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On the class of starlike meromorphic function of complex order

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ABSTRACT: *We give some results for certain subclass of meromorphic function f of complex order defined on the punctured unit disk. A necessary and sufficient condition for functions to belong to the class will be discussed. Further, differential subordinations are also obtained.*

1 – Introduction

Let Σ denotes the class of functions f normalized by

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the punctured open unit disk $U^* = \{z \in: 0 < |z| < 1\} = U - \{0\}$, where U is the open unit disk $U = \{z \in: \ |z| < 1\}$.

A function $f \in \Sigma$ is said to be meromorphic starlike of order $\alpha(0 \leq \alpha < 1)$, if

$$-\Re \frac{zf'(z)}{f(z)} > \alpha \qquad (z \in U^*),$$

and we denote this class by $\Sigma^*(\alpha)$.

KEY WORDS AND PHRASES: *Analytic function – Meromorphic function – Starlike function – Differential subordination*
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Let A denoted the class of functions f normalized by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disc U and let S be the subclass of A consisting of functions which are also univalent in U.

Let $\phi(z)$ be an analytic function with positive real part on U that satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Ma and Minda [10] introduced and studied the class $S^*(\phi)$ consists of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z \in U).$$

Following Ma and Manda [10], Ravichandran *et.al* [9] defined a more general class $S_b^*(\phi)$ of starlike functions of complex order consists of functions $f \in S$

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z),$$

where $b \neq 0$ is a complex number.

Analogous to the class $S_b^*(\phi)$, for $f \in A$, the authors [1] defined the class $M_b^*(\phi)$ of meromorphic functions as the following:

Definition 1.1. Let $\phi(z)$ be an analytic function with positive real part on U which satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis.Let $M_b^*(\phi)$ be the class $f \in \Sigma$ satisfying

$$1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec \phi(z). \tag{2}$$

We shall write $M_1^*(\phi)$ by $\Sigma^*(\phi)$. In the case

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \qquad 0 \leq \alpha < 1,$$

it is obvious that $M_1^*(\phi)$ is the class of meromorphic starlike functions of order α.

Motivated by a similar result of Silverman et. al [2] for $f \in \Sigma^*(\phi)$, the authors[1] obtained the following theorem for $f \in M_b^*(\phi)$.

Theorem 1.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1) belongs to $M_b^*(\phi)$, then for any complex number τ*

$$(i) \quad \left| a_1 - \tau a_0^2 \right| \leq \frac{1}{2} \left| b \right| \left| B_1 \right| \max\left\{1, \left| \frac{B_2}{B_1} - (1 - 2\tau) b B_1 \right| \right\}, \; B_1 \neq 0$$

$$(ii) \quad \left| a_1 - \tau a_0^2 \right| \leq |b| \, , \quad B_1 = 0.$$

The bounds are sharp for the functions $G_1(z)$ and $G_2(z)$ *defined by*

$$1+\frac{1}{b}\left(-\frac{zG_1'(z)}{G_1(z)}-1\right)=\phi(z^2), \quad \text{where} \quad G_1(z)=\frac{1+z^2}{z(1-z^2)},$$

$$1+\frac{1}{b}\left(-\frac{zG_2'(z)}{G_2(z)}-1\right)=\phi(z), \quad \text{where} \quad G_2(z)=\frac{1+z}{z(1-z)}.$$

EXAMPLE 1.1. By taking $b=(1-\beta)e^{-i\lambda}\cos\lambda,\ 0\leq\beta<1,\ |\lambda|<\frac{\Pi}{2}$, and $\phi(z)=\frac{1+z}{1-z}$, we obtain the following sharp inequality

$$\left|a_1-\tau a_o^2\right|<(1-\beta)\cos\lambda\max\left\{1,\left|e^{i\lambda}-2(1-2\tau)(1-\beta)\cos\lambda\right|\right\}$$

Putting $b=1$ in Theorem 1.1, we get the following result obtained by Silverman *et al.* [2].

COROLLARY 1.1. *Let* $\phi(z)=1+B_1z+B_2z^2+\dots$. *If* $f(z)$ *given by (1) belongs to* $\Sigma^*(\phi)$ *, then for any complex number* τ

$$(i)\ \left|a_1-\tau a_0^2\right|\leq\frac{|B_1|}{2}\max\left\{1,\left|\frac{B_2}{B_1}-(1-2\tau)B_1\right|\right\},\ B_1\neq 0$$
$$(ii)\ \left|a_1-\tau a_0^2\right|\leq 1,\quad B_1=0.$$

For $b=1$ in Theorem 1.1, we can also get the following result obtained by Ali and Ravichandran [4].

COROLLARY 1.2. *Let* $\phi(z)=1+B_1z+B_2z^2+\dots$. *If* $f(z)$ *given by (1) belong to* $\Sigma^*(\phi)$ *, then for any complex number* τ

$$\left|a_1-a_0^2\right|\leq\frac{B_1}{2}\max\left\{1,\left|B_1-2\tau B_1-\frac{B_2}{B_1}\right|\right\}$$

The object of this paper is to obtain some results for the class $M_b^*(\phi)$, using mainly the method of subordination. In that sense, we give some definitions, notations and lemmas we need in the next part.

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G written $F\prec G$ if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions F and G which are analytic in U, the function F is said to be subordinate to G if there exist a function w analytic in U with $w(0)=0$ and $(\forall z\in U):\ |w(z)|<1$, such that $F(z)=G(w(z))$.

The general theory of differential subordinations was introduced by Miller and Mocanu [6] (see also [7] and [8]). Namely let $\psi : C^2 \to C$ be analytic in a domain D, let h be univalent in U, and let $p(z)$ be analytic in U with $(p(z); zp'(z)) \in D$ when $z \in U$, then $p(z)$ is said to satisfy the first-order differential subordination if

$$\psi(p(z); zp'(z)) \prec h(z). \tag{3}$$

The univalent function q is said to be a dominant of the differential subordination (3) if $p \prec q$ for all p satisfying (3). If $\widetilde{q}$ is a dominant of (3) and $\widetilde{q} \prec q$ for all dominants q of (3), then $\widetilde{q}$ is said to be the best dominant of (3).

Our results and their proofs are motivated by a similar result of Ravichandran et al. [9], Ali and Ravichandran [4] and Srivastava and Lashin [3] (see also Ibrahim and Darus [5]).

First we cite the following lemmas require to prove our results.

Lemma 1.1. [8]. Let ϕ be a convex univalent function defined on U and $\phi(0) = 1$. Define $F(z)$ by

$$F(z) = z \exp\left(\int_0^z \frac{\phi(\eta) - 1}{\eta} d\eta\right).$$

Let $q(z)$ be analytic in U and $q(0) = 1$. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z),$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(ts)}{tF(sz)}.$$

Lemma 1.2. [7]. Let $q(z)$ be univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$ with $\varphi(z) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \vartheta(q(z)) + Q(z)$. Suppose that either $h(z)$ is convex, or $Q(z)$ is starlike univalent in U. In addition, assume that $\Re[\frac{zh'(z)}{Q(z)}] > 0$ for $z \in U$. If $p(z)$ is analytic in u with $p(0) = q(0), p(U) \subseteq D$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2 – Main Results

We first prove a representation formula for functions in the class $M_b^*(\phi)$,

Theorem 2.1. *A function $f(z) \in M_b^*(\phi)$ if and only if*

$$[zf(z)]^{\frac{1}{b}} = \exp\left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta}d\zeta\right), \tag{4}$$

where $w(z)$ is analytic in U satisfying $w(0) = 0$ and $|w(z)| \leq 1$.

Proof. Let $f \in M_b^*(\phi)$. Then (4) holds and therefore there is a function $w(z)$ analytic in U with $w(0) = 0$ and $|\mathrm{w(z)}| \leq 1$ such that

$$1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) = \phi(w(z)).$$

Rewriting the above equation in the form

$$\frac{1}{b}\left(\frac{f'(z)}{f(z)} + \frac{1}{z}\right) = \frac{1-\phi(w(z))}{z},$$

and integrating from 0 to z, we obtain

$$\frac{1}{b}\left[\ln zf(z)\right] = \left(\int_0^z \frac{1-\phi(w(\zeta))}{\zeta}d\zeta\right),$$

which gives the desired assertion upon exponentiation. The converse follows directly by differentiation. □

Using Lemma 1.1, we obtain the following necessary and sufficient conditions for functions to belong to $M_b^*(\phi)$.

Theorem 2.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. A function f belongs to $M_b^*(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,*

$$\left(\frac{sf(sz)}{tf(tz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}.$$

PROOF. Define the function $p(z)$ by

$$\frac{1}{p(z)} = [zf(z)]^{\frac{1}{b}}.$$

Then a computation show that

$$1 + \frac{zp'(z)}{p(z)} = 1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right).$$

The result now follows from Lemma 1.1. □

In the next theorem we prove a differential subordination result for the class $M_b^*(\phi)$. The motivation of our result here is to generalize the result obtained by Ali and Ravichandran [4].

THEOREM 2.3. *Let α be a nonzero complex number. Let $q(z)$ be univalent in U, $q(0) = 1$. Assume that $q(z)$ or $\gamma + b(3\alpha - 2\alpha b - 1)q(z) + \alpha b^2 q^2(z) - \alpha bzq'(z)$ is convex univalent and*

$$\Re\left\{\frac{1 + 2\alpha b - 3\alpha}{\alpha} - 2bq(z) + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{5}$$

If $f \in \Sigma$ satisfies

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \gamma + b(3\alpha - 2\alpha b - 1)q(z) + \alpha b^2 q^2(z) - \alpha bzq'(z),$$

where $\gamma = \alpha b^2 - 3\alpha b + b + 2\alpha - 1$. Then

$$1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec q(z)$$

and $q(z)$ is the best dominant.

PROOF. Define the function $p(z)$ by

$$p(z) = 1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right). \tag{6}$$

Differentiating (6), we obtain

$$\frac{zp'(z)}{p(z) - 1} = \frac{z^2 f''(z) + 2zf'(z)}{zf'(z) + f(z)} - \frac{zf'(z)}{f(z)},$$

From the above equality, we have

$$1+\frac{1}{b}\left(\frac{zp'(z)}{p(z)-1}\right)=\frac{1}{b}\left[\frac{z^2f''(z)+2zf'(z)}{zf'(z)+f(z)}\right]+1-\frac{1}{b}\left(\frac{zf'(z)}{f(z)}+1\right)+\frac{1}{b}$$

So that

$$\frac{1}{b}\left(\frac{zp'(z)}{p(z)-1}\right)=\frac{1}{b}\left[\frac{z^2f''(z)+2zf'(z)}{zf'(z)+f(z)}\right]+p(z)-1+\frac{1}{b}.$$

A simple computation gives us

$$\frac{z^2f''(z)}{f(z)}+\frac{2zf'(z)}{f(z)}=b^2\left(p(z)-1\right)^2+b\left(p(z)-1\right)-bzp'(z).$$

This relation is equivalent with:

$$\alpha\frac{z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}=b(3\alpha-2\alpha b-1)p(z)+\alpha b^2p^2(z)-\alpha bzp'(z)+\gamma,$$

where $\gamma=\alpha b^2-3\alpha b+b+2\alpha-1$. Define the function $\theta(w)$ and $\phi(w)$ by

$$\theta(w)=b(3\alpha-2\alpha b-1)w+\alpha b^2w^2+\gamma \quad \text{and} \quad \phi(\mathrm{w})=-\alpha \mathrm{b}.$$

The functions $\theta(w)$ and $\phi(w)$ are analytic in C and $\phi(w)\neq 0$. Also,by setting

$$Q(z)=zq'(z)\phi(q(z))=-\alpha bzq'(z),$$
$$h(z)=\theta(q(z))+Q(z)=b(3\alpha-2\alpha b-1)q(z)+\alpha b^2q^2(z)+\gamma+Q(z).$$

we find that Q is starlike univalent in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\}=\Re\left\{\frac{1+2\alpha b-3\alpha}{\alpha}-2bq(z)+1+\frac{zq''(z)}{q'(z)}\right\}>0.$$

Then the relation (5) follows by an application of Lemma 1.2. For b = 1 in Theorem 2.3, we get the following result obtained by Ali and Ravichandran [4].

Corollary 2.1. *Let α be a nonzero complex number. Let $q(z)$ be univalent in U, $q(0)=1$. Assume that $q(z)$ or $(\alpha-1)q(z)+\alpha q^2(z)-\alpha zq'(z)$ is convex univalent and*

$$\Re\left\{\frac{1-\alpha}{\alpha}-2q(z)+1+\frac{zq''(z)}{q'(z)}\right\}>0. \tag{7}$$

If $f\in\Sigma$ satisfies

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec(\alpha-1)q(z)+\alpha q^2(z)-\alpha zq'(z).$$

Then

$$-\frac{zf'(z)}{f(z)}\prec q(z)$$

and $q(z)$ is the best dominant.

Setting $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2.3, we get

Corollary 2.2. *Let α be a nonzero complex number. Let*

$$\Re\left\{1 + \frac{1+2\alpha b - 3\alpha}{\alpha} - 2b\left(\frac{1+Az}{1+Bz}\right)\right\} > 0.$$

If $f \in \Sigma$ satisfies

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \gamma + b(3\alpha - 2\alpha b - 1)\left(\frac{1+Az}{1+Bz}\right) + \\ + \alpha b^2\left(\frac{1+Az}{1+Bz}\right)^2 - \frac{\alpha b(A-B)z}{(1+Bz)^2}.$$

Then

$$1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Setting $q(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we get

Corollary 2.3. *Let α be a nonzero complex number. Let*

$$\Re\left\{1 + \frac{1+2\alpha b - 3\alpha}{\alpha} - 2b\left(\frac{1+z}{1-z}\right)\right\} > 0.$$

If $f \in \Sigma$ satisfies

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \gamma + b(3\alpha - 2\alpha b - 1)\left(\frac{1+z}{1-z}\right) + \alpha b^2\left(\frac{1+z}{1-z}\right)^2 - \frac{2\alpha bz}{(1-z)^2}.$$

Then

$$1 - \frac{1}{b}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ is the best dominant.

Some other works related to meromorphic functions can be found in ([11] - [13]).

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