Rendiconti di Matematica, Serie VII
Volume 35, Roma (2014), 115 – 130

Unofficial history of a joint work with Dieter Happel and of two unexpected quotations

GABRIELLA D'ESTE

Dedicated to the memory of Dieter Happel

Abstract: *This survey contains a recollection of results, problems and conversations which go back to the early years of Representation Theory and Tilting Theory.*

– Introduction

I have many reasons to be very grateful to Dieter Happel. In this note I will describe some facts still vivid in my memory, as a personal account of his mathematical vision and (at the same time) of his attention to the work of other people, even those he had never previously met. Apart from informal conversations with some colleagues in Italy, I remember just one meeting when I said a few words about that, namely in Bielefeld (December 2010). Indeed, in that occasion, I used the last minutes of my talk to speak about two examples of very special and unexpected quotations coming from Bielefeld. The first example is that of an Auslander-Reiten quiver with 28 vertices, contained in Ringel's survey [R1, page 93]. On the other hand, the second example, that of a talk, is contained in Happel-Ringel's paper on derived categories [HR, pages 164 and 180]. This note is organized as follows. In Section 1, I will recall that I met Representation Theory through Happel's handwritten notes [H], based on a course given by Ringel in Bielefeld. In Section 2, I will describe the unofficial history of my joint work [DH] with Happel. Next, in the first part of Section 3, I will collect some remarks and/or conjectures about

Key Words and Phrases: *Auslander-Reiten sequences and quivers – Tilting modules and representable equivalences*
A.M.S. Classification: Primary 16G20, Secondary 16G70, 16D90.

Auslander-Reiten quivers and sequences. To this end, I will use many pictures concerning either the quiver with 28 vertices, mentioned in [R1], or other quivers. Finally, at the end of Section 3, I will sum up the prehistory of my Oberwolfach talk, mentioned in [HR].

1 – Bielefeld, October 1979 : Happel's notes on Representation Theory

I first met Dieter Happel on October 1979 during my first day at Bielefeld Universitaet. On my second day there Claus Ringel gave me a copy of the notes of his lectures of the previous semester written by Happel. The long title of these notes is "Vorlesungausarbeitung Darstellungstheorie endlich – dimensionaler Algebren". I should confess that I read these notes very slowly, and with the help of my two dictionaries. Both of them, a rather small one and a medium sized volume, contained many useful words, but I was unable to find those typically German long words, formed by gluing together two or three words in some order. Hence I often had to guess possible meanings and/or make conjectures. It was a new experience with respect to both the mathematical subject and the language. A direct proof of the last assertion is the exercise-book, where I wrote the German words I hoped to learn. There are both technical terms from Representation Theory and more common words from daily life. After many years, in 1994, my 10 hours course ("Introduction to the representation theory of finite dimensional algebras"), addressed to PhD students of the University of Padova, was nothing else but a presentation with slides of a rather small part of Happel's Vorlesungausarbeitung, translated into Italian. Moreover, I have always suggested these notes to colleagues interested to a complete introduction to quivers, for both theory and applications, and sometimes – the other way round – applications and theory. A look at the index indicates the presence of both these aspects in Happel's notes. Indeed, the second section (pages 7-19) is dedicated to examples. Next, the fifth section (pages 39-58) is dedicated to the Auslanderkonstruktion, that is the direct construction of the dual of the transpose $\tau(M)$ of an indecomposable non projective module M, denoted there by $A(M)$, as a tribute to Auslander. After a section on Ext, there is a section describing the central theorem of the Auslanderkonstruktion and some of its applications (Der Hauptsatz und Anwendungen, pages 64-73), ending with the pictures of two Auslander-Reiten quivers. Finally, the last section deals with the proof of the main theorem (Der Beweis des Hauptsatzes, pages 78-88). The order of the various sections and the algorithmic approach mentioned above were very important for me at that time, and later on. Of course, I agree that "the whole is more than its pieces", or that "the Western Wall is more than its pieces", as I read some years ago in the poster of a conference in Analysis in Israel. However, my belief and/or experience is that sometimes also small pieces may be useful to understand the whole. For instance, the maps $\tau(M) \longrightarrow X$ and $X \longrightarrow M$ of an Auslander-Reiten sequence of the form $0 \longrightarrow \tau(M) \longrightarrow X \longrightarrow M \longrightarrow 0$ are more important, from

the functorial point of view, than the irreducible maps (between indecomposable modules), the small ingredients they are made of, but much more complicated to compute and/or guess. Here I have written "guess" for several reasons. First of all, only a few irreducible maps (between rather special indecomposable modules) are well - known on the theoretical level. Next, a kind of concealed "topology" and geometric symmetries, concerning the shape of close irreducible maps already computed, often suggest what should be the shape of the still unknown irreducible maps $\tau(M) \longrightarrow X(i)$ and $X(i) \longrightarrow M$, where the $X(i)$'s are the indecomposable summands of X. To my astonishment, a mention of "intuition" shows up in the following final remarks of Gabriel's paper [G, page 66] : "Since then, various specialists like Bautista, Brenner, Butler, Riedtmann... have hoarded a few hundred examples in their dossiers, thus getting an intuition which no theoretical argument can replace".

2 – Bielefeld, December 9th, 1988: A talk at the Darstellungstheorie Seminar and what happened next

I gave a talk in Bielefeld entitled "Some remarks on representable equivalences", which is also the title of [D]. Concerning this paper, I am (and was) very grateful to the Editors of the volume "Topics in Algebras", and in particular to Professor Daniel Simson, for asking me to submit a paper, an unexpected opportunity. Indeed, the delay of my first flight to Warsaw lead me to cancel at the last moment my participation to the conference organized by the Banach Center in Warsaw in May 1988, at a very difficult time for Poland. Back to my Bielefeld seminar, the equivalences I presented there are the ones studied by Menini and Orsatti and are involved in their Representation Theorem [MO], a generalization of Fuller's Theorem [F]. The aim of my talk was to show toy examples of these equivalences in well – behaved situations, by dealing with algebras of finite representation type. During my talk I could immediately answer a question by Happel ("Is it extension closed?") on one of the classes involved in the equivalences studied by Menini and Orsatti in [MO]. On the other hand, I had no answer for other natural questions and/or conjectures on the relationship between the $*$-modules considered in [MO], and classical tilting modules. In particular, I hoped to find a finite dimensional $*$-module faithful, but not tilting. The introduction of [D] contains the following remark: "Up to now, we do not know whether or not there exist a finite dimensional algebra A and a $*$-module ${}_AM$ such that ${}_AM$ is not a "disguised" tilting module, that is ${}_{\bar{A}}M$ is not a tilting module, where $\bar{A} = A/\operatorname{ann}_A M$." On the other hand, the last words of [D] are as follows: "The proof of Corollary 6 also shows that the algebra A given by the quiver $\circ \longrightarrow \circ \longrightarrow \circ$ has the following property: if ${}_AM$ is a multiplicity-free $*$-module and $\bar{A} = A/\operatorname{ann}_A M$, then ${}_{\bar{A}}M$ is tilting module. As already observed in the Introduction, we do not know any finite dimensional algebra without this property". I viewed this last result as a pathological case, and not as an indica-

tion of a general property. On the other hand, Happel immediately believed that (over a finite dimensional algebra) faithful $*$-modules and tilting modules always coincide. I recall that he told me more or less the following: "We should prove this fact, and write a joint paper for the journal of Padova University" (the "Rendiconti del Seminario Matematico dell'Università di Padova"). This is the first part of the unofficial history of [DH]. At the same time, it is an example of two decisive factors (somehow in the reverse order with respect to the usual way of thinking):

(1) An overall vision of what should be true.
(2) A winning strategy to prove it, by taking into account all possible contributions of the present and future mathematical community.

The second part of this unofficial history is the big role played by the copy of the manuscript that Riccardo Colpi gave me shortly before my departure. He told me that it was still work in progress, and we had never spoken together in advance about his research. I planned to look at Colpi's note only after my visit to Bielefeld. However, I put the envelope in my bag. When Happel asked me whether I had any paper and/or preprint on $*$-modules of Italian colleagues, I said that I had a copy of a preliminary version, still in Italian, of the work a PhD student. After a short look at Colpi's handwritten pages, Happel told me that the results "looked very interesting". Instead of waiting for a final paper written in English, he asked me to translate the whole manuscript during my short visit to Bielefeld. To do this, I spent more time than expected, because at the same time I wanted to understand what I was translating. As Happel had immediately realized, Colpi's notes contained many new ideas and useful tools to deal with $*$-modules and to prove his conjecture on representable equivalences. An evidence of this fact is that we quoted three different results of [C] (that is, Corollary 4.2, Proposition 4.3 and Theorem 4.1) in the proof the main theorem in [DH].

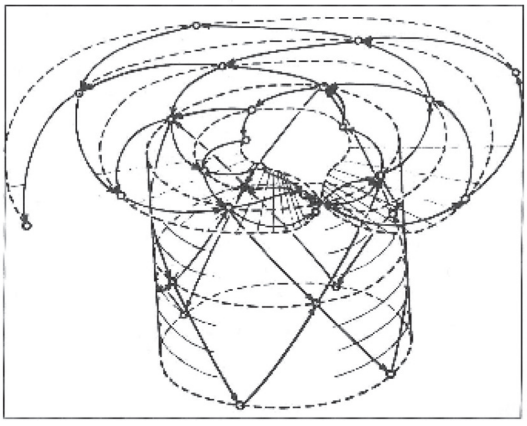
3 – Bielefeld, December 12th, 2010: A colloquium – style talk ending with two examples of more than complete references (and some speculations about them)

Thanks to Claudia Koehler, I had the unexpected opportunity to present some general ideas on my work to the young audience of the Workshop "Women in Representation Theory: Selfinjective Algebras and Beyond" (December 10-12, 2010), open also to "senior mathematicians", as the organizer wrote me. This was the suggestion contained in Claudia's answer to my questions about the style of my talk: "It would be good if you could give a more general talk about something from your research. If possible understandable for PhD students." That's why I prepared a personal survey about recent and less recent results obtained by means of the techniques learnt in Bielefeld. At the end of my talk I read the following remarks in the final part of Ringel's introduction to [R2]: "We have tried to give as

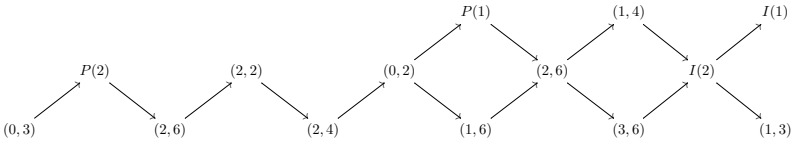
complete references as possible at the end of each chapter, and we apologize for any omission. Of course, it would be easy to trace any omission of a reference to a paper which has appeared in print; however we should point out that some general ideas which have influenced the results and the methods presented here, are not available in official publications, or not even written up." I realize that these mathematical and not mathematical reflections are very important. However, the main reason why I mentioned these remarks is that I want to recall two unexpected examples of quotations (coming from Bielefeld) of quite different results I had obtained (in Bielefeld) long ago. The first one, concerning an Auslander-Reiten quiver, is contained in Ringel's survey [R1]. On the other hand, the second one, concerning more or less large modules, is contained in Happel-Ringel's paper [HR]. I view these two quotations as examples of references which are not only complete, but much more than complete. Both of them are closer to true transfigurations (from the real world to an ideal world) than to usual references. Concerning the first case, that of an Auslander-Reiten quiver, I will try to give an idea of the big gap between real world and ideal world by the means of several pictures. On the other hand, concerning the second case, the gap between my result and its presentation is so big that I do not try to describe any connection between different points of view of the same mathematical object. That's why I will copy the short account prepared for the Bielefeld Workshop. I believe that real facts speak for themselves.

3.1 – An Auslander-Reiten quiver (elegant form and naïve prehistory)

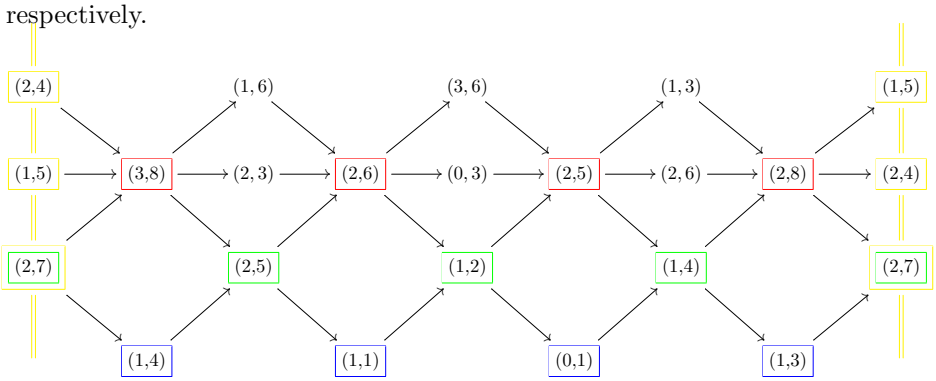
The first example mentioned in my Bielefeld talk was that the Auslander-Reiten quiver of the algebra, say R, given by the quiver $\bullet_1 \xrightarrow{a} \bullet_2 \circlearrowleft b$ with relations $b^4 = 0$ and $b^2a = 0$, described in Ringel's paper [R1, page 93]. I take from Ringel's home page [R3, Abbildung 3] the following picture of the Auslander-Reiten quiver of R.



Only after many years – at the end of my talk at a Beijing conference [FD] – I expressed for the first time my great surprise of finding a quotation (with thanks!) to a quiver I had computed by direct calculations and illustrated in a naïve and simple way. Indeed there is an enormous difference between the topological object (formed a tube with a Möbius strip upstair) presented in Ringel's paper and home page, and my flat picture of the same Auslander-Reiten quiver (formed by two unstable orbits and five stable orbits). I will need several pictures to illustrate some elementary and combinatorial aspects of the Auslander-Reiten quiver in my big sheets. In this way I will give a concrete proof of the big gap between the result I had and its ideal form presented in [R1]. This big gap may be an evident example of the "variety of different appearances" which show up "in mathematics and in daily life", as observed in the section "About Us" of the home page of [CRC 701]. With the usual notation, let $P(x)$ and $I(x)$ denote the indecomposable projective and injective modules corresponding to the vertex x. Moreover, by replacing an indecomposable module which is neither projective nor injective by its dimension type, the next picture describes the two unstable τ-orbits of the above Auslander-Reiten quiver, containing 3 and 5 indecomposable modules respectively, as well as the 6 indecomposable stable modules X in the τ-orbit of the radical of $P(2)$, that is the indecomposable module of dimension type $(0,3)$, such that there is an irreducible map of the form $X \longrightarrow Y$ or $Y \longrightarrow X$ for some indecomposable unstable module Y.



On the other hand, if we identify the left an right vertical lines in order to obtain a Möbius strip, the next picture describes the position of the 20 stable indecomposable modules and their dimension types. More precisely, the τ-orbit of the indecomposable module of dimension type (0,3) contains 8 modules, while the τ-orbits of the indecomposable modules of dimension type $(2,8)$, $(2,7)$ and $(0,1)$ contain 4 modules respectively.



3.2 – An Auslander-Reiten sequence of the previous Auslander-Reiten quiver (= example where all irreducible maps are obvious cancellations or additions)

By comparing dimension types, we deduce from the above pictures that the dimension types of the stable modules considered above – that is (0,1), (0,3), (2,7) and (2,8) – are the dimension types of exactly one indecomposable R-module. This is obvious for the first two cases, which are the dimension types of the simple top of $P(2)$ and of its unique maximal submodule. Let L and M denote the above indecomposable modules of dimension type (2,7) and (2,8) respectively. Then the matrices A and B (resp. A' and B'), describing the injective map a from K^2 to K^7 (resp. K^8) and the endomorphism b of K^7 (resp. K^8), are of the following form:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

I wrote these big matrices just to explain with two reasonably small pictures the conventions, suggested me by Ringel during my stay in Bielefeld, used to describe in a compact (and efficient !) way the indecomposable representations of the form

$$\mathrm{V}(1) \xrightarrow{a} \mathrm{V}(2) \circlearrowleft b$$

with $V(2) \neq 0$, that is different from the simple injective module $I(1)$, for the various algebras given by the quiver

$$\bullet_{1} \xrightarrow{a} \overset{a}{\bullet}_{2} \circlearrowleft b$$

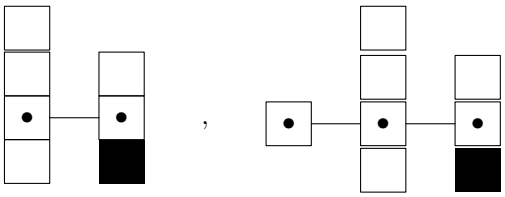
with relations $b^n = 0$ and $b^2a = 0$ for some $n > 1$. Following Ringel's hints, every square in the next pictures indicates a fixed element of a fixed basis of the

vector space $V(2)$, while the various blocks indicate the uniserial summands of the submodule

$$0 \longrightarrow \mathrm{V}(2) \circlearrowright b \ .$$

On the other, every black square indicates a vector v of the fixed basis belonging to the image of the injective linear map a. Finally, every segment connecting finitely many small squares, corresponding to vectors $v(1), v(2), \ldots, v(m)$, indicates that $v(1)+v(2)+\cdots+v(m)$ belongs to the image of a. Following these conventions, the previous modules L and M, of dimension type (2,7) and (2,8) look like as follows.

– Picture 1



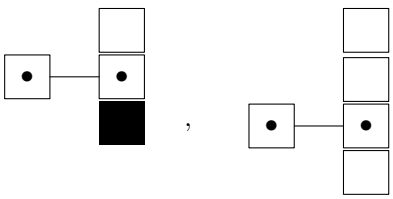
$L = (2,7)$ $\qquad$ $M = (2,8)$

Consequently, the arrow from (2,8) to (2,7) indicates a kind of left cancellation, that is the irreducible epimorphism whose simple kernel is the unique simple summand of

$$0 \longrightarrow K^8 \circlearrowright b \ .$$

On the other hand, the arrow from (2,8) to (2,4) (resp. (1,5)) describes a kind of central (resp. right) cancellation of the unique summands of dimension 4 (resp. 3) of the above submodule of dimension type (0,8). Indeed, these two modules have the following shape:

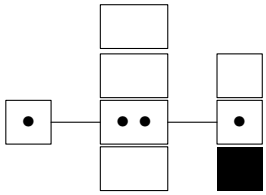
– Picture 2



(2,4) $\qquad$ (1,5)

Dually, keeping the above conventions, the indecomposable module of dimension type (3,8) has the following shape

– Picture 3



Consequently, the three irreducible maps arriving at (3,8) look like additions of different types, that is either left or right or central additions.

3.3 – Comparing (obvious and less obvious) Auslander – Reiten sequences

We list some other properties of the Auslander-Reiten sequence ending at the above indecomposable module of dimension type (3,8), briefly denoted by

$$0 \longrightarrow (2,8) \xrightarrow{f} X \xrightarrow{g} (3,8) \longrightarrow 0. \qquad (\sharp)$$

First of all, for every indecomposable non projective module M of dimension type different from (3,8), we have $\dim_K M + \dim_k \tau(M) < 21$. Hence it is reasonable to expect that ($\sharp$) is one of the most complicated Auslander-Reiten sequences of the above Auslander-Reiten quiver. Secondly, X is the direct sum of 3 indecomposable modules, neither projective nor injective. Therefore, by the four-in-the-middle theorem [BB], X has the largest possible number of indecomposable and stable direct summands. However, even in this complicated case, the maps f and g in ($\sharp$) consist of three irreducible maps which are very natural "cancellations" and "additions" respectively. Hence, we may roughly speaking say that the functorial and global properties of f and g are somehow concealed. On the other hand the local properties of the irreducible components of f and g are evident, at least "intuitively", to repeat the words used in Gabriel's remark [G, page 66] (see the end of section 1). In the above sequence ($\sharp$) we may see at a glance that all the three irreducible components of f and g are the obvious ones, by looking at exactly one visualization of the first and last non-zero modules. This looks like a nice situation, but I do not view it as an exceptional one. Indeed, if Z is an indecomposable summand of the middle term of an Auslander-Reiten sequence, say $(*)$, and h is an irreducible component of one of the two non-zero maps in $(*)$ arriving at Z or ending in Z, then only one of the following cases seems to occur.

The obvious good case: The map h looks like an evident injective or surjective map, that is a kind of translation of the unique well-known irreducible maps at the theoretical level, of the form $X \longrightarrow P$ and $I \longrightarrow Y$, where the modules X, P, I, Y are indecomposable, P is projective and I is injective.

The concealed good case: Up to a change of the basis of the first or last non-zero terms of $(*)$, h becomes an obvious embedding or epimorphism. Moreover, after that change, h looks like as a shift of other irreducible maps close to it.

3.4 – A more complicated Auslander-Reiten sequence (= example where all irreducible maps become obvious cancellations or additions after changing a basis)

After an example of the good case, given by the previous sequence $(\sharp)$, I will describe an example of a concealed good case by means of an Auslander-Reiten sequence, say $(\sharp\sharp)$, ending at an indecomposable module of dimension type (6,24), briefly denoted by

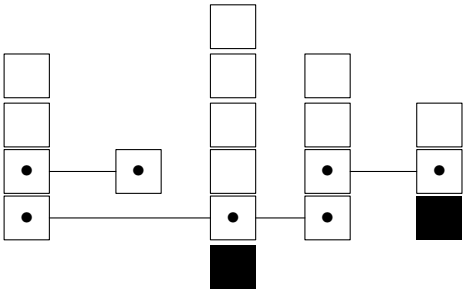
$$0 \longrightarrow (7,27) \longrightarrow X \longrightarrow (6,24) \longrightarrow 0\,. \qquad (\sharp\sharp)$$

In this case the algebra into the game is the path algebra given by the quiver

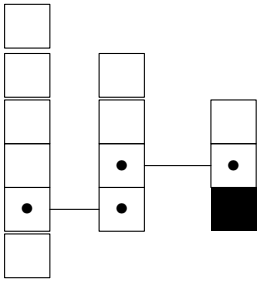
$$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \circlearrowright b \qquad \text{with relations} \quad b^6 = 0 \ \text{ and } \ b^2 a = 0.$$

Also in this case the middle term X is the direct sum of three indecomposable non projective summands of dimension types (5, 18), (3, 13) and (5, 20) with the following shapes respectively:

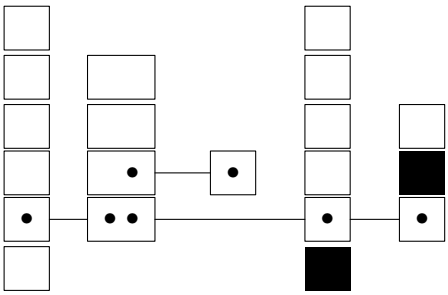
– (5,18)



– (3,13)

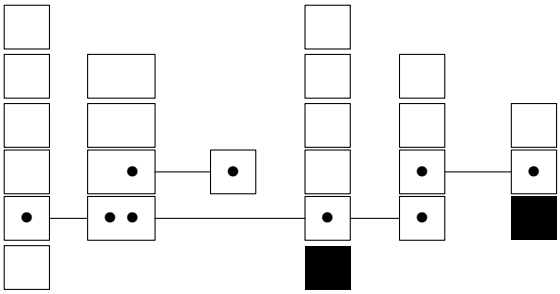


– (5,20)

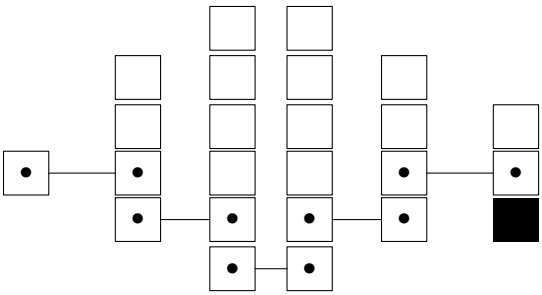


We illustrate in the sequel the original shapes (indexed by A) of the first and the last non-zero term of (♯♯), and those later obtained after a change of the basis (indexed by B and C).

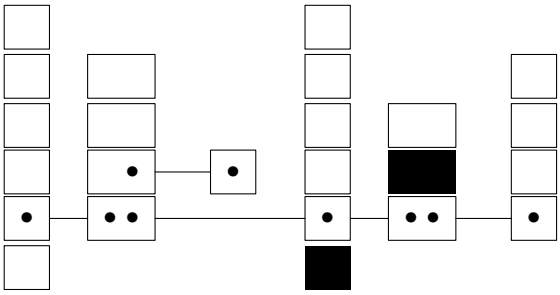
– (6,24) A



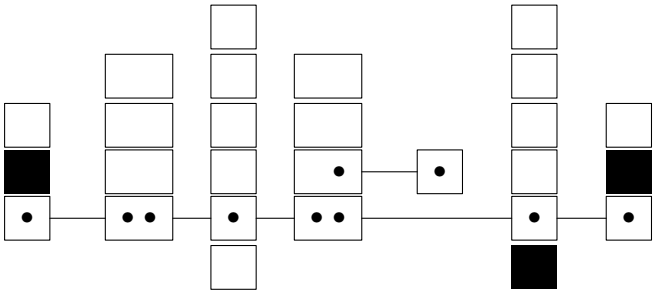
– (6,24) B



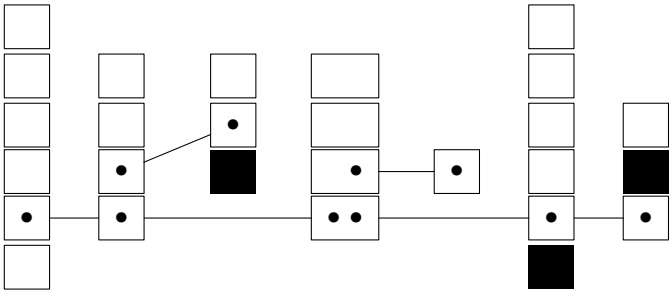
– (6,24) C



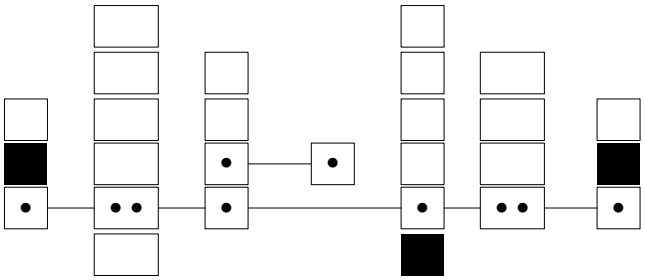
– (7,27) A



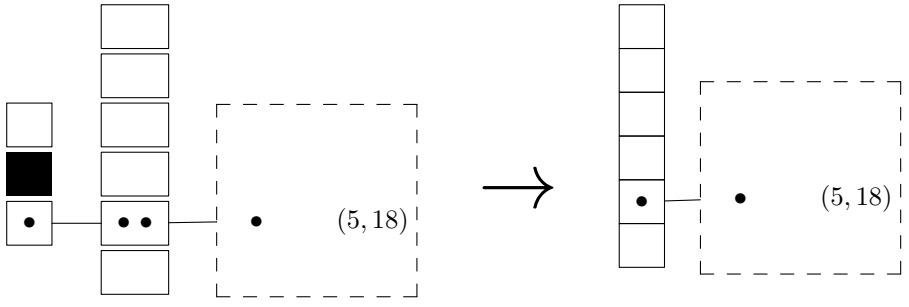
– (7,27) B



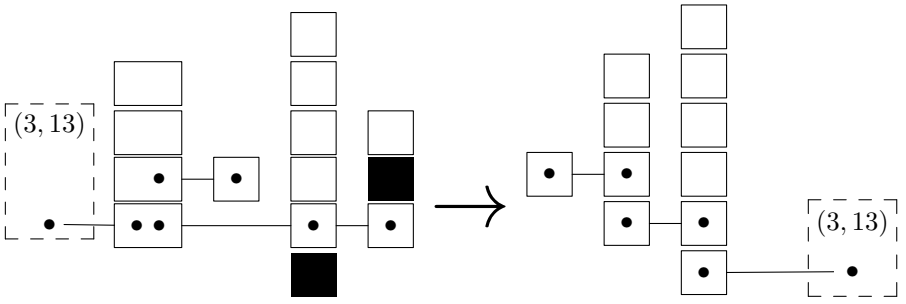
– (7,27) C



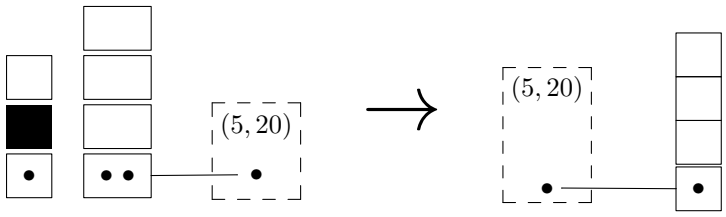
The calculation of (♯♯) goes back to the first problem suggested me by Ringel (with many useful hints and suggestions) at the end of my first year in Bielefeld. By looking at my old pictures of $(6, 24)$ and $(7, 27)$, that is $(6, 24)A$ and $(7, 27)A$), only two out of six irreducible maps, that is $(5, 18) \longrightarrow (6, 24)A$ and $(7, 27)A \longrightarrow (5, 20)$, look like obvious "additions" and "cancellations". In order to see that the same holds for the other four irreducible maps, it suffices to use the visualizations of type B and C of the first and the last non zero modules in (♯♯). We illustrate in the sequel the behaviour of the three reducible maps obtained by making use of the obvious form of the six irreducible components of the two non -zero maps in (♯♯). First of all, the composition $(7, 27)C \longrightarrow (5, 18) \longrightarrow (6, 24)A$ acts as a left cancellation followed by a left addition.



Secondly, the composition $(7, 27)B \longrightarrow (3, 13) \longrightarrow (6, 24)B$ acts as a right cancellation followed by a left addition.



Finally, the composition $(7,27)A \longrightarrow (5,20) \longrightarrow (6,24)C$ acts as a left cancellation followed by a right addition.



3.5 – A preliminary result and a general vision

The second example of unexpected quotation mentioned in my Bielefeld talk deals with Happel – Ringel's paper [HR]. Indeed, the second result in the references of [HR] has the following form:

> **[2]** D'Este, G. Talk at Oberwolfach conference on representation theory 1981, unpublished.

Moreover, the first lines of [HR, page 164] say the following:

> "We remark that our account on the decomposition of $\hat{C}$-*mod* into the module classes M_m and $M_{m,m+1}$ follows closely the treatment given by Gabriella d'Este in her Oberwolfach talk 1981 [2]."

I am not able to find the slides with the example described in my Oberwolfach talk and/or the handwritten pages containing a more general result, suggested by the same example. I remember that I had a long list of hypotheses on quivers, algebras and modules into the game, used to obtain new algebras with very few new (and not too big) indecomposable modules. On the other hand, complexes, functors and all the mysteries of derived categories did not show up at all, neither in what I proved nor in what I ever hoped to prove. The best way to sum up all what I can say about my Oberwolfach talk 1981 will be to copy, without any change, the last slides prepared for my Bielefeld talk 2010, and actually shown at the end. Almost always I omit the final part of my talks, but this did not happened on that occasion. Indeed, before the beginning of the lectures, the chairman opened the files of the various presentations. When the slide of my talk (with a photo of the University of Bielefeld) appeared on the screen, Ringel come to my seat. Since he was leaving shortly afterwards, I showed him all my final slides about the official/unofficial history mentioned in this note. Thanks to a few pictures and even less words, I needed a very short time to illustrate all what I wanted to say. I was happy to see that my recollections were correct. That's why at the end of my talk in Bielefeld I presented the following telegraphic slides.

– **What I remember about my talk (Slide-3)**

A result on the support of the indecomposable modules over an algebra obtained from a tame algebra after two operations:

- a one – point extension
- a one – point coextension

by means of two regular modules (suggested by an example which gave me the idea that only few new indecomposable modules could appear).

– **What I remember BEFORE my talk (Slide-2)**

- A long conversation with Dieter HAPPEL and Claus RINGEL.
- Their hint to present only my original example.
- Their mysterious comments ????? (sometimes in German) that I didn't try to understand.

– **What I believe now (Slide -1)**

Behind my words, calculations done by hands and pictures, they were able to SEE and/or IMAGINE completely different objects:

- quivers with infinitely many vertices (and not just finitely many);
- complexes instead of modules;
- .....

Acknowledgements

I would like to express my thanks to "The BIREP Group", to keep the acronym used in the welcome message to participants contained in the Conference Guide of ICRA 2012. I am very grateful to all the mathematicians of the present and the future generation working and studying in Bielefeld for their inspiration and help during international conferences, seminars and informal talks.

REFERENCES

[BB] R. Bautista – S. Brenner: *On the number of terms in the middle of an almost split sequence*, Repr. Theory of algebras (Puebla 1980), Springer LMN, **933** (1981), 1–8.

[C] R. Colpi: *Tilting modules and $*$-modules*, Comm. Algebra, **21** (4) (1993), 1095–1102.

[CRC 701] Collaborative Researche Center 701: *Spectral Structures and Topological Methods in Mathematics*, Faculty of Mathematics, University of Bielefeld.

[D] G. D'Este: *Some remarks on representable representations, Topics in Algebra*, Parts 1-2; Stanisław Balcerzyk, Tadeusz Józefiak, Jan Krempa, Daniel Simson, Wolfgang Vogel (eds.), Banach Center Publications, **26** (1990), 223–232.

[DH] G. D'Este – D. Happel: *Representable equivalences are represented by tilting modules*, Rend. Sem. Mat. Univ. Padova, **83** (1990), 77–80.

[F] K. R. Fuller: *Density and equivalence*, J. Algebra, **29** (1974), 528–550.

[FD] FDLIST: *International Asia Link Conference on Algebras and Representations*, Beijing, May 23–28, 2005.

[G] P. Gabriel: *Auslander-Reiten sequences and representation-finite algebras*, Springer LNM, **831** (1080), 1–71.

[H] D. Happel: *Vorlesungausarbeitung Darstellungstheorie endlich – dimensionaler Algebren*, Sommersemester, Universitaet Bielefeld, 1979.

[HR] D. Happel – C. M. Ringel: *The derived category of a tubular algebra*, Proceedings ICRA IV, Springer LNM, **1177** (1986), 156–180.

[MO] C. Menini – A. Orsatti: *Representable equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova, **82** (1989), 203–231.

[R1] C. M. Ringel: *Unzerlegbare Darstellungen endlich - dimensionaler Algebren*, Jber. d. Dt. Math. Verein., **85** (1983), 86–105.

[R2] C. M. Ringel: *Tame algebras and integral quadratic forms*, Springer LMN, **1099** (1984).

[R3] C. M. Ringel:`www.mathematik.uni-bielefeld.de/~ringel/opus/diskret/3.jpg`.

Lavoro pervenuto alla redazione il 8 settembre 2014
ed accettato per la pubblicazione il 22 settembre 2014
bozze licenziate il 16 dicembre 2014

Indirizzo dell'Autore:
Gabriella D'Este – Department of Mathematics – University of Milano – Via Saldini, 50 – 20133 – Milano – Italy
E-mail: gabriella.deste@unimi.it