(y,v) &= \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2\\ g(y,v) &= b(v_1+v_2) \end{aligned}$$

with a fixed vector $b = \mathbb{R}^n$. The controls v_i are scalar valued and θ_i are parameters (*non-compact control coupling*).

We want to calculate a Nash point of the L_i. For this we have to set $\frac{\partial}{\partial v_i} L_i = 0$ and calculate the solution of this system:

$$v_i + \theta_i v_k + p_i b - \lambda_i v_i = 0\,, \qquad i \neq k\,.$$

This yields a solution v_1^*, v_2^*

$$\begin{aligned} v_1^* &= \big[(1-\lambda_1)(1-\lambda_2) - \theta_1\theta_2\big]^{-1}\{-p_1 - b(1-\lambda_2) + \theta_1 p_2 \cdot b\}\\ v_2^* &= \big[(1-\lambda_1)(1-\lambda_2) - \theta_1\theta_2\big]^{-1}\{-p_2 - b(1-\lambda_1) + \theta_2 p_1 \cdot b\}\\ v_i^*(u,\nabla u) &= v_i^*|_{\lambda_i = u_i,\, p_i = \nabla u_i}\,. \end{aligned}$$

For the Hamiltonian H_ν we obtain

$$
\begin{aligned}
H_i(x,u,\nabla u) = L_i(x,u,\nabla u_i, v^*) = \\
&= \frac{1}{2}\big|v_i^*(u,\nabla u)\big|^2 + \theta_i v_1^*(u,\nabla u)\cdot v_2^*(u,\nabla u) + \\
&\quad + \nabla u_i \cdot b\big(v_1^*(u,\nabla u) + v_2^*(u,\nabla u)\big) - u_i\big[\frac{1}{2}|v^*(u,\nabla u)|^2\big] + f_i(x)\,.
\end{aligned}
$$

From the above formula, we see that H_i has the form

$$
\begin{aligned}
H_i(x,u,\nabla u) = \hat{H}_{i0}(x,u,\nabla u) + L(x,u,\nabla u)\nabla u_i - \\
- u_i F(x,u,\nabla u) + f_i(x)
\end{aligned}
$$

with $F = \frac{1}{2}|v^*(u,\nabla u)|^2 \geq 0$, quadratic in ∇u, $L(x,u,\nabla u)$ linear growth in ∇u and a term

$$
\hat{H}_{i0}(x,u,\nabla u) = \frac{1}{2}|v_i^*(x,u,\nabla u)|^2 + \theta_i v_1^*(x,u,\nabla u) v_2^*(x,u,\nabla u)\,.
$$

For general θ_1, θ_2, it is not clear that the regularity theorem of this paper covers all cases.

Our theory covers the case $\theta_1 = 0$, θ_2 arbitrary since then

$$
|v_1^*|^2 \leq K|\nabla u_1|^2
$$

One has to arrange a setting so that

$$
\big[(1-u_1)(1-u_2) - \theta_1\theta_2\big]^{-1}
$$

exists, as in our paper [6], and the structure condition (1.9), (1.10) is satisfied. In [2], [3] we have treated the case $\theta_1 = \theta_2$, in absence of discount control. It is an interesting task to generalize the corresponding theorem [2], [3] to cases presented here.

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