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Stabilization and control of partial differential equations of evolution

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– Preface

The present manuscript collects the most significative results achieved during my PhD program. The problems I have been working on derive from two different branches in control theory for partial differential equations: the stabilization of systems of hyperbolic equations and the controllability of degenerate/singular parabolic equations.

Coherently, the first part of the thesis is focused on stabilization properties for systems of two weakly coupled hyperbolic equations. As examples, we imagine to study the stabilization problem for a system of two wave equations, or of two plate equations, or a wave-plate system. While the stabilization issue is well-understood for a single (linear) hyperbolic equation, where exponential decay rate can be ensured by a damping acting locally on the boundary or in the interior of the domain, much less clear is the situation for a system of two equations, even though this might occur in several application to mechanical flexible structures.

I first present the analysis for the indirect stabilization of systems with globally distributed coupling and damping. In this case, a compatibility conditions between the operators involved in the system ensures the polynomial decay rate for the total energy of the system. The abstract setting relies on the use of semigroup and interpolation theory. The main purpose is to prove general energy estimates, depending on the initial conditions, achieved by means of suitable (operator) multipliers.

Moreover, the polynomial decay rate of some systems to which the previous compatibility condition applies can be improved by means of sharp estimates of the norm of the resolvent operator along the imaginary axis.

On the other hand, the multiplier technique allows to show polynomial stabilization rate for systems with coupling and damping located at the boundary. But, in this situation, the coupling operator is unbounded in the energy space, and so the exponential decay rate cannot be ruled out as before.

The second part of the thesis treats parabolic equations with operator degenerating at the interior or at the boundary of the domain.

I consider the null controllability problem for the generalized Grushin operator $Au = \partial_t u - \partial^2_{xx} u - |x|^{2\gamma} \partial^2_{yy} u$ in dimension two, for positive values of the parameter γ, in the domain $D = (-1, 1) \times (0, 1)$, so that the operator degenerates inside the domain. By a duality argument, the null controllability is equivalent to an observability estimate for the adjoint system. By means of a Fourier series decomposition, the problem reduces to deduce a uniform observability inequality for a one dimensional equation, that can be proven thanks to an appropriate Carleman estimate. Thus, I show that a null controllability result in an arbitrary time holds for $\gamma \in (0, 1)$, while null controllability fails for $\gamma > 1$. Interesting is the behaviour in the transition state $\gamma = 1$: a minumum time $T^* > 0$ is needed to achieve null controllability, that fails for small times. The negative result requires explicit supersolution and comparison estimates.

Motivated by a recent result on the Laplace-Beltrami operator in almost Riemannian manifolds, I have started developing the analysis of controllability properties for the Grushin operator with a singular potential. I show null controllability in large time for the operator $Lu = \partial_t u - \partial^2_{xx} u - |x|^2 \partial^2_{yy} u - \frac{\lambda}{x^2} u$, in the domain $\Omega = (0,1) \times (0,1)$, that is, with both degeneracy and singularity occurring at the boundary of the domain, for all coefficients $\lambda < 1/4$, the constant in the Hardy-Poincaré inequality.

– **Resumé**

Le présent manuscrit rassemble les résultats les plus significatifs obtenus au cours de mon doctorat. Les problèmes sur lesquels j'ai travaillé dérivent de deux branches différentes de la théorie du contrôle des équations aux dérivées partielles: la stabilisation des systèmes d'équations hyperboliques et la contrôlabilité des équations paraboliques dégénérées/singulières.

En particulier, la première partie de la thèse se concentre sur les propriétés de stabilisation de certains systèmes de deux équations hyperboliques faiblement couplées. Par exemple, on étudie le problème de stabilisation pour un système de deux équations des ondes, ou de deux équations des plaques, ou un système d'onde-plaque. Bien que la question de la stabilisation soit bien comprise pour une seule équation hyperbolique (linéaire), où le taux de décroissance exponentiel peut être assuré par un amortissement agissant localement sur la frontière ou à l'intérieur du domaine, le cas d'un système de deux équations est beaucoup moins clair, même si cela pourrait se produire dans plusieurs applications à la mécanique des structures flexibles.

J'ai d'abord présenté l'analyse pour la stabilisation indirecte de systèmes avec couplage et amortissement distribués globalement. Dans ce cas, une condition de compatibilité entre les opérateurs impliqués dans le système garantit le taux de décroissance polynomial de l'énergie totale du système. Le cadre abstrait repose sur l'utilisation de semi-groupes et la théorie de l'interpolation. Le but principal est de prouver des estimations générales sur l'énergie, en fonction des conditions initiales, obtenues par des multiplicateurs appropriés (de l'opérateur).

Par ailleurs, le taux de décroissance polynomial de certains systèmes sur lesquels la condition de compatibilité précédente s'applique peut être amélioré au moyen d'estimations optimales de la norme de l'opérateur résolvant le long de l'axe imaginaire.

D'autre part, la technique des multiplicateurs permet de montrer la stabilisation avec taux polynomial pour des systèmes ayant le couplage et l'amortissement situés sur la frontière. Mais, dans cette situation, l'opérateur de couplage n'est pas borné dans l'espace de l'énergie, et donc le taux de décroissance exponentiel ne peut pas être exclu comme avant.

La deuxième partie de la thèse traite d'équations paraboliques où l'opérateur dégénère à l'intérieur ou à la frontière du domaine.

Je considère le problème de la contrôlabilité à zéro pour l'opérateur généralisé de Grushin $Au = \partial_t u - \partial^2_{xx} u - |x|^{2\gamma} \partial^2_{yy} u$ en dimension deux, pour les valeurs positives du paramètre γ, dans le domaine $D = (-1, 1) \times (0, 1)$, de sorte que l'opérateur dégénère à l'intérieur du domaine. Par un argument de dualité, la contrôlabilité à zéro équivaut à une estimation d'observabilité pour le système adjoint. Au moyen d'une décomposition en série de Fourier, le problème se réduit à déduire une inégalité d'observabilité uniforme pour une équation unidimensionnelle, qui peut être prouvée grâce à une estimation de Carleman appropriée. Ainsi, je montre qu'un résultat de contrôlabilité nulle dans un temps arbitraire est valable pour $\gamma \in (0, 1)$, alors que contrôlabilité à zéro n'est vraie pour aucun temps quand $\gamma > 1$. Le comportement de l'état de transition $\gamma = 1$ est particulièrement intéressant : un temps minimum $T^* > 0$ est nécessaire pour réaliser la contrôlabilité à zéro, qui n'est pas vraie en temps petit. Le résultat négatif nécessite une supersolution explicite et des estimations de comparaison.

Enfin, motivé par un résultat récent concernant l'opérateur de Laplace-Beltrami sur les variétés presque riemanniennes, j'ai commencé à développer l'analyse des propriétés de contrôlabilité pour l'opérateur de Grushin avec un potentiel singulier. Je montre la contrôlabilité à zéro en temps grand pour l'opérateur $Lu = \partial_t u - \partial^2_{xx} u - x^2 \partial^2_{yy} u - \frac{\lambda}{x^2} u$, dans le domaine $\Omega = (0, 1) \times (0, 1)$, c'est-à-dire, avec la dégénérescence et la singularité qui se produisent à la frontière du domaine, pour tous les coefficients $\lambda < 1/4$, la constante dans l'inégalité de Hardy-Poincaré.

– Introduction to Part I

The interest of the scientific community in the stabilization and control of systems of partial differential equations has remarkably increased during last decades. This is probably due to the fact that such systems arise in several applied mathematical models, such as those used for studying the vibrations of flexible structures and networks (see, for example, the book by Dager and Zuazua [62] and references therein), or fluids and fluid-structure interactions (see, for instance, [17, 16, 34, 100, 131, 142]).

When dealing with systems involving quantities described by several components, pretending to control or observe all the state variables might be irrealistic, or too expensive. In applications to mathematical models for the vibrations of flexible structures (see [5] and [10]), electromagnetism (see, for instance, [105]), or fluid control (see [55] and the references therein), it may happen that only part of such components can be observed. This is why it becomes essential to study whether controlling only a reduced number of state variables suffices to ensure the stability of the full system. As an example, vibrations of elastic or visco-elastic structures are described by reversible PDEs. In such applications, the main goal is to reduce oscillations through a feedback law implemented within the system, in order to stabilize the system as time increases.

The asymptotic behavior of wave-like equations and, in particular, the derivation of optimal decay rates for the energy when the geometry of the domain and damping region allow rays to be trapped, have been intensively studied by several authors over last years. For such questions and results, we refer the reader to Lebeau [111] and Burq [39] (and the references therein). In [111], Lebeau considered a locally damped wave equation and proved optimal logarithmic decay rates for the energy, provided that damping is active on a nonempty open set. The proof relies on optimal resolvent estimates for the corresponding infinitesimal generator of the associated semigroup. Later on these results were completed by Burq in [39] in exterior domains, in particular for cases in which rays may be trapped by the obstacle.

Indirect stabilization for symmetric hyperbolic systems was first considered by Alabau-Boussouira in [1], and further developed in [4, 2, 6, 8] for more general systems, using energy type methods, together with some new ideas such as the new integral inequality given in [1, 2] (see also Lemma 1.8 in Chapter 1). In this approach, the purpose is rather to focus on the properties of the operators involved in the system, in order to allow the transfer of the damping action of the feedback to the undamped equation.

Indeed, it turns out that certain systems possess an internal structure that compensates for the aforementioned lack of control/feedback variables. Such a phenomenon is referred to as *indirect stabilization* or *indirect control* (see [135]).

Let us describe this feature through an example. Let β, $\kappa > 0$, and consider the system

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u - \kappa v = 0 \\ \partial_t^2 v - \Delta v - \kappa u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{1}$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \text{ on } \Gamma \quad \forall t > 0 \tag{2}$$

and initial conditions

$$\begin{cases} u(x,0) = u^0(x) & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x) & v'(x,0) = v^1(x) \end{cases} \quad x \in \Omega\,. \tag{3}$$

As explained in [2, Example 6.1], the above system describes the evolution of two elastic membranes subject to an elastic force that attracts one membrane to the other with coefficient $\kappa > 0$. Moreover, the term $\beta \partial_t u$ acts on the first membrane as a stabilizer, whereas no direct feedback takes effect on the second membrane. As shown in [2], for sufficiently small κ and enough regular initial conditions (3), the natural energy associated to system (1) decays in time at a polynomial rate. Therefore, we might consider the first membrane as an *indirect stabilizer* for the conservative membrane.

Subsequently, indirect stabilization of coupled systems was investigated in [22], where resolvent estimates were obtained and spectral analysis was used to prove polynomial decay of abstract semigroup, with applications to system (1) or to similar examples of symmetric systems. The works by Liu-Rao [116, 117] and Loreti-Rao [120], used spectral conditions and a Riesz basis approach, achieving polynomial decay rates for the energy of a simplified case of coupled systems, where the dynamics of each component is led by the same operator (wave-wave for example) and the damping operator is a nonpositive fractional power of it (in [120]), so that a dispersion relation for eigenvalues can be precisely exploited.

More recently, inspired by the methods in [111] and [39] and encouraged by previous results on decay rate estimates for weakly coupled systems [22, 1, 4, 2], the works [23] and [32] addressed the optimality of spectral-analysis-derived decay rates, taking into account the asymptotic behaviour of the resolvent along the imaginary axis. However, optimality for the PDE evolution system is subject to the optimality of the required resolvent estimate, that remains usually uncertain.

In the context of indirect stabilization for coupled systems, we would like to stress the fact that checking the asymptotic behaviour of the resolvent norm for the operators involved in the system may be a difficult task. In particular, resolvent estimates may be hard to obtain when the two operators A_1 and A_2 that rule the dynamics of the first and second component do not commute, or damping and coupling operators do not commute with A_1 and A_2. For results when the operators A_1, A_2 are not necessarily equal and B does not commute with them, we refer the reader to [1, 4, 6]. The case of localized or boundary damping, together with localized coupling, is analyzed in [8], supposing the geometric control condition holds for both subdomains. In this case, since the coupling acts locally, the corresponding operator is no longer coercive, and this feature generates additional difficulties. Under the same geometric condition, the controllability problem for a system of two wave-type equation has been addressed in [7, 9].

– Structure of Part I

The Part I of the thesis is organized as follow: in Chapter 1 we propose a new compatibility condition in order to ensure polynomial stabilization for systems of evolution equations in Hilbert space, with several applications to systems of hyperbolic PDEs. In Chapter 2 we show how to take advantage of resolvent estimates to improve the decay rate for some systems considered in Chapter 1, through a general criterion introduced in [32]. Chapter 3 takes into account the stabilization problem for a system of two wave equation coupled at the boundary. This kind of coupling produces relevant consequences in the abstract framework introduced in Chapter 1, Section 1.1.1. Indeed, in this situation the coupling operator loses its compactness in the energy space, thus the exponential stability is not ruled out as in the distributed coupling case. However, by means of energy methods and suitable multipliers, we will show that the total energy of the system decays at a polynomial rate.

1 – A compatibility condition for indirect stabilization of evolution equations with compact coupling

The present chapter reproduces the article *Indirect stabilization of weakly coupled systems with hybrid boundary conditions*, in collaboration with Fatiha Alabau-Boussouira and Piermarco Cannarsa, published in Mathematical Control and Related Fields, Volume 1, Number 4, December 2011.

1.1 – Introduction

An example of indirect stabilization occurs with the hyperbolic system

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u = 0 = v & \text{on} \quad \partial\Omega \times \mathbb{R}\,, \end{cases} \tag{4}$$

where Ω is a bounded open domain of $\mathbb{R}^d$. We observe that, in case $\alpha = 0$, system (4) reduces to two decoupled equations, an exponentially stable wave equation (component u) and a conservative wave equation (in v). Indeed, the "frictional" term $\partial_t u$ acts as a stabilizer for the first equation. The indirect stabilization problem consists in evaluating under which conditions such frictional term $\partial_t u$ might suffice to stabilize the whole system, through a weak (zero order terms) coupling with coefficient α, and, if so, at which decay rate. A general result proved in [2] ensures that, for sufficiently smooth initial conditions and $|\alpha| > 0$ small enough, the energy of the solution (u, v) of (4) decays to zero at a polynomial rate as $t \to \infty$.

The above indirect stabilization property holds true for more general systems of partial differential equations, under the compatibility assumption (13) below, introduced in [2]. For applications to problems in mechanical engineering, however, it is extremely important to consider also boundary conditions that fail to satisfy the assumption (13). This is the case of Neumann or Robin boundary conditions, which describe different physical situations such as hinged or clamped devices. For instance, let us change the boundary conditions in (4) as follows:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ \partial_t^2 v - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u + \frac{\partial u}{\partial \nu} = 0 = v & \text{on} \quad \partial\Omega \times \mathbb{R}\,. \end{cases} \tag{5}$$

Then, as is shown in Proposition 1.26 below, the compatibility assumption (13) is not satisfied. Nevertheless, in this chapter we will prove polynomial stability for system (5), using a new hypothesis (see condition (14)) which is specially designed to handle boundary conditions as above—that we call *hybrid*.

1.1.1 – Abstract setting

More generally, in a real Hilbert space H, with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|_H$, we shall study the system of evolution equations

$$\begin{cases} u''(t)+A_1u(t)+Bu'(t)+\alpha v(t)=0 & \text{in } H \\ v''(t)+A_2v(t)+\alpha u(t)=0 & \text{in } H \end{cases} \tag{6}$$

where

(H1) $A_i : D(A_i)\subset H\to H$ $(i=1,2)$ are densely defined closed linear operators such that

$$A_i=A_i^*\,,\qquad \langle A_iu,u\rangle\geq \omega_i|u|_H^2 \qquad \forall u\in D(A_i)\quad \text{for some } \omega_1,\omega_2>0\,,$$

(H2) B is a bounded linear operator on H such that

$$B=B^*\,,\qquad \langle Bu,u\rangle\geq\beta|u|_H^2\qquad \forall u\in H\quad\text{for some }\beta>0\,,$$

(H3) α is a real number such that $0<|\alpha|<\sqrt{\omega_1\omega_2}\,$.

System (6), with the initial conditions

$$\begin{cases} u(0)=u^0\,, & u'(0)=u^1\,, \\ v(0)=v^0\,, & v'(0)=v^1\,, \end{cases} \tag{7}$$

can be formulated as a Cauchy problem for a certain first order evolution equation in the product space

$$\mathcal{H}:=D(A_1^{1/2})\times H\times D(A_2^{1/2})\times H\,.$$

More precisely, let us define the energies associated to operators A_1, A_2 by

$$E_i(u,p)=\frac{1}{2}\left(|A_i^{1/2}u|_H^2+|p|_H^2\right)\quad \forall (u,p)\in D(A_i^{1/2})\times H\ (i=1,2)\,, \tag{8}$$

and the total energy of the system as

$$\mathcal{E}(U):=E_1(u,p)+E_2(v,q)+\alpha\langle u,v\rangle \tag{9}$$

for every $U=(u,p,v,q)\in\mathcal{H}$. Then, assumption (H1) yields, for $i=1,2$,

$$|u|_H^2\leq\frac{2}{\omega_i}\,E_i(u,p)\qquad \forall u\in D(A_i^{1/2}),\ \forall p\in H\,. \tag{10}$$

Moreover, in view of (H3), for all $U = (u, p, v, q) \in \mathcal{H}$

$$\mathcal{E}(U) \geq \nu(\alpha)\Big[E_1(u, p) + E_2(v, q)\Big], \tag{11}$$

where $\nu(\alpha) = 1 - |\alpha|(\omega_1\omega_2)^{-1/2} > 0$.

Let us introduce the bilinear form on $\mathcal{H}$

$$(U|\widehat{U}) = \langle A_1^{1/2}u, A_1^{1/2}\widehat{u}\rangle + \langle p, \widehat{p}\rangle + \langle A_2^{1/2}v, A_2^{1/2}\widehat{v}\rangle + \langle q, \widehat{q}\rangle + \alpha\langle u, \widehat{v}\rangle + \alpha\langle v, \widehat{u}\rangle .$$

Since

$$(U|U) = 2\mathcal{E}(U) \quad \forall U \in \mathcal{H},$$

thanks to (11) the above form is a scalar product on $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space with such a product. Let now $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (\ p\ ,\ -A_1u - Bp - \alpha v\ ,\ q\ , -A_2v - \alpha u\ ) \qquad \forall U \in D(\mathcal{A}) . \end{cases}$$

Then, problem (6) takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1) . \end{cases} \tag{12}$$

As will be proved in Lemma 1.16, $\mathcal{A}$ is a maximal dissipative operator. Then, from classical results (see, for instance, [130]), it follows that $\mathcal{A}$ generates a C_0-semigroup, $e^{t\mathcal{A}}$, on $\mathcal{H}$. Also,

$$e^{t\mathcal{A}}U_0 = (u(t), p(t), v(t), q(t)) ,$$

where (u, v) is the solution of problem (6)-(7), and $(p, q) = (u', v')$.

1.1.2 – Strategy for the indirect stabilization

In order to introduce our asymptotic analysis of system (6)-(7)—or, equivalently, (12)—let us observe that, as is explained in [2], no exponential stability can be expected. Therefore, weaker decay rates at infinity, such as polynomial ones, are to be sought for. Polynomial decay results for (6) were obtained in [2] assuming that, for some integer $j \geq 2$,

$$|A_1u|_H \leq c|A_2^{j/2}u|_H \qquad \forall u \in D(A_2^{j/2}) . \tag{13}$$

Similar decay estimates for the case of boundary damping (that is, when operator B is unbounded) were derived in [4]. Also, we refer the reader to [31, 30] for indirect stabilization with localized damping, and to [11] for the study of a one-dimensional wave system coupled through velocities.

In this chapter we will replace (13) by the compatibility condition

$$D(A_2) \subset D(A_1^{1/2}) \qquad \text{and} \qquad |A_1^{1/2}u|_H \le c|A_2 u|_H \quad \forall u \in D(A_2), \tag{14}$$

which is satisfied by a large class of systems including (5) as a special case (see Section 1.5 below). Under such a condition we will show that any solution U of (12) satisfies the integral inequality

$$\int_0^T \mathcal{E}(U(t))dt \le c_1 \sum_{k=0}^{4} \mathcal{E}(U^{(k)}(0)) \qquad \forall T > 0\,,\ U_0 \in D(\mathcal{A}^4)\,. \tag{15}$$

Moreover, since the energy of solutions is decreasing in time, and thanks to a abstract result due to Alabau-Boussouira [1] (see also Lemma 1.8), (15) implies, in turn, the polynomial decay of order n of $\mathcal{E}$, that is,

$$\mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{16}$$

for all $n \ge 1$ and $U_0 \in D(\mathcal{A}^{4n})$ (see Corollary 1.11 below). Notice that (16) yields, in particular, the strong stability of $e^{t\mathcal{A}}$.

The compatibility condition (14) is equivalent to the boundedness of $A_1^{1/2}A_2^{-1}$. Let us point out that this hypothesis is sufficient but not necessary. For example, let us consider $A_2 = A_1^\tau$ with $\tau \in (0, 1/2)$. In this case, condition (14) is violated, but it is easy to check that condition (13) holds for the smallest integer j such that $j > 2/\tau$. On the other hand, condition (14) is satisfied for all $\tau \ge 1/2$. This example shows that the present results and those of [2] are in some sense complementary —and, for $A_2 = A_1^\tau$ ($\tau \ge 0$), exactly complementary. One should also note that, for general operators A_1 and A_2, the two compatibility conditions (13) and (14) do not cover all possible cases.

Passing from polynomial to a general power-like decay estimate is quite natural, once (16) has been established. Indeed, in Section 1.4, using *interpolation theory*, we obtain the fractional decay rate

$$\mathcal{E}(U(t)) \le \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{17}$$

for all $n \ge 1$ and $U_0 \in D(\mathcal{A}^n)$ (see Corollary 1.19 below). Moreover, taking initial data in $\big(\mathcal{H}, D(\mathcal{A}^n)\big)_{\theta,2}$ for any $0 < \theta < 1$, we deduce the continuous decay rate

$$\|U(t)\|_{\mathcal{H}}^2 \le \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0 \tag{18}$$

(compare also with [22, Proposition 3.1], where a comparable result is achieved using different techniques). In particular, for $n = 1$, relation (17) implies that, for every $U_0 \in D(\mathcal{A})$, the solution U of problem (12) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \leq \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0\,, \tag{19}$$

and there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \leq c_1 \left(|A_1 u^0|^2_H + |A_1^{1/2} u^1|^2_H + |A_2 v^0|^2_H + |A_2^{1/2} v^1|^2_H \right) .$$

Thus, interpolation theory applied to systems satisfying (14) allows to prove continuous energy decay rates, together with decay rates under explicit smoothness conditions on the initial data. Furthermore, we would like to point out that it also yields stronger results in the framework studied in [2], that is, under condition (13). We describe such applications in Section 1.6, where we show how to deduce power-like decay rates from the energy estimates of [2], thus recovering, in a more general set-up, related asymptotic estimates that can be obtained by spectral analysis.

1.1.3 – Open questions

Let us now mention some open questions. One interesting problem is to derive optimal decay rates for the energy of an indirectly damped coupled system in geometric situations for which trapped rays may exist for the uncoupled damped equation. More precisely, it would be very interesting to generalize Lebeau's resolvent analysis in [111] to such coupled systems obtaining optimal energy estimates. In a somewhat different spirit, another open question would be to determine if it is possible to combine the results of [111] and [39] with the techniques developed in [4, 2, 8] in order to derive sharp upper decay rates for the energy. In all the examples we discuss in the present chapter—as well as in [1, 4, 2, 8]—operators A_1 and A_2 happen to have compact resolvents. It would be very interesting to see if explicit energy decay rates can be derived in different situations. For instance, it would be nice to extend Burq's approach [39] in order to obtain indirect damping of coupled systems in exterior domains, and prove decay of the local total energy of solutions.

1.1.4 – Structure of the chapter

This chapter is organized as follows. In Section 1.1.1 we have introduced the abstract setting that fits the weakly coupled systems we are concerned with in a standard semigroup framework, providing well-posedness of the abstract Cauchy problem. Section 1.2 recalls preliminary notions, mainly related to interpolation theory which is so relevant in the method here developed. Section 1.3 is devoted to the polynomial decay result and its proof. In Section 1.4, we complete the analysis with estimates in interpolation spaces. In Section 1.5, we describe several applications to systems of partial differential operators. Finally, in Section 1.6, we show how to improve the results of [2] by interpolation.

1.2 – Preliminaries

In this section we introduce the main tools required to deal with interpolation theory between Banach spaces. For a general exposition of this theory the reader is referred to [138] and [122]. Interesting introductions are also given in [29] from the point of view of control theory, and [121] for the specific case of analytic semigroups.

Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ two real Banach spaces. We say that Y is continuously embedded into X, and we write $Y \hookrightarrow X$, if $Y \subset X$ and

$$|x|_X \leq c|x|_Y \qquad \forall x \in Y$$

for some constant $c > 0$.

We denote by $\mathcal{L}(Y;X)$ the Banach space of all bounded linear operators $T : Y \to X$ equipped with the standard operator norm. If $Y = X$, we refer to such a space as $\mathcal{L}(X)$. For any given subspace D of X, we denote by $T_{|D}$ the restriction of T to D.

Definition 1.1. Let $(D, |\cdot|_D)$ be a closed subspace of X. A subspace $(Y, |\cdot|_Y)$ of X is said to be an interpolation space between D and X if

(a) $D \hookrightarrow Y \hookrightarrow X$, and
(b) for every $T \in \mathcal{L}(X)$ such that $T_{|D} \in \mathcal{L}(D)$, we have that $T_{|Y} \in \mathcal{L}(Y)$.

Let X, D be Banach spaces, with D continuously embedded into X. For any $\alpha \in [0,1]$, we denote by $J_\alpha(X,D)$ the family of all subspaces Y of X containing D such that

$$|x|_Y \leq c|x|_D^\alpha \, |x|_X^{1-\alpha} \quad \forall x \in D$$

for some constant $c > 0$.

Let us introduce, for each $x \in X$ and $t > 0$, the quantity

$$K(t,x,X,D) := \inf_{\substack{x=a+b,\\ a\in X,\, b\in D}} \left(|a|_X + t|b|_D\right). \tag{20}$$

Let $0 < \theta < 1$ be fixed. We define

$$(X,D)_{\theta,2} := \left\{ x \in X : \int_0^{+\infty} |t^{-\theta-\frac{1}{2}} K(t,x,X,D)|^2 \, dt < +\infty \right\} \tag{21}$$

and

$$|x|^2_{\theta,2} := \int_0^{+\infty} |t^{-\theta-\frac{1}{2}} K(t,x,X,D)|^2 \, dt \, .$$

The space $(X,D)_{\theta,2}$, endowed with the norm $|\cdot|_{\theta,2}$, is a Banach space.

The reader is referred to [122] for the proof of the following results.

Theorem 1.2. *Let X_1, X_2, D_1, D_2 be Banach spaces such that D_i is continuously embedded in X_i, for $i = 1, 2$. If $T \in \mathcal{L}(X_1; X_2) \cap \mathcal{L}(D_1; D_2)$, then we have that $T \in \mathcal{L}((X_1, D_1)_{\theta,2}; (X_2, D_2)_{\theta,2})$ for every $\theta \in (0, 1)$. Moreover,*

$$\|T\|_{\mathcal{L}((X_1,D_1)_{\theta,2};(X_2,D_2)_{\theta,2})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_1;X_2)} \|T\|^{\theta}_{\mathcal{L}(D_1;D_2)} \,.$$

Consequently, the space $(X, D)_{\theta,2}$ belongs to $J_\theta(X, D)$ for every $\theta \in (0, 1)$. Let $\alpha \in [0, 1]$ and denote by $K_\alpha(X, D)$ the family of all subspaces $(Y, |\cdot|_Y)$ of X containing D such that

$$\sup_{t>0,\, x\in Y} \frac{K(t, x, X, D)}{t^\alpha |x|_Y} < +\infty \,.$$

Theorem 1.3 (Reiteration Theorem). *Let $0 < \theta_0 < \theta_1 < 1$. Fix $\theta \in \,]0, 1[$ and set $\omega = (1-\theta)\theta_0 + \theta\theta_1$.*

1) *If $E_i \in K_{\theta_i}(X, D)$, $i = 0, 1$, then $(E_0, E_1)_{\theta,2} \subset (X, D)_{\omega,2}$.*
2) *If $E_i \in J_{\theta_i}(X, D)$, $i = 0, 1$, then $(X, D)_{\omega,2} \subset (E_0, E_1)_{\theta,2}$.*

Consequently, if $E_i \in J_{\theta_i}(X, D) \cap K_{\theta_i}(X, D)$, $i = 0, 1$, then $(E_0, E_1)_{\theta,2} = (X, D)_{\omega,2}$, with equivalence between the respective norms.

Remark 1.4. Since $(X, D)_{\theta,2}$ is contained in $J_\theta(X, D) \cap K_\theta(X, D)$, for every $0 < \theta_0, \theta_1 < 1$ we have

$$((X, D)_{\theta_0,2}, (X, D)_{\theta_1,2})_{\theta,2} = (X, D)_{(1-\theta)\theta_0+\theta\theta_1,2} \,. \tag{22}$$

Since $X \in J_0(X, D) \cap K_0(X, D)$ and $D \in J_1(X, D) \cap K_1(X, D)$, we also have

$$(X, (X, D)_{\theta_1,2})_{\theta,2} = (X, D)_{\theta\theta_1,2} \quad \text{and} \tag{23}$$

$$((X, D)_{\theta_0,2}, D)_{\theta,2} = (X, D)_{(1-\theta)\theta_0+\theta,2} \,. \tag{24}$$

1.2.1 – Interpolation spaces and fractional powers of operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, with norm $|\cdot|_H$. Let $A : D(A) \subset H \to H$ be a densely defined closed linear operator on H such that

$$\langle Ax, x\rangle \geq \delta |x|^2_H \,, \quad \forall\, x \in D(A) \tag{25}$$

for some $\delta > 0$. As usual, we denote by A^θ the fractional power of A for any $\theta \in \mathbb{R}$ (see, for instance, [29, Chapter 1 - Section 5]), and by A^* the adjoint of A. We recall that A is self-adjoint if $D(A) = D(A^*)$ and $\langle Ax, y\rangle = \langle x, Ay\rangle$ for every x, $y \in D(A)$. For the proof of the following result we refer to [122, Theorem 4.36].

Theorem 1.5. *Let A be a self-adjoint operator satisfying* (25)*. Then, for every $\theta \in (0,1)$, α, $\beta \in \mathbb{R}$ such that $\beta > \alpha \geq 0$,*

$$(D(A^\alpha), D(A^\beta))_{\theta,2} = D(A^{(1-\theta)\alpha+\theta\beta})\,. \tag{26}$$

In particular,

$$(H, D(A^\beta))_{\theta,2} = D(A^{\beta\theta})\,. \tag{27}$$

We say that A is an m-accretive operator if

$$\begin{cases} \langle Ax, x\rangle \geq 0 & \forall x \in D(A) \quad (\textit{accretivity}) \\ (\lambda I + A)D(A) = H & \text{for some } \lambda > 0 \quad (\textit{maximality}) \end{cases}$$

Notice that, if the above maximality condition is satisfies for some $\lambda > 0$, then the same condition holds for every $\lambda > 0$. Moreover, we say that A is m-dissipative if $-A$ is m-accretive.

We refer the reader to [122, Section 4.3] for the proof of the next result.

Proposition 1.6. *Let $(A, D(A))$ be an m-accretive operator on a Hilbert space H, with A^{-1} bounded in H. Then for every α, $\beta \in \mathbb{R}$, $\beta > \alpha \geq 0$, $\theta \in (0,1)$, A satisfies* (26) *and* (27)*. In particular,*

$$D(A^\theta) = (H, D(A))_{\theta,2} \quad \forall\, 0 < \theta < 1\,. \tag{28}$$

Corollary 1.7. *If A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on H, with A^{-1} bounded in H, then $D(A^m) = (H, D(A^k))_{\theta,2}$ for every $k \in \mathbb{N}$, $\theta \in (0,1)$ such that $m = \theta k$ is an integer.*

1.2.2 – An abstract decay result

We recall an abstract result obtained in [1] in a slightly different form, and in [2, Theorem 2.1] in the current version.
Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on H.

Lemma 1.8. *Let $L : H \to [0, +\infty)$ be a continuous function such that, for some integer $K \geq 0$ and some constant $c \geq 0$,*

$$\int_0^T L(e^{tA}x)dt \leq c \sum_{k=0}^{K} L(A^k x) \qquad \forall\, T \geq 0\,,\ \forall\, x \in D(A^K)\,. \tag{29}$$

Then, for any integer $n \geq 1$, any $x \in D(A^{nK})$ and any $0 \leq s \leq T$

$$\int_s^T L(e^{tA}x)\,\frac{(t-s)^{n-1}}{(n-1)!}\,dt \leq c^n(1+K)^{n-1}\sum_{k=0}^{nK} L(e^{sA}A^k x)\,. \tag{30}$$

If, in addition, $L(e^{tA}x) \le L(e^{sA}x)$ *for any* $x \in H$ *and any* $0 \le s \le t$*, then*

$$L(e^{tA}x) \le c^n (1+K)^{n-1}\,\frac{n!}{t^n}\sum_{k=0}^{nK} L(A^k x) \qquad \forall t > 0 \tag{31}$$

for any integer $n \ge 1$ *and any* $x \in D(A^{nK})$*.*

1.3 – Main result

We are now ready to state and prove the polynomial decay of solutions to weakly coupled systems. In addition to the standing assumptions (H1), (H2), (H3), we will assume that

$$D(A_2) \subset D(A_1^{1/2}) \qquad \text{and} \qquad |A_1^{1/2}u|_H \le c|A_2 u|_H \quad \forall u \in D(A_2) \tag{32}$$

for some constant $c > 0$. Condition (32) can be formulated in the following equivalent ways.

LEMMA 1.9. *Under assumption* (H1) *the following properties are equivalent.*

(a) *Assumption* (32) *holds.*
(b) $A_1^{1/2}A_2^{-1} \in \mathcal{L}(H)$*.*
(c) *For some constant* $c > 0$

$$|\langle A_1 u, v\rangle| \le c|A_2 v|_H \langle A_1 u, u\rangle^{1/2} \qquad \forall u \in D(A_1)\,,\ \forall v \in D(A_2)\,. \tag{33}$$

PROOF. The implications (a)⇔(b)⇒(c) being straightforward, let us proceed to show that (c)⇒(a). Consider the Hilbert space $V_1 = D(A_1^{1/2})$ with the scalar product

$$\langle u, v\rangle_{V_1} = \langle A_1^{1/2}u, A_1^{1/2}v\rangle$$

and recall that $D(A_1)$ is a dense subspace of V_1. Let $v \in D(A_2)$ and define the linear functional $\phi_v : D(A_1) \to \mathbb{R}$ by

$$\phi_v(u) = \langle A_1 u, v\rangle \qquad \forall u \in D(A_1)\,.$$

Owing to (c), ϕ_v can be extended to a bounded linear functional on V_1 (still denoted by ϕ_v) satisfying $\|\phi_v\| \le c|A_2 v|_H$. Therefore, by the Riesz Theorem, there is a unique vector $w \in V_1$ such that

$$\phi_v(u) = \langle A_1^{1/2}u, A_1^{1/2}w\rangle \qquad \forall u \in V_1\,.$$

Hence, $\langle A_1 u, (v - w)\rangle = 0$ for all $u \in D(A_1)$, and so $v = w \in V_1$ since A_1 is invertible. Moreover, $|A_1^{1/2} v|_H = |w|_{V_1} \leq c|A_2 v|_H$. $\square$

The main result of this section is the following.

THEOREM 1.10. *Assume* (H1), (H2), (H3) *and* (32). *If* $U_0 \in D(\mathcal{A}^4)$, *then the solution* U *of problem* (12) *satisfies*

$$\int_0^T \mathcal{E}(U(t))dt \leq c_1 \sum_{k=0}^{4} \mathcal{E}(U^{(k)}(0)) \qquad \forall T > 0 \tag{34}$$

for some constant $c_1 > 0$.

The proof of Theorem 1.10 will be given in several steps. First, let us recall that, as showed in [2, Lemma 3.3], system (12) is dissipative. Indeed, under the only assumptions (H1) and (H2), the energy of the solution $U = (u, u', v, v')$ of problem (12) with $U_0 \in D(\mathcal{A})$ satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|_H^2 \qquad \forall t \geq 0. \tag{35}$$

In particular, $t \mapsto \mathcal{E}(U(t))$ is nonincreasing on $[0, \infty)$.

COROLLARY 1.11. *Assume* (H1), (H2), (H3) *and* (32).

(a) *If* $U_0 \in D(\mathcal{A}^{4n})$ *for some integer* $n \geq 1$, *then the solution* U *of problem* (12) *satisfies*

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{36}$$

for some constant $c_n > 0$.

(b) *For every* $U_0 \in \mathcal{H}$ *we have*

$$\mathcal{E}(U(t)) \to 0 \quad \textit{as} \quad t \to +\infty.$$

PROOF. Statement (a) follows by combining the dissipation relation (35), Theorem 1.10, and Lemma 1.8. To prove part (b), we fix $U_0 \in \mathcal{H}$ and consider a sequence $(U_0^n)_{n\in\mathbb{N}}$ such that $U_0^n \in D(\mathcal{A}^{4n})$ for every $n \geq 1$ and $U_0^n \to U_0$ in $\mathcal{H}$ for $n \to +\infty$. We set $U^n(t) = e^{t\mathcal{A}}U_0^n$ and $U(t) = e^{t\mathcal{A}}U_0$ for $t \geq 0$. Then, by linearity and the contraction property of $(e^{t\mathcal{A}})_{t\geq 0}$, we have

$$||U^n(t) - U(t)|| \leq ||U_0^n - U_0||, \quad \forall\, t \geq 0, n \in \mathbb{N}.$$

Therefore, recalling the definition of $\mathcal{E}$, we deduce that $\mathcal{E}(U^n(.))$ converges to $\mathcal{E}(U(.))$ as $n \to +\infty$, uniformly on $[0, \infty)$. Since, for any fixed $n \in \mathbb{N}$, $\mathcal{E}(U^n(t))$ converges to 0 as $t \to \infty$, we easily obtain the conclusion. $\square$

We now proceed with the proof of Theorem 1.10. Hereafter, C will denote a generic positive constant, independent of α. To begin with, let us recall that, thanks to [2, Lemma 3.4], the solution of (12) with $U_0 \in D(\mathcal{A})$ verifies

$$\int_0^T \mathcal{E}(U(t))dt \leq \int_0^T |v'(t)|_H^2 dt + C\,\mathcal{E}(U(0)) \tag{37}$$

for some constant $C \geq 0$ and every $T \geq 0$. Hence, the main technical point of the proof is to bound the right-hand side of (37) by the total energy of U (and a finite number of its derivatives) at 0.

Lemma 1.12. *Let $U = (u, u', v, v')$ be the solution of problem (12) with $U_0 \in D(\mathcal{A})$. Then*

$$\int_0^T |A_1^{-1/2} v|_H^2 dt \leq C \int_0^T |A_2^{-1/2} u|_H^2 dt + \frac{C}{\alpha^2}\Big[\mathcal{E}(U(0)) + \mathcal{E}(U'(0))\Big]\,. \tag{38}$$

Proof. Rewrite (12) as system (6) to obtain

$$\int_0^T \langle u'' + A_1 u + Bu' + \alpha v, A_1^{-1} v\rangle dt - \int_0^T \langle v'' + A_2 v + \alpha u, A_2^{-1} u\rangle dt = 0\,.$$

Hence, by straightforward computations,

$$\begin{aligned}\alpha \int_0^T |A_1^{-1/2} v|_H^2 dt \leq \alpha &\int_0^T |A_2^{-1/2} u|_H^2 dt \\ &- \int_0^T \langle Bu', A_1^{-1} v\rangle dt + \int_0^T \big[\langle v'', A_2^{-1} u\rangle - \langle u'', A_1^{-1} v\rangle\big] dt\,.\end{aligned}$$

Integration by parts transforms the last inequality into

$$\begin{aligned}\alpha \int_0^T |A_1^{-1/2} v|_H^2 dt \leq \alpha &\int_0^T |A_2^{-1/2} u|_H^2 dt - \int_0^T \langle A_1^{-1/2} Bu', A_1^{-1/2} v\rangle dt \\ &+ \int_0^T \big[\langle A_1^{-1/2} v, A_1^{1/2} A_2^{-1} u''\rangle - \langle A_1^{-1/2} u'', A_1^{-1/2} v\rangle\big] dt \\ &+ \Big[\langle v', A_2^{-1} u\rangle - \langle v, A_2^{-1} u'\rangle\Big]_0^T\,.\end{aligned} \tag{39}$$

We now proceed to bound the right-hand side of (39). We have

$$\left|\int_0^T \langle A_1^{-1/2} Bu', A_1^{-1/2} v\rangle dt\right| \leq \frac{\alpha}{4}\int_0^T |A_1^{-1/2} v|_H^2 dt + \frac{C}{\alpha}\int_0^T |B^{1/2} u'|_H^2 dt\,.$$

Similarly, thanks to assumption (32) and the fact that B is positive definite,

$$\left|\int_0^T \langle A_1^{-1/2}v, A_1^{1/2}A_2^{-1}u''\rangle dt\right| \le \frac{\alpha}{4}\int_0^T |A_1^{-1/2}v|_H^2 dt + \frac{C}{\alpha}\int_0^T |B^{1/2}u''|_H^2 dt\,.$$

Also,

$$\left|\int_0^T \langle A_1^{-1/2}u'', A_1^{-1/2}v\rangle dt\right| \le \frac{\alpha}{4}\int_0^T |A_1^{-1/2}v|_H^2 dt + \frac{C}{\alpha}\int_0^T |B^{1/2}u''|_H^2 dt\,.$$

Finally, observe that the last term in (39) can be bounded as follows

$$\left|\Big[\langle v', A_2^{-1}u\rangle - \langle v, A_2^{-1}u'\rangle\Big]_0^T\right| \le C\mathcal{E}(U(0))\,.$$

Combining the above estimates with (39), we obtain

$$\begin{aligned}\int_0^T |A_1^{-1/2}v|_H^2 dt \le C\int_0^T |A_2^{-1/2}u|_H^2 dt + \frac{C}{\alpha}\mathcal{E}(U(0))& \\ + \frac{C}{\alpha^2}\int_0^T \big[|B^{1/2}u'|_H^2 + |B^{1/2}u''|_H^2\big] dt&\,.\end{aligned}$$

The conclusion follows from the above inequality and the dissipation identity (35) applied to u and u'. □

LEMMA 1.13. *Let $U = (u, u', v, v')$ be the solution of problem* (12) *with $U_0 \in D(\mathcal{A})$. Then*

$$\int_0^T |v|_H^2 dt \le C\alpha^2\int_0^T |u|_H^2 dt + \frac{C}{\alpha^2}\sum_{k=1}^{3}\mathcal{E}(U^{(k)}(0))\,. \tag{40}$$

PROOF. Since $\langle v'' + A_2 v + \alpha u, A_2^{-1}v\rangle = 0$, integrating over $[0, T]$ we have

$$\int_0^T |v|_H^2 dt = -\alpha\int_0^T \langle v, A_2^{-1}u\rangle dt - \int_0^T \langle v'', A_2^{-1}v\rangle dt\,. \tag{41}$$

The last term in the above identity can be bounded using assumption (32) and Lemma 1.9 as follows

$$\begin{aligned}\left|\int_0^T \langle v'', A_2^{-1}v\rangle dt\right| &= \left|\int_0^T \langle A_1^{-1/2}v'', A_1^{1/2}A_2^{-1}v\rangle dt\right| \\ &\le \frac{1}{4}\int_0^T |v|_H^2 dt + C\int_0^T |A_1^{-1/2}v''|_H^2 dt\,.\end{aligned} \tag{42}$$

Now, applying (38) to v'' and (35) to u', we obtain

$$\begin{aligned}\int_0^T |A_1^{-1/2} v''|_H^2 dt &\le C \int_0^T |A_1^{-1/2} u''|_H^2 dt + \frac{C}{\alpha^2}\left[\mathcal{E}(U''(0)) + \mathcal{E}(U'''(0))\right] \\ &\le C\mathcal{E}(U'(0)) + \frac{C}{\alpha^2}\left[\mathcal{E}(U''(0)) + \mathcal{E}(U'''(0))\right].\end{aligned} \tag{43}$$

On the other hand,

$$\left|\alpha \int_0^T \langle v, A_2^{-1} u\rangle dt\right| \le \frac{1}{4}\int_0^T |v|_H^2 dt + C\alpha^2 \int_0^T |u|_H^2 dt\,. \tag{44}$$

The conclusion follows combining (41), ..., (44). □

Let us now complete the proof of Theorem 1.10.

Proof of Theorem 1.10. To prove (34) it suffices to apply (40) to v' and use the resulting estimate to bound the right-hand side of (37). Since B is positive definite, the conclusion follows by the dissipation identity (35). □

Remark 1.14. (i) Similar results can be obtained for systems of equations coupled with different coefficients, such as

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0 \\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0\,. \end{cases} \tag{45}$$

In this case, (H3) should be replaced with

(H3') α_1, α_2 are two real numbers such that $0 < \alpha_1\alpha_2 < \omega_1\omega_2$.

Let us explain how to adapt our approach to the case of $\alpha_1 \neq \alpha_2$, when α_1, $\alpha_2 > 0$. The total energy is defined by

$$\mathcal{E}(U) := \alpha_2 E_1(u,p) + \alpha_1 E_2(v,q) + \alpha_1\alpha_2\langle u, v\rangle\,,$$

where E_1 and E_2 are the energies of the two components, defined in (8). Moreover, for each $U \in \mathcal{H}$,

$$\mathcal{E}(U) \ge \nu(\alpha_1,\alpha_2)\Big[\alpha_2 E_1(u,p) + \alpha_1 E_2(v,q)\Big]\,,$$

with $\nu(\alpha_1,\alpha_2) = 1 - (\alpha_1\alpha_2)^{1/2}(\omega_1\omega_2)^{-1/2} > 0$. Finally, for each $U_0 \in D(\mathcal{A})$, the solution $U(t) = (u(t), p(t), v(t), q(t))$ of the first order evolution equation associated with system (45) satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|_H^2 \qquad \forall t \ge 0. \tag{46}$$

In particular, $t \mapsto \mathcal{E}(U(t))$ is nonincreasing on $[0, \infty)$. From this point, reasoning as in the above proof, the reader can easily derive the conclusion of Theorem 1.10.

(ii) Another interesting situation occurs when $\alpha_1 = 0$, that is, when the first equation of system (45) is damped, whereas the second component is undamped and weakly coupled with the first one. In this case there is no hope to stabilize the full system by one single feedback. Indeed, let $A_1 = A_2 =: A$ and consider the sequence of positive eigenvalues $(\omega_k)_{k\geq 1}$ of A, satisfying $\omega_k \to +\infty$, with associated eigenspaces $(Z_k)_{k\geq 1}$. Moreover, let $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and $\lambda_k = \sqrt{\omega_k - \beta^2}$. Then, the equation

$$u''(t) + Au(t) + 2\beta u'(t) = 0 \tag{47}$$

with initial conditions

$$u(0) = u^0 = \sum_{k\geq 1} u_k^0 \,, \quad u'(0) = u^1 = \sum_{k\geq 1} u_k^1 \,,$$

where $u_k^i \in Z_k$ for every $k \geq 1$, $(i = 1, 2)$, admits the solution

$$u(t) = e^{-\beta t} \sum_{k\geq 1} \left[u_k^0 \cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k} \sin(\lambda_k t) \right] .$$

In particular, choosing $u^0 \in Z_1$ and $u^1 \in Z_1$, we have that $u(t)$ lies in Z_1 for every $t \geq 0$. On the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \tag{48}$$

is coupled with (47) only in the component in Z_1, while it is conservative in $Z_1^\perp$. More precisely, writing $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^\perp$, equation (48) implies that

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0 \\ v_2''(t) + Av_2(t) = 0 \,. \end{cases} \tag{49}$$

Therefore, taking $v(0) = v^0 \notin Z_1$ and $v'(0) = v^1 \notin Z_1$,

$$E(v_2(t), v_2'(t)) = \frac{1}{2} \left(|v_2'(t)|_H^2 + \langle Av_2(t), v_2(t) \rangle \right) = E(v(0), v'(0)) > 0$$

for all $t \geq 0$. So, system (45) is not stabilizable.

1.4 – Results with data in interpolation spaces

When the initial data belong to an interpolation space between $\mathcal{H}$ and the domain of a power of $\mathcal{A}$ we can improve Corollary 1.11 as follows.

THEOREM 1.15. *Assume* (H1), (H2), (H3) *and* (32). *If* $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}$ *for some* $n \geq 1$ *and* $0 < \theta < 1$, *then the solution* U *of problem* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta}} \|U_0\|_{(\mathcal{H},D(\mathcal{A}^{4n}))_{\theta,2}}^2 \qquad \forall t > 0 \tag{50}$$

for some constant $c_{n,\theta} > 0$.

PROOF. The proof easily follows from the interpolation results recalled in Section 2 applied to the operator $\Lambda_t : \mathcal{H} \to \mathcal{H}$ defined by

$$\Lambda_t(U_0) = e^{t\mathcal{A}} U_0 \in \mathcal{H}$$

for each $U_0 \in \mathcal{H}$. □

Although $(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}$ is usually difficult to identify explicitly, we can single out important special cases where such an identification is possible. We need a preliminary result.

LEMMA 1.16. *The operator* $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ *is invertible, with* $\mathcal{A}^{-1}$ *bounded. Moreover,* $\mathcal{A}$ *is m-dissipative (thus,* $\mathcal{A}$ *generates a* $\mathcal{C}_0$*-semigroup of contractions on* $\mathcal{H}$*).*

PROOF. For any $U = (u, p, v, q)$, $\widehat{U} = (\hat{u}, \hat{p}, \hat{v}, \hat{q}) \in \mathcal{H}$, the identity $\mathcal{A}U = \widehat{U}$ is equivalent to

$$p = \hat{u}\,, \quad -A_1 u - Bp - \alpha v = \hat{p}\,, \quad q = \hat{v}\,, \quad -A_2 v - \alpha u = \hat{q}\,.$$

Hence, $p = \hat{u} \in D(A_1^{1/2})$, $q = \hat{v} \in D(A_2^{1/2})$. So, in order to compute $\mathcal{A}^{-1}$ it suffices to solve the system

$$\begin{cases} A_1 u + \alpha v = f \\ A_2 v + \alpha u = g\,, \end{cases} \tag{51}$$

for suitably chosen f, $g \in H$. Since $I - \alpha^2 A_1^{-1} A_2^{-1}$ is invertible thanks to $(H3)$, it is easy to check that (51) admits the solution

$$\begin{cases} \bar{u} = \left(I - \alpha^2 A_1^{-1} A_2^{-1}\right)^{-1} A_1^{-1}(f - \alpha A_2^{-1} g) \in D(A_1) \\ \bar{v} = A_2^{-1}(g - \alpha \bar{u}) \in D(A_2)\,. \end{cases}$$

Thus, $\mathcal{A}$ is invertible, and $\mathcal{A}^{-1}$ is bounded. Moreover, $\mathcal{A}$ is dissipative, since

$$(\mathcal{A}U|U) \leq -\langle Bp, p\rangle_H \leq -\beta |p|_H^2 \leq 0 \quad \forall\, U \in D(\mathcal{A})\,.$$

In addition, it is easy to check that there exists $\lambda > 0$ such that the range of $\lambda I - \mathcal{A}$ equals $\mathcal{H}$. Thus, by the Lumer-Phillips Theorem (see, *e.g.*, [130, Theorem 4.3]), $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contractions on $\mathcal{H}$. □

Applying Corollary 1.7, we obtain the following result.

Corollary 1.17. *If $\theta k = m$, for some $0 < \theta < 1$ and k, $m \in \mathbb{N}$, then*

$$D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}\,. \tag{52}$$

Remark 1.18. In particular, let us take $k = 4n$ $(n \geq 1)$ and $\theta_j = \frac{j}{4n}$ for $j = 1, \ldots, 4n-1$. Then, (52) yields

$$(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta_j,2} = D(\mathcal{A}^j) \quad (j = 1, \ldots, 4n-1)\,. \tag{53}$$

Thus, applying Theorem 1.15 to the above values of θ_j, one can show that, if $U_0 \in D(\mathcal{A}^j)$, then the associated solution $U(t)$ of problem (12) satisfies

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_{n,j}}{t^{j/4}} \|U_0\|^2_{D(\mathcal{A}^j)} \qquad \forall t > 0$$

for some constant $c_{n,j} > 0$. Moreover, we claim that $c_{n,j}$ can be chosen independent of n. Indeed, since $j \neq 4n$, one can take the smallest positive n_j such that $j < 4n_j$, and use (53) with $\theta_j = j/(4n_j)$ to conclude that $c_{n_j,j} = c_j$. As already mentioned in the introduction, this result can be compared with the one in [22, Proposition 3.1], which was obtained by a different method.

Corollary 1.19. *Assume* (H1), (H2), (H3) *and* (32).

i) *If $U_0 \in D(\mathcal{A}^n)$ for some $n \geq 1$, then the solution of* (12) *satisfies*

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/4}} \sum_{k=0}^{n} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{54}$$

for some constant $c_n > 0$.

ii) *If $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $n \geq 1$ where $0 < \theta < 1$, then the solution of* (12) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0 \tag{55}$$

for some constant $c_{n,\theta} > 0$.

iii) *If $U_0 \in D((-\mathcal{A})^\theta)$ for some $0 < \theta < 1$, then the solution of problem* (12) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_\theta}{t^{\theta/4}} \|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \tag{56}$$

for some constant $c_\theta > 0$.

Proof. Points $i)$ and $ii)$ derive from Corollary 1.11 and following the proof of Theorem 1.15, thanks to Remark 1.18. In order to prove point $iii)$, first we deduce from Lemma 1.16 that $-\mathcal{A}$ is invertible with bounded inverse. Moreover, it is m-accretive on $\mathcal{H}$, hence (28) yields

$$(\mathcal{H}, D(\mathcal{A}))_{\theta,2} = (\mathcal{H}, D(-\mathcal{A}))_{\theta,2} = D((-\mathcal{A})^{\theta})$$

for every $0 < \theta < 1$. The conclusion follows applying $ii)$ with $n = 1$. □

Under further assumptions, the norm in $(\mathcal{H}, D(\mathcal{A}))_{\theta,2}$ can be given a more explicit form. For this purpose, for each $k \geq 0$ consider the space

$$\mathcal{H}_k = D(A_1^{(k+1)/2}) \times D(A_1^{k/2}) \times D(A_2^{(k+1)/2}) \times D(A_2^{k/2})\,.$$

We recall the following result (see [2, Lemma 3.1]).

Lemma 1.20. *Assume* (H1)*, and* (H2)*. Let $n \geq 1$ be such that*

$$BD(A_1^{(k+1)/2}) \subset D(A_1^{k/2}) \tag{57}$$

$$D(A_1^{(k/2)+1}) \subset D(A_2^{k/2}) \tag{58}$$

$$D(A_2^{(k/2)+1}) \subset D(A_1^{k/2}) \tag{59}$$

for every integer k satisfying $0 < k \leq n-1$. (no assumption is made if $n = 1$). Then $\mathcal{H}_k \subset D(\mathcal{A}^k)$ for every $0 \leq k \leq n$.

In [2], it is also shown that $\mathcal{H}_k = D(\mathcal{A}^k)$ for every $0 \leq k \leq n$, provided (58) and (59) are replaced by the stronger assumptions

$$\begin{array}{l} D(A_1^{(k+1)/2}) \subset D(A_2^{k/2}) \\ D(A_2^{(k+1)/2}) \subset D(A_1^{k/2}) \end{array} \qquad \text{for every} \quad 0 < k \leq n-1\,.$$

Let $0 < \theta < 1$ and $k \geq 1$ be fixed. As a direct consequence of Theorem 1.2, choosing appropriate spaces and operator T, one can show that, if $\mathcal{H}_k$ is contained in $D(\mathcal{A}^k)$, then $(\mathcal{H}, \mathcal{H}_k)_{\theta,2}$ is contained in $(\mathcal{H}, D(\mathcal{A}^k))_{\theta,2}$. Moreover, $(\mathcal{H}, \mathcal{H}_k)_{\theta,2}$ equals

$$\begin{aligned} \mathcal{H}_{k,\theta} \quad := \quad & (D(A_1^{1/2}), D(A_1^{(k+1)/2}))_{\theta,2} \times (H, D(A_1^{k/2}))_{\theta,2} \\ & \times (D(A_2^{1/2}), D(A_2^{(k+1)/2}))_{\theta,2} \times (H, D(A_2^{k/2}))_{\theta,2}\,. \end{aligned}$$

Notice that, since A_i is self-adjoint and (25) holds for $i = 1, 2$, applying Theorem 1.5 we have, for every $0 \leq \alpha < \beta$ ($i = 1, 2$),

$$(D(A_i^{\alpha}), D(A_i^{\beta}))_{\theta,2} = D(A_i^{(1-\theta)\alpha+\theta\beta})\,.$$

Therefore, $\mathcal{H}_{k,\theta}$ equals $D(A_1^{\frac{1}{2}+\frac{k}{2}\theta}) \times D(A_1^{\frac{k}{2}\theta}) \times D(A_2^{\frac{1}{2}+\frac{k}{2}\theta}) \times D(A_2^{\frac{k}{2}\theta})$.

Observing that, for initial data in $\mathcal{H}_{n,\theta}$, we can bound (above and below) the norm of U_0 by the norms of its components, we have the following.

Corollary 1.21. *Assume* (H1), (H2), (H3) *and* (32).

1) *If* $\mathcal{H}_n \subset D(\mathcal{A}^n)$ *for some* $n \geq 2$, *then for each* $U_0 \in \mathcal{H}_n$ *the solution* U *of problem* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_n}{t^{n/4}} \|U_0\|_{\mathcal{H}_n}^2 \qquad \forall t > 0 \tag{60}$$

for some constant $c_n > 0$, *where*

$$\|U_0\|_{\mathcal{H}_n}^2 = |u^0|^2_{D(A_1^{(n+1)/2})} + |u^1|^2_{D(A_1^{n/2})} + |v^0|^2_{D(A_2^{(n+1)/2})} + |v^1|^2_{D(A_2^{n/2})} \,.$$

2) *Let* $n \geq 1$ *and* $0 < \theta < 1$ *be fixed. If* $\mathcal{H}_n \subset D(\mathcal{A}^n)$, *then for every* $U_0 \in \mathcal{H}_{n,\theta}$ *the solution* U *of* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|_{\mathcal{H}_{n,\theta}}^2 \qquad \forall t > 0 \tag{61}$$

for some constant $c_{n,\theta} > 0$, *with*

$$\|U_0\|_{\mathcal{H}_{n,\theta}}^2 \asymp |u^0|^2_{D(A_1^{(1+n\theta)/2})} + |u^1|^2_{D(A_1^{n\theta/2})} + |v^0|^2_{D(A_2^{(1+n\theta)/2})} + |v^1|^2_{D(A_2^{n\theta/2})} \,,$$

where $\asymp$ *stands for the equivalence between norms.*

1.5 – Applications to PDEs

In this section we describe some examples of systems of partial differential equations that can be studied by the results of this paper, but fail to satisfy the compatibility condition (13).

Notation 1.22. We will hereafter denote by Ω a bounded domain in $\mathbb{R}^d$ with a sufficiently smooth boundary Γ. For $i = 1, \ldots, d$ we will denote by ∂_i the partial derivative with respect to x_i and by ∂_t the derivative with respect to the time variable. We will also use the notation $H^k(\Omega), H_0^k(\Omega)$ for the usual Sobolev spaces with norm

$$\|u\|_{k,\Omega} = \left[\int_\Omega \sum_{|p| \leq k} |D^p u|^2 dx \right]^{\frac{1}{2}} ,$$

where we have set $D^p = \partial_1^{p_1} \cdots \partial_d^{p_d}$ for any multi-index $p = (p_1, \ldots, p_d)$. Finally, we will denote by $C_\Omega > 0$ the largest constant such that Poincaré's inequality

$$C_\Omega \|u\|_{0,\Omega}^2 \leq \|\nabla u\|_{0,\Omega}^2 \tag{62}$$

holds true for any $u \in H_0^1(\Omega)$, or, for simplicity with the same notation, the constant C_Ω such that

$$C_\Omega \|u\|_{0,\Omega}^2 \leq \|\nabla u\|_{0,\Omega}^2 + \|u\|_{0,\Gamma}^2 \,, \tag{63}$$

for all $u \in H^1(\Omega)$, where $\|u\|_{0,\Gamma}^2 := \left(\int_\Gamma u^2 d\Sigma\right)^{1/2}$ stands for the L^2-norm of u on the boundary Γ of Ω.

In the following examples we take $H = L^2(\Omega)\,,\ B = \beta I$.

EXAMPLE 1.23. Let β, $\lambda > 0$, $\alpha \in \mathbb{R}$, and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \lambda u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{64}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) = 0 \text{ on } \Gamma\,, \qquad v(\cdot, t) = 0 \text{ on } \Gamma \quad \forall t > 0 \tag{65}$$

and initial conditions

$$\begin{cases} u(x,0) = u^0(x) & u'(x,0) = u^1(x) \\ v(x,0) = v^0(x) & v'(x,0) = v^1(x) \end{cases} \quad x \in \Omega\,. \tag{66}$$

The above system can be rewritten in abstract form taking

$$\begin{gathered} D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\}\,, \quad A_1 u = -\Delta u + \lambda u\,, \\ D(A_2) = H^2(\Omega) \cap H_0^1(\Omega)\,, \quad A_2 v = -\Delta v\,. \end{gathered} \tag{67}$$

Notice that, in order to verify assumption $(H3)$, we shall choose α such that $0 < |\alpha| < (C_\Omega(C_\Omega + \lambda))^{1/2}$. Then,

$$\begin{aligned} |\langle A_1 u, v\rangle| &= \left| \int_\Omega \nabla u \nabla v\, dx + \lambda \int_\Omega uv\, dx \right| \\ &\le \left(\int_\Omega |\nabla u|^2\, dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2\, dx \right)^{1/2} + \lambda \left(\int_\Omega u^2 dx \right)^{1/2} \left(\int_\Omega v^2 dx \right)^{1/2} \\ &\le c\, \langle A_1 u, u\rangle^{1/2}\, |A_2 v|_H\,, \end{aligned}$$

where we have used the coercivity of A_2 and the well-known inequality

$$\int_\Omega v^2 + |\nabla v|^2\, dx \le c \int_\Omega |\Delta v|^2\, dx \quad \forall\, v \in H^2(\Omega) \cap H_0^1(\Omega)\,.$$

Since condition (33) is fulfilled, we get the following conclusions.

i) If $(u^0, u^1, v^0, v^1) \in D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2})$, then the solution U of problem (64)-(65)-(66) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \le \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0 \tag{68}$$

for some constant $c > 0$. Moreover, there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \leq c_1 \left(\|u^0\|^2_{2,\Omega} + \|u^1\|^2_{1,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega}\right) .$$

ii) By point *ii*) of Corollary 1.19, if $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $0 < \theta < 1$, $n \geq 1$, then the solution of (64)-(65)-(66) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \leq \frac{c_{n,\theta}}{t^{n\theta/4}} \|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \tag{69}$$

for every $t > 0$ and some constant $c_{n,\theta} > 0$. Moreover, point *iii*) of Corollary 1.19 ensures that, if $U_0 \in D((-\mathcal{A})^\theta)$ for some $0 < \theta < 1$, then

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \leq \frac{c_\theta}{t^{\theta/4}} \|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \tag{70}$$

for some constant $c_\theta > 0$.

Of interest is the case when an operator fulfills different boundary conditions on proper subsets of Γ. For instance, let Γ_0 be an open subset of Γ (with respect to the topology of Γ) and set $\Gamma_1 = \Gamma \backslash \Gamma_0$. We assume that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Consider the system (64) with boundary conditions

$$\begin{array}{c} u(\cdot, t) = 0 \text{ on } \Gamma_0 \,, \ \dfrac{\partial u}{\partial \nu}(\cdot, t) = 0 \text{ on } \Gamma \setminus \Gamma_0 \\ v(\cdot, t) = 0 \text{ on } \Gamma \end{array} \qquad \forall t > 0 \tag{71}$$

and initial conditions (66). Let us set

$$\begin{gathered} D(A_1) = \left\{ u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0 \,, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \setminus \Gamma_0 \right\} , \\ A_1 u = -\Delta u \,. \end{gathered}$$

Then, $|\langle A_1 u, v\rangle| \leq c \langle A_1 u, u\rangle^{1/2} |A_2 v|_H$. So, for $0 < |\alpha| < (C_\Omega(C_\Omega + \lambda))^{1/2}$, condition (32) is fulfilled, and the same conclusions *i*)$-$*ii*) hold for problem (64)-(71)-(66).

Example 1.24. Another interesting situation occurs while coupling two equations of different orders. Let $d = 1$ or $d = 2$, $\beta > 0$, $\alpha \in \mathbb{R}$, and (for $d = 2$) denote by $\nu(x) = (\nu_1, \nu_2)$ the unit normal vector in $x \in \Gamma$ and by τ the tangential unit vector for point at the boundary of the domain. Consider the system

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{72}$$

with boundary conditions on Γ

$$\begin{aligned} \Delta u(\cdot,t) + (1-\mu) B_1 u(\cdot,t) &= 0 \\ \frac{\partial \Delta u}{\partial \nu}(\cdot,t) - u(\cdot,t) + (1-\mu)\frac{\partial B_2 u}{\partial \tau}(\cdot,t) &= 0 \quad \forall t>0\,, \\ v(\cdot,t) &= 0 \end{aligned} \tag{73}$$

where the constant $\mu \in (0,1/2)$ is the Poisson coefficient, and the operators B_1 and B_2 are zero for $d=1$, while for $d=2$ are defined by

$$\begin{aligned} B_1 v &= 2\nu_1\nu_2 \partial^2_{xy} v - \nu_1^2 \partial^2_{yy} v - \nu_2^2 \partial^2_{xx} v\,, \\ B_2 v &= \frac{\partial}{\partial \tau}\left[(\nu_1^2-\nu_2^2)\partial^2_{xy} v + \nu_1\nu_2(\partial^2_{yy} v - \partial^2_{xx} v)\right] \end{aligned} \tag{74}$$

(we refer to [108] for a detailed description of this model), and initial conditions (66). Define the operators

$$D(A_1) = \left\{ u \in H^4(\Omega) : \begin{array}{c} \Delta u + (1-\mu)B_1 u = 0 \\ \dfrac{\partial \Delta u}{\partial \nu} - u + (1-\mu)\dfrac{\partial B_2 u}{\partial \tau} = 0 \end{array} \text{ on } \Gamma \right\},$$

$$A_1 u = \Delta^2 u\,,$$

$$D(A_2) = H^2(\Omega) \cap H^1_0(\Omega)\,, \quad A_2 v = -\Delta v\,.$$

Suppose $0 < |\alpha| < C_\Omega^{1/2}$, as required by (H3). Thanks to [110, Lemma 3C.2], for any $u \in D(A_1)$ and $v \in D(A_2)$ we have

$$\begin{aligned} \langle A_1 u, v\rangle &= \int_\Omega \Delta u \Delta v dx + \int_\Gamma (1-\mu) B_1 u \frac{\partial v}{\partial \nu} d\Sigma \\ &= \int_\Omega \Delta u \Delta v dx + (1-\mu)\int_\Omega \left[2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}\right] dx\,. \end{aligned} \tag{75}$$

Moreover, owing to [110, Equation (3C.28)], we know that $D(A_1^{1/2}) = H^2(\Omega)$, with equivalence between the norms $\|\cdot\|_{D(A_1^{1/2})}$ and $\|\cdot\|_{H^2(\Omega)}$, and for every $u \in D(A_1^{1/2})$

$$\begin{aligned} \|A_1^{1/2} u\|^2_{L^2(\Omega)} &= \langle A_1 u, u\rangle \\ &= \int_\Omega \left[\mu|\Delta u|^2 + (1-\mu)(u_{xx}^2 + u_{yy}^2) + 2(1-\mu)u_{xy}^2\right] dx + \int_\Gamma u^2 d\Sigma\,. \end{aligned} \tag{76}$$

From relations (75) and (76) we deduce that

$$|\langle A_1 u, v\rangle| \le c\, \langle A_1 u, u\rangle^{1/2}\, |A_2 v|_H\,,$$

so condition (33) is fulfilled. Thus, for every $U_0 \in D(\mathcal{A})$, the solution U of problem (72)-(73)-(66) satisfies

$$E_1(u(t), u'(t)) + E_2(v(t), v'(t)) \leq \frac{c}{t^{1/4}} \|U_0\|^2_{D(\mathcal{A})} \qquad \forall t > 0 \tag{77}$$

for some constant $c > 0$. Moreover, there exists $c_1 > 0$ such that

$$\|U_0\|^2_{D(\mathcal{A})} \leq c_1 \left(\|u^0\|^2_{4,\Omega} + \|u^1\|^2_{2,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega} \right) .$$

Note that we give in Example 1.31 another set of boundary conditions for the same symbols for the operators. It is interesting to see that both examples are treated for different classes of compatibility conditions, namely the present example satisfies the compatibility condition (14), whereas the example (1.31) satisfies the compatibility condition (13).

Example 1.25. Let $\beta > 0$, $\alpha \in \mathbb{R}$, and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{78}$$

with boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial \nu} + u \right)(\cdot, t) &= 0 \text{ on } \Gamma \\ v(\cdot, t) &= 0 \text{ on } \Gamma \end{aligned} \qquad \forall t > 0 \tag{79}$$

and initial conditions (66). Let us define

$$\begin{gathered} D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\} , \ A_1 u = -\Delta u \, , \\ D(A_2) = H^2(\Omega) \cap H_0^1(\Omega) \, , \ A_2 v = -\Delta v \, , \end{gathered} \tag{80}$$

and assume $0 < |\alpha| < C_\Omega$. Observe that

$$\begin{aligned} |\langle A_1 u, v \rangle| &= \left| \int_\Omega \nabla u \nabla v \, dx \right| \\ &\leq \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 \, dx \right)^{1/2} \leq c \, \langle A_1 u, u \rangle^{1/2} |A_2 v|_H \, , \end{aligned}$$

since

$$\langle A_1 u, u \rangle = \int_\Omega |\nabla u|^2 \, dx + \int_\Gamma |u|^2 \, dS \, , \qquad \int_\Omega |\nabla v|^2 \, dx \leq c \int_\Omega |\Delta v|^2 \, dx \, .$$

Thus, condition (32) is fulfilled. So, the energy of the solution of problem (78)-(79)-(66) satisfies

$$E_1(u(t),u'(t))+E_2(v(t),v'(t))\le \frac{c}{t^{1/4}}\|U_0\|^2_{D(\mathcal{A})}\qquad \forall t>0 \tag{81}$$

for some constant $c>0$. Moreover, there exists $c_1>0$ such that

$$\|U_0\|^2_{D(\mathcal{A})}\le c_1\left(|A_1u^0|^2_H+|A_1^{1/2}u^1|^2_H+|A_2v^0|^2_H+|A_2^{1/2}v^1|^2_H\right).$$

Our next result show that the operators in Example 1.25 do not fulfill the compatibility condition (13).

Proposition 1.26. *Let A_1, A_2 be defined as in* (80)*. Then for every $k\in\mathbb{N}$, $k\ge 2$, $D(A_2^{k/2})$ is not included in $D(A_1)$.*

Proof. Since $D(A_2^k)\subset D(A_2^{k/2})$ for every $k\in\mathbb{N}$, it is sufficient to prove that $D(A_2^k)$ is not included in $D(A_1)$ for every $k\in\mathbb{N}$, $k\ge 1$. For this purpose, let us fix $k\in\mathbb{N}$, $k\ge 1$, and consider the problem

$$\begin{cases}(-\Delta)^k v_0=1\\ v_0=0=\Delta v_0=\cdots=\Delta^{k-1}v_0 \qquad \text{on } \Gamma\,.\end{cases} \tag{82}$$

Define the sequence $v_1,\, v_2,\,\ldots,v_{k-1}$ by

$$\begin{cases}-\Delta v_0=v_1\\ v_{0_{|\Gamma}}=0\end{cases}\quad\cdots\quad \begin{cases}-\Delta v_{k-2}=v_{k-1}\\ v_{k-2_{|\Gamma}}=0\end{cases}\quad \begin{cases}-\Delta v_{k-1}=1\\ v_{k-1_{|\Gamma}}=0\,.\end{cases} \tag{83}$$

We will argue by contradiction, assuming $D(A_2^k)\subset D(A_1)$. Since v_0 belongs to $D(A_2)\cap D(A_1)$, we have $v_{0_{|\Gamma}}=0=\frac{\partial v_0}{\partial\nu}_{|\Gamma}$. Moreover, from the first system in (83), it follows that

$$\int_\Omega v_1dx=\int_\Omega(-\Delta v_0)dx=-\int_\Gamma\frac{\partial v_0}{\partial\nu}dS=0\,.$$

Hence, $\displaystyle\int_\Omega v_1dx=0$. Let us prove by induction that

$$\int_\Omega \nabla v_{k-i}\nabla v_i dx=0\quad \forall\, i=1,\,2,\ldots,k-1\,. \tag{84}$$

For $i=1$ we have

$$\int_\Omega\nabla v_{k-1}\nabla v_1dx=\int_\Omega(-\Delta v_{k-1})v_1dx=\int_\Omega v_1dx=0\,,$$

since $v_{k-1_{|\Gamma}} = 0 = v_{1_{|\Gamma}}$. Now, let $i > 1$ and suppose

$$\int_\Omega \nabla v_{k-i} \nabla v_i dx = 0 .$$

Then,

$$\begin{aligned} 0 &= \int_\Omega v_{k-i}(-\Delta v_i)dx = \int_\Omega v_{k-i}v_{i+1}dx \\ &= \int_\Omega (-\Delta v_{k-i-1})v_{i+1}dx = \int_\Omega \nabla v_{k-(i+1)} \nabla v_{i+1} dx . \end{aligned}$$

Thus, (84) holds for $i+1$. Moreover, from (84) follows that

$$\int_\Omega v_{k-i}v_{i+1}dx = 0 \quad \forall\, i = 1, 2, \ldots, k-1 , \tag{85}$$

since

$$\int_\Omega v_{k-i}v_{i+1}dx = \int_\Omega v_{k-i}(-\Delta v_i)dx = \int_\Omega \nabla v_{k-i} \nabla v_i dx = 0 .$$

Now, let k be even, say $k = 2p$, $p \in \mathbb{N}^*$. Then, by (84) with $i = p$ we obtain

$$\int_\Omega |\nabla v_p|^2 dx = 0 , \text{ whence } v_p = 0 .$$

So, by a cascade effect,

$$v_{p+1} = -\Delta v_p = 0 \Rightarrow v_{p+2} = -\Delta v_{p+1} = 0 \Rightarrow \cdots \Rightarrow v_{k-1} = -\Delta v_{k-2} = 0 .$$

Since $-\Delta v_{k-1} = 1$, we get a contradiction. If, on the contrary, k is odd, *i.e.* $k = 2p+1$, then, applying (85) with $i = p$, we conclude that

$$\int_\Omega |v_{p+1}|^2 dx = 0 , \text{ whence } v_{p+1} = 0 .$$

Finally, we have that $v_{p+1} = v_{p+2} = \cdots = v_{k-1} = 0$. Since $-\Delta v_{k-1} = 1$, we get a contradiction again. Therefore, $D(A_2^k)$ is not included in $D(A_1)$. $\square$

Example 1.27. Given $\beta > 0$, $\alpha \in \mathbb{R}$, let us now consider the undamped Petrowsky equation coupled with the damped wave equation,

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v + \Delta^2 v + \alpha u = 0 \end{cases} \qquad \text{in } \Omega \times (0, +\infty) \tag{86}$$

with Robin boundary conditions

$$\left(\frac{\partial u}{\partial \nu} + u \right)(\cdot, t) = 0 \text{ on } \Gamma \qquad \forall t > 0 \tag{87}$$

on u and either

$$v(\cdot, t) = \Delta v(\cdot, t) = 0 \text{ on } \Gamma \qquad \forall t > 0 \tag{88}$$

or

$$v(\cdot, t) = \frac{\partial v}{\partial \nu}(\cdot, t) = 0 \text{ on } \Gamma \qquad \forall t > 0 \tag{89}$$

on v, with initial conditions (66). Define

$$\begin{aligned} D(A_1) &= \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \quad A_1 u = -\Delta u \,, \\ D(A_2) &= \left\{ v \in H^4(\Omega) : v = \Delta v = 0 \text{ on } \Gamma \right\}, \quad A_2 v = \Delta^2 v \end{aligned}$$

(with boundary conditions (88) on v), or

$$\tilde{D}(A_2) = \left\{ v \in H^4(\Omega) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma \right\}, \quad A_2 v = \Delta^2 v$$

(with boundary conditions (89) on v). Once again, we have

$$\begin{aligned} |\langle A_1 u, v \rangle| &= \left| \int_\Omega \nabla u \nabla v \, dx \right| \\ &\leq \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 \, dx \right)^{1/2} \leq c \, \langle A_1 u, u \rangle^{1/2} \, |A_2 v|_H \,. \end{aligned}$$

Thus, condition (32) is fulfilled and, for $0 < |\alpha| < C_\Omega^{3/2}$, the polynomial decay of the energy of solution to (86)-(87)-(88)-(66) and (86)-(87)-(89)-(66) follows as in Example 1.23.

1.6 – Improvement of previous results

In this section we apply interpolation theory to extend the polynomial stability result of [2] to a larger class of initial data. We will denote by $j \geq 2$ the integer for which (13) is satisfied. As is shown in [2, Theorem 4.2], under assumptions $(H1), (H2), (H3)$ and (13), if $U_0 \in D(\mathcal{A}^{nj})$ for some integer $n \geq 1$, the solution U of problem (12) satisfies

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0)) \qquad \forall t > 0 \tag{90}$$

for some constant $c_n > 0$. We recall that assumption (13) covers many situations of interest for applications to systems of evolution equations. Indeed (see [2, Section 5] for further details), this is the case for

i) $(A_1, D(A_1)) = (A_2, D(A_2))$, where (13) is fulfilled with $j = 2$;
ii) $D(A_1) = D(A_2)$, with $j = 2$;
iii) $(A_2, D(A_2)) = (A_1^2, D(A_1^2))$, again with $j = 2$;
iv) $(A_1, D(A_1)) = (A_2^2, D(A_2^2))$, with $j = 4$.

The following result completes the analysis of [2], taking the initial data in suitable interpolation spaces.

THEOREM 1.28. *Assume* (H1), (H2), (H3) *and* (13), *and let* $0 < \theta < 1$, $n \geq 1$. *Then for every* U_0 *in* $(\mathcal{H}, D(\mathcal{A}^{nj}))_{\theta,2}$, *the solution* U *of* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta}} \|U_0\|^2_{(\mathcal{H}, D(\mathcal{A}^{nj}))_{\theta,2}} \qquad \forall t > 0 \tag{91}$$

for some constant $c_{n,\theta} > 0$.

Reasoning as in Remark 1.18, one can derive estimate (90) also for $U_0 \in D(\mathcal{A}^k)$, for every $k = 1, \dots, nj - 1$, with decay rate k/j.

COROLLARY 1.29. *Assume* (H1), (H2), (H3) *and* (13).

i) If $U_0 \in D(\mathcal{A}^n)$ *for some* $n \geq 1$, *then the solution of* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_n}{t^{n/j}} \|U_0\|^2_{D(\mathcal{A}^n)} \qquad \forall t > 0 \tag{92}$$

for some constant $c_n > 0$.

ii) If $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ *for some* $n \geq 1$ *and* $0 < \theta < 1$, *then the solution of* (12) *satisfies*

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta/j}} \|U_0\|^2_{(\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0 \tag{93}$$

for some constant $c_{n,\theta} > 0$.

iii) *If* $U_0 \in D((-\mathcal{A})^\theta)$ *for some* $0 < \theta < 1$, *then the solution of problem* (12) *satisfies*

$$\|U(t)\|^2_{\mathcal{H}} \leq \frac{c_\theta}{t^{\theta/j}} \|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \tag{94}$$

for some constant $c_\theta > 0$.

In particular, the previous fractional decay rates can be achieved for initial data in $\mathcal{H}_n$ or in $\mathcal{H}_{n,\theta}$, whenever $\mathcal{H}_n \subset D(\mathcal{A}^n)$, as in Corollary 1.21. This happens, for instance, if any of the following conditions is satisfied:

i) $(A_1, D(A_1)) = (A_2, D(A_2))$;
ii) $D(A_1) = D(A_2)$;
iii) $(A_2, D(A_2)) = (A_1^2, D(A_1^2))$.

Let us apply Corollary 1.29 to two examples from [2].

EXAMPLE 1.30. Given $\beta > 0$, $\kappa > 0$, $\alpha \in \mathbb{R}$, let us study the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \beta \partial_t u + \kappa u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \kappa v + \alpha u = 0 \end{cases} \qquad \text{in} \qquad \Omega \times (0, +\infty) \tag{95}$$

with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t) \qquad \text{on} \quad \Gamma \quad \forall t > 0 \tag{96}$$

and initial conditions

$$\begin{cases} u(x,0) = u^0(x)\,, \quad u'(x,0) = u^1(x) \\ v(x,0) = v^0(x)\,, \quad v'(x,0) = v^1(x) \end{cases} \qquad x \in \Omega\,. \tag{97}$$

Let $H = L^2(\Omega)$, $B = \beta I$, and $A_1 = A_2 = A$ be defined by

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega)\,, \qquad Au = -\Delta u + \kappa u \quad \forall u \in D(A)\,.$$

Notice that (13) is fulfilled with $j = 2$, and condition $0 < |\alpha| < C_\Omega + \kappa =: \omega$ is required in order to fulfill (H3).

As showed in [2, Example 6.1], if $u^0, v^0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u^1, v^1 \in H^1_0(\Omega)$, then

$$\begin{aligned} &\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \\ &\leq \frac{c}{t} \Big(\|u^0\|^2_{2,\Omega} + \|u^1\|^2_{1,\Omega} + \|v^0\|^2_{2,\Omega} + \|v^1\|^2_{1,\Omega} \Big) \qquad \forall t > 0\,. \end{aligned}$$

Moreover, if $u^0, v^0 \in H^{n+1}(\Omega)$ and $u^1, v^1 \in H^n(\Omega)$ are such that

$$\begin{aligned} & u^0 = \cdots = \Delta^{[\frac{n}{2}]} u^0 = 0 = v^0 = \cdots = \Delta^{[\frac{n}{2}]} v^0 \quad \text{on} \quad \Gamma, \\ & u^1 = \cdots = \Delta^{[\frac{n-1}{2}]} u^1 = v^1 = \cdots = \Delta^{[\frac{n-1}{2}]} v^1 = 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

then

$$\begin{aligned} & \int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \\ & \le \frac{c_n}{t^n} \Big(\|u^0\|^2_{n+1,\Omega} + \|u^1\|^2_{n,\Omega} + \|v^0\|^2_{n+1,\Omega} + \|v^1\|^2_{n,\Omega} \Big) \qquad \forall t > 0 \,. \end{aligned}$$

Furthermore, applying Corollary 1.29, we conclude that if U_0 belongs to $\mathcal{H}_{n,\theta} = (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $0 < \theta < 1$, $n \ge 1$, then the solution to (95)-(96)-(97) satisfies

$$\int_\Omega \Big(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \le \frac{c_{n,\theta}}{t^{n\theta/2}} \|U_0\|^2_{\mathcal{H}_{n,\theta}} \qquad \forall t > 0 \tag{98}$$

for some constant $c_{n,\theta} > 0$, with

$$\|U_0\|^2_{\mathcal{H}_{n,\theta}} \asymp |u^0|^2_{D(A_1^{\frac{1}{2}+\frac{n}{2}\theta})} + |u^1|^2_{D(A_1^{\frac{n}{2}\theta})} + |v^0|^2_{D(A_2^{\frac{1}{2}+\frac{n}{2}\theta})} + |v^1|^2_{D(A_2^{\frac{n}{2}\theta})} \,.$$

EXAMPLE 1.31. Taking $\beta > 0$, $0 < |\alpha| < C_\Omega^{3/2}$, and the same operators A_1 and A_2 as in Example 1.24, but with different boundary conditions, we can consider the system

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \beta \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \qquad \text{in} \qquad \Omega \times (0, +\infty) \tag{99}$$

with boundary conditions

$$v(\cdot, t) = u(\cdot, t) = \Delta u(\cdot, t) = 0 \qquad \text{on} \quad \Gamma \quad \forall t > 0 \tag{100}$$

and initial conditions as in (97). Let us set $H = L^2(\Omega)$, $B = \beta I$, and

$$\begin{aligned} & D(A_1) = \big\{ u \in H^4(\Omega) : \Delta u = 0 = u \text{ on } \Gamma \big\} \,, \quad A_1 u = \Delta^2 u \,, \\ & D(A_2) = H^2(\Omega) \cap H_0^1(\Omega) \,, \quad A_2 v = -\Delta v \,. \end{aligned}$$

In this case, since $A_1 = A_2^2$, condition (13) holds with $j = 4$. Consequently, as is shown in [2, Example 6.4], for initial condition $U_0 \in D(\mathcal{A}^4)$

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2 \Big) dx \le \frac{C}{t} \|U_0\|^2_{D(\mathcal{A}^4)} \qquad \forall t > 0 \,,$$

for some constant $C > 0$. By point $i)$ of Corollary 1.29, we can generalize this result to initial data $U_0 \in D(\mathcal{A}^n)$ for some $n \geq 1$. Indeed, in this case the solution to (99)-(100)-(97) satisfies

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{c_n}{t^{n/4}}\|U_0\|^2_{D(\mathcal{A}^n)} \qquad \forall t > 0\,,$$

for some constant $c_n > 0$. Moreover, thanks to point $ii)$ of Corollary 1.29, if $U_0 \in (\mathcal{H}, D(\mathcal{A}^n))_{\theta,2}$ for some $n \geq 1$ and $0 < \theta < 1$, then

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{c_{n,\theta}}{t^{n\theta/4}}\|U_0\|^2_{(\mathcal{H},D(\mathcal{A}^n))_{\theta,2}} \qquad \forall t > 0$$

for some constant $c_{n,\theta} > 0$. Furthermore, thanks to point $iii)$ of Corollary 1.29, if U_0 belongs to $\mathcal{H}_{1,\theta} = D((-\mathcal{A})^\theta)$ for some $0 < \theta < 1$, then the solution to (99)-(100)-(97) satisfies

$$\int_\Omega \Big(|\partial_t u|^2 + |\Delta u|^2 + |\partial_t v|^2 + |\nabla v|^2\Big)dx \leq \frac{c_\theta}{t^{\theta/4}}\|U_0\|^2_{D((-\mathcal{A})^\theta)} \qquad \forall t > 0 \qquad (101)$$

for some constant $c_\theta > 0$, with

$$\|U_0\|^2_{D((-\mathcal{A})^\theta)} \asymp |u^0|^2_{D(A_1^{\frac{1}{2}+\frac{1}{2}\theta})} + |u^1|^2_{D(A_1^{\frac{1}{2}\theta})} + |v^0|^2_{D(A_2^{\frac{1}{2}+\frac{1}{2}\theta})} + |v^1|^2_{D(A_2^{\frac{1}{2}\theta})}\,.$$

2 – Resolvent condition for indirect stabilization

The present chapter collects results from the preprint *Resolvent condition for indirect stabilization of systems of weakly coupled hyperbolic equations*, submitted.

2.1 – Introduction

The stabilization of weakly coupled systems starts with the pioneering works of Lagnese and Lions [107] and Russell [135]. In their approach, the multiplier method is the main tool to reach the desired estimates on the energy of each component of the system. This techniques has been further developed in [106] and later in [1] for systems of hyperbolic equations.

In particular, [135] address the indirect stabilization problem, that occurs when the damping (or the control) acts on a reduced number of equations of the system. In this situation the (uniformly) exponential decay rate is usually out of the range of possible targets. This is the case in [2] as well as in Chapter 1, where polynomial stability is achieved for the whole system, by means of multipliers properly adapted to the peculiar structure of the system under investigation. Indeed, it turns out that different multipliers are required to cope with systems with "homogeneous" [2] or "hybrid" (see Chapter 1) boundary conditions.

In [22] a different method has been proposed to prove polynomial stabilization for the solution of abstract first order Cauchy problems. As far as indirect stabilization for weakly coupled systems is concerned, this technique has revealed successful for few systems of hyperbolic equations, requiring strong compatibility conditions among the operators involved in the system in order to perform the needed spectral analysis.

An operator-theoretical approach has been recently proposed by [23] and [32], referring to the issue of optimality for the established decay rate. Indeed, as first noted by Lebeau [111], the decay rate of the system ruled by the dynamics $\mathcal{A}$ is related to the size of the resolvent operator of $\mathcal{A}$ on the imaginary axis. Thus, the growth of the resolvent operator norm along the imaginary axis gives an explicit rate for the decay of the total energy associated to the system (with the optimality of the decay related to the optimality of the growth estimate for the resolvent operator). This technique has been successfully applied in [137] to systems of Euler-Bernoulli and wave equations with a globally distributed coupling. Let us point out that, however, multipliers cannot be completely avoided, since again suitable multipliers (adapted to the operators of the system) are needed to deal with elliptic estimates for the resolvent operator.

In this chapter we study the indirect stabilization problem for several systems of hyperbolic-type equations, by means of the general criterion given in [32]. Since these systems fall into the general description given in Chapter 1, polynomial stabilization is already ensured (see equation (111)). Here we succeed in improving the stabilization decay rate, thanks to a sharp analysis of the behaviour of the resolvent operator along the imaginary axis.

More precisely, let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be open and bounded, with sufficiently smooth boundary Γ. Let $\lambda > 0$ and $\alpha, \beta \in L^\infty(\Omega)$, strictly positive. We will first consider the weakly coupled system of wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \end{cases} \tag{102}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) = 0 = v(\cdot, t) \quad \text{on } \Gamma\,, \ t > 0 \tag{103}$$

and initial conditions

$$u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 \quad \text{in } \Omega\,, \tag{104}$$

for functions u^i, v^i $(i = 0, 1)$ in suitable spaces (see (115)). Also operators with different boundary conditions on separated portions of the boundary can be treated. Indeed, let Γ_0 and Γ_1 be open subsets of Γ such that

$$\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1\,, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset\,.$$

In this situation, we can consider system (102) with boundary conditions

$$\begin{array}{c} u(\cdot,t)=0 \text{ on } \Gamma_0, \quad \dfrac{\partial u}{\partial \nu}(\cdot,t)=0 \quad \text{on } \Gamma_1 \\ v(\cdot,t)=0 \quad \text{on } \Gamma \end{array} \qquad t>0 \tag{105}$$

and initial conditions (104). Then we will address the system of two wave equations with respectively Robin and Dirichlet boundary conditions,

$$\begin{cases} u_{tt}-\Delta u+\beta u_t+\alpha v=0 & \text{in } \Omega\times(0,+\infty) \\ v_{tt}-\Delta v+\alpha u=0 & \text{in } \Omega\times(0,+\infty) \\ \dfrac{\partial u}{\partial \nu}+\sigma u=0=v & \text{on } \Gamma\times(0,+\infty)\,, \end{cases} \tag{106}$$

for some $\sigma>0$, with initial conditions (104). Finally, we will focus on the Petrowsky-wave system

$$\begin{cases} u_{tt}-\Delta u+\beta u_t+\alpha v=0 & \text{in } \Omega\times(0,+\infty) \\ v_{tt}+\Delta^2 v+\alpha u=0 & \text{in } \Omega\times(0,+\infty)\,, \end{cases} \tag{107}$$

with Robin boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot,t)+\sigma u(\cdot,t)=0 \quad \text{on } \Gamma\,,\ t>0 \tag{108}$$

on u and either

$$v(\cdot,t)=0=\Delta v(\cdot,t) \quad \text{on } \Gamma\,,\ t>0 \tag{109}$$

or

$$v(\cdot,t)=0=\frac{\partial v}{\partial \nu}(\cdot,t) \quad \text{on } \Gamma\,,\ t>0 \tag{110}$$

on v, with initial conditions (104).

We first point out that the three systems presented above fulfill the compatibility condition (32) introduced in Chapter 1, so by point i) of Corollary 1.19 we deduce that

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/4}}|U_0|^2_{\mathcal{H}} \tag{111}$$

for every initial condition $U_0=(u^0,u^1,v^0,v^1)\in D(\mathcal{A})$ and for some constant $C>0$ (see the section below for precise definitions of $D(\mathcal{A})$ and $\mathcal{H}$). However, in the present chapter we will show that the total energy $\mathcal{E}(t)$ decays faster, indeed, it decays polynomially in time with a decay rate 1/2 for initial condition in $D(\mathcal{A})$, that is,

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/2}}|U_0|^2_{\mathcal{H}}$$

for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$ and for some $C > 0$ (see also equation (132)). In this way, we succeed to improve the decay rate of an exponential factor 2.

In the next section we introduce the abstract setting that fits the previous PDEs systems into a first order Cauchy problem.

2.2 – Abstract setting

We briefly recall the abstract setting we have introduced in Section 1.1.1, but on the field $\mathbb{C}$ of complex numbers. Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space on the field $\mathbb{C}$ of complex numbers with associated norm $|\ |_H$. We consider the abstract system of evolution equations

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha v = 0 & \text{in } H \\ v''(t) + A_2 v(t) + \alpha u = 0 & \text{in } H \end{cases} \tag{112}$$

with hypotheses (H1)-(H2)-(H3) on the operators A_1, A_2, B and on the coefficient α. We associate to the operator A_i the energy

$$E_i(u,p) = \frac{1}{2}\left(|A_i^{1/2}u|_H^2 + |p|_H^2\right) \qquad \forall (u,p) \in D(A_i^{1/2}) \times H \quad (i = 1,2)\,, \tag{113}$$

so that assumption $(H1)$ yields

$$|u|_H^2 \le \frac{2}{\omega_i} E_i(u,p) \qquad \forall (u,p) \in D(A_i^{1/2}) \times H \quad (i = 1,2)\,. \tag{114}$$

System (112), with the initial conditions

$$\begin{cases} u(0) = u^0 \in D(A_1^{1/2})\,, \quad u'(0) = u^1 \in H\,, \\ v(0) = v^0 \in D(A_2^{1/2})\,, \quad v'(0) = v^1 \in H\,, \end{cases} \tag{115}$$

can be formulated as a first order Cauchy problem in the space

$$\mathcal{H} = D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H\,,$$

that becomes a Hilbert space on $\mathbb{C}$ endowed with the scalar product

$$(U, \hat{U}) := \langle A_1^{1/2}u, A_1^{1/2}\hat{u}\rangle + \langle p, \hat{p}\rangle + \langle A_2^{1/2}v, A_2^{1/2}\hat{v}\rangle + \langle q, \hat{q}\rangle + \alpha\langle u, \hat{v}\rangle + \alpha\langle v, \hat{u}\rangle \tag{116}$$

for every $U = (u,p,v,q)$, $\hat{U} = (\hat{u}, \hat{p}, \hat{v}, \hat{q}) \in \mathcal{H}$ and associated norm $|U|_{\mathcal{H}} := (U,U)^{1/2}$. Indeed, introducing the operator

$$\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2 v - \alpha u) \quad \forall U \in D(\mathcal{A})\,, \end{cases} \tag{117}$$

problem (112) can be recast as

$$
\begin{cases}
U'(t) = \mathcal{A}U(t) = \begin{pmatrix} 0 & I & 0 & 0 \\ -A_1 & -B & -\alpha I & 0 \\ 0 & 0 & 0 & I \\ -\alpha I & 0 & -A_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} \\
U(0) = U_0 = (u^0, u^1, v^0, v^1) \in \mathcal{H} ,
\end{cases}
\tag{118}
$$

where I stands for the identity operator on H. Moreover, for every $U \in \mathcal{H}$, we define the total energy of system (118) by

$$
\mathcal{E}(U(t)) := E_1(u,p) + E_2(v,q) + 2\alpha\mathcal{R}\langle u, v\rangle = 2|U|^2_{\mathcal{H}} , \tag{119}
$$

where $\mathcal{R}(z)$ stands for the real part of $z \in \mathbb{C}$. By assumption $(H3)$, the total energy $\mathcal{E}$ satisfies

$$
\nu_1(\alpha)\,[E_1(u,p) + E_2(v,q)] \le \mathcal{E}(U(t)) \le \nu_2(\alpha)\,[E_1(u,p) + E_2(v,q)] \ , \tag{120}
$$

where $\nu_1(\alpha) = 1 - |\alpha|(\omega_1\omega_2)^{-1/2} > 0$ and $\nu_2(\alpha) = 1 + |\alpha|(\omega_1\omega_2)^{-1/2}$. The operator $\mathcal{A}$ generates a C_0-semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$ (see Lemma 1.16 in Chapter 1), that satisfies $e^{t\mathcal{A}}U_0 = (u(t), p(t), v(t), q(t))$, where (u, v) is the solution of problem (112) with initial conditions (115) and $(p, q) = (u', v')$.

2.2.1 – Stability properties

In [2] the authors prove that system (112)-(115), or, equivalently, system (118), fails to be exponentially (uniformly) stable. This feature is a consequence of the compactness of the coupling operator $K\binom{u}{u'} = \binom{0}{\alpha u}$ in the energy space $D(A_i^{1/2}) \times H$ $(i = 1, 2)$. Thus, concerning the asymptotic behaviour of system (112)-(115), we look for decay rates weaker than exponentials, such as polynomial ones. A first step in this direction relies on the dissipation relation fulfilled by the total energy of the system, whose proof is achieved by a straightforwards computation.

Proposition 2.1. *For every $U_0 \in D(\mathcal{A})$ we have that*

$$
\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}p|^2_H = \mathcal{R}(\mathcal{A}U, U)_{\mathcal{H}} . \tag{121}
$$

The next result investigates the spectrum of the operator $\mathcal{A}$.

Proposition 2.2. *Under hypotheses* (H1)-(H2)-(H3) *holds $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Let $b \in \mathbb{R}$ and $U = (u, p, v, q) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = ibU\,. \tag{122}$$

We claim that $U = 0$. We first address the case $b \neq 0$. In this case, equation (122) implies

$$\begin{cases} ibu = p \\ ibp = -A_1 u - Bp - \alpha v \\ ibv = q \\ ibq = -A_2 v - \alpha u\,. \end{cases} \tag{123}$$

Multiplying both sides of (122) for U and taking the real part, owing to (121) we deduce that $|B^{1/2}p|_H = 0$. Thanks to hypothesis (H2) we get $p = 0$, by the first equation in (123) we have $u = 0$ (since $b \neq 0$), thus from the second equation in (123) we find $v = 0$ (since $\alpha \neq 0$), finally from the third equation in (123) we have $q = 0$, so we conclude that $U = 0$. On the other hand, if $b = 0$, system (123) reduces to $p = q = 0$ and

$$\begin{cases} A_1 u + \alpha v = 0 \\ A_2 v + \alpha u = 0\,. \end{cases} \tag{124}$$

We now argue by contradiction. Suppose there exist $u \neq 0$, $v \neq 0$ satisfying system (124). Thus, multiplying the first equation therein by u and the second by v, thanks to hypothesis (H1), we have

$$\begin{aligned} \omega_1 |u|_H^2 &\leq \langle A_1 u, u\rangle = -\alpha\langle v, u\rangle\,, \\ \omega_2 |v|_H^2 &\leq \langle A_2 v, v\rangle = -\alpha\langle u, v\rangle\,. \end{aligned} \tag{125}$$

Since the left upper terms in (125) are positive, multiplying both sides together and thanks to the Cauchy-Schwarz inequality we have

$$\omega_1\omega_2 |u|_H^2 |v|_H^2 \leq \alpha^2 |\langle u, v\rangle|^2 \leq \alpha^2 |u|_H^2 |v|_H^2\,, \tag{126}$$

so infering $\omega_1\omega_2 \leq \alpha^2$, that negates hypothesis (H3). Therefore $u = v = 0$ and we conclude again that $U = 0$. $\square$

As a consequence, we deduce that system (118) is strongly stable, relying on a characterization due to Benchimol [28].

Corollary 2.3. *Under hypotheses* (H1)-(H2)-(H3), *the semigroup* $e^{t\mathcal{A}}$ *is strongly stable, that is*

$$\lim_{t\to+\infty} |e^{t\mathcal{A}} U_0|_{\mathcal{H}} = 0 \qquad \forall U_0 \in \mathcal{H}\,. \tag{127}$$

We now analyze the asymptotic behaviour of the contraction semigroup $e^{t\mathcal{A}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$. For this purpose, we recall a result due to Borichev and Tomilov [32, Theorem 2.4], which gives a necessary and sufficient condition for the polynomial decay of the semigroup norm. Given two functions f and g, we use the notation $f(t) = O(g(t))$ as $t \to \infty$ when the function $|f(t)/g(t)|$ is bounded for large t; we denote $f(t) = o(g(t))$ as $t \to \infty$ if the function $f(t)/g(t)$ tends to 0 as $t \to \infty$.

THEOREM 2.4. *Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that the imaginary axis $i\mathbb{R}$ lies in the resolvent set $\rho(A)$ of A. For every fixed $\gamma > 0$, the following conditions are equivalent:*

$$\|R(ib, A)\|_{\mathcal{L}(H)} = O(|b|^{\gamma}) \qquad \text{as } b \to +\infty\,; \tag{128}$$

$$\|T(t)A^{-1}\|_{\mathcal{L}(H)} = O(|t|^{-1/\gamma}) \quad \text{as } t \to +\infty\,; \tag{129}$$

$$\|T(t)A^{-1}x\|_{H} = o(|t|^{-1/\gamma}) \quad \text{as } t \to +\infty \quad \forall x \in H\,. \tag{130}$$

In particular, we are interested in the case $\gamma = 4$, that is the decay rate we will show for all the systems addressed in Section 2.1. Indeed, in the following sections we will prove for each of those systems the next stabilization result.

THEOREM 2.5. *Suppose hypotheses* (H1)-(H2)-(H3) *hold. Assume moreover that*

$$\|R(ib, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} = O(|b|^4) \quad \text{as } b \to +\infty\,. \tag{131}$$

Then, for every integer $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|U(t)|_{\mathcal{H}} = |e^{t\mathcal{A}}U_0|_{\mathcal{H}} \le \frac{C_m}{(1+t)^{m/4}} |U_0|_{D(\mathcal{A}^m)} \quad \forall t \ge 0\,,\ U_0 \in D(\mathcal{A}^m)\,. \tag{132}$$

Once condition (131) has been proved for each system under consideration, we can conclude that the total energy decays polynomially at infinity, with respect to the regularity of the initial condition U_0. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/2}} |U_0|^2_{\mathcal{H}}\,. \tag{133}$$

REMARK 2.6.

i) Whether the decay rate in (133) is sharp or not, is related to the optimality of the estimate we give in the relation (131). Since, along the computations of the proof below, we try to get the finest estimate on the exponent γ (see equations (145)-(175)-(195)), we conjecture that the estimate (133) is optimal for the systems under consideration in Section 2.3.

ii) We can consider systems with more general coupling operators such as

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 P v = 0 & \text{in } H \\ v''(t) + A_2 v(t) + \alpha_2 P^* u = 0 & \text{in } H \end{cases} \tag{134}$$

where P is a bounded linear coercive operator on H, P^* is its adjoint operator, and hypothesis $(H3)$ is replaced by

(H3)' α_1 and α_2 are two real numbers such that $0 < \alpha_1\alpha_2 < \dfrac{\omega_1\omega_2}{\|P\|^2_{\mathcal{L}(H)}}$.

In the case $\alpha_i > 0$, the total energy of system (134) is defined by

$$\mathcal{E}(U(t)) := \alpha_2 E_1(u,p) + \alpha_1 E_2(v,q) + \alpha_1\alpha_2\langle Pu, v\rangle + \alpha_1\alpha_2\langle P^*v, u\rangle$$

and still verifies the estimate of Theorem 2.5.

iii) In all the systems introduced in Section 2.1, the constants α, β and σ can be replaced by bounded functions of the space variable, provided they are strictly positive.

2.3 – Indirect stabilization by resolvent estimate

Let $\Omega \subset \mathbb{R}^d$ be open and bounded, with sufficiently smooth boundary Γ. For $i = 1, \dots, d$ we will denote by ∂_i the partial derivative with respect to the component x_i and by $\partial_t u$ or u_t the derivative with respect to the time variable of function u. We will also use the notation $H^k(\Omega), H^k_0(\Omega)$ for the usual Sobolev spaces with norm

$$|u|_{H^k} = \left(\int_\Omega \sum_{|p|\le k} |D^p u|^2 dx\right)^{1/2}$$

where we have set $D^p = \partial_1^{p_1} \dots \partial_d^{p_d}$ for any multi-index $p = (p_1, \dots, p_d)$. Moreover, we will refer to $C_\Omega > 0$ as the largest constant such that Poincaré's inequality

$$C_\Omega |u|^2_{L^2} \le |\nabla u|^2_{L^2} \tag{135}$$

holds true for any $u \in H^1_0(\Omega)$.

In the following sections we set $H := L^2(\Omega)$ and we define the bounded operator $B : H \to H$ by $Bu = \beta u$ for all $u \in H$, for some positive constant $\beta > 0$.

2.3.1 – Stabilization for a first wave-wave system

Consider the weakly coupled system of wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \\ \dfrac{\partial u}{\partial \nu} = 0 = v & \text{on } \Gamma \times (0, +\infty) \\ u(0) = u^0 \,,\ u_t(0) = u^1 \,,\ v(0) = v^0 \,,\ v_t(0) = v^1 & \text{in } \Omega \,, \end{cases} \tag{136}$$

where $\lambda > 0$ and $\alpha > 0$.

We can rewrite system (136) as (112) (or (118), equivalently) introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu}(\cdot, t) = 0 \text{ on } \Gamma \,,\ t > 0\} \,,\ A_1 u = -\Delta u + \lambda u \,, \tag{137}$$

$$D(A_2) := H^2(\Omega) \cap H_0^1(\Omega) \,, \quad A_2 v = -\Delta v \,. \tag{138}$$

Let λ_0^2 be the least eigenvalue of $-\Delta$ on $H^2(\Omega) \cap H_0^1(\Omega)$, and μ_0^2 be the least eigenvalue of $-\Delta + \lambda I$ with Neumann boundary condition. So the assumption (H3) on α yields

$$0 < |\alpha| < \lambda_0 \mu_0 \,. \tag{139}$$

For every $U \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, the energy associated to system (136) is

$$\mathcal{E}(U(t)) = \frac{1}{2} \int_\Omega \left[u_t^2 + |\nabla u|^2 + \lambda u^2 + v_t^2 + |\nabla v|^2 + 2\alpha \mathcal{R}(u\bar{v}) \right] dx = \frac{1}{2} |U|_{\mathcal{H}}^2 \,, \tag{140}$$

which, for every $U \in D(\mathcal{A})$, satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \mathcal{R}(\mathcal{A}U(t), U(t))_{\mathcal{H}} \,. \tag{141}$$

Corollary 2.3 ensures that system (136) is strongly stable. In order to achieve a better understanding of the asymptotic behaviour of the contraction semigroup $e^{t\mathcal{A}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$, in the sequel we show that condition (131) holds. In this way we will prove that the total energy of system (136) decays polynomially at infinity, with respect to the regularity of the initial condition U_0. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/2}} \left(|u^0|_{H^2}^2 + |u^1|_{H^1}^2 + |v^0|_{H^2}^2 + |v^1|_{H_0^1}^2 \right) \,. \tag{142}$$

Proof of Theorem 2.5 for system (136). Thanks to Proposition 2.2 we know that $i\mathbb{R} \subset \rho(\mathcal{A})$. Thus, we need to show that

$$\|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = O(|b|^4) \quad \text{as } s \to +\infty \,. \tag{143}$$

Let $U \in \mathcal{H}$ and $b \in \mathbb{R}$ such that $|b| \geq \max(1, \beta, \lambda)$. Since $Rank(ibI - \mathcal{A}) = \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U \qquad \text{in } \mathcal{H} \,. \tag{144}$$

Thus, the estimate (143) will hold once provided that there exists $C_\alpha > 0$ (depending on Ω and α but not on b) such that

$$|Z| \leq C_\alpha |b|^4 |U| \,, \tag{145}$$

where C_α blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$. Denoting $Z = (u, p, v, q) \in D(\mathcal{A})$ and $U = (f, g, h, k) \in \mathcal{H}$, equation (144) reads as

$$\begin{cases} ibu - p = f & \text{in } H^1(\Omega) \\ ibp - \Delta u + \lambda u + \beta p + \alpha v = g & \text{in } L^2(\Omega) \\ ibv - q = h & \text{in } H_0^1(\Omega) \\ ibq - \Delta v + \alpha u = k & \text{in } L^2(\Omega) \,. \end{cases} \tag{146}$$

We will proceed in several steps, evaluating each term of the norm $|Z|$.

Step 1 : Estimate of $|p|_H$ and $|bu|_H$.

We first multiply by Z both sides of equation (144) and then take the real part of it. Thanks to the right identity in (141), we deduce that $|\beta p|_H^2 = \mathcal{R}(U, Z)$, so

$$\beta |p|_H^2 \leq |U|\,|Z| \,. \tag{147}$$

Then, from the first equation of system (146) we deduce that

$$\begin{aligned} |bu|_H^2 \leq 2|p|_H^2 + 2|f|_H^2 &\leq \frac{2}{\beta}|U|\,|Z| + \frac{2}{\mu_0^2}|f|_{H^1(\Omega)}^2 \\ &\leq \frac{2}{\beta}|U|\,|Z| + \frac{2\mu_0^{-2}}{1 - |\alpha|(\lambda_0\mu_0)^{-1}}|U|^2 \,, \end{aligned}$$

so

$$|bu|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2 \,, \tag{148}$$

where here and in the following K_α denotes a generic constant depending on α and Ω, which blows up as $\alpha \nearrow \lambda_0\mu_0$.

Step 2 : Estimate of $|\nabla u|_H^2 + \lambda |u|_H^2$.

Consider the scalar product in H of the second identity in system (146) with u

$$\int_\Omega (ibp - \Delta u + \lambda u + \beta p + \alpha v)\overline{u}dx = \int_\Omega g\overline{u}dx\,.$$

Integration by parts leads to

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)dx = \mathcal{R}\int_\Omega (g - (ib+\beta)p - \alpha v)\overline{u}dx\,. \tag{149}$$

We now evaluate each terms in the right-hand side integral.

$$\left|\int_\Omega g\overline{u}dx\right| \le |g|_H|u|_H \le \frac{1}{\mu_0}|g|_H\left[\int_\Omega (|\nabla u|^2 + \lambda u^2)dx\right]^{1/2}$$

$$\le \frac{1}{2\mu_0^2}|g|_H^2 + \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx \le K_\alpha |U|^2 + \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx\,.$$

Thanks to (147)-(148), and keeping in mind that $|b| \ge \beta$,

$$\left|\int_\Omega (ib+\beta)p\overline{u}dx\right| \le \frac{b^2+\beta^2}{4}|u|_H^2 + |p|_H^2 \le \frac{1}{2}|bu|_H^2 + |p|_H^2 \le \frac{2}{\beta}|U|\,|Z| + K_\alpha|U|^2\,.$$

Finally, thanks to (148),

$$\left|\int_\Omega \alpha v\overline{u}dx\right| \le |\alpha||u|_H|v|_H \le \frac{|\alpha|}{|b|}K_\alpha|Z|\,|bu|_H$$

$$\le \frac{|\alpha|K_\alpha}{|b|}|Z|\left(|U|^{1/2}|Z|^{1/2} + K_\alpha|U|\right) \le \frac{|\alpha|K_\alpha}{|b|}\left(|U|^{1/2}|Z|^{3/2} + K_\alpha|U|\,|Z|\right)\,.$$

Plugging the last three inequalities in equation (149), we derive that

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)dx \le K_\alpha\left(|U|\,|Z| + |U|^2 + \frac{|\alpha|}{|b|}|U|^{1/2}|Z|^{3/2}\right)\,, \tag{150}$$

or

$$\int_\Omega (|\nabla u|^2 + \lambda u^2) dx \leq \frac{4}{\beta}|U|\,|Z| + K_\alpha |U|^2 + 2|\alpha||u|_H |v|_H \,. \tag{151}$$

Step 3 : Partial estimate of $|\nabla v|_H$.

From the third identity in system (146) we derive that $q = ibv - h$, and using this relation in the fourth equation of (146) we obtain $-b^2 v - \Delta v + \alpha u = ibh + k$. The scalar product by v of both sides of this relation gives

$$\int_\Omega (-b^2 v - \Delta v + \alpha u)\overline{v} dx = \int_\Omega (ibh + k)\overline{v} dx \,,$$

so that, after integration by parts,

$$\int_\Omega |\nabla v|^2 dx = \int_\Omega b^2 v^2 dx + \mathcal{R} \int_\Omega (ibh + k - \alpha u)\overline{v} dx \,. \tag{152}$$

Since

$$\begin{aligned} &\left| \int_\Omega (ibh + k - \alpha u)\overline{v} dx \right| \leq |h|_H |bv|_H + \frac{|k|_H}{|b|}|bv|_H + |\alpha| \frac{|u|_H}{|b|}|bv|_H \\ &\leq |bv|_H^2 + \frac{1}{4}(|h|_H^2 + |k|_H^2) + \frac{|\alpha|^2}{2b^4}|bu|_H^2 \leq |bv|_H^2 + \frac{|\alpha|^2}{\beta b^4}|U|\,|Z| + K_\alpha |U|^2 \,, \end{aligned}$$

from equation (152) we deduce that

$$\int_\Omega |\nabla v|^2 dx \leq 2|bv|_H^2 + \frac{|\alpha|^2}{\beta b^4}|U|\,|Z| + K_\alpha |U|^2 \,. \tag{153}$$

Step 4 : Estimate of $|bv|_H$.

Taking the inner product between the second equation in (146) and $b^2 v$, we have

$$\int_\Omega \{[(ib + \beta)p - \Delta u + \lambda u + \alpha v] b^2 \overline{v}\} dx = \int_\Omega b^2 g \overline{v} dx \,,$$

so we have that

$$\int_\Omega b^2 v^2 dx = \frac{b^2}{\alpha} \mathcal{R} \int_\Omega [(g - (ib + \beta)p - \lambda u)\overline{v} - \nabla u.\, \nabla \overline{v}] dx \,. \tag{154}$$

First, note that, thanks to (147), (148) and $|b| \geq \lambda$,

$$\left|\frac{b^2}{\alpha}\int_\Omega (g-(ib+\beta)p-\lambda u)\overline{v}dx\right| \leq \int_\Omega\left[\frac{3b^2v^2}{4}+\frac{b^2g^2}{\alpha^2}+\frac{b^4p^2}{\alpha^2}+\frac{\lambda^2b^2u^2}{\alpha^2}\right]dx \\ \leq \frac{3}{4}|bv|_H^2+\frac{K_\alpha}{\alpha^2}b^2|U|^2+\frac{C}{\alpha^2}b^4|U|\,|Z|\,, \tag{155}$$

where here and in the following C denotes a generic positive constant, independent from α and b. Second, observe that, thanks to (151) and (153), we find out that

$$\begin{aligned}\left|\frac{b^2}{\alpha}\int_\Omega \nabla u\nabla\overline{v}dx\right| &\leq \frac{b^2}{|\alpha|}|\nabla u|_H|\nabla v|_H \\ &\leq \frac{b^2}{|\alpha|}\left(\sqrt{2}|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}+\frac{2}{\sqrt{\beta}}|U|^{1/2}|Z|^{1/2}+K_\alpha|U|\right)\cdot \\ &\quad\cdot\left(\sqrt{2}|bv|_H+\frac{|\alpha|}{\sqrt{\beta}b^2}|U|^{1/2}|Z|^{1/2}+K_\alpha|U|\right) \\ &\leq C\bigg(\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2}+\frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H+\frac{K_\alpha}{|\alpha|}b^2|U||bv|_H \\ &\quad+|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2}+|U|\,|Z|+K_\alpha|U|^{3/2}|Z|^{1/2} \\ &\quad+\frac{K_\alpha}{|\alpha|^{1/2}}b^2|u|_H^{1/2}|v|_H^{1/2}|U|+\frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}+\frac{K_\alpha}{|\alpha|}b^2|U|^2\bigg).\end{aligned} \tag{156}$$

Let ε be a positive real number. We recall Young's inequality in the form

$$|xy| \leq \frac{|x|^p}{pq^{1/q}\varepsilon^{p/q}}+\varepsilon|y|^q \tag{157}$$

that holds for every real numbers x, y and for a suitable pair of conjugate exponents (p,q). So we have

$$\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2} \leq \varepsilon|bv|_H^2+C_\varepsilon\frac{b^4}{\alpha^2}|bu|_H^2 \\ \leq \varepsilon|bv|_H^2+C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z|+\frac{K_{\alpha,\varepsilon}}{\alpha^2}b^4|U|^2\,, \tag{158}$$

where C_ε and $K_{\alpha,\varepsilon}$ are positive constants that diverge as ε goes to 0^+. Similarly, thanks to standard Young's inequality with parameter ε, we deduce that

$$\frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H \leq \varepsilon|bv|_H^2 + C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z|\,, \tag{159}$$

$$\frac{K_\alpha}{|\alpha|}b^2|U|\,|bv|_H \leq \varepsilon|bv|_H^2 + \frac{K_{\alpha,\varepsilon}b^4}{\alpha^2}|U|^2\,, \tag{160}$$

$$\begin{aligned} &|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2} \\ &\qquad = (|bv|_H^{1/2}|U|^{1/4}|Z|^{1/4})\left(\frac{|\alpha|^{1/2}}{|b|}|bu|_H^{1/2}|U|^{1/4}|Z|^{1/4}\right) \\ &\qquad \leq |bv|_H|U|^{1/2}|Z|^{1/2} + \frac{|\alpha|}{4b^2}|bu|_H|U|^{1/2}|Z|^{1/2} \\ &\qquad \leq \varepsilon|bv|_H^2 + C_\varepsilon|U|\,|Z| + \frac{1}{b^4}|bu|_H^2 \\ &\qquad \leq \varepsilon|bv|_H^2 + C_\varepsilon|U|\,|Z| + \frac{K_\alpha}{b^4}|U|^2\,, \end{aligned} \tag{161}$$

$$\begin{aligned} \frac{K_\alpha}{|\alpha|^{1/2}}b^2|u|_H^{1/2}|v|_H^{1/2}|U| &\leq \frac{K_\alpha}{|\alpha|^{1/2}}|b||U|\left(\frac{|bu|_H}{2}+\frac{|bv|_H}{2}\right) \\ &\leq \varepsilon|bv|_H^2 + \frac{K_{\alpha,\varepsilon}}{|\alpha|}b^2|U|^2 + |bu|_H^2 + \frac{K_\alpha}{|\alpha|}b^2|U|^2 \\ &\leq \varepsilon|bv|_H^2 + \frac{K_{\alpha,\varepsilon}}{|\alpha|}b^2|U|^2 + \frac{2}{\beta}|U|\,|Z|\,. \end{aligned} \tag{162}$$

Gathering estimates (158)-...-(162) in relation (156), we end up with

$$\begin{aligned} \left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\,\nabla\bar{v}dx\right| &\leq 5C\varepsilon|bv|_H^2 + C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z| + \frac{K_{\alpha,\varepsilon}b^4}{\alpha^2}|U|^2 \\ &\quad + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \end{aligned} \tag{163}$$

Back to relation (154), owing to (155) and (163), we have

$$(1-20C\varepsilon)|bv|_H^2 \leq \frac{K_{\alpha,\varepsilon}b^4}{\alpha^2}|U|^2 + \frac{C_\varepsilon}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,,$$

that, for a sufficiently small $\varepsilon > 0$, ensures that

$$|bv|_H^2 \leq \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \tag{164}$$

Step 5 : Estimate of $|\nabla v|_H$ and $|q|_H$.

Using estimate (164), relation (153) yields

$$|\nabla v|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2}\,. \tag{165}$$

On the other hand, by the third equation in system (146), we conclude that

$$|q|_H^2 \le 2|bv|_H^2 + 2|h|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2}\,. \tag{166}$$

Thanks to equations (147)-(150)-(165)-(166), we deduce that

$$\begin{aligned} |Z|^2 &\le \nu_2(\alpha) \left[\int_\Omega \left(p^2 + |\nabla u|^2 + \lambda u^2 + q^2 + |\nabla v|^2 \right) dx \right] \\ &\le C_\alpha \left[b^4 |U|^2 + b^4 |U|\,|Z| + b^2 |U|^{3/2} |Z|^{1/2} + |U|^{1/2} |Z|^{3/2} \right], \end{aligned}$$

where C_α is a positive constant depending only on Ω and α (but not on b) that blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$. Applying again Young's inequality with suitable choices of conjugate exponents (p, q), we infer that

$$|Z|^2 \le C_\alpha |b|^8 |U|^2\,,$$

that completes the proof of relation (145). □

2.3.2 – Stabilization for a second wave-wave system

We consider now the weakly coupled system of wave equations

$$\begin{cases} u_{tt} - \Delta u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 = v & \text{on } \Gamma \times (0, +\infty) \\ u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 & \text{in } \Omega\,, \end{cases} \tag{167}$$

where $\sigma > 0$ and $\alpha \in \mathbb{R}$. We can rewrite system (167) as (112) (or (118), equivalently) introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \left(\frac{\partial u}{\partial \nu} + \sigma u \right)(\cdot, t) = 0 \text{ on } \Gamma\,,\ t > 0\},\ A_1 u = -\Delta u \tag{168}$$

$$D(A_2) := H^2(\Omega) \cap H_0^1(\Omega)\,, \quad A_2 v = -\Delta v\,. \tag{169}$$

Let λ_0^2 be the least eigenvalue of $-\Delta$ on $H^2(\Omega) \cap H_0^1(\Omega)$, and μ_0^2 be the least eigenvalue of $-\Delta$ with Robin boundary condition. So the assumption (H3) implies

$0 < |\alpha| < \lambda_0\mu_0$. For every $U \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, the energy associated to system (167) is

$$\mathcal{E}(U(t)) = \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2 + v_t^2 + |\nabla v|^2 + 2\alpha\mathcal{R}(u\bar{v})\right] dx + \int_\Gamma \sigma u^2 d\Sigma\,, \tag{170}$$

which, for every $U \in D(\mathcal{A})$, satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \mathcal{R}(\mathcal{A}U(t), U(t))_{\mathcal{H}}\,. \tag{171}$$

From Corollary 2.3 we deduce that system (167) is strongly stable. We further analyze the polynomial stabilization for system (167), by means of Theorem 2.5. Thus, the total energy of system (167) decays polynomially at infinity, with respect to the regularity of the initial condition U_0. In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \leq \frac{C}{(1+t)^{1/2}} \left(|u^0|_{H^2}^2 + |u^1|_{H^1}^2 + |v^0|_{H^2}^2 + |v^1|_{H_0^1}^2\right)\,. \tag{172}$$

Proof of Theorem 2.5 for system (167). Thanks to Proposition 2.2 we know that $i\mathbb{R} \subset \rho(\mathcal{A})$. Thus, we need to show that

$$\|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = O(|b|^4) \quad \text{as } s \to +\infty\,. \tag{173}$$

Let $U \in \mathcal{H}$ and $b \in \mathbb{R}$ such that $|b| \geq \max(1, \beta)$. Since $Rank(ibI - \mathcal{A}) = \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U \qquad \text{in } \mathcal{H}\,. \tag{174}$$

Thus, the estimate (173) will hold once provided that there exists $C_\alpha > 0$ (depending on Ω and α but not on b) such that

$$|Z| \leq C_\alpha |b|^4 |U|\,, \tag{175}$$

where C_α blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$.

Denoting $Z = (u, p, v, q) \in D(\mathcal{A})$ and $U = (f, g, h, k) \in \mathcal{H}$, equation (174) reads as

$$\begin{cases} ibu - p = f & \text{in } H^1(\Omega) \\ ibp - \Delta u + \beta p + \alpha v = g & \text{in } L^2(\Omega) \\ ibv - q = h & \text{in } H_0^1(\Omega) \\ ibq - \Delta v + \alpha u = k & \text{in } L^2(\Omega)\,. \end{cases} \tag{176}$$

We will proceed as in Section 2.3.1, evaluating each term of the norm $|Z|$.

We first estimate the terms $|p|_H$ and $|bu|_H$ as in the Step 1 of the previous section, finding that

$$|p|_H^2 \leq \frac{1}{\beta}|U|\,|Z|\,. \tag{177}$$

and

$$|bu|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2\,, \tag{178}$$

where here and in the following K_α denotes a generic constants depending on α and Ω, which blows up as $|\alpha| \nearrow \lambda_0\mu_0$.

Then, we need an estimate of $\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma$. To this aim, consider the scalar product in H of the second identity in system (176) with u

$$\int_\Omega (ibp - \Delta u + \beta p + \alpha v)\overline{u} dx = \int_\Omega g\overline{u} dx\,.$$

Integration by parts leads to

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma = \mathcal{R}\int_\Omega (g - (ib+\beta)p - \alpha v)\overline{u} dx\,. \tag{179}$$

We now evaluate each terms in the right hand side integral.

$$\begin{aligned}
\left|\int_\Omega g\overline{u} dx\right| &\leq |g|_H |u|_H \leq \frac{1}{\mu_0}|g|_H \left[\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma\right]^{1/2} \\
&\leq \frac{1}{2\mu_0^2}|g|_H^2 + \frac{1}{2}\left[\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma\right] \\
&\leq K_\alpha |U|^2 + \frac{1}{2}\left[\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma\right]\,.
\end{aligned}$$

Thanks to (177)-(178), and keeping in mind that $|b| \geq \beta$,

$$\left|\int_\Omega (ib+\beta)p\overline{u} dx\right| \leq \frac{b^2+\beta^2}{4}|u|_H^2 + |p|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2\,.$$

Finally, thanks to (178),

$$\begin{aligned}
\left|\int_\Omega \alpha v\overline{u} dx\right| &\leq |\alpha||u|_H |v|_H \leq \frac{|\alpha|}{|b|}K_\alpha |Z|\,|bu|_H \\
&\leq \frac{|\alpha|K_\alpha}{|b|}|Z|\Big(|U|^{1/2}|Z|^{1/2} + K_\alpha|U|\Big) \leq \frac{|\alpha|K_\alpha}{|b|}\left(|U|^{1/2}|Z|^{3/2} + K_\alpha |U||Z|\right)\,.
\end{aligned}$$

Plugging the last three inequalities in equation (179), we derive that

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \le K_\alpha \left(|U|\,|Z| + |U|^2 + \frac{|\alpha|}{|b|} |U|^{1/2} |Z|^{3/2} \right), \tag{180}$$

or

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \le \frac{4}{\beta} |U|\,|Z| + K_\alpha |U|^2 + 2|\alpha| |u|_H |v|_H \,. \tag{181}$$

Proceedings as for the previous example, we can deduce estimates of $|\nabla v|_H$ and $|q|_H$, more precisely that

$$|\nabla v|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \tag{182}$$

and

$$|q|_H^2 \le 2|bv|_H^2 + 2|h|_H^2 \le \frac{K_\alpha}{\alpha^2} b^4 |U|^2 + \frac{C}{\alpha^2} b^4 |U|\,|Z| + \frac{K_\alpha}{|\alpha|} b^2 |U|^{3/2} |Z|^{1/2} \,. \tag{183}$$

Thanks to equations (177)-(180)-(182)-(183), we deduce that

$$\begin{aligned} |Z|^2 &\le \nu_2(\alpha) \left[\int_\Omega \left(p^2 + |\nabla u|^2 + q^2 + |\nabla v|^2 \right) dx + \int_\Gamma \sigma u^2 d\Sigma \right] \\ &\le C_\alpha \left[b^4 |U|^2 + b^4 |U|\,|Z| + b^2 |U|^{3/2} |Z|^{1/2} + |U|^{1/2} |Z|^{3/2} \right], \end{aligned}$$

where C_α is a positive constant depending only on Ω and α (but not on b) that blows up as $|\alpha|$ goes to 0 or to $\lambda_0 \mu_0$. Applying again Young's inequality with suitable choices of conjugate exponents (p, q), we have that

$$|Z|^2 \le C_\alpha |b|^8 |U|^2 \,,$$

and so we conclude that the claimed relation (175) holds true. □

2.3.3 – Stabilization for a wave-Petrowsky system

We now focus on the stabilization problem for the weakly coupled system

$$\begin{cases} u_{tt} - \Delta u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0, +\infty) \\ v_{tt} + \Delta^2 v + \alpha u = 0 & \text{in } \Omega \times (0, +\infty) \\ u(0) = u^0 \,,\ u_t(0) = u^1 \,,\ v(0) = v^0 \,,\ v_t(0) = v^1 & \text{in } \Omega \,, \end{cases} \tag{184}$$

with Robin boundary conditions

$$\frac{\partial u}{\partial \nu}(\cdot, t) + \sigma u(\cdot, t) = 0 \quad \text{on } \Gamma \,,\ t > 0 \tag{185}$$

on u (for some $\sigma > 0$) and either clamped boudary conditions

$$v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \quad \text{on } \Gamma\,,\ t>0 \tag{186}$$

or hinged boundary conditions

$$v(\cdot,t) = 0 = \Delta v(\cdot,t) \quad \text{on } \Gamma\,,\ t>0 \tag{187}$$

on v. We can rewrite systems (184)-(185)-(186) and (184)-(185)-(187) as (112) (or (118), equivalently) introducing the operators

$$D(A_1) := \{u \in H^2(\Omega) : \left(\frac{\partial u}{\partial \nu} + \sigma u\right)(\cdot,t) = 0 \text{ on } \Gamma\,,\ t>0\}\,,\ A_1 u = -\Delta u \tag{188}$$

and

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \frac{\partial v}{\partial \nu}(\cdot,t) \text{ on } \Gamma\,,\ t>0\}\,,\ A_2 v = \Delta^2 v \tag{189}$$

or

$$D(A_2) := \{v \in H^4(\Omega) : v(\cdot,t) = 0 = \Delta v(\cdot,t) \text{ on } \Gamma\,,\ t>0\}\,,\ A_2 v = \Delta^2 v\,, \tag{190}$$

and defining $(\mathcal{A}, D(\mathcal{A}))$ as in (117). Let λ_0^2 be the least eigenvalue of $-\Delta$ with Robin boundary conditions, and μ_0^2 be the least eigenvalue of Δ^2 with either clamped or hinged boundary conditions. So the assumption (H3) on α yields $0 < |\alpha| < \lambda_0\mu_0$. For every $U \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times L^2(\Omega)$, the energy associated to systems (184)-(185)-(186) or (184)-(185)-(187) is

$$\mathcal{E}(U(t)) = \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2 + v_t^2 + |\Delta v|^2 + 2\alpha\mathcal{R}(u\bar{v})\right] dx + \frac{1}{2}\int_\Gamma \sigma u^2 d\Sigma = \frac{1}{2}|U|_{\mathcal{H}}^2\,, \tag{191}$$

which, for every $U \in D(\mathcal{A})$, satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \mathcal{R}(\mathcal{A}U(t), U(t))_{\mathcal{H}}\,. \tag{192}$$

We now analyze the asymptotic behaviour of the contraction semigroup $e^{t\mathcal{A}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$.

Proof of Theorem 2.5 for systems (184)-(185)-(186) and (184)-(185)-(187). Thanks to Proposition 2.2 we know that $i\mathbb{R} \subset \rho(\mathcal{A})$. Thus, systems (184)-(185)-(186) and (184)-(185)-(187) are strongly stable and we are left to show that

$$\|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = O(|b|^4) \quad \text{as } s \to +\infty\,. \tag{193}$$

Let $U \in \mathcal{H}$ and $b \in \mathbb{R}$ such that $|b| \geq \max(1, \beta)$. Since $Rank(ibI - \mathcal{A}) = \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ such that

$$ibZ - \mathcal{A}Z = U \qquad \text{in } \mathcal{H}\,. \tag{194}$$

Thus, the estimate (193) will hold once provided that there exists $C_\alpha > 0$ (depending on Ω and α but not on b) such that

$$|Z| \leq C_\alpha |b|^4 |U|\,, \tag{195}$$

where C blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$.

Denoting $Z = (u, p, v, q) \in D(\mathcal{A})$ and $U = (f, g, h, k) \in \mathcal{H}$, equation (194) reads as

$$\begin{cases} ibu - p = f & \text{in } H^1(\Omega) \\ ibp - \Delta u + \beta p + \alpha v = g & \text{in } L^2(\Omega) \\ ibv - q = h & \text{in } H_0^2(\Omega) \\ ibq + \Delta^2 v + \alpha u = k & \text{in } L^2(\Omega)\,. \end{cases} \tag{196}$$

We will proceed in several steps, evaluating each terms of the norm $|Z|$.

Step 1 : Estimate of $|p|_H$ and $|bu|_H$.

We first multiply by Z both sides of equation (194) and then take the real part of it. Thanks to the right identity in (192), we deduce that $|\beta p|_H^2 = \mathcal{R}(U, Z)$, so

$$|p|_H^2 \leq \frac{1}{\beta} |U|\,|Z|\,. \tag{197}$$

Then, from the first equation of system (196) we deduce that

$$|bu|_H^2 \leq 2|p|_H^2 + 2|f|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + \frac{2\mu_0^{-2}}{1 - |\alpha|(\lambda_0\mu_0)^{-1}} |U|^2\,,$$

so

$$|bu|_H^2 \leq \frac{2}{\beta}|U|\,|Z| + K_\alpha |U|^2\,, \tag{198}$$

where here and in the following K_α denotes a generic constant depending on α and Ω, which blows up as $|\alpha| \nearrow \lambda_0\mu_0$.

Step 2 : Estimate of $\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma$.

Consider the scalar product in H of the second identity in system (196) with u

$$\int_\Omega (ibp - \Delta u + \beta p + \alpha v)\overline{u} dx = \int_\Omega g\overline{u} dx\,.$$

Integration by parts leads to

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma = \mathcal{R}\int_\Omega (g-(ib+\beta)p-\alpha v)\overline{u}dx\,. \tag{199}$$

We now evaluate each terms in the right hand side integral.

$$\begin{aligned}\left|\int_\Omega g\overline{u}dx\right| &\le |g|_H|u|_H \le \frac{1}{\mu_0}|g|_H\left[\int_\Omega |\nabla u|^2dx+\int_\Gamma \sigma u^2 d\Sigma\right]^{1/2}\\ &\le \frac{1}{2\mu_0^2}|g|_H^2+\frac{1}{2}\left[\int_\Omega |\nabla u|^2dx+\int_\Gamma \sigma u^2 d\Sigma\right]\\ &\le K_\alpha|U|^2+\frac{1}{2}\left[\int_\Omega |\nabla u|^2dx+\int_\Gamma \sigma u^2 d\Sigma\right]\,.\end{aligned}$$

Thanks to (197)-(198), and keeping in mind that $|b|\ge \max(1,\beta)$,

$$\begin{aligned}\left|\int_\Omega (ib+\beta)p+\alpha v)\overline{u}dx\right| &\le \frac{b^2+\beta^2}{4}|u|_H^2+|p|_H^2+|\alpha||u|_H|v|_H\\ &\le \frac{1}{2}|bu|_H^2+|p|_H^2+\frac{|\alpha|}{|b|}K_\alpha|Z|\,|bu|_H\\ &\le \frac{2}{\beta}|U|\,|Z|+K_\alpha|U|^2+\frac{|\alpha|K_\alpha}{|b|}\Big(|U|^{1/2}|Z|^{3/2}+K_\alpha|U||Z|\Big)\,.\end{aligned}$$

Plugging the last two inequalities in equation (199), we derive that

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \le K_\alpha\left(|U|\,|Z|+|U|^2+\frac{|\alpha|}{|b|}|U|^{1/2}|Z|^{3/2}\right)\,, \tag{200}$$

or

$$\int_\Omega |\nabla u|^2 dx + \int_\Gamma \sigma u^2 d\Sigma \le \frac{4}{\beta}|U|\,|Z|+K_\alpha|U|^2+|\alpha||u|_H|v|_H\,. \tag{201}$$

Step 3 : Partial estimate of $|\Delta v|_H$.

From the third identity in system (196) we derive that $q=ibv-h$, and using this relation in the fourth equation of (196) we obtain $\Delta^2 v-b^2v+\alpha u=ibh+k$. The scalar product by v of both sides of this relation gives, after integrations by parts,

$$\int_\Omega |\Delta v|^2 dx = \int_\Omega b^2v^2dx+\mathcal{R}\int_\Omega (ibh+k-\alpha u)\overline{v}dx\,. \tag{202}$$

Since $|b|\ge 1$, and thanks to (198),

$$\begin{aligned}&\left|\int_\Omega (ibh+k-\alpha u)\overline{v}dx\right| \le |h|_H|bv|_H+\frac{|k|_H}{|b|}|bv|_H+|\alpha|\frac{|bu|_H}{b^2}|bv|_H\\ &\le |bv|_H^2+|h|_H^2+\frac{|k|_H^2}{b^2}+\frac{|\alpha|^2}{2b^4}|bu|_H^2\le |bv|_H^2+\frac{|\alpha|^2}{\beta b^4}|U|\,|Z|+\frac{K_\alpha}{b^2}|U|^2\,,\end{aligned}$$

from equation (202) we deduce that

$$\int_\Omega |\Delta v|^2 dx \leq 2|bv|_H^2 + \frac{|\alpha|^2}{\beta b^4}|U|\,|Z| + \frac{K_\alpha}{b^2}|U|^2 \,. \tag{203}$$

Step 4 : Estimate of $|bv|_H$.

Taking the inner product between the second equation in (196) and $b^2 v$, we have

$$\int_\Omega \{[(ib+\beta)p - \Delta u + \alpha v] b^2 \overline{v}\} dx = \int_\Omega b^2 g \overline{v} dx \,,$$

so we have that

$$\int_\Omega b^2 v^2 dx = \frac{b^2}{\alpha}\mathcal{R}\int_\Omega [(g - (ib+\beta)p)\overline{v} - \nabla u.\, \nabla\overline{v}] dx \,. \tag{204}$$

First, note that, after integrations by parts, we get

$$\begin{aligned}\left|\frac{b^2}{\alpha}\int_\Omega (g-(ib+\beta)p)\overline{v} dx\right| &\leq \int_\Omega \left[\frac{3b^2v^2}{4} + \frac{b^2g^2}{\alpha^2} + \frac{b^4p^2}{\alpha^2}\right] dx \\ &\leq \frac{3}{4}|bv|_H^2 + \frac{K_\alpha}{\alpha^2}b^2|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| \,.\end{aligned} \tag{205}$$

Afterwards, thanks to the well-known inequality

$$\int_\Omega (v^2 + |\nabla v|^2) dx \leq \int_\Omega |\Delta v|^2 dx \qquad \forall v \in H^2(\Omega)\cap H_0^1(\Omega)\,, \tag{206}$$

and owing to (201) and (203), we find out that

$$\begin{aligned}&\left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\nabla\overline{v} dx\right| \leq \frac{b^2}{|\alpha|}|\nabla u|_H |\nabla v|_H \leq |\nabla u|_H |\Delta v|_H \\ &\leq \frac{b^2}{|\alpha|}\left(|\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2} + \frac{2}{\sqrt{\beta}}|U|^{1/2}|Z|^{1/2} + K_\alpha|U|\right)\cdot \\ &\quad\cdot\left(\sqrt{2}|bv|_H + \frac{|\alpha|}{\sqrt{\beta}b^2}|U|^{1/2}|Z|^{1/2} + \frac{K_\alpha}{b}|U|\right) \\ &\leq C\left(\frac{b}{|\alpha|^{1/2}}|bv|_H^{3/2}|bu|_H^{1/2} + \frac{b^2}{|\alpha|}|U|^{1/2}|Z|^{1/2}|bv|_H + \frac{K_\alpha}{|\alpha|}b^2|U|\,|bv|_H \right. \\ &\quad + |\alpha|^{1/2}|u|_H^{1/2}|v|_H^{1/2}|U|^{1/2}|Z|^{1/2} + |U|\,|Z| + K_\alpha|U|^{3/2}|Z|^{1/2} \\ &\quad \left. + \frac{K_\alpha}{|\alpha|^{1/2}}b^2|u|_H^{1/2}|v|_H^{1/2}|U| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2} + \frac{K_\alpha}{|\alpha|}b^2|U|^2\right) \,.\end{aligned} \tag{207}$$

Let ε be a positive real number. Applying Young's inequality together with estimates (158)-...-(162), from (207) we conclude that

$$\left|\frac{b^2}{\alpha}\int_\Omega \nabla u.\,\nabla\overline{v}dx\right| \le 5C\varepsilon|bv|_H^2 + \\ + C_\varepsilon\frac{b^4}{\alpha^2}|U|\,|Z| + \frac{K_{\alpha,\varepsilon}b^4}{\alpha^2}|U|^2 + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \tag{208}$$

Back to relation (204), owing to (205) and (208), we have

$$(1-20C\varepsilon)|bv|_H^2 \le \frac{K_{\alpha,\varepsilon}b^4}{\alpha^2}|U|^2 + \frac{C_\varepsilon}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,,$$

that, fixing a sufficiently small $\varepsilon > 0$, ensures that

$$|bv|_H^2 \le \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \tag{209}$$

Step 5 : Estimate of $|\nabla v|_H$ and $|q|_H$.

Using (209), relation (203) yields

$$|\nabla v|_H^2 \le \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \tag{210}$$

On the other hand, by the third equation in system (196), we conclude that

$$|q|_H^2 \le 2|bv|_H^2 + 2|h|_H^2 \le \frac{K_\alpha}{\alpha^2}b^4|U|^2 + \frac{C}{\alpha^2}b^4|U|\,|Z| + \frac{K_\alpha}{|\alpha|}b^2|U|^{3/2}|Z|^{1/2}\,. \tag{211}$$

Thanks to equations (197)-(200)-(210)-(211), we deduce that

$$\begin{aligned}|Z|^2 &\le \nu_2(\alpha)\left[\int_\Omega \left(p^2+|\nabla u|^2+q^2+|\nabla v|^2\right)dx + \int_\Gamma \sigma u^2 d\Sigma\right]\\ &\le C_\alpha\left[b^4|U|^2 + b^4|U|\,|Z| + b^2|U|^{3/2}|Z|^{1/2} + |U|^{1/2}|Z|^{3/2}\right]\,,\end{aligned}$$

where C_α is a positive constant depending only on Ω and α (but not on b) that blows up as $|\alpha|$ goes to 0 or to $\lambda_0\mu_0$. Applying again Young's inequality with suitable choices of conjugate exponents (p,q), we conclude that

$$|Z|^2 \le C_\alpha |b|^8|U|^2\,,$$

completing the proof of relation (195). □

Thus, the total energy of system (184)-(185) decays polynomially at infinity, with respect to the regularity of the initial condition U_0.
In particular, for every $U_0 = (u^0, u^1, v^0, v^1) \in D(\mathcal{A})$,

$$\mathcal{E}(U(t)) \le \frac{C}{(1+t)^{1/2}}\left(\|u^0\|_{H^2}^2 + \|u^1\|_{H^1}^2 + \|v^0\|_{H^4}^2 + \|v^1\|_{H_0^2}^2\right) \quad \forall t > 0\,. \tag{212}$$

2.4 – Prospectives

2.4.1 – Localizing the damping/coupling regions

A first open issue concerns the extension of the previous results for systems with damping and/or coupling acting locally in Ω. Indeed, in our approach both damping and coupling are globally distributed in Ω, but it is natural to consider the case where they acts only in two open subset ω_d and ω_c of Ω. In this setting, one challenging question arises: is it possible to have polynomial stabilization of the system, even in the case $\omega_d \cap \omega_c = \emptyset$? In this case, there would be region of Ω where the two equations involved in the system evolve completely independently and without any dissipation of energy. A first result in this direction can be found in [8], where the authors prove a polynomial indirect stabilization result in the case of a system with the same operator acting on both components u and v, provided the two subset ω_d and ω_c satisfy the geometrical optical control condition, that is, each optical ray travelling at speed one must intersect both subset ω_d and ω_c in finite time. The problem of extending this property for larger classes of operators is widely open, and relies on the possibility to show some observability inequalities through the geometrical optical control condition.

In Chapter 3 we will address a different configuration of the stabilization problem for two wave equations, with damping and coupling acting only on the boundary of the domain. In this case, it happens that in the domain Ω the two components are uncoupled and both conservative. However, we will give positive answer for the indirect stabilization of the whole system (see Theorem 3.6).

2.4.2 – Different boundary conditions

Let us consider the system

$$\begin{cases} u_{tt} + \Delta^2 u + \beta u_t + \alpha v = 0 & \text{in } \Omega \times (0,+\infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in } \Omega \times (0,+\infty) \\ u(0) = u^0\,,\ u_t(0) = u^1\,,\ v(0) = v^0\,,\ v_t(0) = v^1 & \text{in } \Omega\,, \end{cases} \tag{213}$$

with Robin boundary conditions

$$\frac{\partial v}{\partial \nu}(\cdot,t) + \sigma v(\cdot,t) = 0 \quad \text{on } \Gamma\,,\ t > 0 \tag{214}$$

on v and either clamped boudary conditions

$$u(\cdot,t) = 0 = \frac{\partial u}{\partial \nu}(\cdot,t) \quad \text{on } \Gamma\,,\ t > 0 \tag{215}$$

or hinged boundary conditions

$$u(\cdot,t) = 0 = \Delta u(\cdot,t) \quad \text{on } \Gamma\,,\ t > 0 \tag{216}$$

on u. Also in this case we can rewrite systems (213)-(214)-(215) and (213)-(214)-(216) with suitable initial conditions as system (112) (or (118), equivalently) introducing the operators

$$D(A_2) := \{v \in H^2(\Omega) : \left(\frac{\partial v}{\partial \nu} + \sigma v\right)(\cdot, t) = 0 \text{ on } \Gamma\,,\ t > 0\}\,,\ A_2 v = -\Delta v \tag{217}$$

and

$$D(A_1) := \{u \in H^4(\Omega) : u(\cdot, t) = 0 = \frac{\partial u}{\partial \nu}(\cdot, t) \text{ on } \Gamma\,,\ t > 0\}\,,\ A_1 u = \Delta^2 u \tag{218}$$

or

$$D(A_1) := \{u \in H^4(\Omega) : u(\cdot, t) = 0 = \Delta u(\cdot, t) \text{ on } \Gamma\,,\ t > 0\}\,,\ A_1 u = \Delta^2 u\,, \tag{219}$$

and defining the operator $(\mathcal{A}, D(\mathcal{A}))$ as in (117).

Moreover, for every $U \in \mathcal{H} = H_0^2(\Omega) \times L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, the energy associated to systems (213)-(214)-(215) or (213)-(214)-(216) is

$$\begin{aligned} \mathcal{E}(U(t)) = \frac{1}{2}\int_\Omega \left[u_t^2 + |\Delta u|^2 + v_t^2 + |\nabla v|^2 + 2\alpha\mathcal{R}(u\bar{v})\right] dx \\ + \frac{1}{2}\int_\Gamma \sigma v^2 d\Sigma = \frac{1}{2}|U|^2_{\mathcal{H}}\,, \end{aligned} \tag{220}$$

which, for every $U \in D(\mathcal{A})$, still satisfies

$$\frac{d}{dt}\mathcal{E}(U(t)) = -\int_\Omega \beta u_t^2 dx = \mathcal{R}(\mathcal{A}U(t), U(t))_{\mathcal{H}}\,. \tag{221}$$

But, while trying to get an estimate of the norm of the resolvent operator along the imaginary axis of the type

$$\exists \gamma > 0 \quad \text{s.t.} \quad \|(ibI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|b|^\gamma \quad \forall b \in \mathbb{R}\,,$$

we get stucked in controlling the norm

$$\left\|\frac{\partial \Delta u}{\partial \nu}\right\|_{L^2(\Gamma)} \qquad \forall u \in D(A_1)\,, \tag{222}$$

preventing to achieve the desired estimate. Thus, the indirect polynomial stabilization for systems (213)-(214)-(215) and (213)-(214)-(216) is still an open problem. At least two different approaches might be pursued: perform a better estimate of the term (222) to solve this specific problem, or, more in general, look for new compatibility conditions between operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ in order to reach polynomial stabilization with a certain decay rate related to the compatibility condition, as has been done in Chapter 1.

Moreover, observe that the structure of the proof of the resolvent estimate in Sections 2.3.1 - 2.3.2 - 2.3.3 is similar. Therefore, having improved (thanks to the resolvent estimate) the decay rate established by Theorem 1.10, we expect that for these systems a compatibility condition sharper than (32) shall hold, ensuring the decay rate 1/2.

3 – Indirect stabilization for two wave equations with boundary coupling

The present chapter is extracted from the preprint *Indirect stabilization of two wave equations with boundary coupling*, in collaboration with Fatiha Alabau-Boussouira and Piermarco Cannarsa, in progress.

3.1 – Introduction

From the point of view of applications to mechanical vibrating systems, it appears natural to consider two membranes (or two plates, or one membrane and one plate) connected at the boundary only, or possibly on a subdomain of it. In this case, we shall exploit the transmission of information between the two components, that occurs only at the boundary, and nowhere else inside the domain. As we will show below, this feature has major consequences.

More precisely, let Ω be a nonempty bounded open subset of $\mathbb{R}^d$, $d \geq 1$, with sufficiently smooth boundary Γ.

Notation 3.1. Let Γ_0 and Γ_1 be open subsets of Γ such that

$$\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \,, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset \,.$$

Suppose the existence of a point $x_0 \in \mathbb{R}^d$ such that $(x - x_0).\, \nu \leq 0$ for every $x \in \Gamma_0$ and $(x - x_0).\, \nu \geq m_0 > 0$ for every $x \in \Gamma_1$, where ν stands for the outward unit vector at the boundary Γ of Ω, and set $m(x) = x - x_0$ for every $x \in \mathbb{R}^d$, $R = R(x_0) = \sup_{x\in\Omega} |m(x)|$.

With the same notations as in Notation 1.22, we recall the well-known Rellich's identity, that holds for every $u \in H^2(\Omega)$,

$$\begin{aligned} 2\mathcal{R}\int_\Omega \Delta u(m.\, \nabla u)dx = (d-2)\int_\Omega |\nabla u|^2 dx \\ + 2\mathcal{R}\int_\Gamma \frac{\partial u}{\partial \nu}(m.\, \nabla u)d\Sigma - \int_\Gamma (m.\, \nu)|\nabla u|^2 d\Sigma \,, \end{aligned} \tag{223}$$

where $\mathcal{R}(z)$ denotes the real part of $z \in \mathbb{C}$. We will consider the following system of hyperbolic equations with mixed boundary conditions

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}\,, \\ \partial_t^2 v - \Delta v = 0 & \text{in } \Omega \times \mathbb{R}\,, \\ \dfrac{\partial u}{\partial \nu} + \sigma_1 u + a\partial_t u + \alpha v = 0 & \text{on } \Gamma_1 \times \mathbb{R}\,, \\ \dfrac{\partial v}{\partial \nu} + \sigma_2 v + \alpha u = 0 & \text{on } \Gamma_1 \times \mathbb{R}\,, \\ u = 0 = v & \text{on } \Gamma_0 \times \mathbb{R}\,, \end{cases} \tag{224}$$

where $\alpha, a, \sigma_1, \sigma_2 \in L^\infty(\Gamma_1)$ and $\alpha \geq \overline{\alpha}$, $a(x) \geq \overline{a}$, $\sigma_i(x) \geq \overline{\sigma}_i$ for some positive constant $\overline{\alpha}$, $\overline{a}$ and $\overline{\sigma}_i$ $(i = 1, 2)$. For simplicity, in the following we will consider $\alpha, a, \sigma_1, \sigma_2$ positive constants.

Let us first note that system (224) do not fit into the abstract system of evolution equation (6) introduced in Chapter 1. Indeed, in system (224) the damping operator is no longer bounded in H, thus it does not satisfy hypothesis (H2). Most important, the coupling operator is no compact anymore in the energy space and, as a matter of fact, it is not even bounded in H. For this reason, in the next section we introduce a suitable abstract setting for this model.

3.2 – Abstract setting and well-posedness

Following [106] and [4], we consider V_1, V_2, H separable real Hilbert spaces, such that the embeddings $V_i \subset H$ are dense, compact and continuous, for $i = 1, 2$.

We identify H with its dual space, so that the inclusions $V_i \subset H \subset V_i'$ are continuous, dense and compact, $i = 1, 2$. We denote by $(\;,\;)_{V_i}$ and $(\;,\;)_H$ (respectively $| \;|_{V_i}$ and $| \;|_H$) the scalar products (resp. norms) on V_i, $i = 1, 2$, and H. The symbol $\langle \;,\; \rangle_{V_i',V_i}$ stands for the duality product between V_i and V_i', whereas A_i is the duality map from V_i to V_i' defined by

$$\langle A_i u, v\rangle_{V_i',V_i} = (u, v)_{V_i} \qquad \forall u, v \in V_i\,,\ i = 1,\, 2\,. \tag{225}$$

Thanks to the Riesz-Frćhet rapresentation theorem, each operator A_i is an isometric isomorphism of V_i onto V_i' $(i = 1, 2)$. Moreover, relation (225) ensures that

$$(A_i u, v)_H = (u, v)_{V_i} \qquad \forall u \in V_i \ \ s.t.\ \ A_i u \in H, \forall v \in V_i\,,\ i = 1,\, 2\,, \tag{226}$$

that, in turn, implies that

$$(A_i^{-1} u, v)_{V_i} = (u, v)_H \qquad \forall u, v \in V_i\,,\ i = 1,\, 2\,. \tag{227}$$

Relation (226) leads us to introduce the domain

$$D(A_i) := \{u \in V_i : A_i u \in H\} \subset V_i \qquad (i = 1, 2)$$

and, by abuse of notation, we denote by the same symbol A_i the operator $A_i : D(A_i) \subset H \to H$.

Let B be a linear continuous operator from V_1 to V_1', which satisfies

$$\langle Bu, u\rangle_{V_1',V_1} \geq 0\,, \quad \langle Bu, z\rangle_{V_1',V_1} = \langle Bz, u\rangle_{V_1',V_1} \quad \forall u,\, z \in V_1\,.$$

We moreover consider a nonzero parameter $\alpha \in \mathbb{R}$ and two linear continuous operators $P_1 : V_2 \to V_1'$ and $P_2 : V_1 \to V_2'$ such that

$$\langle P_1 u_2, u_1\rangle_{V_1',V_1} = \langle P_2 u_1, u_2\rangle_{V_2',V_2} \quad \forall u_1 \in V_1,\, u_2 \in V_2\,.$$

We are interested in studying the stabilization properties of the following weakly coupled system of evolution equations

$$\begin{cases} u_1'' + A_1 u_1 + B u_1' + \alpha P_1 u_2 = 0 & \text{in } V_1'\,, \\ u_2'' + A_2 u_2 + \alpha P_2 u_1 = 0 & \text{in } V_2'\,, \\ (u_1, u_1')(0) = (u_1^0, u_1^1) \in V_1 \times H\,, \\ (u_2, u_2')(0) = (u_2^0, u_2^1) \in V_2 \times H\,. \end{cases} \tag{228}$$

We further assume the existence of a subspace $V_0 \subset V_i$ which is closed with respect to the norm induced by $(\ ,\ )_{V_i}$, $i = 1, 2$, and dense in $(H, (\ ,\ )_H)$. For any $i = 1, 2$, denoting by π_i the canonical injection from V_0 to V_i and by Π_i the projection from V_i to V_0, we can characterize Π_i by

$$\begin{cases} \langle A_i \pi_i(\Pi_i u_i), \pi_i(\phi)\rangle_{V_1',V_1} = \langle A_i u_i, \pi_i(\phi)\rangle_{V_1',V_1} \quad \forall \phi \in V_0\,,\ u_i \in V_i\,, \\ \Pi_i u_i \in V_0\,. \end{cases} \tag{229}$$

Moreover, we ask that for every $\phi \in V_0\,,\ u_i \in V_i\,,\ i = 1, 2\,,$

$$\langle B u_1, \pi_1(\phi)\rangle_{V_1',V_1} = \langle P_1 u_2, \pi_1(\phi)\rangle_{V_1',V_1} = \langle P_2 u_1, \pi_2(\phi)\rangle_{V_2',V_2} = 0\,. \tag{230}$$

We set $V = V_1 \times V_2$ equipped with the scalar product

$$(u, \tilde{u})_V = (u_1, \tilde{u}_1)_{V_1} + (u_2, \tilde{u}_2)_{V_2} \quad \forall u = (u_1, u_2),\ \tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in V$$

and with the corresponding norm $\|\ \|_V$. The embeddings $V \subset H \times H \subset V'$ are continuous, dense and compact. Define a linear continuous operator $A : V \to V'$ by

$$Au = (A_1 u_1 + \alpha P_1 u_2, A_2 u_2 + \alpha P_2 u_1) \quad \forall u = (u_1, u_2) \in V,$$

and consider the bilinear continuous form on V

$$(u, \tilde{u})_\alpha = (u, \tilde{u})_V + \alpha \langle P_1 u_2, \tilde{u}_1\rangle_{V_1',V_1} + \alpha \langle P_2 u_1, \tilde{u}_2\rangle_{V_2',V_2}$$

for all $u = (u_1, u_2)$, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in V$.

Proposition 3.2. *There exists $\alpha_0 > 0$ such that for all $0 \le |\alpha| < \alpha_0$ there exist two positive constants $\nu_1(\alpha), \nu_2(\alpha)$ such that*

$$\nu_1(\alpha)\|u\|_V \le (u,u)_\alpha^{1/2} \le \nu_2(\alpha)\|u\|_V \quad \forall u \in V. \tag{231}$$

Hence, for all $0 \le |\alpha| < \alpha_0$, the application

$$u \in V \longmapsto \|u\|_\alpha := (u,u)_\alpha^{1/2}$$

defines a norm on V equivalent to $\| \ \|_V$. Moreover, for all $0 \le |\alpha| < \alpha_0$, the operator A is the duality map from $(V, \| \ \|_\alpha)$ to V'.

Proof. A straightforward calculation yields the estimates

$$(1 - |\alpha|\|P_1\|_{\mathcal{L}})\|u\|_V^2 \le |(u,u)_\alpha| \le (1 + |\alpha|\|P_1\|_{\mathcal{L}})\|u\|_V^2 \,,$$

where $\|P_1\|_{\mathcal{L}} = \|P_1\|_{\mathcal{L}(V_2,V_1')}$ is the operator norm of P_1 from V_2 to V_1'. Hence, fixing $\alpha_0 = \|P_1\|_{\mathcal{L}}^{-1}$, the conditions in (231) hold for every $0 \le |\alpha| < \alpha_0$ with $\nu_1(\alpha) = \sqrt{1 - |\alpha|\|P_1\|_{\mathcal{L}}}$ and $\nu_2(\alpha) = \sqrt{1 + |\alpha|\|P_1\|_{\mathcal{L}}}$.

Moreover, for every $u = (u_1, u_2)$, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in V$, we have

$$\begin{aligned}\langle Au, \tilde{u}\rangle_{V',V} = \langle (A_1u_1 + \alpha P_1 u_2, A_2 u_2 + \alpha P_2 u_1), (\tilde{u}_1, \tilde{u}_2)\rangle_{V',V} \\ (u,\tilde{u})_V + \alpha\langle P_1 u_2, \tilde{u}_1\rangle_{V_1',V_1} + \alpha\langle P_2 u_1, \tilde{u}_2\rangle_{V_2',V_2} = (u,\tilde{u})_\alpha \,,\end{aligned}$$

thus A is the duality map between V endowed with the scalar product $(\ , \ )_\alpha$ and V'. □

We introduce the space $\mathcal{H} = V_1 \times H \times V_2 \times H$, equipped with the bilinear form

$$(U, \tilde{U})_{\mathcal{H}} = ((u_1,u_2),(v_1,v_2))_\alpha + ((p_1,p_2),(q_1,q_2))_{H\times H} \tag{232}$$

for every $U = (u_1, p_1, u_2, p_2)$, $\tilde{U} = (v_1, q_1, v_2, q_2) \in \mathcal{H}$. Then, for every $0 \le |\alpha| < \alpha_0$, the application $(\ , \ )_{\mathcal{H}}$ is a scalar product on $\mathcal{H}$ that satisfies

$$\nu_1(\alpha)(\|(u_1,u_2)\|_V^2 + \|(p_1,p_2)\|_{H\times H}^2) \le (U,U)_{\mathcal{H}} \le \nu_2(\alpha)(\|(u_1,u_2)\|_V^2 + \|(p_1,p_2)\|_{H\times H}^2)$$

for every $U \in \mathcal{H}$, with related norm $\|U\|_{\mathcal{H}} := (\|(u_1,u_2)\|_\alpha^2 + \|(p_1,p_2)\|_{H\times H}^2)^{1/2}$, and $(\mathcal{H}, (\ , \ )_{\mathcal{H}})$ is a Hilbert space. We consider the unbounded linear operator $\mathcal{A}$ on $\mathcal{H}$ defined by

$$\begin{aligned} D(\mathcal{A}) &= \{U = (u_1,p_1,u_2,p_2) \in (V_1 \times V_2)^2 : A(u_1,u_2) + (Bp_1, 0) \in H \times H\}\,, \\ \mathcal{A}U &= (p_1, -A_1u_1 - Bp_1 - \alpha P_1 u_2, p_2, -A_2 u_2 - \alpha P_2 u_1) \quad \forall U \in D(\mathcal{A})\,. \end{aligned} \tag{233}$$

We can now reformulate the system (228) as the abstract first order equation

$$\begin{cases} U'(t) = \mathcal{A}U(t)\,, \\ U(0) = U_0 \in \mathcal{H} \end{cases} \tag{234}$$

and deduce a well-posedness result by standard semigroup theory.

Proposition 3.3. *For every $0 \le |\alpha| < \alpha_0$ the operator $\mathcal{A}$ is maximal dissipative on $\mathcal{H}$. Thus, for every $U_0 \in \mathcal{H}$, problem* (234) *admits a unique solution $U \in C([0,+\infty),\mathcal{H})$. In addition, if $U_0 \in D(\mathcal{A}^k)$ for some $k \in \mathbb{N}$, then $U \in C^{k-j}([0,+\infty),D(\mathcal{A}^j))$ for all $j = 0,\dots,k$. Moreover, the energy of the solution*

$$\mathcal{E}(U(t)) := \frac{1}{2}\|U(t)\|^2_{\mathcal{H}} \tag{235}$$

is locally absolutely continuous and, for every $U_0 \in D(\mathcal{A})$, it satisfies the dissipation relation

$$\mathcal{E}'(U(t)) = -\langle Bp_1, p_1\rangle_{V_1',V_1}\,.$$

Proof. Note that for every $U = (u_1,p_1,u_2,p_2) \in D(\mathcal{A})$

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{H}} &= ((p_1,p_2),(u_1,u_2))_\alpha - (A(u_1,u_2) + (Bp_1,0),(p_1,p_2))_{H\times H}\\ &= -\langle Bp_1,p_1\rangle_{V_1',V_1} \le 0\,,\end{aligned}$$

thus $\mathcal{A}$ is a dissipative operator. We now prove that $\mathcal{I} - \mathcal{A}$ is onto $\mathcal{H}$. For this purpose, fix $\widehat{U} = (\widehat{u}_1,\widehat{p}_1,\widehat{u}_2,\widehat{p}_2) \in \mathcal{H}$. Denoting $U = (u_1,p_1,u_2,p_2) \in D(\mathcal{A})$, the identity $(\mathcal{I}-\mathcal{A})U = \widehat{U}$ reduces to the system

$$\begin{cases}(I + A_1 + B)u_1 + \alpha P_1 u_2 = f \in V_1'\\ (I + A_2)u_2 + \alpha P_2 u_1 = g \in V_2'\,,\end{cases} \tag{236}$$

that, thanks to the Lax-Milgram theorem, admits a unique solution $(\bar{u}_1,\bar{u}_2)$ that satisfies

$$\|(\bar{u}_1,\bar{u}_2)\|_V \le \frac{1}{1-|\alpha|\|P_1\|}\|(f,g)\|_{V'}\,.$$

Indeed, consider the bilinear form $a : V\times V \to \mathbb{R}$ defined by

$$\begin{aligned}&a((u_1,u_2),(\varphi_1,\varphi_2)) =\\ &\qquad \langle (I+A_1+B)u_1 + \alpha P_1u_2,\varphi_1\rangle_{V_1',V_1} + \langle (I+A_2)u_2 + \alpha P_2 u_1,\varphi_2\rangle_{V_2',V_2}\end{aligned}$$

for every (u_1,u_2), $(\varphi_1,\varphi_2) \in V$. Then

i) a is continuous, that is, there exists a constant $C > 0$ such that

$$|a((u_1,u_2),(\varphi_1,\varphi_2))| \le C\|(u_1,u_2)\|_V\|(\varphi_1,\varphi_2)\|_V$$

for every (u_1,u_2), $(\varphi_1,\varphi_2) \in V$;

ii) a is coercive, since

$$a((u_1,u_2),(u_1,u_2)) \ge (1-|\alpha|\|P_1\|)\|(u_1,u_2)\|_V$$

for every $(u_1,u_2) \in V$.

Then, by the Lumer-Phillips theorem (for instance, see [130]), we conclude that the problem (234) admits an unique solution. Moreover, a direct calculation shows that

$$\mathcal{E}'(U(t)) = (\mathcal{A}U,U)_{\mathcal{H}} = -\langle Bp_1,p_1\rangle_{V_1',V_1}\,. \qquad \square$$

3.3 – Two wave equations with boundary coupling

Let assume Notations 1.22 and 3.1. We consider the system of two wave equations with mixed boundary conditions and coupling acting at the part Γ_1 of the boundary

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}\,, \\ \partial_t^2 v - \Delta v = 0 & \text{in } \Omega \times \mathbb{R}\,, \\ \dfrac{\partial u}{\partial \nu} + \sigma_1 u + a\partial_t u + \alpha v = 0 & \text{on } \Gamma_1 \times \mathbb{R}\,, \\ \dfrac{\partial v}{\partial \nu} + \sigma_2 v + \alpha u = 0 & \text{on } \Gamma_1 \times \mathbb{R}\,, \\ u = 0 = v & \text{on } \Gamma_0 \times \mathbb{R}\,, \end{cases} \tag{237}$$

where $\alpha, a, \sigma_1, \sigma_2$ are positive constants.

The total energy $\mathcal{E}(t)$ associated with system (237) is defined by

$$\frac{1}{2}\int_\Omega \left(|u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2\right) dx + \frac{1}{2}\int_{\Gamma_1} (\sigma_1 u^2 + \sigma_2 v^2 + 2\alpha uv) d\Sigma\,. \tag{238}$$

We set $H = (L^2(\Omega), \|\ \|_{L^2(\Omega)})$, $V_0 = (H_0^1(\Omega), \|\ \|_{H_0^1(\Omega)})$ and, for every $i = 1, 2$, $V_i = H^1_{\Gamma_0}(\Omega)$ endowed with the scalar products

$$(u, v)_{V_i} = \int_\Omega \nabla u . \nabla v dx + \int_{\Gamma_1} \sigma_i uv d\Sigma \qquad \forall u,\ v \in H^1_{\Gamma_0}(\Omega)$$

and with the corresponding norms. We define the duality maps A_1 and A_2 as in (225).

Remark 3.4. Note that, for every $w \in \mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$ and $u \in V_1$, we have

$$\langle A_1 u, w\rangle_{V_1', V_1} = \int_\Omega \nabla u . \nabla w dx = \int_\Omega -\Delta u w dx = \langle -\Delta u, w\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}\,,$$

that yields $A_1 u = -\Delta u$ in $\mathcal{D}'(\Omega)$. Thanks to relation (226), the same reasoning ensures that, if $u \in D(A_1)$, then for every $w \in \mathcal{D}(\Omega)$

$$(A_1 u, w)_H = \int_\Omega -\Delta u w dx = \langle -\Delta u, w\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}\,,$$

thus $A_1 u = -\Delta u \in H$ by density of $\mathcal{D}(\Omega)$ in H.

Moreover, we introduce a linear continuous operator $B : V_1 \to V_1'$ defined by

$$\langle Bu, v\rangle_{V_1', V_1} = \int_{\Gamma_1} a uv d\Sigma \qquad \forall u,\ v \in V_1$$

and two linear continuous operators $P_1 : V_2 \to V_1'$ and $P_2 : V_1 \to V_2'$ as

$$\langle P_2 u, v\rangle_{V_2', V_2} = \int_{\Gamma_1} uv d\Sigma = \langle P_1 v, u\rangle_{V_1', V_1} \qquad \forall u \in V_1,\, v \in V_2\,.$$

Hence, system (237) with initial conditions

$$\begin{cases} u(0) = u^0 \in H^1_{\Gamma_0}(\Omega)\,, & u'(0) = u^1 \in L^2(\Omega)\,, \\ v(0) = v^0 \in H^1_{\Gamma_0}(\Omega)\,, & v'(0) = v^1 \in L^2(\Omega)\,, \end{cases} \tag{239}$$

can be read as system (234). Thus, fixed any $U_0 = (u^0, u^1, v^0, v^1) \in \mathcal{H}$, system (237)-(239) admits a unique solution, provided $0 \le |\alpha| < \alpha_0 = \|P_1\|_{\mathcal{L}}$. We can further characterize the domain $D(\mathcal{A})$ as follows.

Proposition 3.5. *Let us consider*

$$\begin{aligned} W := \Big\{ &(u,p,v,q) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \times H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) : \\ &\frac{\partial u}{\partial \nu} + \sigma_1 u + a\partial_t u + \alpha v = 0 = \frac{\partial v}{\partial \nu} + \sigma_2 v + \alpha u \ \textit{on}\ \Gamma_1 \times \mathbb{R} \Big\}\,. \end{aligned} \tag{240}$$

Then $D(\mathcal{A}) = W$.

Proof. We first show that $D(\mathcal{A}) \subset W$. Indeed, let $U = (u,p,v,q) \in D(\mathcal{A})$, then we have $\varphi = A_1 u + Bp + \alpha P_1 v \in H$ and $\psi = A_2 v + \alpha P_2 u \in H$. So, for every $w \in \mathcal{D}(\Omega) \subset V_0$,

$$(\varphi, w)_H = \langle \varphi, w\rangle_{V_1', V_1} = \int_\Omega \nabla u . \nabla w dx = \int_\Omega -\Delta u w dx\,, \tag{241}$$

where the last identity holds in the sense of distributions. Thus (see also Remark 3.4), $A_1 u = -\Delta u = \varphi$ in $\mathcal{D}'(\Omega)$, and $\varphi \in L^2(\Omega)$. By density of $\mathcal{D}(\Omega)$ in H, relation (241) yields

$$A_1 u = -\Delta u = \varphi \quad \text{in } H\,, \tag{242}$$

that is, $u \in D(A_1)$. Moreover, for every $w \in D(A_1)$,

$$\begin{aligned} (\varphi, w)_H &= \int_\Omega \nabla u . \nabla w dx + \int_{\Gamma_1} (\sigma_1 u + ap + \alpha v) w d\Sigma \\ &= \int_\Omega -\Delta u w dx + \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu} + \sigma_1 u + ap + \alpha v \right) w d\Sigma\,, \end{aligned}$$

that, thanks to (242), the arbitrariness of $w \in D(A_1)$ and the density of $D(A_1)$ in V_1, implies

$$\frac{\partial u}{\partial \nu} + \sigma_1 u + ap + \alpha v = 0 \quad \text{on } \Gamma_1\,. \tag{243}$$

Finally, since $\sigma_1 u + ap + \alpha v \in H^{1/2}(\Gamma_1)$, by elliptic regularity we can conclude that $u \in H^2(\Omega)$, so that $D(A_1) = H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$. By the same reasoning for $\psi \in H$, we can show that $v \in D(A_2) = H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and satisfies

$$\frac{\partial v}{\partial \nu} + \sigma_2 v + \alpha u = 0 \quad \text{on } \Gamma_1 \,, \tag{244}$$

that completes the proof of the first inclusion.

On the other hand, chosen $U = (u, p, v, q) \in W$, we only need to show that $\varphi = A_1 u + Bp + \alpha P_1 v \in H$ and $\psi = A_2 v + \alpha P_2 u \in H$.

Since $A_1 u + Bp + \alpha P_1 v \in V_1'$, for every $w \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$

$$\begin{aligned} \langle A_1 u + Bp + \alpha P_1 v, w \rangle_{V_1', V_1} &= \int_\Omega \nabla u . \nabla w dx + \int_{\Gamma_1} (\sigma_1 u w + a p w + \alpha v w) d\Sigma \\ &= \int_\Omega -\Delta u w dx = (-\Delta u, w)_H \,. \end{aligned}$$

This relation, together with the density of $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ in H, ensures that $A_1 u + Bp + \alpha P_1 v \in H$. In a similar way, considering $w \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, we prove that $\psi = A_2 v + \alpha P_2 u \in H$. □

In the next section we focus on the indirect stabilization problem for system (237).

3.3.1 – Indirect stabilization

Main result We first remark that, even though the feedback acts on only one component of system (237), for every $U_0 \in D(\mathcal{A})$, the total energy of the system defined in (235) satisfies the dissipation relation

$$\frac{d}{dt} \left(\mathcal{E}(U(t)) \right) = - \int_{\Gamma_1} a |u_t|^2 d\Sigma \,. \tag{245}$$

For this reason it is reasonable to expect stabilization for system (237), with a proper decay rate, as shown in the following result.

THEOREM 3.6. *Let α be a real number such that $0 < |\alpha| < \alpha_0$.*

i) If $U_0 \in D(\mathcal{A}^m)$ for some $m \geq 1$, then there exists $C > 0$ such that

$$\mathcal{E}(U(t)) \leq \frac{C}{(1+t)^m} \sum_{k=0}^{m} \mathcal{E}(U^{(k)}(0)) \quad \forall t > 0 \,. \tag{246}$$

ii) For every $U_0 \in \mathcal{H}$

$$\lim_{t \to \infty} \mathcal{E}(U(t)) = 0 \,, \tag{247}$$

that is, the semigroup $e^{t\mathcal{A}}$ generated by the operator $\mathcal{A}$ is strongly stable.

Proof of Theorem 3.6 In view of the abstract Lemma 1.8, we look for an estimate of the integral in time of the total energy $\mathcal{E}(U(t))$. In the following we denote by c and $C(\alpha)$ two generic positive constants, with $C(\alpha)$ depending on α.

We will proceed in three steps. We first address the problem of the transmission of information between u (the damped component) and v (the undamped one) on the part Γ_1 of the boundary.

LEMMA 3.7. *Let $0 < |\alpha| < \alpha_0$. For every $U_0 \in D(\mathcal{A})$ holds*

$$\frac{|\alpha|}{2}\int_S^T\int_{\Gamma_1} v^2 d\Sigma dt \le C(\alpha)\int_S^T\int_{\Gamma_1}\left(u^2+u_t^2\right)d\Sigma dt + C(\alpha)\mathcal{E}(U(S)), \tag{248}$$

for some constant $C(\alpha) > 0$.

PROOF. Multiplying the first equation of (237) for v, respectively the second for u, and integrating over $\Omega \times [S,T]$, for every $0 \le S < T$, we obtain

$$\begin{aligned} 0 &= \int_S^T\int_\Omega (u_{tt}-\Delta u)v dxdt = [\langle u_t, v\rangle_{L^2}]_S^T - \int_S^T\int_\Omega u_t v_t dxdt \\ &\quad - \int_S^T\int_{\Gamma_1}\frac{\partial u}{\partial\nu} v d\Sigma dt + \int_S^T\int_\Omega \nabla u.\nabla v dxdt\,, \\ 0 &= \int_S^T\int_\Omega (v_{tt}-\Delta v)u dxdt = [\langle v_t, u\rangle_{L^2}]_S^T - \int_S^T\int_\Omega u_t v_t dxdt \\ &\quad - \int_S^T\int_{\Gamma_1}\frac{\partial v}{\partial\nu} u d\Sigma dt + \int_S^T\int_\Omega \nabla u.\nabla v dxdt\,. \end{aligned}$$

Using the boundary conditions in (237) and subtracting the two equations we have

$$\int_S^T\int_{\Gamma_1}[(\sigma_2 v+\alpha u)u - (\sigma_1 u + a u_t + \alpha v)v]\,d\Sigma dt + [\langle v_t, u\rangle_{L^2} - \langle u_t, v\rangle_{L^2}]_S^T = 0\,,$$

and so we deduce that

$$\begin{aligned} \alpha\int_S^T\int_{\Gamma_1} v^2 d\Sigma dt &= [\langle v_t, u\rangle_{L^2} - \langle u_t, v\rangle_{L^2}]_S^T \\ &\quad + \int_S^T\int_{\Gamma_1}\left[(\sigma_2-\sigma_1)uv + \alpha u^2 - a u_t v\right]d\Sigma dt\,. \end{aligned} \tag{249}$$

We now estimate each term in the right-hand side. Owing to (245) the total energy $\mathcal{E}(U(t))$ is decreasing, so by Young's inequality we have

$$\begin{aligned} &\left|[\langle v_t, u\rangle_{L^2} - \langle u_t, v\rangle_{L^2}]_S^T\right| \\ &\le \frac{1}{2}\Big(|v_t(T)|^2_{L^2(\Omega)} + |u(T)|^2_{L^2(\Omega)} + |u_t(T)|^2_{L^2(\Omega)} + |v(T)|^2_{L^2(\Omega)} + |v_t(S)|^2_{L^2(\Omega)} \\ &\qquad + |u(S)|^2_{L^2(\Omega)} + |u_t(S)|^2_{L^2(\Omega)} + |v(S)|^2_{L^2(\Omega)}\Big) \le C(\alpha)\mathcal{E}(U(S))\,. \end{aligned} \tag{250}$$

Thanks to Young's inequalities with appropriate constants,

$$
\begin{aligned}
&\left|\int_S^T \int_{\Gamma_1} \left[(\sigma_2-\sigma_1)uv + \alpha u^2 - a u_t v\right] d\Sigma dt\right| \\
&\leq \int_S^T \int_{\Gamma_1} \left[\frac{1}{|\alpha|}(\sigma_2-\sigma_1)^2 u^2 + \frac{|\alpha|}{4} v^2 + |\alpha| u^2 + \frac{1}{|\alpha|} a^2 u_t^2 + \frac{|\alpha|}{4} v^2\right] d\Sigma dt \qquad (251)\\
&\leq \frac{|\alpha|}{2} \int_S^T \int_{\Gamma_1} v^2 d\Sigma dt + K_\alpha \int_S^T \int_{\Gamma_1} \left(u^2 + u_t^2\right) d\Sigma dt\,,
\end{aligned}
$$

where $K_\alpha := \max\left(\frac{1}{|\alpha|}(\sigma_2-\sigma_1)^2 + |\alpha|, \frac{1}{|\alpha|} a^2\right)$. From (249), (250) and (251) we conclude that

$$
\frac{|\alpha|}{2} \int_S^T \int_{\Gamma_1} v^2 d\Sigma dt \leq C(\alpha) \int_S^T \int_{\Gamma_1} \left(u^2 + u_t^2\right) d\Sigma dt + C(\alpha)\mathcal{E}(U(S))\,. \qquad (252)
$$

□

The second lemma gives a simultaneous estimate of u and v on the boundary of Ω.

Lemma 3.8. *Let $0 < |\alpha| < \min(\alpha_0, (\sigma_1\sigma_2)^{1/2})$. Then, for every $U_0 \in D(\mathcal{A})$ and for every $\varepsilon > 0$,*

$$
\begin{aligned}
\int_S^T \int_{\Gamma_1} (u^2+v^2) d\Sigma dt \leq c\mathcal{E}(U(S)) &+ \varepsilon \int_S^T \mathcal{E}(U(t)) dt \\
&+ \frac{c}{\varepsilon} \int_S^T \int_{\Gamma_1} (|u_t|^2 + |v_t|^2) d\Sigma dt\,. \qquad (253)
\end{aligned}
$$

Proof. Let $U_0 \in D(\mathcal{A})$ and $U(t)$ be the corresponding solution of (234). From Proposition 3.3, we have that $U(t) = (u(t), v(t), p(t), q(t)) \in D(\mathcal{A})$ for all $t \geq 0$. In particular, $u(t)$, $v(t) \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and $u(t)_{|\Gamma}$, $v(t)_{|\Gamma} \in H^{3/2}(\Gamma)$ for all $t \geq 0$. For any fixed time $t \geq 0$, we consider two functions z and w (depending on the parameter t) such that

$$
\begin{cases} \Delta z = 0 & \text{in } \Omega\,, \\ z = u(t) & \text{on } \Gamma\,, \end{cases} \qquad \begin{cases} \Delta w = 0 & \text{in } \Omega\,, \\ w = v(t) & \text{on } \Gamma\,. \end{cases} \qquad (254)
$$

Thus z, $w \in H^2(\Omega)$ and moreover

$$
\begin{aligned}
\int_\Omega \nabla z . \nabla(u(t) - z) dx &= -\int_\Omega (u(t) - z)\Delta z dx + \int_\Gamma (u(t) - z)\frac{\partial z}{\partial \nu} d\Sigma = 0\,, \\
\int_\Omega \nabla w . \nabla(v(t) - w) dx &= -\int_\Omega (v(t) - w)\Delta w dx + \int_\Gamma (v(t) - w)\frac{\partial w}{\partial \nu} d\Sigma = 0\,,
\end{aligned}
$$

so

$$\int_\Omega \nabla z.\nabla u(t)dx = \int_\Omega |\nabla z|^2 dx \geq 0\,, \quad \int_\Omega \nabla w.\nabla v(t)dx = \int_\Omega |\nabla w|^2 dx \geq 0\,. \tag{255}$$

Moreover, since u_t, $v_t \in H^1_{\Gamma_0}(\Omega)$, then $u_t(t)_{|\Gamma}$, $v_t(t)_{|\Gamma} \in H^{1/2}(\Gamma)$ for all $t \geq 0$, so z_t, w_t are weak solutions of systems

$$\begin{cases} \Delta z_t = 0 & \text{in } \Omega\,, \\ z_t = u_t(t) & \text{on } \Gamma\,, \end{cases} \qquad \begin{cases} \Delta w_t = 0 & \text{in } \Omega\,, \\ w_t = v_t(t) & \text{on } \Gamma\,, \end{cases} \tag{256}$$

respectively. By elliptic regularity theory, and since $\sigma_i > 0$, $i = 1, 2$, we have that

$$\begin{aligned} &\int_\Omega z^2 dx \leq c\int_{\Gamma_1} |u(t)|^2 d\Sigma \leq C(\alpha)\mathcal{E}(U(t))\,, \\ &\int_\Omega w^2 dx \leq c\int_{\Gamma_1} |v(t)|^2 d\Sigma \leq C(\alpha)\mathcal{E}(U(t))\,, \end{aligned} \tag{257}$$

and

$$\int_\Omega |z_t|^2 dx \leq c\int_{\Gamma_1} |u_t(t)|^2 d\Sigma\,, \quad \int_\Omega |w_t|^2 dx \leq c\int_{\Gamma_1} |v_t(t)|^2 d\Sigma\,. \tag{258}$$

We multiply the first two equations in (237) for z and for w, respectively, and integrate over $[S,T]\times\Omega$, for every $0 \leq S < T$,

$$\begin{aligned} 0 = \int_S^T \int_\Omega (u_{tt} - \Delta u) z dx dt &= [\langle u_t, z\rangle_{L^2}]_S^T - \int_S^T \int_\Omega u_t z_t dx dt \\ &- \int_S^T \int_\Gamma \frac{\partial u}{\partial \nu} z d\Sigma dt + \int_S^T \int_\Omega \nabla u.\nabla z dx dt\,, \end{aligned} \tag{259}$$

$$\begin{aligned} 0 = \int_S^T \int_\Omega (v_{tt} - \Delta v) w dx dt &= [\langle v_t, w\rangle_{L^2}]_S^T - \int_S^T \int_\Omega v_t w_t dx dt \\ &- \int_S^T \int_\Gamma \frac{\partial v}{\partial \nu} w d\Sigma dt + \int_S^T \int_\Omega \nabla v.\nabla w dx dt\,. \end{aligned} \tag{260}$$

Adding identities (259) and (260), thanks to the boundary conditions in (237) and relations (255), we obtain

$$\begin{aligned} \int_S^T \int_{\Gamma_1} (\sigma_1 u^2 + \sigma_2 v^2) d\Sigma dt &\leq \left[\int_\Omega (u_t z + v_t w) dx\right]_T^S \\ &+ \int_S^T \int_\Omega (u_t z_t + v_t w_t) dx dt - \int_S^T \int_{\Gamma_1} (a u u_t + 2\alpha u v) d\Sigma dt\,. \end{aligned} \tag{261}$$

We now estimate each term on the right-hand side of (261).

Thanks to estimates (257) and the dissipation relation (245) of the total energy, we have

$$\begin{aligned}
&\left|\left[\int_\Omega (u_t z + v_t w)dx\right]_T^S\right| \le \frac{1}{2}\int_\Omega \left(|u_t(S)|^2 + |z(S)|^2\right) dx \\
&\quad + \frac{1}{2}\int_\Omega \left(|v_t(S)|^2 + |w(S)|^2 + |u_t(T)|^2 + |z(T)|^2 + |v_t(T)|^2 + |w(T)|^2\right) dx \\
&\le C(\alpha)(\mathcal{E}(U(S)) + \mathcal{E}(U(T))) \le C(\alpha)\mathcal{E}(U(S))\,,
\end{aligned}$$

where $z(t)$ and $w(t)$ refer to the solutions of systems (254) associated at the fixed time t. Thanks to Young's inequality and relations (258), we deduce that for every $\varepsilon > 0$

$$\begin{aligned}
&\left|\int_S^T \int_\Omega (u_t z_t + v_t w_t) dx dt\right| \\
&\le \varepsilon \int_S^T \int_\Omega \left(|u_t|^2 + |v_t|^2\right) dx dt + \frac{c}{\varepsilon}\int_S^T \int_\Omega \left(|z_t|^2 + |w_t|^2\right) dx dt \\
&\le 2\varepsilon \int_S^T \mathcal{E}(U(t)) dt + \frac{c}{\varepsilon}\int_S^T \int_{\Gamma_1} \left(|u_t|^2 + |v_t|^2\right) d\Sigma dt\,.
\end{aligned}$$

Again thanks to Young's inequality, for every $\varepsilon_2 > 0$ we have

$$\left|\int_S^T \int_{\Gamma_1} a u u_t d\Sigma dt\right| \le \varepsilon_2 \int_S^T \int_{\Gamma_1} u^2 d\Sigma dt + \frac{c}{\varepsilon_2}\int_S^T \int_{\Gamma_1} |u_t|^2 d\Sigma dt\,.$$

Finally, since $|\alpha| < (\sigma_1\sigma_2)^{1/2}$,

$$\begin{aligned}
\left|2\int_S^T \int_{\Gamma_1} \alpha u v d\Sigma dt\right| &= \left|2\int_S^T \int_{\Gamma_1} \frac{(\sigma_1\alpha)^{1/2}}{(\sigma_1\sigma_2)^{1/4}} u \frac{(\sigma_2\alpha)^{1/2}}{(\sigma_1\sigma_2)^{1/4}} v d\Sigma dt\right| \\
&\le \int_S^T \int_{\Gamma_1} \left(\delta\sigma_1 u^2 + \delta\sigma_2 v^2\right) d\Sigma dt\,,
\end{aligned}$$

where $0 < \delta = |\alpha|/(\sigma_1\sigma_2)^{-1/2} < 1$. Back to equation (261), combining the last four inequalities ensures that for all ε, $\varepsilon_2 > 0$

$$\begin{aligned}
&\int_S^T \int_{\Gamma_1} ((1-\delta)\sigma_1 - \varepsilon_2)u^2 + (1-\delta)\sigma_2 v^2) d\Sigma dt \le C(\alpha)\mathcal{E}(U(S)) \\
&\qquad + \varepsilon \int_S^T \mathcal{E}(U(t)) dt + \left(\frac{c}{\epsilon} + \frac{c}{\epsilon_2}\right)\int_S^T \int_{\Gamma_1} (|u_t|^2 + |v_t|^2) d\Sigma dt\,.
\end{aligned}$$

Fix $\varepsilon_2 > 0$ such that $(1-\delta)\sigma_1 - \varepsilon_2 > 0$. Then relation (253) holds for every $\varepsilon > 0$. □

Remark 3.9. It shall be possible to compare the two quantities bounding $|\alpha|$, the hypothesis $|\alpha| < (\sigma_1\sigma_2)^{1/2}$ from Lemma 3.8 with the "well-posedness" hypothesis $0 \leq |\alpha| < \alpha_0 = \|P_1\|_{\mathcal{L}}^{-1}$. Indeed, $\|P_1\|_{\mathcal{L}}$ depends on the values of σ_1 and σ_2, since they appear in the norm of the spaces $(V_i, |\cdot|_{V_i})$, $i = 1, 2$.

The next lemma links the integral of the total energy on a time interval to the contributions of u, v and their derivatives on the part Γ_1 of the boundary Γ.

Lemma 3.10. *Let $0 < |\alpha| < \alpha_0$, for some positive constant α_0. Then, for every $U_0 \in D(\mathcal{A})$,*

$$\int_S^T \mathcal{E}(U(t))dt \leq c\mathcal{E}(U(S)) + c\int_S^T \int_{\Gamma_1} (u^2 + |u_t|^2 + v^2 + |v_t|^2) d\Sigma dt\,. \tag{262}$$

Proof. We now introduce the multiplier $Mu = m.\nabla u + \frac{1}{2}(d-1)u$. Then, we multiply the first equation of (237) for Mu, respectively the second for Mv, and integrate over $\Omega \times [S, T]$, for every $0 \leq S < T$. Integrations by parts and Rellich's identity (223) lead to

$$\frac{1}{2}\int_S^T \int_\Omega \left[|u_t|^2 + |\nabla u|^2\right] dxdt = [\langle u_t, Mu\rangle_{L^2}]_T^S + \int_S^T \int_{\Gamma_0} \frac{m.\nu}{2}\left|\frac{\partial u}{\partial \nu}\right|^2 d\Sigma dt$$
$$+ \int_S^T \int_{\Gamma_1} \left[\frac{m.\nu}{2}\left(|u_t|^2 - |\nabla u|^2\right) - (\sigma_1 u + a u_t + \alpha v)Mu\right] d\Sigma dt$$

and

$$\frac{1}{2}\int_S^T \int_\Omega \left[|v_t|^2 + |\nabla v|^2\right] dxdt = [\langle v_t, Mv\rangle_{L^2}]_T^S + \int_S^T \int_{\Gamma_0} \frac{m.\nu}{2}\left|\frac{\partial v}{\partial \nu}\right|^2 d\Sigma dt$$
$$+ \int_S^T \int_{\Gamma_1} \left[\frac{m.\nu}{2}\left(|v_t|^2 - |\nabla v|^2\right) - (\sigma_2 v + \alpha u)Mv\right] d\Sigma dt\,.$$

So, we deduce that

$$\begin{aligned}
\int_S^T \mathcal{E}(U(t))dt =& \frac{1}{2}\int_S^T \int_\Omega \left[|u_t|^2 + |\nabla u|^2 + |v_t|^2 + |\nabla v|^2\right] dxdt \\
&+ \frac{1}{2}\int_S^T \int_{\Gamma_1} \left[\sigma_1 u^2 + \sigma_2 v^2 + 2\alpha uv\right] d\Sigma dt \\
=& \left[\int_\Omega (u_t Mu + v_t Mv)\, dx\right]_T^S + \frac{1}{2}\int_S^T \int_{\Gamma_0} \left(\left|\frac{\partial u}{\partial \nu}\right|^2 + \left|\frac{\partial v}{\partial \nu}\right|^2\right) m.\nu d\Sigma dt \\
&+ \int_S^T \int_{\Gamma_1} \left[\frac{m.\nu}{2}\left(|u_t|^2 - |\nabla u|^2 + |v_t|^2 - |\nabla v|^2\right) + \alpha uv\right] d\Sigma dt \\
&+ \int_S^T \int_{\Gamma_1} \left[\frac{1}{2}(\sigma_1 u^2 + \sigma_2 v^2) - (\sigma_1 u + a u_t + \alpha v)Mu - (\sigma_2 v + \alpha u)Mv\right] d\Sigma dt
\end{aligned}$$

Since $m.\nu \geq m_0 > 0$ on Γ_1, we have that

$$\frac{m.\nu}{2}\left(|u_t|^2 - |\nabla u|^2 + |v_t|^2 - |\nabla v|^2\right) - (\sigma_1 u + a u_t + \alpha v)Mu + \frac{1}{2}\sigma_1 u^2 - (\sigma_2 v + \alpha u)Mv + \frac{1}{2}\sigma_2 v^2 + \alpha uv \leq c(u^2 + |u_t|^2 + v^2 + |v_t|^2) \quad \text{on } \Gamma_1 .$$

Moreover, the integral on Γ_0 is non-positive and

$$\left|\left[\int_\Omega (u_t Mu + v_t Mv)\, dx\right]_T^S\right| \leq C(\alpha)\mathcal{E}(U(S)) .$$

Then we conclude that

$$\int_S^T \mathcal{E}(U(t))dt \leq C(\alpha)\mathcal{E}(U(S)) + c\int_S^T \int_{\Gamma_1} (u^2 + |u_t|^2 + v^2 + |v_t|^2) d\Sigma dt . \qquad \square$$

Proof of Theorem 3.6. Thanks to (253), for every $\varepsilon > 0$ we have

$$\int_S^T \mathcal{E}(U(t))dt \leq C(\alpha)\mathcal{E}(U(S)) + \varepsilon \int_S^T \mathcal{E}(U(t))dt + \frac{c}{\varepsilon}\int_S^T \int_{\Gamma_1} (|u_t|^2 + |v_t|^2) d\Sigma dt .$$

Morover, the transmission relation (252) applied to v_t gives

$$\frac{|\alpha|}{2}\int_S^T \int_{\Gamma_1} |v_t|^2 d\Sigma dt \leq C(\alpha)\int_S^T \int_{\Gamma_1} \left(|u_t|^2 + |u_{tt}|^2\right) d\Sigma dt + C(\alpha)\mathcal{E}(U'(S)) . \tag{263}$$

So, choosing $0 < \varepsilon < 1$ and thanks to (263) we conclude that

$$\int_S^T \mathcal{E}(U(t))dt \leq C(\alpha)\left(\mathcal{E}(U(S)) + \mathcal{E}(U'(S))\right) + c\int_S^T \int_{\Gamma_1} (|u_t|^2 + |u_{tt}|^2) d\Sigma dt . \tag{264}$$

Moreover, applying the equation (245) to $U'(t) = (u_t, v_t, u_{tt}, v_{tt})$ yields

$$\frac{d}{dt}\left(\mathcal{E}(U'(t))\right) = -\int_{\Gamma_1} a|u_{tt}|^2 d\Sigma . \tag{265}$$

Thanks to (245), (265) and condition $a > 0$, equation (264) implies

$$\int_S^T \mathcal{E}(U(t))dt \leq C(\alpha)(\mathcal{E}(U(S)) + \mathcal{E}(U'(S))) \tag{266}$$

for every $0 \leq S < T$. Thus, by Lemma 1.8 we deduce that

$$\mathcal{E}(U(t)) \leq \frac{C}{(1+t)^m}\sum_{k=0}^m \mathcal{E}(U^{(k)}(0)) \quad \forall t > 0 . \qquad \square$$

3.4 – Direct proof of strong stability

We now consider $\mathcal{H}$ on the field of complex numbers $\mathbb{C}$. A well-known characterization of strong stability (see [28]) ensures the equivalence with the spectral condition

$$i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset\,; \quad \text{(strongly stable)} \tag{267}$$

We will show that system (237) satisfies the condition (267) for every coupling coefficient $0 < |\alpha| < \alpha_0 = \|P_1\|_{\mathcal{L}}^{-1}$.

Indeed, let $b \in \mathbb{R}$ and $U = (u, v, p, q) \in D(\mathcal{A})$ such that $\mathcal{A}U = ibU$. By definition (233) yields

$$\begin{cases} p = ibu & \text{in } \Omega \\ q = ibv & \text{in } \Omega \\ \Delta u = ibp & \text{in } \Omega \\ \Delta v = ibq & \text{in } \Omega \\ u = 0 = v & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} + \sigma_1 u + \beta p + \alpha v = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial v}{\partial \nu} + \sigma_2 v + \alpha u = 0 & \text{on } \Gamma_1\,. \end{cases} \tag{268}$$

Our aim is to show that $U = 0$.

Case $b = 0$. From the first two equations in (268) we deduce that $p = 0 = q$. Moreover, multiplying the third and fourth equation of (268) for $\bar{u}$ and $\bar{v}$ respectively, integrating over Ω, summing the two relations and integrating by parts, we get

$$0 = \int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx + \int_{\Gamma_1} [\sigma_1 u^2 + \sigma_2 v^2 + \alpha(u\bar{v} + \bar{u}v)]d\Sigma\,, \tag{269}$$

so $u = 0 = v$ for α sufficiently small, thus $U = 0$.

Case $b \neq 0$. Multiplying for $-U$ both terms of the identity $\mathcal{A}U = ibU$, then taking the real part, we have that

$$\int_{\Gamma_1} \beta p^2 d\Sigma = \langle Bp, p\rangle_{V_1', V_1} = -\mathcal{R}(\mathcal{A}U, U)_{\mathcal{H}} = 0\,,$$

so $p = 0$ on Γ_1 and, from the first identity in (268), also $u = 0$ on Γ_1. Thus, the boundary conditions on Γ_1 in (268) become

$$\begin{cases} \dfrac{\partial u}{\partial \nu} + \alpha v = 0 \\ \dfrac{\partial v}{\partial \nu} + \sigma_2 v = 0 \end{cases} \quad \text{on } \Gamma_1\,, \tag{270}$$

whereas, combining the first with the third equations and the second with the forth, the equations in Ω reduce to

$$\begin{cases} \Delta u + b^2 u = 0 \\ \Delta v + b^2 v = 0 \end{cases} \quad \text{in } \Omega\,. \tag{271}$$

Multiplying the first equation for $\bar{v}$ and the second for $\bar{u}$ in the $L^2(\Omega)$-scalar product, then integrating by parts, we deduce

$$\begin{aligned} 0 &= \int_\Omega (b^2 v\bar{u} - \nabla v.\, \nabla \bar{u})dx + \int_{\Gamma_1} v \frac{\partial \bar{u}}{\partial \nu} d\Sigma\,, \\ 0 &= \int_\Omega (b^2 v\bar{u} - \nabla v.\, \nabla \bar{u})dx\,. \end{aligned}$$

Then, subtracting these two relations, and using the first equation in (270), we have that $\alpha \int_{\Gamma_1} v^2 d\Sigma = 0$, so $v = 0$ on Γ_1, and again from (270) we have that $\partial_\nu u = 0 = \partial_\nu v$ on Γ_1. So we conclude that $\nabla u = 0 = \nabla v$ on Γ_1. By applying the multiplier $M\bar{u} = d\bar{u} + 2m.\, \nabla \bar{u}$ to the first equation in (271), we deduce the identity

$$2\int_\Omega |\nabla u|^2 dx = \int_{\Gamma_0} m.\, \nu |\frac{\partial u}{\partial \nu}|^2 d\Sigma \leq 0\,,$$

so $\nabla u = 0$ in Ω and $u = 0$ by Poincaré's inequality. We obtain also that $v = 0$, multiplying the second equation of (271) for $M\bar{v}$. Thus, we have $U = 0$.

Finally, we have shown that $i\mathbb{R} \subset \rho(\mathcal{A})$, that is, condition (267) holds. We point out that condition (267), together with relation

$$\sup\{\|(i\beta - \mathcal{A})^{-1}\|_{L(\mathcal{H})} : \beta \in \mathbb{R}\} < +\infty\,, \tag{272}$$

gives a characterization of (uniform) exponential stability for the linear dynamical system (237) (see [98]), that remains, for the time being, an open issue. Indeed, exponential stability for system (237) cannot be ruled out by the same compactness argument as in the case of distributed coupling (see Chapter 1). It would be interesting to look to disprove condition (272) for system (237), in this way ensuring that exponential stability cannot occur for system (237).

– **Introduction to Part II**

One of the most fascinating aspects of the theory of parabolic equations lies on the wide connection and interaction of several different mathematical subjects, either abstract in nature such as evolution equations, harmonic analysis, stochastic processes, or toward applications such as fluid models, population dynamics and mathematical finance.

In this prospective, one very challenging topic in the theory of parabolic operators concerns the issue of control theory for parabolic equations, ranging from approximate and null controllability to optimality conditions.
In view of optimal control problems, thanks to the regularizing effect of the heat operator, most of the results connected with the Pontryagin Maximum Principle (see, *e.g.*, [76, 77]) extends to parabolic control systems, as well as a substantial part of the dynamic programming approach for both linear quadratic problems (see, *e.g.*, [109]; see also [29]) and nonlinear control problems (see, *e.g.*, [57, 58]; see also [49, 51]).

On the other hand, as far as controllability and stabilization are concerned, the case of parabolic operators differs strongly from the finite-dimensional one or other kinds of partial differential equations like the wave equation, owing to peculiar aspects of parabolic operators, for example, with regards to the infinite speed of propagation and the subsequent instantaneous regularizing effect.

Pioneering works on the controllability of parabolic equations are mainly due to Fattorini and Russell [79, 134, 78, 67]. Their approach was essentially based on Riesz basis expansion techniques, proving very effective to treat operators with constant coefficients.

Thereafter, new substantial developments were achieved in the nineties by the systematic use of Carleman type estimates. Such estimates are weighted energy estimates in suitable Sobolev norms, with weights of exponential type. First introduced in early works by Carleman [52] for the quantification of the unique continuation property for elliptic operators in dimension two, Carleman estimates were then extended to large classes of partial differential operators in arbitrary space dimensions by Hörmander [96, 97] and other authors (see, *e.g.*, [145]), still in a unique continuation context. In the prospective of control theory, Lebeau and Robbiano [112] applied Carleman estimates to control problems for parabolic operators, by combining local Carleman estimates (*i.e.*, for solutions with compact support) with Riesz basis techniques. Then, Fursikov and Imanuvilov [102, 90] performed global estimates for solutions satisfying boundary conditions, and applied them in order to derive null controllability results directly.

More recently, controllability theory for (uniformly) parabolic equations has grown in various directions, such as:

- semilinear parabolic problems (see for example [12, 13, 56, 68, 69, 73, 82, 85, 84, 143]),
- problems in unbounded domains (see [40, 64, 65, 125, 126] and also [50] and [119]),
- fluid models such as Euler, Stokes and Navier-Stokes equations (see [20, 53, 54, 74, 75, 101, 99, 83, 87, 90]), and
- equations with discontinuous coefficients (see [26, 133]).

On the other hand, up to now fewer results are known for degenerate parabolic operator, even though this class of problems has received increasing attention in recent years, being associated with both important theoretical subject, such as stochastic diffusion processes, and interesting applications to engineering, physics, biology, and economics. Compared to nondegenerate parabolic problems, degenerate parabolic equations require major technical adaptations and a frequent use of Hardy type inequalities.

We describe below typical examples where degeneracy of the parabolic equations is entailed in the mathematical modelisation of the problem.

– Stochastic invariance for subset of $\mathbb{R}^d$

The interaction between degenerate parabolic operators and stochastic processes is well-known since Feller's investigations [80, 81]. Moreover, in recent years, several authors have singled out the class of degenerate elliptic operators which may degenerate at the boundary of the space domain, in the normal direction to the boundary. Such class of operators is related with the study of invariant sets for stochastic diffusion processes. Given Lipschitz continuous maps $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^m)$, with $d, m \in \mathbb{N}$, let $X(x, \cdot)$ denote the unique solution of

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \,, \end{cases}$$

where $W(t)$ is a standard $m-$dimensional Brownian motion on a complete filtered probability space. A set $S \subset \mathbb{R}^d$ is said to be invariant for $X(\cdot, \cdot)$ if and only if

$$x \in S \Rightarrow X(x, t) \in S \quad \mathbb{P} - \text{a.s. } \forall t \geq 0 \,.$$

Several investigations and results were obtained for the problem of finding conditions for the invariance of a closed domain $\overline{\Omega}$ for the stochastic flow $X(\cdot, \cdot)$, and even for more general problems such as stochastic differential inclusions and control systems (see [89, 14, 15, 36, 21, 61, 60, 59]). Within the conditions for the invariance of a set, a main role is played by the distance function and the elliptic operator

$$Lu(x) = \frac{1}{2} Tr[a(x)\nabla^2 u(x)] + \langle b(x), \nabla u(x) \rangle \,, \tag{273}$$

where $a(x) = \sigma(x)\sigma^*(x)$. More precisely, one can show that, assuming the boundary $\Gamma := \partial\Omega$ to be regular, the domain $\overline{\Omega}$ is invariant if and only if for all $x \in \Gamma$

$$\begin{aligned} &(i) && Ld_{\Gamma,\Omega}(x) \geq 0 \\ &(ii) && \langle a(x)\nabla d_{\Gamma,\Omega}(x), \nabla d_{\Gamma,\Omega}(x) \rangle = 0 \end{aligned} \tag{274}$$

where

$$d_{\Gamma,\Omega}(x) := \begin{cases} d(x;\Gamma) & \text{if } x \in \Omega \\ -d(x;\Gamma) & \text{if } x \in \Omega^c \end{cases}$$

is the so-called oriented distance from Γ. In particular, observe that condition (ii) in (274) forces $a(x)$ to be a singular matrix for all $x \in \Gamma$, with $\nabla d_{\Gamma,\Omega}(x)$–the inward unit normal to $\overline{\Omega}$ at x–eigenvector of $a(x)$ associated with the zero eigenvalue. The same conditions (274) show to be necessary and sufficient for the invariance of the open set Ω. Using the invariance of Ω, one can then show that, for any sufficiently smooth function $\varphi : \overline{\Omega} \to \mathbb{R}$, the transition semigroup

$$u(x,t) = \mathbb{E}[\varphi(X(x,t))]$$

is the unique solution of the parabolic equation

$$\begin{cases} u_t = Lu & \text{in } \Omega \times (0,T) \\ \langle a\nabla u, \nabla d_{\Gamma,\Omega}\rangle = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = \varphi(x) & x \in \Omega\,, \end{cases}$$

where the above boundary condition is a direct consequence of (274).

– Laminar flow

Another example of a degenerate parabolic operator arises from a completely different domain, in fluid dynamics models. Consider the Prandtl equations (see, *e.g.*, [129]), that describes the velocity field of a laminar flow on a flat plate. Applying the so-called "Crocco change of variables", these equations transform into a nonlinear degenerate parabolic equation–the Crocco equation–on the plane domain $\Omega = (0,L) \times (0,1)$. At this point, in order to study properties of equilibria of the system, we focus our interest on the linearization of the Crocco equation at a stationary solution

$$\begin{cases} u_t + bu_x - au_{yy} + cu = f & (x,y,t) \in \Omega \times (0,T)\,, \\ u_y(x,0,t) = u(x,1,t) = 0 & (x,t) \in (0,L) \times (0,T)\,, \\ u(0,y,t) = u_1(y,t) & (y,t) \in (0,1) \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) & (x,y) \in \Omega\,, \end{cases} \tag{275}$$

where f and u_1 depend on the incident velocity of the flow. Moreover, the coefficients a, b and c are regular but degenerate at the boundary, indeed hold the conditions

$$0 < b_1 \le \frac{b(y)}{y} \le b_2\,, \quad 0 < a_1 \le \frac{a(x,y)}{-(y-1)^2 ln(\mu(1-y))} \le a_2\,, \quad c(x,y) \ge 0\,,$$

for suitable constants a_i, b_i $(i = 1, 2)$ and $\mu \in (0, 1)$ (see [35]). Clearly, another degeneracy for problem (275) occurs owing to the second derivative u_{xx}, that is missing throughout the whole domain.

– Budyko-Sellers climate models

Two classical models for climate changes go back to the end of sixties, independently due to Budyko [37, 38] and Sellers [136] (see also [66]). Both of them investigate the evolution of the (sea level zonally averaged) temperature $u(x, t)$ on the Earth surface, for a long period of time, taking into account the interaction between large ice masses and solar radiation. The mathematical model describing such evolution takes the form of a semilinear parabolic equation defined on a compact manifold in $\mathbb{R}^3$. By considering the temperature constant along a parallel, we obtain a simplified one-dimensional model, representing the evolution of temperature along a fixed meridian. Then, the heat-balance equation for the temperature u is given by

$$cu_t - (k(1 - x^2)u_x)_x = \frac{1}{4}S_0 s(x)\alpha(x, u) - I(u)\,, \quad (x, t) \in (-1, 1) \times (0, T)\,, \quad (276)$$

where c is the thermal capacity of the Earth, k the horizontal thermal conductivity which may be a function of x, S_0 the solar constant, $s(x)$ the normalized distribution of solar input, α the coalbedo and $I(u)$ the outgoing infrared radiation which, in Budyko's model, is an affine function, that is, $I(u) = a + bu$. Notice that the diffusion coefficient of the operator in equation (276) degenerates at the boundary of the space domain. The above equation shall be endowed with the following boundary conditions

$$(1 - x^2)u_x = 0 \quad \text{at} \quad x = \pm 1\,. \quad (277)$$

– Structure of Part II

The Part II of this monograph addresses controllability properties of degenerate parabolic operators. In Chapter 4 we prove a null controllability result for a generalized Grushin operator, and lack thereof, with dependence of a parameter that characterizes the degeneracy order. This operator has been first introduced in works due to Baouendi [19] and Grushin [93, 94], in the contest of hypoelliptic operators and elliptic pseudodifferential operators degenerating on a submanifold of the domain, further developed by Hörmander [95, 97]. Subsequently, Grushin equation has been investigated concerning strong unique continuation [91] (see Section 4.1.2).

Chapter 5 is devoted to prove the null controllability in large time of a Grushin operator with a singular potential, that locally models the Laplace-Beltrami operator

as recently suggested by the study of the Grushin metric on a two dimensional compact manifold endowed with an almost Riemannian structure [33].

4 – The generalized Grushin operator in dimension two

The present chapter is based on the article *Null controllability of Grushin-type operators in dimension two*, Journal of European Mathematical Society, 67-101 (16) 2014, in collaboration with Karine Beauchard and Piermarco Cannarsa.

4.1 – Introduction

4.1.1 – Main result

We consider the generalized Grushin equation

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(x,y,t)1_\omega(x,y) & (x,y,t) \in \Omega \times (0,\infty)\,, \\ u(x,y,t) = 0 & (x,y,t) \in \partial\Omega \times (0,\infty)\,, \end{cases} \tag{278}$$

where $\Omega := (-1,1) \times (0,1)$, $\omega \subset \Omega$, and $\gamma > 0$. Problem (278) is a linear control system in which

- the state is u,
- the control f is supported in the subset ω.

It is a degenerate parabolic equation, since the coefficient of $\partial_y^2 u$ vanishes on the line $\{x = 0\} \times \{y \in (0,1)\}$. We will investigate the null controllability of (278).

Definition 4.1 (Null controllability). Let $T > 0$. System (278) is *null controllable in time* T if, for every $u_0 \in L^2(\Omega)$, there exists $f \in L^2(\Omega \times (0,T))$ such that the solution of

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(x,y,t)1_\omega(x,y) & (x,y,t) \in \Omega \times (0,T)\,, \\ u(t,x,y) = 0 & (t,x,y) \in \partial\Omega \times (0,T)\,, \\ u(0,x,y) = u_0(x,y) & (x,y) \in \Omega\,, \end{cases} \tag{279}$$

satisfies $u(\cdot,\cdot,T) = 0$.

System (278) is *null controllable* if there exists $T > 0$ such that it is null controllable in time T.

The main result of this chapter is the following one.

Theorem 4.2. *Let ω be an open subset of* $(0,1) \times (0,1)$.

1. *If $\gamma \in (0,1)$, then system* (278) *is null controllable in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ where $0 < a < b \leqslant 1$, then there exists $T^* \geqslant \frac{a^2}{2}$ such that*
   - *for every $T > T^*$ system* (278) *is null controllable in time T,*
   - *for every $T < T^*$ system* (278) *is not null controllable in time T.*
3. *If $\gamma > 1$, then* (278) *is not null controllable.*

Remark 4.3. Indeed, in Section 4.3.3 we will prove that there exists $T^* \geqslant \frac{a^2}{2}$ such that point 2. is verified (compare with Theorem 4.17). The identity $T^* = \frac{a^2}{2}$ has been recently proved by Miller [128] (see also [71]).

By a duality argument, the null controllability of (278) is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma}\partial_y^2 g = 0 & (x,y,t) \in \Omega \times (0,\infty)\,, \\ g(x,y,t) = 0 & (x,y,t) \in \partial\Omega \times (0,\infty)\,. \end{cases} \tag{280}$$

Definition 4.4 (Observability). Let $T > 0$. System (280) is *observable in ω in time T* if there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega)$, the solution of

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma}\partial_y^2 g = 0 & (x,y,t) \in \Omega \times (0,T)\,, \\ g(x,y,t) = 0 & (x,y,t) \in \partial\Omega \times (0,T)\,, \\ g(x,y,0) = g_0(x,y) & (x,y) \in \Omega\,, \end{cases} \tag{281}$$

satisfies

$$\int_\Omega |g(x,y,T)|^2 dxdy \leqslant C \int_0^T \int_\omega |g(x,y,t)|^2 dxdydt\,.$$

System (280) is *observable in ω* if there exists $T > 0$ such that it is observable in ω in time T.

Thus, Theorem 4.2 is equivalent to the following observability/lack of observability result.

Theorem 4.5. *Let ω be an open subset of $(0,1) \times (0,1)$.*

1. *If $\gamma \in (0,1)$, then system (281) is observable in ω in any time $T > 0$.*
2. *If $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ where $0 < a < b \leqslant 1$, then there exists $T^* \geqslant \frac{a^2}{2}$ such that*
   - *for every $T > T^*$ system (281) is observable in ω in time T,*
   - *for every $T < T^*$ system (281) is not observable in ω in time T.*
3. *If $\gamma > 1$, then system (281) is not observable in ω.*

Remark 4.6. When $\gamma = 1$, the geometric restriction on the control domain ω only affects our positive result. Indeed, Theorem 4.2 trivially implies that (278) fails to be null controllable (if $\gamma = 1$ and T is small) when ω is any connected open set at positive distance from the degeneracy region $\{x = 0\}$. It is also straightforward to observe that, if ω contains a strip containing $\{x = 0\}$, then null controllability holds for any $\gamma > 0$ thanks to standard localization arguments (see the Appendix 4.6).

4.1.2 – Motivation and bibliographical comments

Null controllability of the heat equation The null and approximate controllability of the heat equation are essentially well understood subjects for both linear and semilinear equations, and for bounded or unbounded domains (see, for instance, [68, 73, 84, 85, 86, 92, 104, 112, 118, 126, 127, 143, 144]). Let us summarize one of the existing main results. Consider the linear heat equation

$$\begin{cases} \partial_t u - \Delta u = f(x,t) 1_\omega(x) & (x,t) \in \Omega \times (0,T)\,, \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T)\,, \\ u(x,0) = u_0(x) & x \in \Omega\,, \end{cases} \tag{282}$$

where Ω is an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, and ω is a subset of Ω. The following theorem is due, for the case $d = 1$, to H. Fattorini and D. Russell [79, Theorem 3.3], and, for $d \geqslant 2$, to O. Imanuvilov [102, 103] (see also the book [90] by A. Fursikov and O.Imanuvilov) and G. Lebeau and L. Robbiano [112] (see also [113]).

Theorem 4.7. *Let Ω be a bounded connected open set with boundary of class C^2 and ω be a nonempty open subset of Ω. Then the control system* (282) *is null controllable in any time $T > 0$.*

So, the heat equation on a smooth bounded domain is null controllable

– in arbitrarily small time;
– with an arbitrarily small control support ω.

Recently, null controllability results have also been obtained for uniformly parabolic operators with discontinuous (see, *e.g.* [69, 26, 27, 133]) or singular ([139] and [72]) coefficients.

It is then natural to wonder whether null controllability also holds for degenerate parabolic equations such as (278) (see the Introduction to Part II for application fields of this kind of equations). Let us compare the known results for the heat equation with the results proved in this chapter. The first difference concerns the geometry of Ω: a more restrictive configuration is assumed in Theorem 4.2 than in Theorem 4.7. The second difference concerns the structure of the controllability results. Indeed, while the heat equation is null controllable in arbitrarily small time, the same result holds for the Grushin equation only when degeneracy is not too strong (*i.e.* $\gamma \in (0,1)$). On the contrary, when degeneracy is too strong (*i.e.* $\gamma > 1$), null controllability does not hold any more. Of special interest is the transition regime ($\gamma = 1$), where the 'classical' Grushin operator appears: here, both behaviours live together, and a positive minimal time is required for the null controllability, a feature more suited for hyperbolic equations. To our knowledge, the existence of a minimal time for the null controllability of the Grushin equation is a novelty for parabolic operators.

Boundary-degenerate parabolic equations The null controllability of parabolic equations degenerating on the boundary of the domain in one space dimension is well-understood, much less so in higher dimension.
Given $0 < a < b < 1$ and $\gamma > 0$, let us consider the 1D equation

$$\partial_t u - \partial_x (x^{2\gamma} \partial_x u) = f(x,t) 1_{(a,b)}(x) , \qquad (x,t) \in (0,1) \times (0,\infty) ,$$

with suitable boundary conditions. Then, it can be proved that null controllability holds if and only if $\gamma \in (0,1)$ (see [44, 45]), while, for $\gamma \geq 1$, the best result one can show is "regional null controllability"(see [43]), which consists in controlling the solution within the domain of influence of the control. Several extensions of the above results are available in one space dimension, see [3, 123] for equations in divergence form, [48, 47] for nondivergence form operators, and [41, 88] for cascade systems. Fewer results are available for multidimensional problems, mainly in the case of two dimensional parabolic operators which simply degenerate in the normal direction to the boundary of the space domain, see [46]. Note that, similarly to the above references, also for the Grushin equation null controllability holds if and only if the degeneracy is not too strong.

Parabolic equations degenerating inside the domain In [124] the authors study a linearized Crocco type equation

$$\begin{cases} \partial_t u + \partial_x u - \partial_{vv} u = f(x,v,t) 1_\omega (x,v) & (x,v,t) \in (0,L) \times (0,1) \times (0,T) , \\ u(x,0,t) = u(x,1,t) = 0 & (x,t) \in (0,L) \times (0,T) , \\ u(0,v,t) = u(L,v,t) & (v,t) \in (0,1) \times (0,T) . \end{cases}$$

For a given open subset ω of $(0,L) \times (0,1)$, they prove regional null controllability. Notice that, in the above equation, diffusion (in v) and transport (in x) are decoupled.

In [24], the authors study the Kolmogorov equation

$$\partial_t u + v \partial_x u - \partial_{vv} u = f(x,v,t) 1_\omega (x,v) , \quad (x,v) \in (0,1)^2 , \tag{283}$$

with periodic-type boundary conditions. They prove null controllability in arbitrarily small time, when the control region ω is a strip, parallel to the x-axis. We note that the above Kolmogorov equation degenerates on the whole space domain, unlike Grushin's equation. However, differently from the linearized Crocco equation, transport (in x at speed v) and diffusion (in v) are coupled. For this reason, the null controllability results differ strongly for these two equations.

Unique continuation and approximate controllability It is well-known that, for evolution equations, approximate controllability can be equivalently formulated as unique continuation for the adjoint system (see [141]). The unique continuation problem for the elliptic Grushin-type operator

$$A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$$

has been widely investigated. In particular, in [91] (see also references therein) unique continuation is proved for every $\gamma > 0$ and every open set ω. For the generalized parabolic Grushin operator studied in this chapter, unique continuation holds for every $\gamma > 0$, $T > 0$, and any open set $\omega \subset \Omega$ (see Proposition 4.13).

Null controllability and hypoellipticity We find interesting to analyze the null controllability problem for the generalized Grushin operator in order to investigate the connections between null controllability of an evolution equation and hypoellipticity of its principal operator.

For a given distribution u in $\Omega \subset \mathbb{R}^d$, the smallest set K for which u is smooth on $\mathbb{R}^d \setminus K$ is the singular support of u, denoted by sing supp$u$. We recall that a linear differential operator P with C^∞ coefficients in an open set Ω is called hypoelliptic if, for every distribution u in Ω, we have

$$\text{sing supp} u = \text{sing supp} Pu \,,$$

that is, u must be a C^∞ function in every open set where so is Pu. The following sufficient condition (which is also essentially necessary) for hypoellipticity is due to Hörmander (see [95]).

THEOREM 4.8. *Let P be a second order differential operator of the form*

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c \,,$$

where $X_0, ..., X_r$ denote first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^d$ with C^∞ coefficients, and $c \in C^\infty(\Omega)$. Assume that there exists d operators among

$$X_{j_1}\,,\ [X_{j_1}, X_{j_2}]\,,\ [X_{j_1}, [X_{j_2}, X_{j_3}]]\,, \dots,\ [X_{j_1}, [X_{j_2}, [X_{j_3}, [..., X_{j_k}]...]]]\,,$$

where $j_i \in \{0, 1, ..., r\}$, which are linearly independent at any given point in Ω. Then, P is hypoelliptic.

Hörmander's condition is satisfied by the Grushin operator $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ for every $\gamma \in \mathbb{N}^*$ (for other values of γ, the coefficients are not C^∞). Indeed, set

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

Then,

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix},\ [X_1, [X_1, X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix},\ \dots$$

Thus, if $\gamma = 1$, Hörmander's condition is satisfied with X_1 and $[X_1, X_2]$. In general, if $\gamma \geq 1$, γ iterated Lie brackets are required to span $\mathbb{R}^2$, precisely X_1 and the $\gamma-$th Lie bracket.

Theorem 4.2 emphasizes that hypoellipticity is not sufficient for null controllability: Grushin's operator is hypoelliptic for all $\gamma \in \mathbb{N}^*$, but null controllability holds only when $\gamma = 1$.

The situation is similar for the Kolmogorov equation (283), where

$$X_0(x,v) := \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad X_1(x,v) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [X_1, X_2](x,v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here again, null controllability holds and the first iterated Lie bracket is sufficient to satisfy Hörmander's condition.

Therefore, the null controllability may be related to the number of iterated Lie bracket necessary to satisfy Hörmander's condition. Indeed, the Grushin equation is null controllable when $\gamma = 1$, *i.e.* when the first Lie bracket is sufficient to achieve Hörmander's condition, but it is not null controllable when $\gamma > 1$, *i.e.* when Lie brackets of order 0 and γ are required, with a gap in between.

A general result which relates null controllability to the number of iterated Lie brackets that are necessary to satisfy Hörmander's condition would be very interesting, but remains—for the time being—a challenging open problem.
An intermediate step in this direction would be to characterize Hörmander's condition as necessary and sufficient (with further hypotheses) condition for approximate controllability.

Sensitivity to singular lower order terms In [33] the authors study the Laplace-Beltrami operator on a two dimensional compact manifold endowed with a 2D almost Riemannian structure. Under very general assumptions, they prove that this operator is essentially selfadjoint. In the particular case of the Grushin metric, their result implies that any solution of

$$\partial_t u - \partial_x^2 u - x^2 \partial_y^2 u - \frac{1}{x}\partial_x u = 0, \quad x \in \mathbb{R}, y \in \mathbb{T} \tag{284}$$

such that $u(\cdot,\cdot,0)$ is supported in $\mathbb{R}_+^* \times \mathbb{T}$ stays supported in this set. As a consequence, with a distributed control as a source term in the right hand side, supported in $\mathbb{R}_+^* \times \mathbb{T}$, this system is not null controllable. This example shows that the control result studied in this chapter is sensitive to the addition of singular lower order terms. See also Chapter 5 for a first glance into controllability properties of equation (284).

4.1.3 – Structure of the chapter

Section 4.2 is devoted to general results about Grushin's equation (278): well posedness in Section 4.2.1, Fourier decomposition of solutions and unique continuation in Section 4.2.2, dissipation rate of the Fourier components in Section 4.2.3.

Section 4.3 presents the proof of the negative statements of Theorem 4.5, (and, equivalently, of Theorem 4.2), when $\gamma > 1$ or $\gamma = 1$ and T is small. In Section 4.3.1 we explain the strategy for the proof, which relies on uniform observability estimates with respect to Fourier frequencies. Then, we show the negative statements of Theorem 4.5, thanks to appropriate test functions to falsify uniform observability, in Section 4.3.2 for $\gamma > 1$ and in Section 4.3.3 for $\gamma = 1$.

In Section 4.4 we perform the proof of the positive statements of Theorem 4.2, (and equivalently of Theorem 4.5) when $\gamma \in (0, 1)$ or $\gamma = 1$ and T is large. In Section 4.4.1 we prove a useful Carleman inequality for 1D heat equations with parameters. In Section 4.4.2, we obtain observability for such equations, uniformly with respect to the parameter. In Section 4.4.3, we prove Theorem 4.5 when $\gamma < 1$. Then, in Section 4.4.4, we conclude the proof of Theorem 4.5.

Finally, in Section 4.5, we shortly outline some open problems related to the problem studied in this chapter. Appendix 4.6 completes the analysis in the case of $\{x = 0\} \subset \omega$.

4.2 – Well-posedness and Fourier decomposition

4.2.1 – Well-posedness of the Cauchy problem

Let $H := L^2(\Omega)$, and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|_H$, respectively, the scalar product and norm in H. Define the product

$$(u, v) := \int_\Omega \big(u_x v_x + |x|^{2\gamma} u_y v_y\big)\, dxdy \tag{285}$$

for every u, v in $C_0^\infty(\Omega)$, and set $V = \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$, where $|u|_V := (u, u)^{1/2}$.

Observe that $H_0^1(\Omega) \subset V \subset H$, thus V is dense in H. Consider the bilinear form a on V defined by

$$a(u, v) = -(u, v) \quad \forall u,\, v \in V\,. \tag{286}$$

Moreover, set

$$D(A) = \{u \in V : \exists\, c > 0 \text{ such that } |a(u, h)| \le c\|h\|_H \ \forall h \in V\}\,, \tag{287}$$

$$\langle Au, h\rangle = a(u, h) \qquad \forall h \in V\,. \tag{288}$$

Then, we can apply a result by Lions [115] (see also Theorem 1.18 in [141]) to conclude that $(A, D(A))$ generates an analytic semigroup $S(t)$ of contractions on H. Note that A is selfadjoint on H, and (288) implies that

$$Au = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u \qquad \text{a.e. in } \Omega\,.$$

So, system (279) can be recast in the form

$$\begin{cases} u'(t) = Au(t) + f(t) & t \in [0,T]\,, \\ u(0) = u_0\,, \end{cases} \tag{289}$$

where $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in H$.

Let us now recall the definition of weak solutions to (289).

Definition 4.9 (Weak solution). Let $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in H$. A function $u \in C([0,T];H) \cap L^2(0,T;V)$ is a weak solution of (289) if for every $h \in D(A)$ the function $\langle u(t), h\rangle$ is absolutely continuous on $[0,T]$ and for a.e. $t \in [0,T]$

$$\frac{d}{dt}\langle u(t), h\rangle = \langle u(t), Ah\rangle + \langle f(t), h\rangle\,. \tag{290}$$

Note that, as showed in [114], condition (290) is equivalent to the definition of solution by transposition, that is,

$$\begin{aligned}&\int_\Omega [u(x,y,t^*)\varphi(x,y,t^*) - u_0(x,y)\varphi(x,y,0)]dxdy \\ &= \int_0^{t^*}\int_\Omega \left\{u\left(\partial_x^2\varphi + |x|^{2\gamma}\partial_y^2\varphi\right) + f\varphi\right\} dxdydt\end{aligned}$$

for every $\varphi \in C^2(\Omega \times [0,T])$ and $t^* \in (0,T)$.

Let us recall that, for every $T > 0$ and $f \in L^2(0,T;H)$, the mild solution of (289) is defined as

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds\,, \quad t \in [0,T]\,. \tag{291}$$

From [18], we have that the mild solution to (289) is also the unique weak solution in the sense of Definition 4.9. The following existence and uniqueness result follows.

Proposition 4.10. *For every $u_0 \in H$, $T > 0$ and $f \in L^2(0,T;H)$, there exists a unique weak solution of the Cauchy problem* (289). *This solution satisfies*

$$|u(t)|_H \leqslant |u_0|_H + \sqrt{T}\|f\|_{L^2(0,T;H)} \quad \forall t \in [0,T]\,. \tag{292}$$

Moreover, $f(t) \in D(A)$ and $f'(t) \in H$ for a.e. $t \in (0,T)$.

Proof. Relation (292) follows from (291). Moreover, since $S(\cdot)$ is analytic, the application $t \mapsto S(t)f_0$ belongs to $C^1((0,T];H) \cap C^0((0,T];D(A))$, and $t \mapsto \int_0^t S(t-s)u(s)ds$ belongs to $H^1(0,T;H) \cap L^2(0,T;D(A))$. In particular $f(t) \in D(A)$ and $f'(t) \in H$ for a.e. $t \in (0,T)$ (see, *e.g.*, [29]). $\square$

4.2.2 – Fourier decomposition and unique continuation

Let us consider the solution of (281) in the sense of Definition 4.9, that is, the solution of system (289) with $u_0 = g_0$ and $f = 0$. Since g belongs to $C([0,T]; L^2(\Omega))$, the function $y \mapsto g(x,y,t)$ belongs to $L^2(0,1)$ for a.e. $(x,t) \in (-1,1) \times (0,T)$, thus it can be developed in Fourier series with respect to y as follows

$$g(x,y,t) = \sum_{n \in \mathbb{N}^*} g_n(x,t)\varphi_n(y)\,, \tag{293}$$

where

$$\varphi_n(y) := \sqrt{2}\sin(n\pi y) \quad \forall n \in \mathbb{N}^*$$

and

$$g_n(x,t) := \int_0^1 g(x,y,t)\varphi_n(y)dy \quad \forall n \in \mathbb{N}^*\,. \tag{294}$$

Proposition 4.11. *For every $n \geq 1$, g_n is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + (n\pi)^2|x|^{2\gamma} g_n = 0 & (x,t) \in (-1,1) \times (0,T)\,, \\ g_n(\pm 1, t) = 0 & t \in (0,T)\,, \\ g_n(x,0) = g_{0,n}(x) & x \in (-1,1)\,, \end{cases} \tag{295}$$

where $g_{0,n} \in L^2(-1,1)$ is given by $g_{0,n}(x) := \int_0^1 g_0(x,y)\varphi_n(y)dy$.

For the proof we need the following characterization of the elements of V. We denote by $L^2_\gamma(\Omega)$ the space of all square-integrable functions with respect to the measure $d\mu = |x|^{2\gamma}dxdy$.

Lemma 4.12. *For every $g \in V$ there exist $\partial_x g \in L^2(\Omega)$, $\partial_y g \in L^2_\gamma(\Omega)$ such that*

$$\begin{aligned} &\int_\Omega \big(g(x,y)\partial_x\phi(x,y) + |x|^{2\gamma} g(x,y)\partial_y\phi(x,y)\big)\,dxdy \\ &= -\int_\Omega \big(\partial_x g(x,y) + |x|^{2\gamma}\partial_y g(x,y)\big)\,\phi(x,y)dxdy \end{aligned} \tag{296}$$

for every $\phi \in C_0^\infty(\Omega)$.

Proof. Let $g \in V$, and consider a sequence $(g^n)_{n\geq 1}$ in $C_0^\infty(\Omega)$ such that $g^n \to g$ in V, that is

$$\int_\Omega \left[(g^n - g)_x^2 + |x|^{2\gamma}(g^n - g)_y^2\right] dxdy \to 0 \quad \text{as } n \to +\infty\,.$$

Thus, $(\partial_x g^n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$ and $(\partial_y g^n)_{n\geq 1}$ is a Cauchy sequence in $L^2_\gamma(\Omega)$. So, there exist $h \in L^2(\Omega)$ and $k \in L^2_\gamma(\Omega)$ such that $\partial_x g^n \to h$ in $L^2(\Omega)$ and $\partial_y g^n \to k$ in $L^2_\gamma(\Omega)$. Hence,

$$\begin{array}{ccc} \displaystyle\int_\Omega \left(g^n \partial_x \phi + |x|^{2\gamma} g^n \partial_y \phi\right) dxdy & = & \displaystyle -\int_\Omega \left(\partial_x g^n \phi + |x|^{2\gamma} \partial_y g^n \phi\right) dxdy \\ \downarrow & & \downarrow \\ \displaystyle\int_\Omega \left(g \partial_x \phi + |x|^{2\gamma} g \partial_y \phi\right) dxdy & = & \displaystyle -\int_\Omega \left(h \phi + |x|^{2\gamma} k \phi\right) dxdy \end{array}$$

as $n \to +\infty$. This yields the conclusion with $\partial_x g = h$ and $\partial_y g = k$. □

For any $n \geq 1$, system (295) is a first order Cauchy problem, that admits the unique weak solution

$$\tilde{g}_n \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_0(-1,1))$$

which satisfies

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-1}^1 \tilde{g}_n(x,t)\psi(x)dx\right) \\ &\quad + \int_{-1}^1 \Big[\tilde{g}_{n,x}(x,t)\psi_x(x) + (n\pi)^2 |x|^{2\gamma} \tilde{g}_n(x,t)\psi(x)\Big] dx = 0 \end{aligned} \tag{297}$$

for every $\psi \in H^1_0(-1,1)$.

Proof of Proposition 4.11. In order to verify that the nth Fourier coefficient of g, defined by (294), satisfies system (295), observe that

$$g_n(\cdot, 0) = g_{0,n}(\cdot)\,, \qquad g_n(\pm 1, t) = 0 \quad \forall t \in (0,T)$$

and

$$g_n \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_0(-1,1))\,.$$

Thus, it is sufficient to prove that g_n fulfills condition (297). Indeed, using the identity (294), for all $\psi \in H^1_0(-1,1)$ we obtain, for a.e. $t \in [0,T]$,

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-1}^1 g_n \psi dx\right) + \int_{-1}^1 \left(g_{n,x}\psi_x + (n\pi)^2 |x|^{2\gamma} g_n \psi\right) dx \\ &= \int_{-1}^1 \int_0^1 \left\{g_t \varphi_n \psi + g_x \varphi_n \psi_x + (n\pi)^2 |x|^{2\gamma} g \varphi_n \psi\right\} dydx\,. \end{aligned} \tag{298}$$

Observe that Proposition 4.10 ensures $g_t(\cdot,t) \in L^2(\Omega)$ and $g(\cdot,t) \in D(A)$ for a.e. $t \in (0,T)$. So, multiplying $g_t - Ag = 0$ by $h(x,y) = \psi(x)\varphi_n(y) \in V$ and integrating over Ω we obtain, for a.e. $t \in (0,T)$,

$$
\begin{aligned}
0 &= \int_0^1 \int_{-1}^1 (g_t - Ag)\psi\varphi_n dxdy \\
&= \int_0^1 \int_{-1}^1 g_t\psi\varphi_n dxdy + \int_0^1 \int_{-1}^1 \left(g_x\psi_x\varphi_n + |x|^{2\gamma} g_y\psi\varphi_{n,y}\right) dxdy \\
&= \int_0^1 \int_{-1}^1 g_t\psi\varphi_n dxdy + \int_0^1 \int_{-1}^1 \left(g_x\psi_x\varphi_n + (n\pi)^2|x|^{2\gamma} g\psi\varphi_n\right) dxdy \,,
\end{aligned}
\tag{299}
$$

where (in the last identity) we have used Lemma 4.12. Combining (298) and (299) completes the proof. □

Proposition 4.13. *Let $T > 0$, $\gamma > 0$, let ω be a bounded open subset of $(0,1) \times (0,1)$, and let $g \in C([0,T];H) \cap L^2(0,T;V)$ be a weak solution of (280). If $g \equiv 0$ on $\omega \times (0,T)$, then $g \equiv 0$ on $\Omega \times (0,T)$.*

Proof. Let $\epsilon > 0$ be such that $\omega \subset (\epsilon,1) \times (0,1)$. By unique continuation for uniformly parabolic 2D equation, we deduce that $g \equiv 0$ on $(\epsilon,1) \times (0,1) \times (0,T)$. Thus, $g_n \equiv 0$ on $(\epsilon,1) \times (0,T)$ for every $n \in \mathbb{N}^*$. Then, by unique continuation for the uniformly parabolic 1D equation (295), we deduce that $g_n \equiv 0$ on $(-1,1)\times(0,T)$ for every $n \in \mathbb{N}^*$, thus $g \equiv 0$ on $\Omega \times (0,T)$ thanks to equation (293). □

4.2.3 – Dissipation speed

Let us introduce, for every $n \in \mathbb{N}^*, \gamma > 0$, the operator $A_{n,\gamma}$ defined on $L^2(-1,1)$ by

$$
D(A_{n,\gamma}) := H^2 \cap H_0^1(-1,1)\,, \quad A_{n,\gamma}\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi\,. \tag{300}
$$

The least eigenvalue of $A_{n,\gamma}$ is given by

$$
\lambda_{n,\gamma} = \min\left\{ \frac{\int_{-1}^1 \left[v'(x)^2 + (n\pi)^2|x|^{2\gamma}v(x)^2\right] dx}{\int_{-1}^1 v(x)^2 dx} ; v \in H_0^1(-1,1),\ v \neq 0 \right\}. \tag{301}
$$

We are interested in the asymptotic behavior (as $n \to +\infty$) of $\lambda_{n,\gamma}$, which quantifies the dissipation speed of the solution of (295).

Lemma 4.14. *Problem*

$$
\begin{cases} -v''_{n,\gamma}(x) + (n\pi)^2|x|^{2\gamma}v_{n,\gamma}(x) = \lambda_{n,\gamma}v_{n,\gamma}(x) & x \in (-1,1)\,, \\ v_{n,\gamma}(\pm 1) = 0\,, \end{cases} \tag{302}
$$

admits a unique positive solution with $L^2(-1,1)$-norm one. Moreover, $v_{n,\gamma}$ is even.

PROOF. Since equation (302) is a Sturm-Liouville problem, it is well-known that its first eigenvalue is simple, and the associated eigenfunction has no zeros. Thus, we can choose $v_{n,\gamma}$ to be strictly positive everywhere. Moreover, by normalization, we can find a unique positive solution satisfying the condition $\|v_{n,\gamma}\|_{L^2(-1,1)} = 1$. Finally, $v_{n,\gamma}$ is even. Indeed, if not so, let us consider the function $w(x) = v_{n,\gamma}(|x|)$. Then, w still belongs to $H^1_0(-1,1)$, it is a weak solution of (302) and it does not increase the functional in (301), *i.e.*

$$\frac{\int_{-1}^{1}\left[w'(x)^2 + (n\pi)^2|x|^{2\gamma}w(x)^2\right]dx}{\int_{-1}^{1} w(x)^2dx} \leq \frac{\int_{-1}^{1}\left[v'_{n,\gamma}(x)^2 + (n\pi)^2|x|^{2\gamma}v_{n,\gamma}(x)^2\right]dx}{\int_{-1}^{1} v_{n,\gamma}(x)^2dx}.$$

The coefficients of the equation in (302) being regular, we deduce that w is a classical solution of (302). Since $\lambda_{n,\gamma}$ is simple, it follows $v_{n,\gamma}(x) = v_{n,\gamma}(|x|)$. □

The following result turns out to be a key point of the proof of Theorem 4.2.

PROPOSITION 4.15. *For every $\gamma > 0$, there are constants $c_* = c_*(\gamma), c^* = c^*(\gamma) > 0$ such that*

$$c_* n^{\frac{2}{1+\gamma}} \leqslant \lambda_{n,\gamma} \leqslant c^* n^{\frac{2}{1+\gamma}} \qquad \forall\, n \in \mathbb{N}^*.$$

PROOF. First, we prove the lower bound. Let $\tau_n := n^{\frac{1}{1+\gamma}}$. With the change of variable $\phi(x) = \sqrt{\tau_n}\varphi(\tau_n x)$, we get

$$\begin{aligned}\lambda_{n,\gamma} &= \inf\left\{\int_{-1}^{1}\left(\phi'(x)^2 + (n\pi)^2|x|^{2\gamma}\phi(x)^2\right)dx\,;\ \phi \in C_c^\infty(-1,1), \|\phi\|_{L^2(-1,1)} = 1\right\}\\ &= \tau_n^2 \inf\left\{\int_{-\tau_n}^{\tau_n}\left(\varphi'(y)^2 + \pi^2|y|^{2\gamma}\varphi(y)^2\right)dy\,;\ \varphi \in C_c^\infty(-\tau_n,\tau_n), \|\varphi\|_{L^2(-\tau_n,\tau_n)} = 1\right\}\\ &\geqslant c_*\tau_n^2\end{aligned}$$

where

$$c_* := \inf\left\{\int_{\mathbb{R}}\left(\varphi'(y)^2 + \pi^2|y|^{2\gamma}\varphi(y)^2\right)dy\ ;\ \varphi \in C_c^\infty(\mathbb{R}), \|\varphi\|_{L^2(\mathbb{R})} = 1\right\} \qquad (303)$$

is positive (see [132] for the case of $\gamma = 1$).

Now, we prove the upper bound in Proposition 4.15. For every $k > 1$ let us consider the function $\varphi_k(x) := (1 - k|x|)^+$, that belongs to $H^1_0(-1,1)$. Easy computations show that

$$\int_{-1}^{1}\varphi_k(x)^2dx = \frac{2}{3k}\,,\ \int_{-1}^{1}\varphi_k'(x)^2dx = 2k\,,\ \int_{-1}^{1}|x|^{2\gamma}\varphi_k(x)^2dx = 2c(\gamma)k^{-1-2\gamma}\,,$$

where

$$c(\gamma) := \left(\frac{1}{2\gamma+1} - \frac{1}{\gamma+1} + \frac{1}{2\gamma+3}\right).$$

Thus, $\lambda_{n,\gamma} \leqslant f_{n,\gamma}(k) := 3[k^2 + (\pi n)^2 c(\gamma) k^{-2\gamma}]$ for all $k > 1$. Since $f_{n,\gamma}$ attains its minimum at $\bar{k} = \tilde{c}(\gamma) n^{\frac{1}{\gamma+1}}$, we have $\lambda_{n,\gamma} \leqslant f_{n,\gamma}(\bar{k}) = C(\gamma) n^{\frac{2}{\gamma+1}}$. □

4.3 – Proof of the negative statements of Theorem 4.5

Let $a := \inf\{x \in (0,1) : (x,y) \in \omega\} > 0$, so that $\omega \subset (a,1) \times (0,1)$. The goal of this section is the proof of the following results:

- if $\gamma = 1$ and $T < \frac{a^2}{2}$, then system (281) is not observable in ω in time T,
- if $\gamma > 1$ and $T > 0$, then system (281) is not observable in ω in time T.

Without loos of generality, one may assume that $\omega = (a,b) \times (0,1)$ with $0 < a < b < 1$.

4.3.1 – Strategy for the proof

Let g be the solution of (281). Then, g can be represented as in (293), and we emphasize that, for a.e. $t \in (0,T)$ and every $-1 \leqslant a_1 < b_1 \leqslant 1$,

$$\int_{(a_1,b_1)\times(0,1)} |g(x,y,t)|^2 dx dy = \sum_{n=1}^{\infty} \int_{a_1}^{b_1} |g_n(x,t)|^2 dx$$

(Bessel-Parseval identity). Thus, in order to prove Theorem 4.5, it is sufficient to study the observability of system (295) uniformly with respect to $n \in \mathbb{N}^*$.

Definition 4.16 (Uniform observability). Let $0 < a < b \leqslant 1$ and $T > 0$. System (295) is *observable in* (a,b) *in time* T *uniformly with respect to* $n \in \mathbb{N}^*$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*$, $g_{0,n} \in L^2(-1,1)$, the solution g_n of (295) satisfies

$$\int_{-1}^{1} |g_n(x,T)|^2 dx \leqslant C \int_0^T \int_a^b |g_n(x,t)|^2 dx \,.$$

System (295) is *observable in* (a,b) *uniformly with respect to* $n \in \mathbb{N}^*$ if there exists $T > 0$ such that it is observable in (a,b) in time T uniformly with respect to $n \in \mathbb{N}^*$.

The negative parts of the conclusion of Theorem 4.5 follow from the result below.

Theorem 4.17. *Let* $0 < a < b \leqslant 1$.

1. *If* $\gamma = 1$ *and* $T < \frac{a^2}{2}$, *then system* (295) *is not observable in* (a,b) *in time* T *uniformly with respect to* $n \in \mathbb{N}^*$.

2. *If* $\gamma > 1$, *then system* (295) *is not observable in* (a, b) *uniformly with respect to* $n \in \mathbb{N}^*$.

The proof of Theorem 4.17 relies on the use of appropriate test functions that falsify uniform observability. This is proved thanks to a well adapted maximum principle (see Lemma 4.18) and explicit supersolutions (see (307)) for $\gamma > 1$, and thanks to direct computations for $\gamma = 1$.

4.3.2 – Proof of Theorem 4.17 for $\gamma > 1$

Let $\gamma \in [1, +\infty)$ be fixed and $T > 0$. For every $n \in \mathbb{N}^*$, we denote by λ_n (instead of $\lambda_{n,\gamma}$) the first eigenvalue of the operator $A_{n,\gamma}$ defined in Section 4.2.3, and by v_n the associated positive eigenvector of norm one, which satisfies

$$\begin{cases} -v_n''(x) + [(n\pi)^2|x|^{2\gamma} - \lambda_n]v_n(x) = 0\,, \quad x \in (-1,1)\,,\ n \in \mathbb{N}^*\,, \\ v_n(\pm 1) = 0\,, \quad v_n \geq 0\,, \\ \|v_n\|_{L^2(-1,1)} = 1\,. \end{cases}$$

Then, for every $n \geq 1$, the function

$$g_n(x,t) := v_n(x)e^{-\lambda_n t} \qquad \forall (x,t) \in (-1,1) \times \mathbb{R}\,,$$

solves the adjoint system (295). Let us note that

$$\int_{-1}^{1} g_n(x,T)^2 dx = e^{-2\lambda_n T}\,,$$

$$\int_0^T \int_a^b g_n(x,t)^2 dx dt = \frac{1 - e^{-2\lambda_n T}}{2\lambda_n} \int_a^b v_n(x)^2 dx\,.$$

So, in order to prove that uniform observability fails, it suffices to show that

$$\frac{e^{2\lambda_n T}}{\lambda_n} \int_a^b v_n(x)^2 dx \to 0 \text{ when } n \to +\infty\,. \tag{304}$$

The above convergence will be obtained comparing v_n with an explicit supersolution of the problem on a suitable subinterval of $[-1, 1]$.

Lemma 4.18. *Let* $0 < a < b < 1$. *For every* $n \in \mathbb{N}^*$, *set*

$$x_n := \left(\frac{\lambda_n}{(n\pi)^2}\right)^{\frac{1}{2\gamma}} \tag{305}$$

and let $W_n \in C^2([x_n, 1], \mathbb{R})$ *be a solution of*

$$\begin{cases} -W_n''(x) + [(n\pi)^2 x^{2\gamma} - \lambda_n]W_n(x) \geqslant 0\,, \quad x \in (x_n, 1)\,, \\ W_n(1) \geqslant 0\,, \\ W_n'(x_n) < -\sqrt{x_n}\lambda_n\,. \end{cases} \tag{306}$$

Then there exists $n_* \in \mathbb{N}^*$ such that, for every $n \geqslant n_*$,

$$\int_a^b v_n(x)^2 dx \leqslant \int_a^b W_n(x)^2 dx \,.$$

Proof. First, observe that, thanks to Proposition 4.15, $x_n \to 0$ as $n \to +\infty$. In particular, there exists $n_* \geqslant 1$ such that $x_n \leqslant a$ for every $n \geqslant n_*$. Now, let us prove that $|v_n'(x_n)| \leqslant \sqrt{x_n}\lambda_n$ for all $n \geqslant n_*$. Indeed, by Lemma 4.14, we have $v_n(x) = v_n(-x)$, thus $v_n'(0) = 0$. Hence, thanks to the Cauchy-Schwarz inequality and the relation $\|v_n\|_{L^2(-1,1)} = 1$,

$$\begin{aligned}|v_n'(x_n)| &= \Big| \int_0^{x_n} v_n''(s) ds \Big| = \Big| \int_0^{x_n} [(n\pi)^2 |s|^{2\gamma} - \lambda_n] v_n(s) ds \Big| \\ &\leqslant \Big(\int_0^{x_n} [(n\pi)^2 |s|^{2\gamma} - \lambda_n]^2 ds \Big)^{1/2} \Big(\int_0^{x_n} v_n(s)^2 ds \Big)^{1/2} \leqslant \sqrt{x_n}\lambda_n \,.\end{aligned}$$

Furthermore, we claim that $v_n(x) \leqslant W_n(x)$ for every $x \in [x_n, 1]$, $n \geqslant n_*$. Indeed, if not, there would exist $x_* \in [x_n, 1]$ such that

$$(W_n - v_n)(x_*) = \min\{(W_n - v_n)(x); x \in [x_n, 1]\} < 0 \,.$$

Since $(W_n - v_n)(1) \geqslant 0$ and $(W_n - v_n)'(x_n) < 0$, we have $x_* \in (x_n, 1)$. Moreover, the function $W_n - v_n$ has a minimum at x_*, thus $(W_n - v_n)'(x_*) = 0$ and $(W_n - v_n)''(x_*) \geqslant 0$. Therefore,

$$-(W_n - v_n)''(x_*) + [(n\pi)^2 |x_*|^{2\gamma} - \lambda_n](W_n - v_n)(x_*) < 0 \,,$$

which is a contradiction. Our claim follows and the proof is complete. □

In order to apply Lemma 4.18, we need an explicit supersolution W_n of (306) of the form

$$W_n(x) = C_n e^{-\mu_n x^{\gamma+1}} \,, \tag{307}$$

where $C_n, \mu_n > 0$. Notice that, in particular, $W_n(1) \geqslant 0$.

First step: let us prove that, for an appropriate choice of μ_n, the first inequality of (306) holds. Since

$$\begin{aligned}W_n'(x) &= -\mu_n(\gamma+1)x^\gamma W_n(x) \,, \\ W_n''(x) &= [-\mu_n \gamma(\gamma+1)x^{\gamma-1} + \mu_n^2(\gamma+1)^2 x^{2\gamma}] W_n(x) \,,\end{aligned}$$

the first inequality of (306) holds if and only if, for every $x \in (x_n, 1)$,

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2] x^{2\gamma} + \mu_n \gamma(\gamma+1) x^{\gamma-1} \geqslant \lambda_n \,. \tag{308}$$

In particular, it holds when

$$\mu_n \leqslant \frac{n\pi}{\gamma+1} \tag{309}$$

and

$$[(n\pi)^2 - \mu_n^2(\gamma+1)^2]x_n^{2\gamma} + \mu_n\gamma(\gamma+1)x_n^{\gamma-1} \geqslant \lambda_n\,. \tag{310}$$

Indeed, in this case, the left hand side of (308) is an increasing function of x. In view of (305), and after several simplifications, inequality (310) can be recast as

$$\mu_n \leqslant \frac{\gamma}{\gamma+1}\left(\frac{(n\pi)^2}{\lambda_n}\right)^{\frac{1}{2}+\frac{1}{2\gamma}}\,.$$

So, recalling (309), in order to satisfy the first inequality of (306) we can take

$$\mu_n := \min\left\{\frac{n\pi}{\gamma+1}; \frac{\gamma}{\gamma+1}\left(\frac{(n\pi)^2}{\lambda_n}\right)^{\frac{1}{2}+\frac{1}{2\gamma}}\right\}. \tag{311}$$

For the following computations, it is important to notice that, thanks to (311) and Proposition 4.15, for n large enough μ_n is of the form

$$\mu_n = C_1(\gamma)n\,. \tag{312}$$

Second step: let us prove that, for an appropriate choice of C_n, the third inequality of (306) holds. Since

$$W_n'(x_n) = -C_n\mu_n(\gamma+1)x_n^{\gamma}e^{-\mu_n x_n^{\gamma+1}}\,,$$

the third inequality of (306) is equivalent to

$$C_n > \frac{\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-\frac{1}{2}}}\,.$$

Therefore, it is sufficient to choose

$$C_n := \frac{2\lambda_n e^{\mu_n x_n^{\gamma+1}}}{(\gamma+1)\mu_n x_n^{\gamma-\frac{1}{2}}}\,. \tag{313}$$

Third step: let us prove condition (304). Thanks to Lemma 4.18 and conditions (307), (312) and (313), for every $n \geqslant n_*$,

$$\frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b v_n(x)^2dx \leqslant \frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b W_n(x)^2dx \leqslant \frac{e^{2\lambda_n T}}{\lambda_n}W_n(a)^2$$
$$\leqslant \frac{e^{2\lambda_n T}}{\lambda_n}C_n^2 e^{-2\mu_n a^{1+\gamma}} \leqslant \frac{e^{2\lambda_n T}}{\lambda_n}\frac{4\lambda_n^2 e^{2\mu_n x_n^{\gamma+1}}}{(\gamma+1)^2\mu_n^2 x_n^{2\gamma-1}}e^{-2\mu_n a^{1+\gamma}}.$$

By identities (305), (312) and Proposition 4.15, we have

$$\mu_n x_n^{\gamma+1} \leqslant C_2(\gamma) \qquad \forall n \in \mathbb{N}^*\,,$$

thus

$$\frac{e^{2\lambda_n T}}{\lambda_n}\int_a^b v_n(x)^2dx \leqslant e^{2n\left(\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma}\right)}\frac{4\lambda_n e^{2C_2(\gamma)}}{(\gamma+1)^2\mu_n^2 x_n^{2\gamma-1}}\,. \tag{314}$$

Since $\gamma > 1$, we deduce from Proposition 4.15 that

$$\frac{\lambda_n}{n} \to 0 \quad \text{as} \quad n \to +\infty\,.$$

So, for every $T > 0$, there exists $n_\sharp \geqslant n_*$ such that, for every $n \geqslant n_\sharp$,

$$\frac{\lambda_n}{n}T - C_1(\gamma)a^{1+\gamma} < -\frac{1}{2}C_1(\gamma)a^{1+\gamma}\,. \tag{315}$$

Then, inequality (314) yields condition (304) (since the term that multiplies the exponential behaves like a rational fraction of n).

4.3.3 – Proof of Theorem 4.17 for $\gamma = 1$

In this section, we take $\gamma = 1$ and keep the abbreviated forms λ_n, v_n for $\lambda_{n,\gamma}$, $v_{n,\gamma}$ introduced in Section 4.2.3. Moreover, given two real sequences $\alpha_n \geqslant 0$ and $\beta_n > 0$, we write $\alpha_n \sim \beta_n$ to mean that $\lim_n \alpha_n/\beta_n = 1$.

With the above notation in mind, we have the following result.

LEMMA 4.19. *Let a and b be real numbers such that $0 < a < b \leqslant 1$. Then*

$$\lambda_n \sim n\pi \tag{316}$$

and

$$\int_a^b v_n(x)^2dx \sim \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}} \tag{317}$$

as $n \to +\infty$.

When $T < \frac{a^2}{2}$, we can easily deduce from the above lemma that (304) holds; thus, system (295) is not observable in (a, b) uniformly with respect to $n \in \mathbb{N}^*$.

Proof of Lemma 4.19. The proof relies on the explicit expression

$$G(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}$$

of the first eigenvector of the harmonic oscillator on the whole line, *i.e.*,

$$\begin{cases} -G''(x) + x^2 G(x) = G(x) & x \in \mathbb{R} \\ \displaystyle\int_{\mathbb{R}} G(x)^2 dx = 1. \end{cases}$$

First step: Let us construct an explicit approximation k_n of v_n. Fix $\varepsilon > 0$ with

$$1 + (1-\varepsilon)^2 > 2a^2\,, \tag{318}$$

and let $\theta \in C^\infty(\mathbb{R})$ be such that

$$\theta(\pm 1) = 1 \quad \text{and} \quad \operatorname{supp}(\theta) \subset (-1-\varepsilon, -1+\varepsilon) \cup (1-\varepsilon, 1+\varepsilon)\,. \tag{319}$$

Define

$$k_n(x) = \frac{\sqrt[4]{n\pi}\, G(\sqrt{n\pi}x) - \sqrt[4]{n} e^{-\frac{n\pi}{2}} \theta(x)}{C_n}\,, \qquad x \in [-1, 1]\,,$$

where $C_n > 0$ is such that $\|k_n\|_{L^2(-1,1)} = 1$. Note that $C_n^2 = C_{n,1} + C_{n,2} + C_{n,3}$ where

$$\begin{aligned}
C_{n,1} &= \sqrt{n} \int_{-1}^{1} e^{-n\pi x^2 dx} = 1 + O\left(\frac{e^{-n\pi}}{\sqrt{n}}\right), \\
C_{n,2} &= \sqrt{n} e^{-n\pi} \int_{-1}^{1} \theta(x)^2 dx \\
C_{n,3} &= -2\sqrt{n} e^{-\frac{n\pi}{2}} \int_{-1}^{1} e^{-\frac{n\pi x^2}{2}} \theta(x) dx = O(\sqrt{n} e^{-\frac{n\pi}{2}\left(1+(1-\varepsilon)^2\right)}).
\end{aligned}$$

Thus,

$$C_n = 1 + O(\sqrt{n} e^{-\frac{n\pi}{2}[1+(1-\varepsilon)^2]}). \tag{320}$$

We have

$$\begin{cases} -k_n''(x) + (n\pi x)^2 k_n(x) = n\pi k_n(x) + E_n(x)\,, & x \in (-1, 1)\,, \\ k_n(\pm 1) = 0, \end{cases}$$

where

$$E_n(x) := \frac{\sqrt[4]{n}e^{-\frac{n\pi}{2}}}{C_n}\left[\theta''(x) - (n\pi x)^2\theta(x) + n\pi\theta(x)\right].$$

Second step: Let us prove (316). As in the proof of Proposition 4.15, we have $\lambda_n \geqslant n\pi$. Moreover,

$$\begin{aligned}\lambda_n &\leqslant \int_{-1}^{1}\left[k_n'(x)^2 + (n\pi x)^2 k_n(x)^2\right]dx = n\pi + \int_{-1}^{1} k_n(x)E_n(x)dx \\ &\leqslant n\pi + O(n^{\frac{9}{4}}e^{-\frac{n\pi}{2}}),\end{aligned}$$

which proves (316).

Third step: *Let us prove that*

$$\int_a^b k_n(x)^2 dx \sim \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}}. \tag{321}$$

Indeed, the left-hand side of (321) is the sum of three terms $(I_j)_{1\leqslant j\leqslant 3}$, that satisfy, thanks to (320)

$$\begin{aligned}I_1 &:= \frac{1}{\sqrt{\pi}C_n^2}\int_{a\sqrt{n\pi}}^{b\sqrt{n\pi}} e^{-y^2}dy = \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}} + O\left(\frac{e^{-a^2 n\pi}}{n^{\frac{3}{2}}}\right), \\ I_2 &:= \frac{\sqrt{n}e^{-n\pi}}{C_n^2}\int_a^b \theta(x)^2 dx = O(\sqrt{n}e^{-n\pi}), \\ I_3 &:= -\frac{2\sqrt{n}e^{-\frac{n\pi}{2}}}{C_n^2}\int_a^b e^{-n\pi x^2}\theta(x)dx = O(\sqrt{n}e^{-\frac{n\pi}{2}[1+(1-\varepsilon)^2]}).\end{aligned}$$

So, (321) follows thanks to (318).

Fourth step: Let us prove that

$$\|v_n - k_n\|^2_{L^2(-1,1)} = O(n^{\frac{9}{2}}e^{-n\pi}), \tag{322}$$

which ends the proof of (317). Let A_n be the operator defined by

$$D(A_n) = H^2 \cap H_0^1(-1,1), \quad A_n\varphi(x) =: -\varphi''(x) + (n\pi x)^2\varphi(x),$$

let $(\lambda_n^j)_{j\in\mathbb{N}^*}$ be its eigenvalues, with associated eigenvectors $(v_n^j)_{j\in\mathbb{N}^*}$, so $A_n v_n^j = \lambda_n^j v_n^j$. We have $k_n = \sum_{j=1}^{\infty} z_j v_n^j$ where $z_j = \langle E_n, v_n^j\rangle/(\lambda_n^j - n\pi)$ for all $j \geqslant 2$. Thus,

$$\sum_{j=2}^{\infty} z_j^2 \leqslant C\|E_n\|^2_{L^2(-1,1)} = O(n^{\frac{9}{2}}e^{-n\pi})$$

and

$$z_1 = \sqrt{1 - \sum_{j=2}^{\infty} z_j^2} = 1 + O(n^{\frac{9}{2}} e^{-n\pi})\,.$$

We can then recover (322) since $\|v_n - k_n\|^2_{L^2(-1,1)} = (1-z_1)^2 + \sum_{j=2}^{\infty} z_j^2$. □

4.4 – Proof of the positive statements of Theorem 4.2

The goal of this section is the proof of the following results:

- if $\gamma \in (0,1)$, then system (278) is null controllable in any time $T > 0$,
- if $\gamma = 1$ and $\omega = (a,b) \times (0,1)$, with $0 < a < b \leqslant 1$, then there exists $T_1 > 0$ such that system (278) is null controllable in any time $T > T_1$ or, equivalently, system (280) is observable in ω in any time $T > T_1$.

The proof of these results relies on a new global Carleman estimate for solutions of equation (295), stated and proved in the next section.

4.4.1 – A global Carleman estimate

Let $n \in \mathbb{N}^*$, and introduce the operator

$$\mathcal{P}_n g := \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2 |x|^{2\gamma} g.$$

Proposition 4.20. *Let $\gamma \in (0,1]$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b \leqslant 1$. Then there exist a weight function $\beta \in C^1([-1,1]; \mathbb{R}^*_+)$ and positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that for every $n \in \mathbb{N}^*$, $T > 0$, and $g \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_0(-1,1))$ the following inequality holds*

$$\begin{aligned} &\mathcal{C}_1 \int_0^T \int_{-1}^1 \left(\frac{M}{t(T-t)} \Big| \frac{\partial g}{\partial x}(x,t) \Big|^2 + \frac{M^3}{(t(T-t))^3} \big| g(x,t) \big|^2 \right) e^{-\frac{M\beta(x)}{t(T-t)}} \, dx dt \\ &\leqslant \int_0^T \int_{-1}^1 |\mathcal{P}_n g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} \, dx dt + \int_0^T \int_a^b \frac{M^3}{(t(T-t))^3} |g(x,t)|^2 e^{-\frac{M\beta(x)}{t(T-t)}} \, dx dt \end{aligned} \tag{323}$$

where $M := \mathcal{C}_2 \max\{T + T^2; nT^2\}$.

Remark 4.21. In the case of $\gamma \in [1/2, 1]$, our weight β will be the classical one (see (325), (326), (327) and (328)). On the other hand, for $\gamma \in (0, 1/2)$ we will follow the strategy of [3, 48, 123], adapting the weight β to the nonsmooth coefficient $|x|^{2\gamma}$ (see (325), (326), (327), (356) and (357)).

Proof of Proposition 4.20. Without loss of generality, we may assume that $b < 1$. Let a', b' be such that $a < a' < b' < b$. All the computations of the proof will be made assuming, first, $g \in H^1(0,T;L^2(-1,1)) \cap L^2(0,T;H^2 \cap H_0^1(-1,1))$. Then, the conclusion of Proposition 4.20 will follow by a density argument.

First case: $\gamma \in [1/2, 1]$ Consider the weight function

$$\alpha(x,t) := \frac{M\beta(x)}{t(T-t)}, \quad (x,t) \in [-1,1] \times (0,T)\,, \tag{324}$$

where $\beta \in C^2([-1,1])$ satisfies

$$\beta \geqslant 1 \text{ on } (-1,1)\,, \tag{325}$$

$$|\beta'| > 0 \text{ on } [-1,a'] \cup [b',1]\,, \tag{326}$$

$$\beta'(1) > 0\,, \quad \beta'(-1) < 0\,, \tag{327}$$

$$\beta'' < 0 \text{ on } [-1,a'] \cup [b',1] \tag{328}$$

and $M = M(T,n,\beta) > 0$ will be chosen later on. We also introduce the function

$$z(x,t) := g(x,t)e^{-\alpha(x,t)}\,, \tag{329}$$

that satisfies

$$e^{-\alpha}\mathcal{P}_n g = P_1 z + P_2 z + P_3 z\,, \tag{330}$$

where

$$\begin{gathered} P_1 z := -\tfrac{\partial^2 z}{\partial x^2} + (\alpha_t - \alpha_x^2)z + (n\pi)^2|x|^{2\gamma}z\,, \quad P_2 z := \tfrac{\partial z}{\partial t} - 2\alpha_x\tfrac{\partial z}{\partial x}\,, \\ P_3 z := -\alpha_{xx}z\,. \end{gathered} \tag{331}$$

We develop the classical proof (see [90]), taking the $L^2(Q)$-norm in the identity (330), then developing the double product, which leads to

$$\int_Q \left(P_1 z P_2 z - \frac{1}{2}|P_3 z|^2\right) dxdt \leqslant \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,, \tag{332}$$

where $Q := (-1,1) \times (0,T)$ and we compute precisely each term, paying attention to the behaviour of the different constants with respect to n and T.

Terms concerning $-\partial_x^2 z$ Integrating by parts, we get

$$-\int_Q \frac{\partial^2 z}{\partial x^2}\frac{\partial z}{\partial t}dxdt = \int_Q \frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial t\partial x}dxdt = \int_0^T \frac{1}{2}\frac{d}{dt}\int_{-1}^1 \left|\frac{\partial z}{\partial x}\right|^2 dxdt = 0\,, \tag{333}$$

because $\partial_t z(t,\pm 1) = 0$ and $z(0) \equiv z(T) \equiv 0$, which is a consequence of assumptions (329), (324) and (325). Moreover,

$$\begin{aligned}\int_Q \frac{\partial^2 z}{\partial x^2} 2\alpha_x \frac{\partial z}{\partial x} dxdt = &- \int_Q \Big|\frac{\partial z}{\partial x}\Big|^2 \alpha_{xx} dxdt \\ &+ \int_0^T \Big(\alpha_x(t,1)\Big|\frac{\partial z}{\partial x}(t,1)\Big|^2 - \alpha_x(t,-1)\Big|\frac{\partial z}{\partial x}(t,-1)\Big|^2 \Big)\, dt\,.\end{aligned} \tag{334}$$

Terms concerning $(\alpha_t - \alpha_x^2)z$ Again integrating by parts, we have

$$\int_Q (\alpha_t - \alpha_x^2) z \frac{\partial z}{\partial t} dxdt = -\frac{1}{2}\int_Q (\alpha_t - \alpha_x^2)_t |z|^2 dxdt\,. \tag{335}$$

Indeed, the boundary terms at $t = 0$ and $t = T$ vanish because, thanks to (329), (324), (325),

$$|(\alpha_t - \alpha_x^2)|z|^2| \leqslant \frac{1}{[t(T-t)]^2} e^{\frac{-M}{t(T-t)}} |M(T-2t)\beta + (M\beta')^2| \cdot |g|^2$$

tends to zero when $t \to 0$ and $t \to T$, for every $x \in [-1,1]$. Moreover,

$$-2\int_Q (\alpha_t - \alpha_x^2) z \alpha_x \frac{\partial z}{\partial x} dxdt = \int_Q [(\alpha_t - \alpha_x^2)\alpha_x]_x |z|^2 dxdt\,, \tag{336}$$

thanks to an integration by parts in the space variable.

Terms concerning $(n\pi)^2|x|^{2\gamma}z$ First, since $z(0) \equiv z(T) \equiv 0$,

$$\int_Q (n\pi)^2 |x|^{2\gamma} z \frac{\partial z}{\partial t} dxdt = \frac{1}{2}\int_0^T \frac{d}{dt}\int_{-1}^1 (n\pi)^2|x|^{2\gamma}|z|^2 dxdt = 0\,. \tag{337}$$

Furthermore, thanks to an integration by parts in the space variable,

$$-2\int_Q (n\pi)^2|x|^{2\gamma} z\alpha_x \frac{\partial z}{\partial x} dxdt = \int_Q [n^2\pi^2|x|^{2\gamma}\alpha_x]_x z^2 dxdt\,. \tag{338}$$

Combining (332), (333), (334), (335), (336), (337) and (338), we conclude that

$$\begin{aligned}&\int_Q |z|^2 \Big\{ -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x + n^2\pi^2[|x|^{2\gamma}\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \Big\} dxdt \\ &+ \int_0^T \Big(\alpha_x(t,1)\Big|\frac{\partial z}{\partial x}(t,1)\Big|^2 - \alpha_x(t,-1)\Big|\frac{\partial z}{\partial x}(t,-1)\Big|^2 \Big)\, dt \\ &- \int_Q \Big|\frac{\partial z}{\partial x}\Big|^2 \alpha_{xx} dxdt \leqslant \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,.\end{aligned} \tag{339}$$

In view of (327), we have $\alpha_x(t,1) \geqslant 0$ and $\alpha_x(t,-1) \leqslant 0$, thus (339) yields

$$\begin{aligned}&\int_Q |z|^2\Big\{ -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 + n^2\pi^2[|x|^{2\gamma}\alpha_x]_x \Big\} dxdt \\ &- \int_Q \Big|\frac{\partial z}{\partial x}\Big|^2 \alpha_{xx} dxdt \leqslant \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,.\end{aligned} \tag{340}$$

Now, in the left hand side of (340) we separate the terms on $(a',b') \times (0,T)$ and those on $[(-1,a') \cup (b',1)] \times (0,T)$. One has

$$\begin{aligned}-\alpha_{xx}(x,t) &\geqslant \frac{C_1 M}{t(T-t)} \qquad \forall x \in [-1,a'] \cup [b',1]\,,\ t \in (0,T)\,, \\ |\alpha_{xx}(x,t)| &\leqslant \frac{C_2 M}{t(T-t)} \qquad \forall x \in [a',b']\,,\ t \in (0,T)\,,\end{aligned} \tag{341}$$

where $C_1 = C_1(\beta) := \min\{-\beta''(x); x \in [-1,a'] \cup [b',1]\}$ is positive thanks to the assumption (328) and $C_2 = C_2(\beta) := \sup\{|\beta''(x)|; x \in [a',b']\}$. Moreover,

$$\begin{aligned}-\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 &= \frac{1}{(t(T-t))^3}\Big\{ M\beta(3Tt - T^2 - 3t^2) \\ &+ M^2\Big[(2t-T)(\beta''\beta + 2\beta'^2) - \frac{t(T-t)\beta''^2}{2}\Big] - 3M^3\beta''\beta'^2 \Big\}\,.\end{aligned}$$

Hence, owing to hypotheses (326) and (328), there exist $m_1 = m_1(\beta) > 0$, $C_3 = C_3(\beta) > 0$ and $C_4 = C_4(\beta) > 0$ such that, for every $M \geqslant M_1$ and $t \in (0,T)$,

$$\begin{aligned}&-\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \geqslant \frac{C_3 M^3}{[t(T-t)]^3} \quad \forall x \in [-1,a'] \cup [b',1]\,, \\ &\Big| -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \Big| \leqslant \frac{C_4 M^3}{[t(T-t)]^3} \quad \forall x \in [a',b']\,.\end{aligned} \tag{342}$$

where

$$M_1 = M_1(T,\beta) := m_1(\beta)(T + T^2). \tag{343}$$

Using estimates (340), (341) and (342), we deduce that, for every $M \geqslant M_1$,

$$\begin{aligned}&\int_0^T \int_{(-1,a')\cup(b',1)} \frac{C_1 M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 dxdt \\ &+ \int_0^T \int_{(-1,a')\cup(b',1)} \Big[\frac{C_3 M^3}{(t(T-t))^3}|z|^2 + (n\pi)^2[|x|^{2\gamma}\alpha_x]_x|z|^2\Big] dxdt \\ &\leqslant \int_0^T \int_{a'}^{b'} \Big[\frac{C_2 M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 + \frac{C_4 M^3}{(t(T-t))^3}|z|^2 - (n\pi)^2[|x|^{2\gamma}\alpha_x]_x|z|^2\Big] dxdt \\ &+ \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,.\end{aligned} \tag{344}$$

Moreover, for every $x \in (-1,1)$, we have

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| = \frac{M(n\pi)^2}{t(T-t)}\Big|2\gamma\mathrm{sign}(x)|x|^{2\gamma-1}\beta'(x) + |x|^{2\gamma}\beta''(x)\Big| \leqslant \frac{C_5 n^2 M}{t(T-t)}\,,$$

where $C_5 = C_5(\beta) := \pi^2 \max\{2\gamma|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)|; x \in [-1,1]\}$ is finite because $2\gamma - 1 \geqslant 0$. Let $M_2 = M_2(T,n,\beta)$ be defined by

$$M_2 = M_2(T,n,\beta) := \sqrt{\frac{2C_5}{C_3} n \left(\frac{T}{2}\right)^2}\,. \tag{345}$$

From now on, we take

$$M = M(T,n,\beta) := \mathcal{C}_2 \max\{T + T^2; nT^2\} \tag{346}$$

where

$$\mathcal{C}_2 = \mathcal{C}_2(\beta) := \max\left\{m_1; \sqrt{\frac{C_5}{8C_3}}\right\}\,,$$

so that $M \geqslant M_1$ and M_2 (see (343) and (345)). From $M \geqslant M_2$, we deduce that

$$|(n\pi)^2[|x|^{2\gamma}\alpha_x]_x| \leqslant \frac{C_3 M^3}{2[t(T-t)]^3} \quad \forall (x,t) \in Q\,.$$

We have

$$\begin{aligned}&\int_0^T \int_{(-1,a')\cup(b',1)} \left(\frac{C_1 M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 + \frac{C_3 M^3}{2(t(T-t))^3}|z|^2\right) dxdt \\ &\leqslant \int_0^T \int_{a'}^{b'} \left(\frac{C_2 M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 + \frac{C_6 M^3}{(t(T-t))^3}|z|^2\right) dxdt + \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,,\end{aligned} \tag{347}$$

where $C_6 = C_6(\beta) := C_4 + C_3/2$. Since for every $\varepsilon > 0$

$$\begin{aligned}&\frac{C_1 M}{t(T-t)}\Big|\frac{\partial g}{\partial x} - \alpha_x g\Big|^2 + \frac{C_3 M^3}{2(t(T-t))^3}|g|^2 \\ &\geqslant \left(1 - \frac{1}{1+\varepsilon}\right)\frac{C_1 M}{t(T-t)}\Big|\frac{\partial g}{\partial x}\Big|^2 + \frac{M^3}{(t(T-t))^3}\left(\frac{C_3}{2} - \varepsilon C_1(\beta')^2\right)|g|^2\,.\end{aligned} \tag{348}$$

Hence, choosing

$$\varepsilon = \varepsilon(\beta) := \frac{C_3}{4C_1\|\beta'\|_\infty^2}\,,$$

from relations (347), (348) and identity (329) we deduce that

$$\begin{aligned}&\int_0^T \int_{(-1,a')\cup(b',1)} \left(\frac{C_7 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3 M^3 |g|^2}{4(t(T-t))^3}\right) e^{-2\alpha} dxdt \\ &\leqslant \int_0^T \int_{a'}^{b'} \left(\frac{C_9 M^3 |g|^2}{(t(T-t))^3} + \frac{C_8 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2\right) e^{-2\alpha} dxdt + \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt\,,\end{aligned} \tag{349}$$

where $C_7 = C_7(\beta) := [1 - 1/(1+\varepsilon)]C_1$, $C_8 = C_8(\beta) := 2C_2$ and $C_9 = C_9(\beta) := C_6 + 2C_2 \sup\{\beta'(x)^2 : x \in [a', b']\}$. So, adding the same quantity to both sides,

$$\begin{aligned}&\int_Q \left(\frac{C_7 M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3 M^3 |g|^2}{4(t(T-t))^3}\right) e^{-2\alpha} dxdt \leqslant \int_Q |e^{-\alpha}\mathcal{P}_n g|^2 dxdt \\ &+ \int_0^T \int_{a'}^{b'} \left(\frac{C_{11} M^3 |g|^2}{(t(T-t))^3} + \frac{C_{10} M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2\right) e^{-2\alpha} dxdt\,,\end{aligned} \tag{350}$$

where $C_{10} = C_{10}(\beta) := C_8 + C_7$ and $C_{11} = C_{11}(\beta) := C_9 + C_3/4$. Let us prove that the third term of the right hand side may be dominated by terms similar to the other two. We consider $\rho \in C^\infty(\mathbb{R};\mathbb{R}_+)$ such that $0 \le \rho \le 1$ and

$$\rho \equiv 1 \text{ on } (a', b')\,, \tag{351}$$

$$\rho \equiv 0 \text{ on } (-1, a) \cup (b, 1)\,. \tag{352}$$

We have

$$\int_Q (\mathcal{P}_n g) \frac{g\rho e^{-2\alpha}}{t(T-t)} dxdt = \int_0^T \int_{-1}^1 \left[\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2 |x|^{2\gamma} g\right] \frac{g\rho e^{-2\alpha}}{t(T-t)} dxdt.$$

Integrating by parts with respect to time and space, we obtain

$$\int_Q \frac{1}{2}\frac{\partial(g^2)}{\partial t}\frac{\rho e^{-2\alpha}}{t(T-t)} dxdt = \int_Q \frac{1}{2}|g|^2 \rho \left(\frac{2\alpha_t}{t(T-t)} + \frac{T-2t}{(t(T-t))^2}\right) e^{-2\alpha} dxdt$$

and

$$\begin{aligned}-\int_Q \frac{\partial^2 g}{\partial x^2}\frac{g\rho e^{-2\alpha}}{t(T-t)} dxdt &= \int_Q \frac{\rho e^{-2\alpha}}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 dxdt \\ &- \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)}\left(\rho'' - 4\rho'\alpha_x + \rho(4\alpha_x^2 - 2\alpha_{xx})\right) dxdt\,.\end{aligned} \tag{353}$$

Thus,

$$\begin{aligned}&\int_Q \mathcal{P}_n g \frac{g\rho e^{-2\alpha}}{t(T-t)} dxdt \geqslant \int_Q \frac{\rho e^{-2\alpha}}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 dxdt \\ &- \int_Q \frac{|g|^2 e^{-2\alpha}}{2t(T-t)}\left(\rho'' - 4\rho'\alpha_x + \rho\left(4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)}\right)\right) dxdt\,.\end{aligned} \tag{354}$$

Therefore,

$$\begin{aligned}
&\int_0^T \int_{a'}^{b'} \frac{C_{10}M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 e^{-2\alpha}dxdt \\
&\leqslant \int_Q \frac{C_{10}M\rho}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 e^{-2\alpha}dxdt \leqslant \int_Q \mathcal{P}_n g \frac{C_{10}Mg\rho e^{-2\alpha}}{t(T-t)}dxdt \\
&\quad + \int_Q \frac{C_{10}M|g|^2e^{-2\alpha}}{2t(T-t)}\left(\rho'' - 4\rho'\alpha_x + \rho\left(4\alpha_x^2 - 2\alpha_{xx} - 2\alpha_t - \frac{T-2t}{t(T-t)}\right)\right)dxdt \\
&\leqslant \int_Q |\mathcal{P}_n g|^2 e^{-2\alpha}dxdt + \int_0^T\int_a^b \frac{C_{12}M^3|g|^2e^{-2\alpha}}{(t(T-t))^3}dxdt
\end{aligned}$$

for some constant $C_{12} = C_{12}(\beta,\rho) > 0$. Combining (350) with the previous inequality, we get

$$\begin{aligned}
&\int_Q \left(\frac{C_7M}{t(T-t)}\left|\frac{\partial g}{\partial x}\right|^2 + \frac{C_3M^3|g|^2}{4(t(T-t))^3}\right)e^{-2\alpha}dxdt \\
&\leqslant \int_Q 2|e^{-\alpha}\mathcal{P}_n g|^2dxdt + \int_0^T\int_a^b \frac{C_{13}M^3|g|^2}{(t(T-t))^3}e^{-2\alpha}dxdt\,,
\end{aligned} \tag{355}$$

where $C_{13} = C_{13}(\beta,\rho) := C_{11} + C_{12}$. Then, the global Carleman estimates (323) holds with

$$\mathcal{C}_1 = \mathcal{C}_1(\beta) := \frac{\min\{C_7; C_3/4\}}{\max\{2; C_{13}\}}\,.$$

Second case: $\gamma \in (0,1/2)$ The previous strategy does not apply to $\gamma \in (0,1/2)$ because the term $(n\pi)^2[|x|^{2\gamma}\alpha_x]_x$ (that diverges at $x=0$) in (344) can no longer be bounded by $\frac{C_3M^3}{(t(T-t))^3}$ (which is bounded at $x=0$). Note that both terms are of the same order as M^3, because of the dependence of M with respect to n in (346). In order to deal with this difficulty, we adapt the choice of the weight β and the dependence of M with respect to the parameter n.

Let $\beta \in C^1([-1,1]) \cap C^2([-1,0)\cup(0,1])$ be such that

$$\beta'' < 0 \text{ on } [-1,0)\cup(0,a']\cup[b',1] \tag{356}$$

and β has the following form on a neighborhood $(-\varepsilon,\varepsilon)$ of 0

$$\beta(x) = \mathcal{C}_0 - \int_0^x \sqrt{\operatorname{sign}(s)|s|^{2\gamma} + \mathcal{C}_1}ds \qquad \forall x \in (-\varepsilon,\varepsilon)\,, \tag{357}$$

where $\mathcal{C}_0$, $\mathcal{C}_1$ are large enough to ensure that $\beta(x) \geqslant 1$, and $\operatorname{sign}(s)|s|^{2\gamma} + \mathcal{C}_1 \geq 0$ on $(-\varepsilon,\varepsilon)$. Notice that

$$\beta'(x) = -\sqrt{\operatorname{sign}(x)|x|^{2\gamma} + \mathcal{C}_1} \qquad \forall x \in (-\varepsilon,\varepsilon)\,, \tag{358}$$

thus β'' diverges at $x = 0$. Performing the same computations as in the previous case, we get to inequality (340). Notice that one obtains (338) even if $\gamma \in (0, 1/2)$: the boundary terms vanish and $x \mapsto |x|^{2\gamma-1}$ is integrable at $x = 0$. Then, owing to (326) and (356), there exist $m_1 = m_1(\beta) > 0$, $C_3 = 1/2$ and $C_4 = C_4(\beta) > 0$ such that, for every $M \geqslant M_1$ and $t \in (0, T)$,

$$\begin{aligned}
&-\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \\
&\geqslant \frac{C_3 M^3}{[t(T-t)]^3}|\beta''(x)|\,\beta'(x)^2 \qquad \forall x \in [-1, 0) \cup (0, a'] \cup [b', 1]\,,
\end{aligned}$$

and

$$\Big| -\frac{1}{2}(\alpha_t - \alpha_x^2)_t + [(\alpha_t - \alpha_x^2)\alpha_x]_x - \frac{1}{2}\alpha_{xx}^2 \Big| \leqslant \frac{C_4 M^3}{[t(T-t)]^3} \qquad \forall x \in [a', b']\,,$$

where $M_1 = M_1(T, \beta)$ is defined by (343). In view of (340) and (356), for every $M \geqslant M_1$,

$$\begin{aligned}
&\int_0^T \int_{(-1,a')\cup(b',1)} \left[\frac{C_3 M^3}{(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2\,|z|^2 + (n\pi)^2(|x|^{2\gamma}\alpha_x)_x|z|^2\right] dxdt \\
&\leqslant \int_0^T\int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)}\Big|\frac{\partial z}{\partial x}\Big|^2 + \frac{C_4 M^3}{(t(T-t))^3}|z|^2\right] dxdt \\
&\quad - (n\pi)^2 \int_0^T\int_{a'}^{b'} (|x|^{2\gamma}\alpha_x)_x|z|^2\,dxdt\,.
\end{aligned} \tag{359}$$

Moreover,

$$\begin{aligned}
|(n\pi)^2(|x|^{2\gamma}\alpha_x)_x| &= (n\pi)^2\frac{M}{t(T-t)}\Big|2\gamma\,\mathrm{sign}(x)|x|^{2\gamma-1}\beta'(x) + |x|^{2\gamma}\beta''(x)\Big| \\
&\leqslant \frac{C_5 n^2 M}{t(T-t)}\Big(|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)|\Big) \quad \forall x \in (-1, 0) \cup (0, 1)\,,
\end{aligned}$$

where $C_5 = \pi^2(2\gamma + 1)$. From now on, we take

$$M = M(T, n, \beta) := \mathcal{C}_2 \max\{T + T^2; nT^2\}\,, \tag{360}$$

where

$$\mathcal{C}_2 = \mathcal{C}_2(\beta) := \max\left\{m_1, \frac{1}{\lambda}\right\}$$

and $\lambda = \lambda(\beta)$ is a (small enough) constant, that will be chosen later on. From $M \geqslant nT^2/\lambda$, we deduce that, for every $x \in (-1,0) \cup (0,1)$,

$$|(n\pi)^2(|x|^{2\gamma}\alpha_x)_x| \leqslant \frac{C_6\lambda^2 M^3}{(t(T-t))^3}\Big(|x|^{2\gamma-1}|\beta'(x)| + |x|^{2\gamma}|\beta''(x)|\Big)\,,$$

where $C_6 = C_6(\gamma) > 0$. Let us verify that, for $\lambda = \lambda(\beta) > 0$ small enough and for every $x \in (-1,0) \cup (0,a') \cup (b',1)$, we have

$$\begin{aligned}\frac{C_6\lambda^2 M^3}{(t(T-t))^3}|x|^{2\gamma-1}|\beta'(x)| &\leqslant \frac{C_3 M^3}{4(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2\,,\\ \frac{C_6\lambda^2 M^3}{(t(T-t))^3}|x|^{2\gamma}|\beta''(x)| &\leqslant \frac{C_3 M^3}{4(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2\,,\end{aligned}$$

or, equivalently, for every $x \in (-1,0) \cup (0,a') \cup (b',1)$,

$$\begin{aligned}C_6\lambda^2|x|^{2\gamma-1} &\leqslant \frac{C_3}{4}|\beta''(x)|\cdot|\beta'(x)|\,,\\ C_6\lambda^2|x|^{2\gamma} &\leqslant \frac{C_3}{4}\beta'(x)^2\,.\end{aligned} \tag{361}$$

The second inequality is easy to satisfy for $\lambda = \lambda(\beta)$ small enough, because $|\beta'| > 0$ on $[-1,a'] \cup [b',1]$. Thanks to (358), for every $x \in (-\varepsilon,\varepsilon)$,

$$\beta'(x)^2 = \operatorname{sign}(x)|x|^{2\gamma} + \mathcal{C}_1\,,$$

so

$$\beta''(x)\beta'(x) = \gamma|x|^{2\gamma-1}\,.$$

Therefore, for every $x \in (-\varepsilon,\varepsilon)\setminus\{0\}$, the first inequality in (361) is equivalent to

$$C_6\lambda^2 \leqslant \frac{C_3}{4}\gamma\,,$$

which is trivially satisfied when $\lambda = \lambda(\beta)$ is small enough. Moreover, the first inequality of (361) holds for every $x \in [-1,-\varepsilon] \cup [\varepsilon,a'] \cup [b',1]$ when $\lambda = \lambda(\beta)$ is small enough, since $|\beta''\beta'| > 0$ on this compact set. Finally, we have

$$\begin{aligned}&\int_0^T \int_{(-1,a')\cup(b',1)} \frac{C_3 M^3}{2(t(T-t))^3}|\beta''(x)|\,\beta'(x)^2\,|z|^2 dxdt\\ &\leqslant \int_0^T\int_{a'}^{b'}\left[\frac{C_2 M}{t(T-t)}\left|\frac{\partial z}{\partial x}\right|^2 + \frac{C_5 M^3}{(t(T-t))^3}|z|^2\right]dxdt\,,\end{aligned} \tag{362}$$

where $C_5 = C_5(\beta) > 0$. Since the function $|\beta''|(\beta')^2$ is bounded from below by some positive constant on $[-1, a'] \cup [b', 1]$, we also have

$$\begin{aligned}
&\int_0^T \int_{(-1,a')\cup(b',1)} \frac{C_6 M^3}{2(t(T-t))^3}|z|^2 dxdt \\
&\leqslant \int_0^T \int_{a'}^{b'} \left[\frac{C_2 M}{t(T-t)}\left|\frac{\partial z}{\partial x}\right|^2 + \frac{C_5 M^3}{(t(T-t))^3}|z|^2\right] dxdt\,,
\end{aligned} \tag{363}$$

where $C_6 = C_6(\beta) > 0$. The rest of the proof goes as for $\gamma \in [1/2, 1]$. □

4.4.2 – Uniform observability

The Carleman estimate of Proposition 4.20 allows to prove the following uniform observability result.

Proposition 4.22. *Let $\gamma \in (0,1)$ and let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $C > 0$ such that for every $T > 0$, $n \in \mathbb{N}^*$, and $g_{0,n} \in L^2(-1,1)$ the solution of* (295) *satisfies*

$$\int_{-1}^1 g_n(x,T)^2 dx \leqslant T^2 e^{C\left(1+T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_a^b g_n(x,t)^2 dxdt.$$

Let us recall that explicit bounds on the observability constant of the heat equation with a potential are already known.

Theorem 4.23. *Let $-1 < a < b < 1$. There exists $c > 0$ such that, for every $T > 0$, α, $\beta \in L^\infty((-1,1)\times(0,T))$, $g_0 \in L^2(-1,1)$, the solution of*

$$\begin{cases} \partial_t g - \partial_x^2 g + \beta\partial_x g + \alpha g = 0 & (x,t) \in (-1,1)\times[0,T]\,, \\ g(\pm 1, t) = 0 & t \in [0,T]\,, \\ g(x,0) = g_0(x) & x \in (-1,1)\,, \end{cases}$$

satisfies

$$\int_{-1}^1 |g(x,T)|^2 dx \leqslant e^{cH(T,\|\alpha\|_\infty,\|\beta\|_\infty)} \int_0^T \int_a^b |g(x,t)|^2 dxdt\,,$$

where $H(T,A,B) := 1 + \frac{1}{T} + TA + A^{2/3} + (1+T)B^2$.

For the proof of the above result we refer to [85, Theorem 1.3] in the case of $\beta \equiv 0$, and to [68, Theorem 2.3] for $\beta \not\equiv 0$. The optimality of the power $2/3$ of A in $H(T,A,B)$ has been proved in [70].

Proposition 4.22 may be seen as an improvement of the above estimate (relatively to the asymptotic behavior as $n \to +\infty$), in the special case of (295).

Proof of Proposition 4.22. We derive an explicit observability constant from the Carleman estimate of Proposition 4.20. For $t \in (T/3, 2T/3)$, we have

$$\frac{4}{T^2} \leqslant \frac{1}{t(T-t)} \leqslant \frac{9}{2T^2}$$

and

$$\int_{-1}^{1} g(x,T)^2 dx \leqslant e^{-\frac{2}{3}\lambda_n T} \int_{-1}^{1} g(x,t)^2 dx \,.$$

Thus,

$$\mathcal{C}_1 \frac{64M^3}{T^6} e^{-\frac{9M\beta^*}{2T^2}} \frac{T}{3} e^{\frac{2}{3}\lambda_n T} \int_{-1}^{1} g(x,T)^2 dx \leqslant \mathcal{C}_3 \int_0^T \int_a^b g(x,t)^2 dx dt$$

where $\beta^* := \max\{\beta(x) : x \in [-1,1]\}$, $\beta_* := \min\{\beta(x) : x \in [-1,1]\}$ and $\mathcal{C}_3 := \max\{x^3 e^{-\beta_* x}\}$. Using the inequality $M \geqslant \mathcal{C}_2[T+T^2]$ and Proposition 4.15, we get

$$\int_{-1}^{1} g(x,T)^2 dx \leqslant \mathcal{C}_4 T^2 e^{c_1 \frac{M}{T^2} - c_2 n^{\frac{2}{1+\gamma}} T} \int_0^T \int_a^b g(x,t)^2 dx dt \qquad (364)$$

for some constants $c_1, c_2, \mathcal{C}_4 > 0$ (independent of n, T and g).

First case: $n < 1 + \frac{1}{T}$. Then, $M = \mathcal{C}_2(T+T^2)$ thus

$$\int_{-1}^{1} g(x,T)^2 dx \leqslant \mathcal{C}_4 T^2 e^{c_1 \mathcal{C}_2 \left(1+\frac{1}{T}\right)} \int_0^T \int_a^b g(x,t)^2 dx dt.$$

Second case: $n \geqslant 1 + \frac{1}{T}$. Then, $M = \mathcal{C}_2 n T^2$. The maximum value of the function $x \mapsto c_1 \mathcal{C}_2 x - c_2 x^{\frac{2}{1+\gamma}} T$ on $(0,+\infty)$ is of the form $c_3 T^{-\frac{1+\gamma}{1-\gamma}}$ for some constant $c_3 > 0$ (independent of T). Thus,

$$\int_{-1}^{1} g(x,T)^2 dx \leqslant \mathcal{C}_4 T^2 e^{c_3 T^{-\frac{1+\gamma}{1-\gamma}}} \int_0^T \int_a^b g(x,t)^2 dx dt.$$

This gives the conclusion. □

In the case of $\gamma = 1$, we also have the following result.

Proposition 4.24. *Assume $\gamma = 1$. Let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$. Then there exists $T_1 > 0$ such that, for every $T > T_1$, system (295) is observable in (a,b) in time T uniformly with respect to $n \in \mathbb{N}^*$.*

Proof. One can follow the lines of the previous proof until (364). Then, for $n \geqslant 1 + \frac{1}{T}$, we have $M = \mathcal{C}_2 n T^2$. Thus,

$$\int_{-1}^{1} g(x,T)^2 dx \leqslant \mathcal{C}_4 T^2 e^{[c_1 \mathcal{C}_2 - c_2 T] n} \int_0^T \int_a^b g(x,t)^2 dx dt.$$

This proves Proposition 4.24 with $T_1 := c_1 \mathcal{C}_2 / c_2$. □

4.4.3 – Construction of the control function for $\gamma \in (0,1)$

The goal of this section is the proof of null controllability in any time $T > 0$ for $\gamma \in (0,1)$. Our construction of the control steering the initial state to zero is the one of [27], which is in turn inspired by [112] (see also [113]).

For $n \in \mathbb{N}^*$, we define $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$ and $H_n := L^2(-1,1) \otimes \varphi_n$, which is a closed subspace of $L^2(\Omega)$. For $j \in \mathbb{N}$, we define $E_j := \oplus_{n \leqslant 2^j} H_n$ and denote by Π_{E_j} the orthogonal projection onto E_j.

Proposition 4.25. *Let $\gamma \in (0,1)$, and let $a, b, c, d \in \mathbb{R}$ be such that $0 < a < b < 1$ and $0 < c < d < 1$. Then there exists a constant $C > 0$ such that for every $T > 0$, every $j \in \mathbb{N}^*$, and every $g_0 \in E_j$ the solution of* (281) *satisfies*

$$\int_\Omega g(x,y,T)^2 dxdy \leqslant T^2 e^{C\left(2^j + T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_\omega g(x,y,t)^2 dxdydt$$

where $\omega := (a,b) \times (c,d)$.

For the proof of Proposition 4.25 we shall need the following inequality obtained in [112] (see also [113]).

Proposition 4.26. *Let $c, d \in \mathbb{R}$ be such that $c < d$. There exists $C > 0$ such that, for every $L \in \mathbb{N}^*$ and $(b_k)_{1 \leqslant k \leqslant L} \in \mathbb{R}^L$,*

$$\sum_{k=1}^{L} |b_k|^2 \leqslant e^{CL} \int_c^d \left| \sum_{k=1}^{L} b_k \varphi_k(y) \right|^2 dy.$$

Proof of Proposition 4.25. Let $(g_{0,n})_{1 \leqslant n \leqslant 2^j} \in L^2(-1,1)^{2^j}$ be such that

$$g_0(x,y) = \sum_{n=1}^{2^j} g_{0,n}(x)\varphi_n(y)\,.$$

Then the solution of (281) is given by

$$g(x,y,t) = \sum_{n=1}^{2^j} g_n(x,t)\varphi_n(y)$$

where, for every $n \in \mathbb{N}^*$, g_n is the solution of (295). Applying Propositions 4.22 and 4.26, and recalling that $(\varphi_n)_{n \in \mathbb{N}^*}$ is an orthonormal sequence of $L^2(0,1)$, we deduce

$$
\begin{aligned}
\int_\Omega g(x,y,T)^2 dxdy &= \sum_{n=1}^{2^j} \int_{-1}^{1} g_n(x,T)^2 dx \\
&\leqslant T^2 e^{C\left(1+T^{-\frac{1+\gamma}{1-\gamma}}\right)} \sum_{n=1}^{2^j} \int_0^T \int_a^b g_n(x,t)^2 dxdt \\
&\leqslant T^2 e^{C\left(2^j+T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_a^b \int_c^d \left| \sum_{n=1}^{2^j} g_n(x,t)\varphi_n(y) \right|^2 dydxdt \\
&= T^2 e^{C\left(2^j+T^{-\frac{1+\gamma}{1-\gamma}}\right)} \int_0^T \int_\omega g(x,y,t)^2 dxdydt\,,
\end{aligned}
$$

where the constant C may change from line to line. □

Let $T > 0$ and $u_0 \in L^2(\Omega)$. We now proceed to construct a control $f \in L^2(0,T;L^2(\Omega))$ such that the solution of (279) satisfies $u(\cdot,T) \equiv 0$. Fix $\rho \in \mathbb{R}$ with

$$0 < \rho < \frac{1-\gamma}{1+\gamma} \tag{365}$$

and let $K = K(\rho) > 0$ be such that $K\sum_{j=1}^{\infty} 2^{-j\rho} = T$. Let $(a_j)_{j\in\mathbb{N}}$ be defined by

$$
\begin{cases}
a_0 = 0 \\
a_{j+1} = a_j + 2T_j\,, \quad j \geqslant 0\,,
\end{cases}
$$

where $T_j := K2^{-j\rho}$ for every $j \in \mathbb{N}$. We now define the control f in the following way. On $[a_j, a_j + T_j]$, we apply a control f such that $\Pi_{E_j} u(\cdot, a_j + T_j) = 0$ and

$$\|f\|_{L^2(a_j,a_j+T_j;L^2(\Omega))} \leqslant \mathcal{C}_j \|u(\cdot,a_j)\|_{L^2(\Omega)}$$

where, in view of Proposition 4.25,

$$\mathcal{C}_j := e^{C\left(2^j+T_j^{-\frac{1+\gamma}{1-\gamma}}\right)}\,.$$

Observe that, in light of (292),

$$\|u(\cdot,a_j+T_j)\|_{L^2(\Omega)} \leqslant (1+\sqrt{T_j}\mathcal{C}_j)\|u(\cdot,a_j)\|_{L^2(\Omega)}.$$

Then, on the interval $[a_j + T_j, a_{j+1}]$ we apply no control in order to take advantage of the natural exponential decay of the solution, thus obtaining

$$\|u(\cdot,a_{j+1})\|_{L^2(\Omega)} \leqslant e^{-\lambda_{2^j}T_j}\|u(\cdot,a_j+T_j)\|_{L^2(\Omega)}\,,$$

where λ_n is defined in (301). Combining the above inequalities, we conclude that

$$\|u(\cdot, a_{j+1})\|_{L^2(\Omega)} \leqslant \exp\Big(\sum_{k=1}^{2^j} \big[\ln(1 + \sqrt{T_k}\mathcal{C}_k) - C(2^k)^{\frac{2}{1+\gamma}} T_k \big] \Big) \|u_0\|_{L^2(\Omega)} \,.$$

The choice of ρ ensures that the sum in the exponential diverges to $-\infty$ as $j \to +\infty$, forcing $u(\cdot, T) \equiv 0$. The fact that $f \in L^2(0,T; L^2(\Omega))$ can be checked by similar arguments. □

4.4.4 – End of the proof of Theorems 4.2 and 4.5

Let ω be an open subset of $(0,1) \times (0,1)$. There exists $a, b, c, d \in \mathbb{R}$ with $0 < a < b < 1$, $0 < c < d < 1$ such that $(a,b) \times (c,d) \subset \omega$.

The first (resp. third) statement of Theorem 4.5 has been proved in Section 4.4.3 (resp. Section 4.3); let us prove the second one.

Let us consider $\gamma = 1$ and $\omega = (a,b) \times (0,1)$. From Proposition 4.24, we deduce that system (280) is observable in ω in any time $T > T_1$. From Theorem 4.17, we deduce that for any time $T < \frac{a^2}{2}$, system (280) is not observable in ω in time T. Thus, the quantity

$$T^* := \inf\{T > 0\,;\ \text{system (280) is observable in } \omega \text{ in time } T\ \}$$

is well defined and belongs to $[\frac{a^2}{2}, +\infty)$. Clearly, observability in some time $T_\sharp$ implies observability in any time $T > T_\sharp$, so

- for every $T > T^*$, (281) is observable in ω in time T,
- for every $T < T^*$, (281) is not observable in ω in time T.

Moreover, Miller [128] has recently proved the equality $T^* = \frac{a^2}{2}$.

4.5 – Conclusion and open problems

In this chapter we have studied the null controllability of the generalized Grushin equation (278), in the rectangle $\Omega = (-1,1) \times (0,1)$, with a distributed control localized on an open subset ω of $(0,1) \times (0,1)$. We have proved that null controllability:

- holds in any positive time, when degeneracy is not too strong, *i.e.* $\gamma \in (0,1)$,
- holds only in large time, when $\gamma = 1$ and ω is a strip parallel to the y-axis,
- does not hold when degeneracy is too strong, *i.e.* $\gamma > 1$.

Null controllability when $\gamma = 1$, T is large enough, and the control region ω is more general is an open problem.

The technique of this chapter should possibly extend to higher dimensional cylindrical domains of the form $(-1,1)\times(0,1)^m$. However, the generalization of this result to other muldimensional configurations (including $x \in (-1,1)^n$, $y \in (0,1)^m$ with $m, n \geqslant 1$) or boundary controls, is widely open.

4.6 – Appendix: the case when $\{x = 0\} \subset \omega$

In this appendix we briefly explain why null controllability holds when degeneracy occurs inside the control region. Consider the control system

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(x,y,t)1_\omega(x,y) & (x,y,t) \in \Omega \times (0,T) \\ u(x,y,t) = 0 & (x,y,t) \in \partial\Omega \times (0,T) \\ u(x,y,0) = u_0(x,y) & (x,y) \in \Omega \end{cases} \tag{366}$$

with $\omega = (-a,a)\times(0,1)$, $0 < a \leq 1$. Fix $b \in (0,a)$ and choose cut-off functions $\xi_i \in C^\infty(\mathbb{R})$, $i = 0,1,2$, such that $0 \leq \xi_i \leq 1$ and

$$\begin{cases} \xi_0 + \xi_1 + \xi_2 \equiv 1 \\ \xi_0(x) = 1 \quad \text{if } |x| \leq b, & \xi_0(x) = 0 \quad \text{if } |x| \geq a \\ \xi_1(x) = 0 \quad \text{if } x \leq b, & \xi_0(x) = 1 \quad \text{if } x \geq a \\ \xi_2(x) = 1 \quad \text{if } x \leq -a, & \xi_2(x) = 0 \quad \text{if } x \geq -b \end{cases} \tag{367}$$

Let $\omega_1 = (b,a)\times(0,1)$ and let $\Omega_1 = (b,1)\times(0,1)$. There exists a control $f_1 \in L^2((0,T)\times\Omega_1)$ such that the solution u_1 of

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f_1(x,y,t)1_{\omega_1}(x,y) & (x,y,t) \in \Omega_1 \times (0,T) \\ u(x,y,t) = 0 & (x,y,t) \in \partial\Omega_1 \times (0,T) \\ u(x,y,0) = u_0(x,y) & (x,y) \in \Omega_1 \end{cases}$$

satisfies $u_1(T,\cdot) \equiv 0$ on Ω_1. Similarly, let $\omega_2 = (-a,-b)\times(0,1)$ and let $f_2 \in L^2((0,T)\times\Omega_2)$, where $\Omega_2 = (-1,-b)\times(0,1)$, be such that the solution u_2 of

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f_2(x,y,t)1_{\omega_2}(x,y) & (x,y) \in \Omega_2 \times (0,T) \\ u(x,y,t) = 0 & (x,y,t) \in \partial\Omega_2 \times (0,T) \\ u(x,y,0) = u_0(x,y) & (x,y) \in \Omega_2 \end{cases}$$

satisfies $u_2(T,\cdot) \equiv 0$ on Ω_2. Finally, let $\Omega_0 = (-a,a)\times(0,1)$ and let u_3 be the solution of

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = 0 & (t,x,y) \in \Omega_0 \times (0,T) \\ u(x,y,t) = 0 & (x,y,t) \in \partial\Omega_0 \times (0,T) \\ u(x,y,0) = \xi_0(x)u_0(x,y) & (x,y) \in \Omega_0\,. \end{cases}$$

Then

$$u(x,y,t) := \xi_1(x)u_1(x,y,t) + \xi_2(x)u_2(x,y,t) + \frac{T-t}{T}u_3(x,y,t)$$

satisfies (366) for a suitable control f, as well as $u(\cdot,T) \equiv 0$ on Ω.

5 – The Grushin operator with singular potential

The present chapter is part of the article *Null controllability in large times for the parabolic Grushin equation with singular potential*, in G. Stefani, U. Boscain, J.-P. Gauthier, A. Sarychev, M. Sigalotti(eds.): Geometric Control Theory and sub-Riemannian Geometry, Springer INdAM Series 5, 87-102, 2013, in collaboration with Piermarco Cannarsa [42].

5.1 – Introduction

In Chapter 4 we have provided a complete range of controllability properties (with respect to the values of $\gamma > 0$ and $T > 0$) for the generalized Grushin equation

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(x,y,t)1_\omega(x,y) & (x,y,t) \in D \times (0,T)\,, \\ u(x,y,t) = 0 & (x,y,t) \in \partial D \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) \in L^2(D)\,, \end{cases} \tag{368}$$

where $D := (-1,1) \times (0,1)$ and $\omega \subset (0,1) \times (0,1)$. We can summarize the controllability result in Chapter 4 as follows:

1. If $\gamma \in (0,1)$, then system (368) is null controllable in any time $T > 0$.
2. If $\gamma = 1$ and $\omega = (a,b) \times (0,1)$ where $0 < a < b \leqslant 1$, then
   - for every $T > \frac{a^2}{2}$ system (368) is null controllable in time T,
   - for every $T < \frac{a^2}{2}$ system (368) is not null controllable in time T.
3. If $\gamma > 1$, then (368) is not null controllable.

On the other hand, the controllability property for the operator in system (368) is in general sensitive to lower order perturbations. Indeed, a result in [33] shows that, for all $\gamma \geq 1$, the dynamics ruled by the operator

$$Lu = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u - \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{x^2}u \tag{369}$$

separates the two connected component of $D \setminus \{0\} \times [0,1]$, where $\{0\} \times [0,1]$ is the singular set for the γ-Grushin metric generated by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ |x|^\gamma \end{pmatrix}, \quad \gamma \geq 1\,.$$

Thus, there is no transmission of information across the singular set. In turn, it implies that in the case $\gamma \geq 1$ no controllability results can be sought for the equation

$$\begin{cases} \partial_t u - Lu = f(x,y,t)1_\omega(x,y) & \text{in } D \times (0,\infty)\,, \\ u(x,y,t) = 0 & \text{on } \partial D \times (0,\infty)\,, \end{cases} \tag{370}$$

when ω lies in only one connected component of $D \setminus \{0\} \times [0,1]$, the case accounted in Chapter 4.

Thus, we are naturally led to face the following question: which controllability properties do hold for the operator L?

In this chapter we establish a partial (positive) answer to the above question. Indeed, we show a null controllability result for all sufficiently large times, in the case $\gamma = 1$, restricting the domain to one side only of the singular set. More precisely, posed $\Omega := (0,1) \times (0,1)$, we address the null controllability problem for the equation

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^2 \partial_y^2 u - \frac{\lambda}{x^2} u = f(x,y,t)1_\omega(x,y) & \text{in } \Omega \times (0,T)\,, \\ u(x,y,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) \in L^2(\Omega)\,, \end{cases} \tag{371}$$

where $T > 0$, $\lambda \in \mathbb{R}$ and ω is an open subset of Ω. The following result holds.

Theorem 5.1. *Let $\omega = (a,b) \times (0,1)$ for some $0 < a < b \leqslant 1$ and $\lambda < 1/4$. Then there exists $T^* > 0$ such that for every $T > T^*$ system (371) is null controllable in time T.*

Thus, also for the Grushin operator with singular potential, the case $\gamma = 1$ turns out to be a transition regime, needing a minimum time for the null controllability, as in the case addressed in Chapter 4.

By a standard duality argument, Theorem 5.1 is equivalent to the observability in large times from ω for the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^2 \partial_y^2 g - \frac{\lambda}{x^2} g = 0 & \text{in } \Omega \times (0,T)\,, \\ g(x,y,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ g(x,y,0) = g_0(x,y) \in L^2(\Omega)\,. \end{cases} \tag{372}$$

Thanks to a suitable Carleman estimate (see Proposition 5.10), we will prove the following result.

Theorem 5.2. *Let $\omega = (a,b) \times (0,1)$ for some $0 < a < b \leqslant 1$ and $\lambda < 1/4$. Then there exists $T^* > 0$ such that for every $T > T^*$ system (372) is observable in ω in time T.*

As a consequence, we deduce null controllability in large times also for equation (370), with a control region located on both sides of the degeneracy, of the type $w = (a_1, b_1) \cup (a_2, b_2) \times (0, 1)$, with $-1 \le a_1 < b_1 < 0 < a_2 < b_2 \le 1$.

For the time being, the Carleman estimate that we implement allows to reach the observability only in the case $\gamma = 1$, though we expect a similar result (without minimum time) also in the case $0 < \gamma < 1$, just like in Chapter 4 (see also Remark 5.12).

For future reference, in Section 5.2 and 5.3 we will treat the general case of an operator $Lu = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u + \frac{\lambda}{x^2}u$, with $\gamma > 0$. From Section 5.4 on we will focus on the case $\gamma = 1$.

5.2 – Well-posedness and Fourier decomposition

5.2.1 – Well-posedness of the Cauchy-problem

Let $H := L^2(\Omega)$, and denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|_H$, respectively, the scalar product and norm in H. We recall the well-known Hardy's inequality [63]

$$\int_0^1 \frac{z^2}{x^2}dx \le 4\int_0^1 z_x^2 dx \qquad \forall z \in H_0^1(0,1). \tag{373}$$

Thanks to (373), the scalar product

$$(u, v) := \int_\Omega \left(u_x v_x + |x|^{2\gamma}u_y v_y - \frac{\lambda}{x^2}uv\right) dxdy \qquad \forall u,\, v \in C_0^\infty(\Omega) \tag{374}$$

is positive for every $\lambda < 1/4$ (as we will assume from now on). Set $W := \overline{C_0^\infty(\Omega)}^{|\cdot|_W}$, where $|u|_W := (u, u)^{1/2}$, and observe that $H_0^1(\Omega) \subset W \subset H$, thus W is dense in H. Introduce the space $V := \overline{C_0^\infty(\Omega)}^{|\cdot|_V}$ as in Chapter 4, where $|u|_V := ((u, u))^{1/2}$ and

$$((u, v)) := \int_\Omega \left(u_x v_x + |x|^{2\gamma}u_y v_y\right) dxdy \quad \forall u,\, v \in C_0^\infty(\Omega).$$

Hardy's inequality (373) ensures that, for all $z \in C_0^\infty(\Omega)$, holds $(z, z) \ge C_\lambda((z, z))$, with $C_\lambda := 1 - 4\lambda > 0$. Thus $W \subset V$, and from Lemma 4.12 we deduce that for every g in W there exist $\partial_x g \in L^2(\Omega)$, $\partial_y g \in L^2_\gamma(\Omega)$ such that for every $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}&\int_\Omega \left(g(x,y)\partial_x\phi(x,y) + |x|^{2\gamma}g(x,y)\partial_y\phi(x,y)\right) dxdy\\ &= -\int_\Omega \left(\partial_x g(x,y) + |x|^{2\gamma}\partial_y g(x,y)\right)\phi(x,y)dxdy.\end{aligned} \tag{375}$$

Define now

$$D(A) = \{u \in W : \exists\, c > 0 \text{ such that } |(u,h)| \le c|h|_H \quad \forall h \in W\}\ , \tag{376}$$

$$\langle Au, h\rangle := -(u,h) \qquad \forall h \in W\,. \tag{377}$$

Then (see [141], Theorem 1.18), the operator $(A, D(A))$ generates an analytic semigroup $S(t)$ of contractions on H. Note that A is selfadjoint on H, and (377) implies that

$$Au = \partial_x^2 u + |x|^{2\gamma}\partial_y^2 u + \frac{\lambda}{x^2}u \qquad \text{a.e. in } \Omega\,.$$

So, system (371) can be recast in the form

$$\begin{cases} u'(t) = Au(t) + f(t) & t \in [0,T]\,, \\ u(0) = u_0\,, \end{cases} \tag{378}$$

where $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in H$.

Definition 5.3 (Weak solution). A function $u \in C([0,T];H) \cap L^2(0,T;W)$ is a weak solution of system (378) if for every $h \in D(A)$ the function $\langle u(t), h\rangle$ is absolutely continuous on $[0,T]$ and for a.e. $t \in [0,T]$

$$\frac{d}{dt}\langle u(t), h\rangle = \langle u(t), Ah\rangle + \langle f(t), h\rangle\,. \tag{379}$$

In [114] it is shown the equivalence between condition (379) and the definition of solution by transposition, that is,

$$\begin{aligned} &\int_\Omega [u(x,y,t^*)\varphi(x,y,t^*) - u_0(x,y)\varphi(x,y,0)]dxdy \\ &= \int_0^{t^*}\int_\Omega \left\{u\left(\partial_x^2\varphi + |x|^{2\gamma}\partial_y^2\varphi + \frac{\lambda}{x^2}\varphi\right) + f\varphi\right\} dxdydt \end{aligned}$$

for every $\varphi \in C^2([0,T]\times\Omega)$ and $t^* \in (0,T)$.

Moreover, the unique weak solution of (378) in the sense of Definition 5.3 is given by the variations-of-constants formula (see [18])

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds\,, \quad t \in [0,T]\,. \tag{380}$$

The following existence and uniqueness result follows.

Proposition 5.4. *For every $u_0 \in H$, $T > 0$ and $f \in L^2(0,T;H)$, there exists a unique weak solution of the Cauchy problem* (378). *This solution satisfies*

$$|u(t)|_H \leqslant |u_0|_H + \sqrt{T}\|f\|_{L^2(0,T;H)} \quad \forall t \in [0,T]\,. \tag{381}$$

Moreover, $u(t) \in D(A)$ and $u'(t) \in H$ for a.e. $t \in (0,T)$.

5.2.2 – Fourier decomposition of the solution

Let $g \in C([0,T];H) \cap L^2(0,T;W)$ be the solution of equation (372) in the sense of Definition 5.3. Thus, the function $y \mapsto g(x,y,t)$ belongs to $L^2(0,1)$ for a.e. $(x,t) \in (0,1) \times (0,T)$, and we can develop g in Fourier series with respect to y

$$g(x,y,t) = \sum_{n\in\mathbb{N}^*} g_n(x,t)\varphi_n(y)\,, \tag{382}$$

where for all $n \in \mathbb{N}^*$ we set $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$ and

$$g_n(x,t) := \int_0^1 g(x,y,t)\varphi_n(y)dy\,. \tag{383}$$

Proposition 5.5. *For every $n \geq 1$, g_n is the unique weak solution of*

$$\begin{cases} \partial_t g_n - \partial_x^2 g_n + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] g_n = 0 & (x,t) \in (0,1)\times(0,T)\,,\\ g_n(0,t) = g_n(1,t) = 0 & t \in (0,T)\,,\\ g_n(x,0) = g_{0,n}(x) & x \in (0,1)\,, \end{cases} \tag{384}$$

where $g_{0,n} \in L^2(0,1)$ is given by $g_{0,n}(x) := \displaystyle\int_0^1 g_0(x,y)\varphi_n(y)dy$.

Proof. First, observe that, for any $n \geq 1$, system (384) is a first order Cauchy problem, that admits the unique weak solution

$$\tilde g_n \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$$

which satisfies

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^1 \tilde g_n(x,t)\psi(x)dx\right)\\ &\quad + \int_0^1 \left[\tilde g_{n,x}(x,t)\psi_x(x) + \left((n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right)\tilde g_n(x,t)\psi(x)\right]dx = 0 \end{aligned} \tag{385}$$

for every $\psi \in H_0^1(0,1)$.

In order to verify that the nth Fourier coefficient of g, defined by (383), satisfies system (384), observe that

$$g_n(\cdot,0) = \int_0^1 g_0(y,\cdot)dy = g_{n,0}(\cdot)\,, \qquad g_n(0,t) = g_n(1,t) = 0 \quad \forall t \in (0,T)$$

and

$$g_n \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))\,.$$

Thus, it is sufficient to prove that g_n fulfills condition (385). Indeed, using the identity (383), for all $\psi \in H_0^1(0,1)$ we obtain, for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 g_n\psi dx\right) + \int_0^1 \left(g_{n,x}\psi_x + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] g_n\psi\right) dx \\
&= \int_0^1\int_0^1 \left\{g_t\varphi_n\psi + g_x\varphi_n\psi_x + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] g\varphi_n\psi\right\} dydx\,.
\end{aligned} \tag{386}$$

Observe that Proposition 5.4 ensures $g_t(\cdot,t) \in L^2(\Omega)$ and $g(\cdot,t) \in D(A)$ for a.e. $t \in (0,T)$. So, multiply $g_t = Ag$ by $h(x,y) = \psi(x)\varphi_n(y) \in W$ and integrate over Ω, in order to obtain, for a.e. $t \in (0,T)$,

$$\begin{aligned}
\int_0^1\int_0^1 g_t\psi\varphi_n dxdy &= \int_0^1\int_0^1 Ag\psi\varphi_n dxdy \\
&= -\int_0^1\int_0^1 \left(g_x\psi_x\varphi_n + |x|^{2\gamma} g_y\psi\varphi_{n,y} - \frac{\lambda}{x^2} g\psi\varphi_n\right) dxdy \\
&= -\int_0^1\int_0^1 \left(g_x\psi_x\varphi_n + (n\pi)^2|x|^{2\gamma} g\psi\varphi_n - \frac{\lambda}{x^2} g\psi\varphi_n\right) dxdy\,,
\end{aligned} \tag{387}$$

where (in the last identity) we have used relation (375). Combining identities (386) and (387) completes the proof. $\square$

The unique continuation result for the adjoint system (372) can be readily derived.

PROPOSITION 5.6. *Let $T > 0$, $\gamma > 0$, $\lambda < 1/4$, ω an open subset of $(0,1)\times(0,1)$, and let $g \in C([0,T];H) \cap L^2(0,T;W)$ be a weak solution of system* (372). *If $g \equiv 0$ on $\omega \times (0,T)$, then $g \equiv 0$ on $\Omega \times (0,T)$.*

PROOF. Let $\epsilon > 0$ be such that $\omega \subset (\epsilon,1) \times (0,1)$. In the rectangle $(\epsilon,1) \times (0,1)$, equation (372) has none degenerate coefficients neither singular potential, so we are in the position to apply the unique continuation for uniformly parabolic $2-$D equation. Thus, the hypothesis $g \equiv 0$ on $\omega \times (0,T)$ implies that $g \equiv 0$ on $(\epsilon,1) \times (0,1) \times (0,T)$. Then, relation (383) ensures that $g_n \equiv 0$ on $(\epsilon,1) \times (0,T)$ for every $n \in \mathbb{N}^*$. Moreover, since $g_n \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$, in particular, for a.e. $t \in (0,T)$, we have $g_n(\cdot,t) \in H_0^1(0,1) \subset C([0,1])$. Thus, by continuity, we conclude that $g_n \equiv 0$ on $(0,1) \times (0,T)$ for every $n \in \mathbb{N}^*$ (compare also with the observability inequality in [139, Lemma 3.2(ii)]). Therefore, back to equation (382), we conclude that $g \equiv 0$ on $\Omega \times (0,T)$. $\square$

Remark 5.7. Thanks to Proposition 5.6, we derive that the Grushin operator with singular potential (371) is approximate controllable by a locally distributed control in an arbitrary open subset ω of Ω, for every $T > 0$, $\gamma > 0$ and $\lambda < 1/4$. In particular, the condition $\lambda < 1/4$ embraces the case of the operator (369) accounted in [33], whose potential coefficient $-\gamma/2(\gamma/2 + 1)$ is smaller than $1/4$ for every $\gamma \neq -1$.

5.3 – Spectral analysis for the 1−D problem

In the prospective of proving null controllability for equation (371), we now focus on the asymptotic behaviour (with respect to n) of the one dimensional eigenvalue problem. For this reason, let us introduce, for every $n \in \mathbb{N}^*$, $\gamma > 0$ and $\lambda < 1/4$, the operator $A_{n,\gamma,\lambda}$ on $L^2(0,1)$ by

$$\begin{aligned} D(A_{n,\gamma,\lambda}) := \big\{\varphi \in H_0^1(0,1) \ : \ &\varphi' \in AC((0,1]) \text{ and} \\ &-\varphi'' + \left[(n\pi)^2|x|^{2\gamma} - \tfrac{\lambda}{x^2}\right]\varphi \in L^2(0,1)\big\}\ , \\ A_{n,\gamma,\lambda}\varphi := -\varphi'' + &\left[(n\pi)^2|x|^{2\gamma} - \tfrac{\lambda}{x^2}\right]\varphi \in L^2(0,1)\,. \end{aligned} \tag{388}$$

The smallest eigenvalue of $A_{n,\gamma,\lambda}$ is given by

$$\mu_{n,\gamma,\lambda} = \min_{\substack{v \in H_0^1(0,1)\\ v \neq 0}} \left\{ \frac{\int_0^1 \left\{v'(x)^2 + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] v(x)^2\right\} dx}{\int_0^1 v(x)^2 dx} \right\}. \tag{389}$$

For simplicity, from now on we will refer to $A_{n,\gamma,\lambda}$ and $\mu_{n,\gamma,\lambda}$ just as A_n and μ_n. We mention that the case $n = 0$ has been investigated in [140], where well-posedness and observability are proven for the operator A_0. Here we would achieve a similar observability result for the general operator A_n, uniformly in n (and in γ and λ as well). We start by characterizing the behaviour of μ_n as $n \to +\infty$, that quantifies the dissipation speed of the solution of (384).

Lemma 5.8. *Problem*

$$\begin{cases} -v''_{n,\gamma,\lambda}(x) + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right] v_{n,\gamma,\lambda}(x) = \mu_n v_{n,\gamma,\lambda}(x) \\ v_{n,\gamma,\lambda}(0) = v_{n,\gamma,\lambda}(1) = 0\,, \end{cases} \tag{390}$$

admits a unique positive solution with $L^2(0,1)$-norm one.

Proof. Observe that the domain $D(A_n)$ of A_n is compactly embedded in $L^2(0,1)$, thus the resolvent operator of A_n is a compact operator. Then, there exists an orthonormal basis of $L^2(0,1)$ consisting of eigenvectors of A_n, and the first eigenvalue is simple. Moreover, the associated eigenfunction v is positive. Indeed, if not so, let us consider the function $w(x) = |v(x)|$. Then, w still belongs to $H_0^1(0,1)$, it is a weak solution of (390) and it does not increase the functional in (389). □

We next provide a precise growth condition for the eigenvalue μ_n, with respect to $n \in \mathbb{N}^*$.

PROPOSITION 5.9. *For every $\gamma > 0$ and $\lambda < 1/4$, there exist two constants $C_* = C_*(\gamma, \lambda)$, $C^* = C^*(\gamma) > 0$ such that*

$$C_* n^{\frac{2}{1+\gamma}} \leqslant \mu_n \leqslant C^* n^{\frac{2}{1+\gamma}} \qquad \forall n \in \mathbb{N}^* .$$

PROOF. We prove first the lower bound. Let $\tau_n := n^{\frac{1}{1+\gamma}}$. With the change of variable $\phi(x) = \sqrt{\tau_n}\varphi(\tau_n x)$, we get

$$\begin{aligned}\mu_n &= \inf_{\phi \in C_c^\infty(0,1)} \left\{ \int_0^1 \left(\phi'(x)^2 + \left[(n\pi)^2 |x|^{2\gamma} - \frac{\lambda}{x^2} \right] \phi(x)^2 \right) dx : \|\phi\|_{L^2(0,1)} = 1 \right\} \\ &= \tau_n^2 \inf_{\varphi \in C_c^\infty(0,\tau_n)} \left\{ \int_0^{\tau_n} \left(\varphi'(y)^2 + \left[\pi^2 |y|^{2\gamma} - \frac{\lambda}{y^2} \right] \varphi(y)^2 \right) dy : \|\varphi\|_{L^2(0,\tau_n)} = 1 \right\} \\ &\geqslant C_* \tau_n^2\end{aligned}$$

where

$$C_* := \inf_{\varphi \in C_c^\infty(0,+\infty)} \left\{ \int_0^{+\infty} \left(\varphi'(y)^2 + \left[\pi^2 |y|^{2\gamma} - \frac{\lambda}{y^2} \right] \varphi(y)^2 \right) dy : \|\varphi\|_{L^2(0,+\infty)} = 1 \right\}$$

is positive since, owing to the Hardy's inequality, it is greater than $(1-4\lambda)c_*$, where c_* is the positive constant in (303). Moreover, C_* goes to 0 as $\lambda \to 1/4$.

Now we prove the upper bound for μ_n. For every $k > 1$ we define the function $\varphi_k \in H_0^1(0,1)$ by

$$\varphi_k(x) = \begin{cases} kx & \text{for } x \in [0, 1/k)\,, \\ 2 - kx & \text{for } x \in [1/k, 2/k)\,, \\ 0 & \text{for } x \in [2/k, 1]\,. \end{cases} \tag{391}$$

Straightforward computations show that

$$\begin{aligned}&\int_0^1 \varphi_k(x)^2 dx = \frac{2}{3k}\,, \quad \int_0^1 |x|^{2\gamma} \varphi_k(x)^2 dx = c(\gamma) k^{-1-2\gamma}\,, \\ &\int_0^1 \varphi_k'(x)^2 dx = 2k\,, \quad \int_0^1 \frac{1}{x^2} \varphi_k(x)^2 dx = 4(1 - \ln 2)k\,,\end{aligned}$$

where

$$c(\gamma) := \frac{2^{2\gamma+3}}{2\gamma+3} + 4\frac{2^{2\gamma+1}-1}{2\gamma+1} - 2\frac{2^{2\gamma+2}-1}{\gamma+1}\,.$$

Thus, $\mu_n \leqslant f_{n,\gamma,\lambda}(k) := 3k^2 + 3/2(\pi n)^2 c(\gamma)k^{-2\gamma} - 6\lambda(1-\ln 2)k^2$ for all $k > 1$. Since $f_{n,\gamma,\lambda}$ attains its minimum at $\bar{k} = \tilde{c}(\gamma,\lambda)n^{\frac{1}{\gamma+1}}$, we have that

$$\mu_n \leqslant f_{n,\gamma,\lambda}(\bar{k}) = C(\gamma,\lambda)n^{\frac{2}{\gamma+1}} .$$

Moreover, since

$$C(\gamma,\lambda) = 3\left(\frac{\pi^2\gamma c(\gamma)}{2}\right)^{1/(\gamma+1)} \frac{\gamma+1}{\gamma}\left[1-2\lambda(1-\ln 2)\right]^{\gamma/(\gamma+1)} ,$$

the constant C^* can be chosen independent from λ; indeed, $1-2\lambda(1-\ln 2) > 0$ for every $\lambda < 1/4$, and the exponent $\gamma/(\gamma+1)$ of the rightmost term is smaller than one. $\square$

5.4 – A global Carleman inequality

We want to prove that, if $\gamma = 1$ and $\omega = (a,b)\times(0,1)$ with $0 < a < b \leq 1$, then there exists a positive time $T^* > 0$ such that system (371) is null controllable in any time $T > T^*$, or, equivalently, system (372) is observable in any time $T > T^*$. For this purpose, we will implement a global Carleman inequality for solutions of (384).

For every $n \in \mathbb{N}^*$, we introduce the operator

$$P_n g = g_t - g_{xx} + \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right] g$$

and the functions $\theta(t) = [t(T-t)]^{-k}$, $t \in (0,T)$, for some $k > 2$, and

$$\beta(x) := \frac{2-x^2}{4}\,, \quad x \in [0,1]\,. \tag{392}$$

We then consider the weight function

$$p(x,t) = M\theta(t)\beta(x)\,, \quad (x,t) \in Q := (0,1)\times(0,T) \tag{393}$$

for a sufficiently large constant M.

Proposition 5.10. *There exist positive constant C_1 and C_2 and $\eta \in (0,2)$ such that for every $n \in \mathbb{N}^*$, $T > 0$ and $g \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H_0^1(0,1))$ we have*

$$\begin{aligned} &C_1 \int_Q \left[M\theta(g_x^2 - \frac{\lambda}{x^2}g^2) + M^3\theta^3 x^2 g^2 + M\theta\frac{g^2}{x^\eta}\right] e^{-2p} dQ \\ &\leq \int_Q |P_n g|^2 e^{-2p} dQ + \int_0^T M\theta(g_x^2 e^{-2p})_{|x=1} dt\,, \end{aligned} \tag{394}$$

where $M := C_2 \max(T^{k/2} + T^{2k}, T^{2k} n)$.

REMARK 5.11. In the following proof, in order to ensure the regularity of function g needed for all integrations by parts, namely, that $g \in H^2(0,1) \cap H_0^1(0,1)$, we will regularize the operator P_n with the relaxed operator $P_{n,\delta}$ with potential $\frac{\lambda}{(x+\delta)^2} g$, and then pass to the limit as $\delta \to 0$. For simplicity, we will perform computations directly on P_n.

PROOF. Let $g \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1))$, and define

$$z(x,t) := g(x,t)e^{-p(x,t)}, \tag{395}$$

with weight function $p(x,t)$ defined as in (393). First, note that

$$\begin{cases} z(0,t) = z(1,t) = z_t(0,t) = z_t(1,t) = 0 \quad \text{for all } t \in (0,T), \\ \theta^2 z,\ \theta_t z \text{ and } z_x \to 0 \quad \text{as } t \to 0^+ \text{ or } t \to T^-. \end{cases} \tag{396}$$

Moreover, one verifies that

$$e^{-p} P_n g = P_n^+ z + P_n^- z,$$

where $P_n^+ z = (p_t - p_x^2)z - z_{xx} + \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right] z$ and $P_n^- z = z_t - 2p_x z_x - p_{xx} z$. Thus, we have

$$\langle P_n^- z, P_n^+ z\rangle \le \frac{1}{2}\int_Q e^{-2p}|P_n g|^2 dQ \tag{397}$$

and $\langle P_n^- z, P_n^+ z\rangle = D + B$, where (after several integration by parts we have that) the distributed part D is given by

$$\begin{aligned} D = &-2\int_Q p_{xx} z_x^2 dQ - \int_Q p_{xxx} z z_x dQ - \int_Q \frac{1}{2}(p_{tt} - 2p_x p_{xt}) z^2 dQ \\ &+ \int_Q (p_t - p_x^2)_x p_x z^2 dQ + \int_Q \left[(n\pi)^2 x^2 - \frac{\lambda}{x^2}\right]_x p_x z^2 dQ \end{aligned} \tag{398}$$

and the boundary terms are

$$\begin{aligned} B = &\left[\int_0^1 \frac{1}{2}\left(p_t - p_x^2 + (n\pi)^2 x^2 - \frac{\lambda}{x^2}\right) z^2 dx\right]_0^T \\ &+ \left[\int_0^T \left(p_x z_x^2 + p_{xx} z z_x - [p_t - p_x^2 + (n\pi)^2 x^2 - \frac{\lambda}{x^2}] p_x z^2\right) dt\right]_0^1. \end{aligned} \tag{399}$$

Observe that, thanks to hypotheses (396), the boundary contribution reduces to

$$B = \left[\int_0^T p_x z_x^2 dt\right]_0^1 .$$

In order to cope with the singular potential, we shall adapt the choice of the spatial weight β. As proposed in [45] and later in [139], we choose $\beta(x) := (2-x^2)/4$, as in (392). Recalling that $p(x,t) = M\theta(t)\beta(x)$, the distributed part becomes

$$\begin{aligned} D = & \int_Q M\theta z_x^2 dQ + \int_Q \frac{M^3}{4} x^2\theta^3 z^2 dQ \\ & + \int_Q \left[\frac{M^2}{2}x^2\theta\theta_t - \frac{M}{8}(2-x^2)\theta_{tt} - (n\pi)^2 M\theta x^2 - M\theta\frac{\lambda}{x^2}\right] z^2 dQ\,, \end{aligned} \tag{400}$$

and

$$B = \int_0^T -\frac{1}{2}M\theta z_x^2(1)dt\,. \tag{401}$$

We now estimate by below the distributed component D, taking advantage of the two coercive terms on the first line in equation (400). To this aim, we need an improved version of the Hardy's inequality, the so-called Hardy-Poincaré's inequality: for all $m > 0$ and $\eta < 2$ there exists a positive constant $C_0 = C_0(\eta, m)$ such that

$$\int_0^1 \left(z_x^2 - \frac{1}{4}\frac{z^2}{x^2}\right) dx \geq m\int_0^1 \frac{z^2}{x^\eta}dx - C_0\int_0^1 z^2 dx\,. \tag{402}$$

Since $\lambda < 1/4$, applying the Hardy-Poincaré's inequality with $m = 2$, we deduce that

$$\begin{aligned} D \geq & \frac{M}{2}\int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2}z^2\right) dQ + \frac{M^3}{4}\int_Q x^2\theta^3 z^2 dQ + \int_Q M\theta\frac{z^2}{x^\eta}dQ \\ & - \frac{C_0}{2}\int_Q M\theta z^2 dQ + \int_Q \left[-\frac{M}{8}(2-x^2)\theta_{tt} + \frac{M^2}{2}x^2\theta\theta_t - (n\pi)^2 x^2 M\theta\right] z^2 dQ\,, \end{aligned}$$

where the three terms on the first line are positive, whereas the integrals in the second line need to be evaluated. Observe that

$$|\theta_t(t)| \leq c_1(T)\theta^{1+1/k} \quad \text{and} \quad |\theta_{tt}(t)| \leq c_2(T)\theta^{1+2/k} \qquad \forall t \in (0,T)\,, \tag{403}$$

where $c_1(T) = kT$ and $c_2(T) = k(k+1)T + k/2T^2$. Moreover,

$$|\theta^{2+1/k}| \le c_3(T)\theta^3$$

with $c_3(T) = cT^{2(k-1)}$, where here and in the following c stands for a generic constant independent from n and T. Thus,

$$\left|\int_Q \frac{M^2}{2} x^2\theta\theta_t z^2 dQ\right| \le \frac{c_1c_3}{2}\int_Q M^2x^2\theta^3z^2dQ\,.$$

So, for $M \ge C_1(T) = cT^{2k-1}$, we deduce that

$$\begin{aligned} D \ge{}& \frac{M}{2}\int_Q \theta\left(z_x^2 - \frac{\lambda}{x^2}z^2\right)dQ + \frac{M^3}{8}\int_Q x^2\theta^3z^2dQ + \int_Q M\theta\frac{z^2}{x^\eta}dQ \\ &- \frac{C_0}{2}\int_Q M\theta z^2dQ + \int_Q\left[-\frac{M}{8}(2-x^2)\theta_{tt} - (n\pi)^2x^2M\theta\right]z^2dQ\,. \end{aligned}$$

On the other hand, fix $k = 1+2/\eta$ and choose $q = k$ and $q' = k/(k-1)$ conjugate exponents. Then, posed $c_4(T) = c(T+T^4)$, for every $\varepsilon > 0$,

$$\begin{aligned} &\left|\int_Q\left[-\frac{C_0}{2}M\theta - \frac{M}{8}(2-x^2)\theta_{tt}\right]z^2dQ\right| \le c_4M\int_Q \theta^{1+2/k}z^2dQ \\ &= c_4M\int_Q\left(\frac{1}{\varepsilon}\theta^{1+2/k-1/q'}x^{\eta/q'}z^{2/q}\right)(\varepsilon\theta^{1/q'}x^{-\eta/q'}z^{2/q'})dQ \\ &\le \frac{cc_4M}{\varepsilon^q}\int_Q \theta^{q(1+2/k-1/q')}x^{\eta q/q'}z^2dQ + \varepsilon^{q'}c_4M\int_Q\theta\frac{z^2}{x^\eta}dQ \\ &= \frac{cc_4M}{\varepsilon^q}\int_Q \theta^{q(1+2/k-1/q')}x^{\eta q/q'}z^2dQ + \varepsilon^{q'}c_4M\int_Q\theta\frac{z^2}{x^\eta}dQ\,. \end{aligned}$$

Note that

$$q(1+2/k-1/q') = 3 \quad \text{and} \quad \eta q/q' = 2\,.$$

Thus,

$$\left|\int_Q\left[-\frac{C_0}{2}M\theta - \frac{M}{8}(2-x^2)\theta_{tt}\right]z^2dQ\right| \le \frac{c_4M}{\varepsilon^q}\int_Q\theta^3x^2z^2dQ + \varepsilon^{q'}c_4M\int_Q\theta\frac{z^2}{x^\eta}dQ\,.$$

Now, choose $\varepsilon > 0$ such that $1 - \varepsilon^{q'} c_4 = 1/2$. So, for all $M \geq c(T^{k/2} + T^{2k})$, we have that

$$D \geq \frac{M}{2} \int_Q \theta \left(z_x^2 - \frac{\lambda}{x^2} z^2 \right) dQ + \frac{M^3}{16} \int_Q x^2 \theta^3 z^2 dQ + \frac{1}{2} \int_Q M\theta \frac{z^2}{x^\eta} dQ - \int_Q (n\pi)^2 x^2 M\theta z^2 dQ \,.$$

Finally, we estimate the last integral, whose coefficient depends from n. Since $\theta \leq c_5 \theta^3$, with $c_5(T) = cT^{4k}$,

$$\left| \int_Q (n\pi)^2 x^2 M\theta z^2 dQ \right| \leq c_5 n^2 M \int_Q x^2 \theta^3 z^2 dQ \,, \tag{404}$$

so, for every $M \geq c \max(T^{k/2} + T^{2k}, T^{2k} n)$, we conclude that

$$D \geq \frac{M}{2} \int_Q \theta \left(z_x^2 - \frac{\lambda}{x^2} z^2 \right) dQ + \frac{M^3}{32} \int_Q x^2 \theta^3 z^2 dQ + \frac{1}{2} \int_Q M\theta \frac{z^2}{x^\eta} dQ \,. \tag{405}$$

Thanks to relation (395) and estimates (397)–(401)–(405), we complete the proof of (394). □

Remark 5.12. We explicitly note that the Carleman estimate with spatial weight (392) does not apply to the case $0 < \gamma < 1$, since the term in the left-hand side of equation (404) would be $\left| \int_Q (n\pi)^2 x^{2\gamma} M\theta z^2 dQ \right|$, that would not be controlled by the coercive contributions of the distributed part D, that is, the three positive terms in the right-hand side of equation (405).

5.5 – Uniform observability

Thanks to the Carleman estimate of Proposition 5.10, we can prove an uniform observability result for the adjoint system (384).

Proposition 5.13. *Let $a, b \in \mathbb{R}$ be such that $0 < a < b \leq 1$. Then there exist $C > 0$, $k > 2$ and $T^* > 0$ such that for every $T > T^*$, $n \in \mathbb{N}^*$ and $g_{0,n} \in L^2(0,1)$ the solution of* (384) *for $\gamma = 1$ satisfies*

$$\int_0^1 g_n(x,T)^2 dx \leqslant T^{2k-1} e^{C\left(1+T^{-3k/2}\right)} \int_0^T \int_a^b g_n(x,t)^2 dx dt \,. \tag{406}$$

Proof. Let $(a', b') \subset\subset (a, b)$, $0 \leq \chi \leq 1$ such that $\chi(x) \equiv 1$ on $(0, a')$ and $\chi(x) \equiv 0$ on $(b', 1)$, and define

$$w(x,t) := \chi(x) g(x,t) \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)) \,.$$

Observe that $supp(\chi_{xx}) \subset supp(\chi_x) \subset (a', b')$, and $P_n w = \chi_{xx} g + 2\chi_x g_x$. By definition of w we deduce that

$$\int_0^T \int_0^a \theta g^2 e^{-2p} dx dt \leq \int_Q \theta w^2 e^{-2p} dQ \,. \tag{407}$$

Moreover, since $w_x(1) = 0$, the Carleman estimate in Proposition 5.10 ensures that for every $n \in \mathbb{N}^*$, $T > 0$ and for some $\eta \in (0, 2)$ we have

$$\begin{aligned} M \int_Q \theta w^2 e^{-2p} dQ &\leq M \int_Q \theta \frac{w^2}{x^\eta} e^{-2p} dQ \\ &\leq c \int_Q |P_n w|^2 e^{-2p} dQ \leq c \int_0^T \int_{a'}^{b'} (g^2 + g_x^2) e^{-2p} dx dt \,, \end{aligned}$$

where $M := C_2 \max(T^{k/2} + T^{2k}, T^{2k} n)$. Thanks to the Caccioppoli's inequality (see [45])

$$\int_0^T \int_{a'}^{b'} g_x^2 e^{-2p} dx dt \leq c \int_0^T \int_a^b g^2 dx dt \,,$$

so

$$M \int_Q \theta w^2 e^{-2p} dQ \leq c \int_0^T \int_a^b g^2 dx dt \,. \tag{408}$$

Combining equations (407)–(408), we have that

$$M \int_0^T \int_0^a \theta g^2 e^{-2p} dx dt \leq c \int_0^T \int_a^b g^2 dx dt \,. \tag{409}$$

By the same argument, choosing a cut-off function that vanishes in a neighbourhood of 0 and is 1 near the point $x = 1$, we deduce a similar inequality and conclude that

$$M \int_0^T \int_0^1 \theta g^2 e^{-2p} dx dt \leq c \int_0^T \int_a^b g^2 dx dt \,. \tag{410}$$

Note that, for every $t \in (T/3, 2T/3)$,

$$\left(\frac{4}{T^2}\right)^k \leqslant \theta(t) \leqslant \left(\frac{9}{2T^2}\right)^k$$

and

$$\int_0^1 g^2(x, T) dx \leqslant e^{-\frac{2}{3}\mu_n T} \int_0^1 g^2(x, t) dx \,.$$

Integrating over $(T/3, 2T/3)$, we deduce that

$$\begin{aligned}\frac{T}{3}\int_0^1 g^2(x,T)dx &\le e^{-\frac{2}{3}\mu_n T}\int_{T/3}^{2T/3}\int_0^1 g^2(x,t)dxdt \\ &\le e^{-\frac{2}{3}\mu_n T}\left(\frac{T^2}{4}\right)^k e^{\left(\frac{9}{2}\right)^k \frac{M}{T^{2k}}}\int_{T/3}^{2T/3}\int_0^1 \theta g^2(x,t)e^{-2p}dxdt\,.\end{aligned}$$

Thanks to relation (410) and Proposition 5.9, we conclude that

$$\int_0^1 g^2(x,T)dx \le \frac{c_1}{T}\left(\frac{T^2}{4}\right)^k e^{-c_2 nT+\left(\frac{9}{2}\right)^k \frac{M}{T^{2k}}}\int_0^T\int_a^b g^2 dxdt\,, \tag{411}$$

for some constants $c_1, c_2 > 0$ (independent of n, T and g).

Recalling that $M := C_2\max(T^{k/2}+T^{2k}, T^{2k}n)$, we consider two different cases.

First case: $n < 1+\frac{1}{T^{3k/2}}$. Then $M = C_2(T^{k/2}+T^{2k})$, thus

$$\int_0^1 g^2(x,T)dx \leqslant cT^{2k-1}e^{c_1\left(1+\frac{1}{T^{3k/2}}\right)}\int_0^T\int_a^b g^2(x,t)dxdt\,.$$

Second case: $n \geqslant 1+\frac{1}{T^{3k/2}}$. Then $M = \mathcal{C}_2 nT^{2k}$, and

$$\int_0^1 g^2(x,T)dx \leqslant cT^{2k-1}e^{\left(\frac{9}{2}\right)^k n-\frac{2}{3}cnT}\int_0^T\int_a^b g^2(x,t)dxdt\,.$$

Finally, observe that $\left(\frac{9}{2}\right)^k n - \frac{2}{3}cnT < 0$ as soon as $T \ge T^* := \left(\frac{9}{2}\right)^k\frac{3}{2c}$, completing the proof of (406). ☐

5.6 – Open problems and prospectives

In this chapter we have shown a first positive controllability result for the Grushin operator with a singular (and critical) potential in the square $\Omega = (0,1)\times(0,1)$: approximate controllability holds for every $\gamma > 0$ and every $\lambda < 1/4$; moreover, exploiting the spectral analysis provided in Section 5.3, we have proven null controllability in large time in the case $\gamma = 1$ and $\lambda < 1/4$. By analogy with the theory in [25], it should be possible to obtain a negative controllability results, if T is too small, as well as positive and negative results depending on the value of the parameter γ. Indeed, for subcritical values of the coefficient of the inverse square potential ($\lambda < 1/4$), we expect a behaviour similar to the case of the generalized Grushin operator without singular potential studied in [25]: null controllability should hold in every time for $\gamma \in (0,1)$, whereas it should fail for $\gamma > 1$. Widely open is the case of a potential term with the critical coefficient $\lambda = 1/4$. In this case, one has to adapt the functional setting in order to supply for the lack of coercivity of the associated bilinear form (see [140]). Furthermore, completely open is the controllability problem for the Grushin operator with singular potential in the domain $D = (-1,1) \times (0,1)$, that is, with degeneracy of the diffusion coefficient and singularity of the potential occurring at the interior of the domain.

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