Rendiconti di Matematica, Serie VII
Volume 35, Roma (2014), 131 – 158

Stratifications of the moduli space of curves and related questions

GABRIELE MONDELLO

Abstract: *We discuss results and open problem on the geometry of the moduli space of complex curves, with special emphasis on vanishing results in cohomology and in the tautological ring. The focus is on the techniques, namely algebraic stratifications, transversely real foliations and q-convex exhaustion functions. A selection of related problems in the field is also quickly mentioned.*

1 – Introduction

Let g and n be non-negative integers such that $2g-2+n>0$. The moduli space $\mathcal{M}_{g,n}$ of curves of genus g with n distinct marked points can be looked at from different point of views: algebro-geometric, complex-analytic, differential-geometric and topological.

- For algebraic geometers, $\mathcal{M}_{g,n}$ classifies families of smooth complete curves of genus g with n distinct sections and it is usually given the structure of smooth Deligne-Mumford stack (see [18, 8]), which is in fact a global quotient of a smooth quasi-projective variety by a finite group (see [42], or [8]). Moreover, the moduli space is an actual variety whenever $n>2g+2$. The merit of this approach is that one deals with a true classifying space, which thus carries a universal family $\pi:\mathcal{C}_{g,n}\to\mathcal{M}_{g,n}$ endowed with n distinct sections $\sigma_1,\dots,\sigma_n$; though occasionally one can also consider the associated coarse space, which is a quasi-projective variety with quotient singularities.
- A complex analyst would focus more on Teichmüller space $\mathfrak{T}_{g,n}$, which is a complex manifold that classifies complex structures on a fixed (compact, connected) oriented surface of S genus g, up to Teichmüller equivalence (*i.e.* up to isotopies

Key Words and Phrases: *Riemann surfaces – Moduli space of curves – Tautological ring – Stratifications – Abelian differentials – Translation surfaces*
A.M.S. Classification: 32G15, 14H10, 30F30.

of S that pointwise fix a subset $P \subset S$ of n distinct points). See for instance [1, 33, 50]. The moduli space[1] $\mathcal{M}_{g,n}$ can be obtained as a quotient of $\mathfrak{T}_{g,n}$ by the mapping class group $\mathrm{MCG}_{g,n}$ of isotopy classes (relative to P) of orientation-preserving diffeomorphisms of S, which acts properly and with finite stabilizers. This endows $\mathcal{M}_{g,n}$ with the structure of complex-analytic orbifold. As $\mathfrak{T}_{g,n}$ is diffeomorphic to a ball, it plays the role of "orbifold universal cover" of $\mathcal{M}_{g,n}$, with "orbifold fundamental group" $\pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}) \cong \mathrm{MCG}_{g,n}$. The advantage of this approach is that one works on a fixed surface S, though one must then take care of the action of the mapping class group (for an introduction to the group MCG, see [28]).

- A differential geometer would look at the space $\mathfrak{Met}_{g,n}$ of Riemannian metrics on a fixed S, and would recover $\mathfrak{T}_{g,n}$ as the quotient of $\mathfrak{Met}_{g,n}$ under conformal and Teichmüller equivalence (see [94]). The structure of $\mathfrak{Met}_{g,n}$ is richer, as it is endowed with a natural Riemannian pairing and points are not defined up to annoying equivalences; on the other hand, $\mathfrak{Met}_{g,n}$ is an infinite-dimensional manifold.
- From the topological point of view, the orbifold $\mathcal{M}_{g,n} = \mathfrak{T}_{g,n}/\mathrm{MCG}_{g,n}$ is a $K(\mathrm{MCG}_{g,n}, 1)$ and so it is again a classifying space in a suitable topological category (for example, the category of orbifolds or of simplicial spaces). Thus, many invariants, such as the singular cohomology ring, can still be computed using more flexible tools coming from topology. On the other hand, a lot of structure (from complex analysis or differential geometry) is not preserved under most topological manipulations.

In this paper, I will discuss some results and problems concerning certain algebro-geometric and complex-analytic properties of $\mathcal{M}_{g,n}$. Because of the vastity of the subject, I will concentrate on the problem of the cohomological dimension of $\mathcal{M}_{g,n}$, namely of vanishing results for high degree cohomology groups, but also of vanishing of high degree tautological classes.

I will put special emphasis on the techniques employed by the authors, namely stratifications by algebraic subvarieties, transversely real-analytic foliations with holomorphic leaves, exhaustion functions with controlled complex Hessian.

I would like to underline that the purpose of this survey paper is to only touch a limited selection of topics; moreover, the treatment as well as the bibliography are by no means intended to be exhaustive.

1.1 – Structure of the paper

In Section 2 some selected problems are briefly discussed, such as

- structure of de Rham cohomology and tautological subring of $\mathcal{M}_{g,n}$, upper bounds on the dimension of compact holomorphic subvarieties of $\mathcal{M}_{g,n}$;

[1] By abuse of notation, we will use the symbol $\mathcal{M}_{g,n}$ both for the moduli space of curves and for the space $\mathcal{M}_{g,n}(\mathbb{C})$ of its complex-valued points.

- cellular decompositions of $\mathcal{M}_{g,n}$, higher vanishing of de Rham cohomology, volumes of $\mathcal{M}_{g,n}$;
- first cohomology group of finite covers of $\mathcal{M}_g$ and existence of relative bicanonical forms on the universal family.

Section 3 is dedicated to the analysis of the techniques involving stratifications, foliations and exhaustion functions.

Finally, in Appendix A formal definitions of de Rham, Dolbeault and algebraic cohomological dimensions are given, together with a short list of immediate properties. The last subsection contains a few naive considerations on the cases of $\mathcal{M}_{0,n}$, $\mathcal{M}_{1,n}$ and $\mathcal{M}_{2,n}$.

2 – Selected questions on the moduli space of curves

2.1 – De Rham cohomology

Being an orbifold, many non-torsion invariants of $\mathcal{M}_{g,n}$ behave like those of a manifold, for instance its singular cohomology with coefficients in a field of characteristic zero or the mixed Hodge structure on $H^*_{dR}(\mathcal{M}_{g,n};\mathbb{C})$. Calculating the de Rham cohomology of $\mathcal{M}_{g,n}$ is a really hard problem, as well as determining the (rational) Chow ring.

The de Rham cohomological dimension of $\mathcal{M}_{g,n}$ (see Appendix A.1) was computed by Harer [48]: he showed that $H^d(\mathcal{M}_{g,n};\mathbb{Q})$ vanishes for $d > \mathrm{vcd}(g,n)$, where

$$\mathrm{vcd}(g,n) = \begin{cases} n-3 & \text{if } g=0 \\ 4g-4+n & \text{if } g,n>0 \\ 4g-5 & \text{if } g\geq 2 \text{ and } n=0 \end{cases}$$

and proved that this bound is optimal.

Another major result by Harer [47] says that $H^d(\mathcal{M}_g;\mathbb{Q})$ stabilizes as g is at least $\approx 3d$. Such a stability bound was later improved by Ivanov [54] to $g \approx 2d$ and by Boldsen [13] to the essentially optimal $g \approx \frac{3}{2}d$. As conjectured by Mumford [76] and proven by Madsen-Weiss [66], this stable cohomology is generated by κ classes.

To spell things with algebro-geometric language, let $\widehat{M}_{g,1}$ be the moduli space of triples (C,p,z), where $(C,p)\in\mathcal{M}_{g,1}$ and z is a formal coordinate on C centered at p, and let Gr^{1-g}_∞ be the infinite Grassmannian of subspaces V of $\mathbb{C}(\!(z)\!)$ such that the homomorphism $V\to\mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!]$ has index $1-g$. Krichever's map $K:\widehat{M}_{g,1}\to \mathrm{Gr}^{1-g}_\infty$ attaches to (C,p,z) the subspace of $\mathbb{C}(\!(z)\!)$ corresponding to meromorphic functions on C which are regular away from p. A consequence of Madsen-Weiss proof is the following.

Corollary 2.1. *The map K induces an isomorphism on rational homology in degrees $\leq \frac{2g}{3}$.*

PROBLEM 2.2. Is there an algebro-geometric proof of Corollary 2.1?

In general, the de Rham cohomology of $\mathcal{M}_{g,n}$ in the unstable range $\frac{2}{3}g \leq d \leq 4g-5$ remains mysterious, though we know the orbifold Euler characteristic by work of Harer-Zagier [44] (then simplified by Penner [80] and Kontsevich [58]).

Exhaustive computations are available for low genera (see Mumford [76], Faber [23], Getzler [34], Looijenga [62] and Getzler-Looijenga [35], Faber [24], Izadi [55], Tommasi [92, 93], Gorinov [39]). Genus-free computations are available in low degrees using topological methods (see Harer [45] and [46]) and a mixture of topological and algebro-geometric methods (see Arbarello-Cornalba [7]).

2.2 – Tautological ring

Instead of dealing with the whole cohomology, many people concentrate on the so-called *tautological* (Chow or cohomology) subring, introduced by Mumford [76], generated by the classes $\psi_i := c_1(\mathcal{L}_i)$ and $\kappa_b := \pi_*\left(c_1(\omega_\pi(D))^{b+1}\right)$, where ω_π is the π-relative (holomorphic) cotangent line bundle, $\mathcal{L}_i := \sigma_i^*(\omega_\pi(D))$, $D = D_1 \cup \cdots \cup D_n$ and D_i is the image of the i-th section of the universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$. As tautological classes are rather ubiquitous in intersection-theoretic computations, their study is quite valuable.

The most relevant conjectures were formulated by Faber [25] (Conjecture 1). Part (b) of such conjecture was proven by Ionel [53], Morita [74] and Boldsen [13]; the vanishing of part (a) was shown by Looijenga [64] and the one-dimensionality in degree $g-2$ was verified by Faber [25]. The relations of part (c) were proven to be a consequence on Givental's proof [37] of Virasoro conjecture for $\mathbb{P}^2$; other proofs are due to Getzler-Pandharipande [36], Liu-Xu [61] and Buryak-Shadrin [15]. Further relations conjectured by Faber-Zagier in 2000 were established by Pandharipande-Pixton [77]. See [26] for a survey of the vast literature on this subject till 2013.

In many of these papers a key role is played by suitable compactifications of the moduli space of interest, like Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ (see [18]), which is again a smooth Deligne-Mumford stack and indeed a global quotient (see [63, 12, 2, 8]), the universal curve $\overline{\pi} : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, the space of admissible covers [49], or the space of stable maps [32]. Indeed, intersection-theoretic computations are often performed on such compactifications, where tautological classes admit a very natural extension.

In the last year, an impressive conjectural description of relations among tautological classes (see Pixton [85]), that generalize the FZ-relations to $\overline{\mathcal{M}}_{g,n}$, was proven by Pandharipande-Pixton-Zvonkine [78]; on the other hand, Petersen-Tommasi [83] showed that such a ring is not Gorenstein in general, thus negatively answering another important question.

2.3 – Compact subvarieties

As $\mathcal{M}_g$ is not compact, it is very natural to wonder what is the largest dimension of a compact holomorphic (and so algebraic) subvariety of $\mathcal{M}_g$. The conjecturally optimal upper bound is $g-2$, which is known to be attained for $g=3$ (because the Satake compactification of $\mathcal{M}_g$ has codimension 2 for $g \geq 3$). All known explicit constructions of compact subvarieties of a certain dimension d inside $\mathcal{M}_g$ essentially rely on Kodaira's idea (see [57, 38, 99, 98]), which is not expected to provide a much better estimate than $d \sim \log_2(g)$.

Some approaches to this problem will be more extensively discussed in Section 3.

2.4 – Cohomology of coherent sheaves

Almost nothing is known about cohomology of coherent sheaves (in the algebraic or analytic sense) on $\mathcal{M}_{g,n}$. Very few computations can be performed using known results on de Rham cohomology and Hodge theory; it would be interesting to be able to answer at least some of the following questions.

Problem 2.3. Compute some algebraic or analytic $H^i(\mathcal{M}_{g,n};\mathcal{F})$, when $\mathcal{F}$ is a tensor product of the canonical bundle $K_{\mathcal{M}_{g,n}}$, the Hodge bundle $\pi_*(\omega_\pi)$ and $\pi_*\left(\omega_\pi^{\otimes N}\right)$, the line bundles $\mathcal{L}_i$ and of their duals.

Being products of Chern classes, tautological classes can be seen as living in the algebraic $H^p(\mathcal{M}_{g,n},\Omega^p_{\mathcal{M}_{g,n}})$ or in the analogous Dolbeault cohomology group. Their vanishing in high degree might be a consequence of a more radical vanishing of the whole (algebraic or Dolbeault) cohomology of coherent sheaves in high degree, namely of an upper bound on the Dolbeault/algebraic cohomological dimension of $\mathcal{M}_{g,n}$ (see Appendices A.2 and A.3 for more precise definitions).

Problem 2.4. Is $\text{coh-dim}_{\text{Dol}}(\mathcal{M}_{g,n}) = \text{coh-dim}_{\text{alg}}(\mathcal{M}_{g,n}) = g-1-\delta_{0,n}$ for $g \geq 3$?

For $g = 0, 1, 2$, the problem can be easily attacked by hand, whereas for $g = 3, 4, 5$ a positive answer follows from the existence of an optimal affine cover [31]. For $g \geq 6$ the problem is still open.

2.5 – Cellularizations

The above mentioned cohomological vanishing in high degree by Harer is part of a stronger result that states that mapping class groups are virtual (*i.e.* up to a finite-index subgroup) duality groups. The technique is purely topological and makes use of a cellularization of $\mathcal{M}_{g,n}$ with $n \geq 1$ by ribbon graphs due to several authors (Harer [48], Mumford, Thurston, Penner [79], Bowditch-Epstein [14]). In all constructions, one can bijectively attach a metrized ribbon graph (G,ℓ) (*i.e.* a finite graph endowed with a cyclic orientation of half-edges incident at each vertex) to a pointed Riemann surface with positive weights $(C, p_1, \dots, p_n, w_1, \dots, w_n)$ at its marked points, so that $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ becomes homeomorphic to a space of ribbon graphs $\mathcal{G}_{g,n}$, which can be decomposed in cells e_G indexed by isomorphism type of unmetrized ribbon graph G. Such a space can be retracted by deformation onto a smaller complex, from which one deduces the vanishing of high degree (co)homology groups. The virtual duality property for MCG follows from the sphericity of the so-called *curve complex*, that describes the combinatorics of the boundary strata of $\overline{\mathcal{M}}_{g,n}$.

The idea behind the homeomorphism $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \to \mathcal{G}_{g,n}$ is to first geometrize a given conformal structure on a surface and then produce a graph embedded in such a surface by exploiting the geometrization. Harer-Mumford-Thurston's construction relies on the existence of quadratic differentials with closed horizontal trajectories and prescribed quadratic residues at the punctures (proven by Strebel [91]), the graph of interest being given by the critical horizontal trajectories; on the other hand, Penner-Bowditch-Epstein's construction uses the uniformization theorem and hyperbolic metrics with cusps, the graph being given by the cut-locus associated to a fixed set of horocycles at the punctures. Though geometrically different, these cellularization are in fact isotopic (see [73]), which explains why we can freely use one or other for most purposes.

The following has remaining unanswered since then.

Problem 2.5. Does a similar cellularization exist for $\mathcal{M}_g$, *i.e.* for the moduli space of unpunctured Riemann surfaces?

Other questions arise very naturally.

Problem 2.6. Can one produce similar cellular model that parametrizes (possibly punctured) Riemann surfaces endowed with extra structure, such as $\mathbb{C}$-linear systems (or $\mathbb{C}$-linear systems of fixed finite order)?

Problem 2.7. Can one produce a similar cellular model for the Hurwitz scheme or, more generally, for spaces of finite branched covers between Riemann surfaces?

2.6 – Volumes via cellularizations

As the above-mentioned cellularization of $\mathcal{M}_{g,n}$ via ribbon graphs is indeed geometric, it can be exploited to decompose volume integrals as sums indexed by ribbon graphs. This strategy allowed Kontsevich [58] to rephrase such sums as Gaussian matrix integrals and so to prove Witten's conjecture [96]: top intersection numbers of ψ classes on the compactified moduli space of curves satisfy certain recursive relations.

Witten conjectured [97] that similar recursions should hold for the moduli space $\overline{\mathcal{M}}_{g,n}^{1/r}$ of curves endowed with an r-th root of the canonical bundle. This was established by Faber-Shadrin-Zvonkine [27] using different tools (reduction to a genus 0 problem using double ramification cycles and then Givental's theory of Gromov-Witten potential [37]), though it is believed that there should exist a proof of this fact in the line of Kontsevich's.

Problem 2.8. Is there a matrix integral that produces top intersection numbers on $\overline{\mathcal{M}}^{1/r}_{g,n}$?

Two-pointed Hurwitz numbers are also conjectured to satisfy similar recursions; the strategy suggested in [40] would be to relate them to top-intersections on a suitable compactification $\overline{\mathcal{J}}_{g,n}$ of the universal Jacobian variety over $\mathcal{M}_{g,n}$.

Problem 2.9. Is there a matrix integral that produces top intersection numbers of tautological classes on $\overline{\mathcal{J}}_{g,n}$?

Variations of this strategy allowed many authors to translate tautological classes into combinatorially defined cycles with closed support: see Penner [82] and Arbarello-Cornalba [6] for κ_1, Igusa [51] and Mondello [68] for the remaining κ classes, and [70] for a survey on this topic. Penner's proof passes through the explicit expression of Weil-Petersson symplectic form in the local simplicial coordinates [81]; similar computations for surfaces with boundary or with conical points are available [71] and [72], and show a tight relation between the different geometrizations of conformal structures on a surface and the different cellularizations and symplectic structures on $\mathcal{M}_{g,n}$.

2.7 – Volumes of spaces of translation surfaces

The Kobayashi metric on $\mathcal{M}_{g,n}$ was shown to agree with the Teichmüller metric [89], which is explicit but still difficult to handle, being Finsler and not smooth. Out of such metric, it is possible to define two different volumes (depending on whether one works on the tangent or cotangent space to $\mathcal{M}_{g,n}$), both of which are finite. In the latter and more usual case, the volume computation can be phrased in terms of volumes of spaces of quadratic differentials q with $\int |q| < 1$, whose volume form is inherited from the integral structure $H^1(\tilde{C}_q, P_q; \mathbb{Z} \oplus i\mathbb{Z}) \subset H^1(\tilde{C}_q, P_q; \mathbb{C})$ on the period domain of the degree 2 cover $\tilde{C}_q \to C$ that pulls q back to the square of an Abelian differential.

The following is deliberately vague.

Problem 2.10. Translate such a volume form into "something understandable" from the algebro-geometric point of view.

More generally, given $g \geq 2$ and a partition $m = (m_1, m_2, \dots, m_n)$ of $2g-2$, consider the moduli space $\mathcal{H}(m)$ of data $(C, p_1, \dots, p_n, \varphi)$, where $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$ and φ is an Abelian differential on C with a zero of order m_i at each p_i. Such a space has an affine structure, coming from the local period map to $H^1(C, P; \mathbb{C})$, which is a local biholomorphism. Subvarieties $\mathcal{A}$ of $\mathcal{H}(m_1, \ldots, m_n)$ that look affine in local period coordinates, and which are invariant under the action of $\mathrm{SL}_2(\mathbb{R})$ on the coefficients of $H^1(C, P; \mathbb{R} \oplus i\mathbb{R})$, are also endowed with a volume form. Moreover, it is known that the volume form thus attached to the projectivization $\mathbb{P}\mathcal{A}$ is finite.

Problem 2.11. Translate the computation of the volume of $\mathbb{P}\mathcal{A}$ coming from the period map in terms of characteristic classes.

The following question is almost implicit in the above.

Problem 2.12. Is there a (smooth) compactification of $\mathbb{P}\mathcal{H}(m_1, \ldots, m_n)$ that supports all "tautological" characteristic classes definable on $\mathbb{P}\mathcal{H}(m_1, \ldots, m_n)$ and on which the action of $\mathrm{SL}_2(\mathbb{R})$ extends?

Notice that the evaluation of volumes of $\mathbb{P}\mathcal{H}(m_1, \ldots, m_n)$ with each m_i divisible by r is tightly related to the computation of top-intersection numbers on the space $\overline{\mathcal{M}}_{g,n}^{1/r}$ of r-spin curves.

Problem 2.13. Does there exist a cellularization via ribbon graphs for $\mathcal{H}(m_1, \ldots, m_n)$? Is there an integration scheme via Gaussian matrix integral for such a space?

2.8 – Finite covers

In the orbifold category, unramified covers of $\mathcal{M}_g$ correspond to subgroups of MCG_g. Many problems, such as the computation of the cohomology ring, can be posed for unramified covers of $\mathcal{M}_{g,n}$: the most elementary one seems to be the following.

Problem 2.14. Is $H^1(\Gamma; \mathbb{C}) = 0$ for every finite-index subgroup $\Gamma \subset \mathrm{MCG}_g$ with $g \geq 3$?

It is known that $H^1(\overline{\mathcal{M}}_g; \mathbb{C}) = 0$ because MCG_g is generated by Dehn twists (see [17]). Moreover, for $g \geq 3$ Mumford [75] showed that $H^1(\mathcal{M}_g; \mathbb{C}) = H^1(\overline{\mathcal{M}}_g; \mathbb{C})$, whereas it is well-known that $H^1(\mathcal{M}_{0,n}; \mathbb{C}) \neq 0$ and that some finite unramified covers of $\mathcal{M}_{1,1}$ and $\mathcal{M}_2$ will have nonvanishing $H^1_{\mathbb{C}}$.

The conjecture was shown to hold for finite-index subgroups of MCG_g that contain all Dehn twists along separating simple closed curves (see Putman [86]), but Problem 2.14 is still open.

Recent work of Boggi-Looijenga [11] shows that the conjecture reduces to a purely topological statement about the topology of a finite branched cover of Riemann surfaces.

In a similar fashion, inspired by Putman's computation [87] of the second rational homology group of $\mathrm{MCG}_g^{(m)}$, one can also speculate about H^2 of an arbitrary finite cover of $\mathcal{M}_g$.

PROBLEM 2.15. Is $H^2(\Gamma;\mathbb{C})$ generated by κ_1 for every finite-index subgroup Γ of MCG_g and $g \geq 3$?

Problems concerning finite unramified covers of $\mathcal{M}_{g,n}$ can be rephrased in terms of local systems, and so of finite-dimensional representations of $\mathrm{MCG}_{g,n}$. It is known that braid groups and so $\mathrm{MCG}_{0,n}$ are linear (see Bigelow [10] and Krammer [60]) and that $\mathrm{MCG}_{1,1} = \mathrm{SL}_2(\mathbb{Z})$. Moreover, MCG_2 is an extension of $\mathcal{M}_{0,6}/\mathfrak{S}_6$ by $\mathbb{Z}/2\mathbb{Z}$, because every curve of genus 2 is hyperelliptic; hence, MCG_2 is linear too. But it is still unknown whether for $g \geq 3$ there are faithful finite-dimensional linear representations of MCG_g, though it was proven than these groups cannot be embedded as lattices inside Lie groups (see [56], for instance). Some time ago Kontsevich suggested a way of attacking this problem by looking at the cohomology of configuration spaces on a compact oriented surface of genus g and analyzing the dynamics of the many pseudo-Anosov elements of MCG_g on such a finite-dimensional vector space. Nobody seems to have tried to implement such ideas at present.

2.9 – Relative pluricanonical sections

Assume $g \geq 3$, so that $\mathcal{M}_g$ has no holomorphic functions (because its Satake compactification is normal and it has boundary of codimension 2 in this case). The vanishing of $H^1(\mathcal{M}_g;\mathbb{C})$ recalled above suggests that $\pi_*\left(\omega_\pi^{\otimes 2}\right) \cong T^*\mathcal{M}_g$ has no holomorphic sections.

PROBLEM 2.16. Show that every finite-degree unramified cover $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ has no nonzero holomorphic $(1,0)$-forms.

Notice that, being $\omega_{\overline{\pi}}$ big and nef on $\overline{\mathcal{C}}_g$ (see Arakelov [4] and Mumford [76]), there will be plenty of sections of $\omega_{\overline{\pi}}^{\otimes N}$, and so of $\pi_*(\omega_{\overline{\pi}}^{\otimes N})$, for N big enough.

PROBLEM 2.17. Find the smallest N such that the line bundle $\omega_\pi^{\otimes N}$ on $\mathcal{C}_g \cong \mathcal{M}_{g,1}$ has a nonzero section.

3 – Stratifications

3.1 – Arbarello's stratification

The principle that $\mathcal{M}_g$ could be better understood by using geometrically meaningful stratifications started to become popular since Arbarello [5] exhibited a filtration

$$\mathcal{M}_g = \overline{\mathcal{W}}_g \supset \overline{\mathcal{W}}_{g-1} \supset \cdots \supset \overline{\mathcal{W}}_2 \supset \overline{\mathcal{W}}_1 = \emptyset$$

satisfying the following properties:

- $\overline{\mathcal{W}}_k$ is a closed, irreducible, algebraic subvariety of $\mathcal{M}_g$ of pure codimension $g-k$;
- $\mathcal{W}_k := \overline{\mathcal{W}}_k \setminus \overline{\mathcal{W}}_{k-1}$ for all $k = 2, \dots, g$ is smooth;
- $\mathcal{W}_2$ concides with the locus of hyperelliptic curves.

The precise definition is as follows

$$\overline{\mathcal{W}}_k := \{C \in \mathcal{M}_g \mid \exists p \in C \text{ such that } h^0(kp) \geq 2\}.$$

Actually, analogous loci on $\mathcal{M}_{g,1}$ can be even more naturally defined as

$$\overline{\mathcal{W}}_k^* := \{(C,p) \in \mathcal{M}_{g,1} \mid h^0(kp) \geq 2\}$$

which determine the filtration

$$\mathcal{M}_{g,1} = \overline{\mathcal{W}}_{g+1}^* \supset \overline{\mathcal{W}}_g^* \supset \cdots \supset \overline{\mathcal{W}}_2^* \supset \overline{\mathcal{W}}_1^* = \emptyset$$

with the properties:

- $\overline{\mathcal{W}}_k^*$ is a closed, irreducible, algebraic subvariety of $\mathcal{M}_{g,1}$ of pure codimension $g-k+1$;
- $\mathcal{W}_k^* := \overline{\mathcal{W}}_k^* \setminus \overline{\mathcal{W}}_{k-1}^*$ for all $k = 2, \dots, g+1$ is smooth;
- $\mathcal{W}_2^*$ concides with the locus of couples (C,p), where C is hyperelliptic and p is a Weierstrass point of C.

The above stratifications attracted the attention of Mumford [76], who computed the cohomology (and Chow) classes of $\overline{\mathcal{W}}_k$ inside $\mathcal{M}_g$ (and of $\overline{\mathcal{W}}_k^*$ inside $\mathcal{M}_{g,1}$).

3.2 – Diaz's stratification

In his paper Arbarello posed the following question.

Problem 3.1 (Arbarello). Can the strata $\mathcal{W}_k$ contain a complete curve?

Arbarello suspected that this is not the case; if confirmed, it would follow that $\mathcal{M}_g$ is allowed to contain complete subvarieties of dimension at most $g-2$. Indeed, the coarse space of $\mathcal{M}_2$ is affine, as it coincides with $\mathcal{M}_{0,6}/\mathfrak{S}_6$, and so $\mathcal{M}_2$ cannot contain a complete curve. (The same argument works for the stratum $\mathcal{W}_2 \subset \mathcal{M}_g$ with $g \geq 2$.) In genus 3, the stratum $\mathcal{W}_3$ can be obtained as a quotient of the space of smooth plane quartics (which is affine) by PGL_3, and so its coarse space is again affine. On the other hand, $\mathcal{M}_3$ contains a complete curve, because its Satake compactification $\overline{\mathcal{M}}_3^{Sat}$ has coarse space which is projective and with boundary of codimension 2.

Diaz [20] managed to prove that indeed $\mathcal{M}_g$ cannot contain a complete subvariety of dimension greater than $g-2$, by suitably modifying Arbarello's stratification as follows. Fix $d \geq g$ and, for every $k = 2, \ldots, g$, let

$$\begin{aligned}\overline{\mathcal{D}}_{k,d} :=&\{C \in \mathcal{M}_g \mid \exists f : C \to \mathbb{P}^1 \text{ of degree } d \text{ such that } \operatorname{div}(f)\\ &\text{is supported on at most } k \text{ points}\}\end{aligned}$$

Diaz shows that

$$\mathcal{M}_g = \overline{\mathcal{D}}_{g,d} \supset \overline{\mathcal{D}}_{g-1,d} \supset \cdots \supset \overline{\mathcal{D}}_{2,d} \supset \overline{\mathcal{D}}_{1,d} = \emptyset$$

is a stratification and that $\mathcal{D}_{k,d} := \overline{D}_{k,d} \setminus \overline{D}_{k-1,d}$ cannot contain a complete curve.

The intuitive idea is rather simple: deforming a curve C inside $\mathcal{D}_{k,d}$ amounts to keeping the ramification profile of the map f frozen over $\{0, \infty\}$ and moving the branch points in $\mathbb{C}$; indeed, sending a branch point to 0 (or to ∞) would result in going to a deeper stratum $\mathcal{D}_{k',d}$ with $k' < k$ or in developing a singularity in C. Thus, the moduli of such a $C \in \mathcal{D}_{k,d}$ move in an open subset of an affine space. Hence, if the base of the deformation is complete, then the moduli of such a C could not move. In order to make such an argument formal, in passing from families of curves to families of maps and in studying the degenerations of such ramified covers, Diaz used the theory of admissible covers, developed by Harris-Mumford [49].

After Diaz's work, one is naturally lead to the following.

Problem 3.2. Does there exist a complete subvariety of $\mathcal{M}_g$ of dimension $g-2$?

It is rather embarassing to admit that at present Problem 3.2 is still open for $g \geq 4$.

3.3 – Looijenga's vanishing

A variant of the above stratifications was considered by Looijenga [64]. He defines the space

$$\mathcal{R} := \left\{ (C, P, x, Q) \,\middle|\, \begin{array}{ll} (C,P) \in \mathcal{M}_g^n,\ x \in C, & Q_\infty = (n+g)x, \\ Q \text{ degenerate linear system on } C, & P \subset \operatorname{supp}(Q_0) \end{array} \right\}$$

which admits a forgetful map $f : \mathcal{R} \to \mathcal{M}_g^n$ to the space of curves with n not necessarily distinct labelled points, and which can be stratified as

$$\mathcal{R} = \overline{\mathcal{R}}^0 \supset \overline{\mathcal{R}}^1 \supset \cdots \supset \overline{\mathcal{R}}^{g+n-1} = \emptyset$$

where

$$\overline{\mathcal{R}}^k := \left\{ (C, P, x, Q) \in \mathcal{R} \,\middle|\, \#\operatorname{supp}(Q_0) \leq g + n - k - 1 \right\}.$$

As in Diaz's work, the locally closed strata $\mathcal{R}^k := \overline{\mathcal{R}}^k \setminus \overline{\mathcal{R}}^{k-1}$ are quasi-affine (and this already reproves Diaz's bound).

Looijenga's key observation is that a suitable power of $f^*\mathcal{L}_i$ trivializes over $\mathcal{R}^k$. Indeed, at each $(C,P,x,Q) \in \mathcal{R}^k$, the pencil Q determines a map to $s : C \to \mathbb{P}^1$ which can be uniquely pinned down by setting the product of the branch values (other than 0 and ∞) to 1, because the ramification pattern over 0 and ∞ is frozen all over $\mathcal{R}^k$. Hence, the differential form dz on $\mathbb{P}^1 \setminus \{\infty\}$ lifts to a differential form $s^*(dz)$ on C which has fixed vanishing order m_i at p_i, and so it determines an everywhere-nonzero section of $f^*\mathcal{L}_i^{\otimes(m_i+1)}$.

As a consequence, $f^*(\psi_1^{a_1}\cdots\psi_n^{a_n})$, viewed as a class in rational cohomology or rational Chow ring, is supported on $\overline{\mathcal{R}}^k$ with $k = a_1+\cdots+a_n$. Because f is proper, this implies that $\psi_1^{a_1}\cdots\psi_n^{a_n}$ is supported on $f\left(\overline{\mathcal{R}}^k\right)$ and so this class rationally vanishes if $k \geq g+n-1$.

In the special case $n=1$, this means that $\psi_1^g = 0$ on $\mathcal{M}_{g,1}$ and so $\kappa_b = 0$ on $\mathcal{M}_g$ for $b > g-2$.

The idea of looking at (suitable compactified or partially compactified) space of maps to $\mathbb{P}^1$ is very classical, still rather fruitful. Though technically more involved, the same idea was implemented by Ionel [52] (who works in cohomology) and Graber-Vakil [41] (who work in the Chow ring) to prove that tautological classes of degree $> g-1+c-\delta_{n,0}$ vanish on the curious partial compactification $\mathcal{M}_{g,n}^{\leq c} \subset \overline{\mathcal{M}}_{g,n}$ consisting of stable curves with at most c rational irreducible components.

It is natural to wonder whether such vanishing theorems come from more general upper bounds on the Dolbeault/algebraic cohomological dimension of $\mathcal{M}_{g,n}^{\leq c}$. The corollaries for de Rham cohomology of these conjectural upper bounds were verified in [69], the proof essentially relying on Harer's result.

3.4 – Direct algebro-geometric approach

Direct ways to approach the problem of compact holomorphic subvarieties of $\mathcal{M}_g$ are the following.

3.4.1 – Affine covers

Try to find $D_1,\dots,D_{g-1}$ (rational) divisors on $\overline{\mathcal{M}}_g$ such that

(a) D_i is ample (in the coarse space of $\overline{\mathcal{M}}_g$) for every $i=1,\dots,g-1$;
(b) $\bigcap_{i=1}^{g-1}\mathrm{supp}(D_i)$ is contained inside the boundary locus $\partial\mathcal{M}_g := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$.

Notice that all divisors of $\overline{\mathcal{M}}_g$ are Cartier, since $\overline{\mathcal{M}}_g$ is smooth; hence, so is the boundary $\partial\mathcal{M}_g$. As a consequence, divisors $D_1,\dots,D_{g-1}$ satisfying (a) and (b) can always be replaced by $D_1+\varepsilon\partial\mathcal{M}_g,\dots,D_{g-1}+\varepsilon\partial\mathcal{M}_g$, where $\varepsilon > 0$ is a suitable small rational number, and so it is harmless to also require

(c) $\bigcup_{i=1}^{g-1}\mathrm{supp}(D_i)$ contains $\partial\mathcal{M}_g$.

Given such divisors, $\mathcal{M}_g$ can be realized as the union of the $g-1$ open subsets $\mathcal{U}_i := \mathcal{M}_g \setminus \mathrm{supp}(D_i)$, whose coarse space is affine. With a slight abuse of terminology, we will say that a Deligne-Mumford stack is affine if its coarse space is, and so that such a $\{\mathcal{U}_i\}$ is an *affine cover* of $\mathcal{M}_g$.

3.4.2 – Affine stratifications

Try to find a stratification made of $g-1$ layers

$$\mathcal{M}_g = \overline{\mathcal{S}}^0 \supset \overline{\mathcal{S}}^1 \supset \cdots \supset \overline{\mathcal{S}}^{g-1} = \emptyset$$

by algebraic subvarieties such that locally closed strata $\mathcal{S}^k$ are affine. Again by abuse of terminology, we will say that such a $\{\mathcal{S}^k\}$ is an *affine stratification* of $\mathcal{M}_g$.

Clearly, given an affine cover $\{\mathcal{U}_i\}$, one can produce an affine stratification just by setting $\mathcal{S}^1 := \mathcal{U}_1$ and $\mathcal{S}^k := (\mathcal{U}_1 \cup \ldots \mathcal{U}_{k-1})^c \cap \mathcal{U}_k$ for all $k = 2, \ldots, g-1$ (remember that $\mathcal{M}_g$ is separated).

Thus, one is naturally led to the following two questions.

Problem 3.3 (Looijenga). Does $\mathcal{M}_g$ admit an affine cover made of $g-1$ open subsets?

Problem 3.4 (Looijenga). Does $\mathcal{M}_g$ admit an affine stratification made of $g-1$ layers?

We have seen above that Problem 3.3 is harder than Problem 3.4, which in turn would imply Diaz's bound because affine algebraic varieties cannot contain complete curves. At present both problems are open for $g \geq 6$.

3.4.3 – q-convex exhaustion functions

Try to find a (smooth) function $\xi : \mathcal{M}_g \to \mathbb{R}$ which satisfies

- ξ is proper and bounded from below;
- the complex Hessian $(i\partial\overline{\partial}\xi)_C$ at the point $C \in \mathcal{M}_g$ is a Hermitean form of signature $(n_+, n_0, n_-)_C$ with the property that $(n_0 + n_-)_C \leq g-2$ for every $C \in \mathcal{M}_g$.

By a result by Andreotti-Grauert [3], the existence of such a function would imply the vanishing of Dolbeault cohomology in degrees $> g-2$, from which Diaz's bound would again follow. Indeed, as $\mathcal{M}_g$ can be endowed with a Kähler $(1,1)$-form α, the volume of a compact holomorphic subvariety X of dimension d would be given by the cap product $0 \neq \mathrm{vol}_\alpha(C) = [X] \cap [\alpha^d]$, which shows that $0 \neq [\alpha^d] \in H^{0,d}_{\overline{\partial}}(\mathcal{M}_{g,n}; \Omega^{d,0}_{\mathcal{M}_{g,n}})$ and so implies $d \leq g-2$.

Problem 3.5 (Looijenga). Does $\mathcal{M}_g$ admit such an exhaustion function with controlled complex Hessian?

3.5 – Approach via foliations

We have seen that, in order to bound the dimension of a compact holomorphic subvariety of $\mathcal{M}_g$, the strategy elaborated by Arbarello and then Diaz was to find a stratification made of $g-1$ layers whose locally closed strata could not contain a complete curve.

Grushevsky-Krichever [43] found a different proof of this result, that uses foliations instead of stratifications, and which we now describe.

First of all, they lift the problem to the $(3g-1)$-dimensional space $\mathcal{M}_g^2$ of triples (C,p,q), with $C \in \mathcal{M}_g$ and $p,q \in C$, namely p and q are not required to be distinct, and they show that $\mathcal{M}_g^2$ cannot contain a compact holomorphic subvariety Y of dimension $g+1$. They proceed by contradiction, assuming that such a Y exists.

They consider the $4g$-dimensional space $\widehat{\mathcal{M}}_g^2$ of quadruples (C,p,q,φ), where φ is a meromorphic differential on C which is regular on $C \setminus \{p,q\}$ and has simple poles at p,q. This space has natural local holomorphic coordinates given by relative periods, *i.e.* by integrals of φ along paths in $C \setminus \{p,q\}$ that are closed or that join two zeroes of φ. This space has a natural holomorphic foliation $\mathfrak{F}$ whose leaves $\mathcal{L}$ are defined by requiring that φ has locally constant absolute periods, *i.e.* integrals along closed paths only. Hence, each $\mathcal{L}$ has (complex) codimension $2g+1$ and so (complex) dimension $2g-1$.

Then they consider a real-analytic section Φ of the bundle $\widehat{\mathcal{M}}_g^2 \to \mathcal{M}_g^2$ by using a very classical tool dating back to Riemann's existence theorem, namely to every (C,p,q) they associate the *unique* differential $\varphi = \Phi(C,p,q)$ satisfying:

- φ has residue 1 at p and -1 at q, if $p \neq q$ (otherwise $\varphi \equiv 0$);
- all periods of φ are real.

Remark 3.6. Call $q_1, \dots, q_{2g} \in C$ the zeroes of φ. Hence, there exists a unique harmonic function $F : C \setminus \{p,q\} \to \mathbb{R}$ such that $\mathrm{Im}(\varphi) = dF$ and $\sum_{i=1}^{2g} F(q_i) = 0$. If we call $t_i := F(q_i)$, then we can reorder the indices in such a way that $t_1 \geq t_2 \geq \cdots \geq t_{2g}$. Notice that the quantities $t_i - t_j$ are relative periods.

One can check that $\Phi(\mathcal{M}_g^2)$ is the union of leaves of $\mathfrak{F}$, so $\mathcal{M}_g^2$ inherits a holomorphic but transversely real-analytic foliation, whose leaves $\mathcal{L}$ have complex dimension $2g-1$ (and so complex codimension g). So the intersection $\mathcal{L} \cap Y$ consists of a (possible empty) disjoint union of complex subvarieties of $\mathcal{M}_g^2$ of dimension ≥ 1.

Strategy. The idea is to construct compact real subvarieties $Y_{2g-1} \subset Y_{2g-2} \subset \cdots \subset Y_1 \subset Y_0 = Y$ such that

(a) $t_k|_{Y_{k-1}}$ constantly attains its maximum μ_k along Y_k for $k = 1, \dots, 2g-1$;
(b) if $\mathcal{L}$ is the leaf through $(C,p,q) \in Y_k$, then the component $(\mathcal{L} \cap Y)_{(C,p,q)}$ of $\mathcal{L} \cap Y$ containing (C,p,q) is completely contained inside Y_k.

Pick a $(C,p,q) \in Y_{2g-1}$. The intersection $(\mathcal{L} \cap Y)_{(C,p,q)}$ is at least (complex) 1-dimensional; on the other hand, the values of $t_1, \dots, t_{2g}$ are frozen on Y_{2g-1} and so all relative periods are frozen along $(\mathcal{L} \cap Y)_{(C,p,q)}$, which contradicts the fact the relative periods are coordinates.

Notice that the subvarieties Y_k are well-defined by property (a), because of the compactness of Y and that $X_1 \subset \mathcal{M}_{g,2}$ (unless $t_1 \equiv \cdots \equiv t_{2g-1} \equiv 0$ along X, which would immediately allow us to conclude). So one only needs to show that the above property (b) holds. The restriction of the function t_1 to a leaf $\mathcal{L}$ is locally the maximum of finitely many harmonic functions (more precisely, they become harmonic after a possible base-change) and so $t_1|_{\mathcal{L}}$ is subharmonic. Let $\mathcal{L}$ be the leaf through $(C,p,q) \in Y_1$. Being $(\mathcal{L} \cap Y)_{(C,p,q)}$ holomorphic and positive-dimensional, as t_1 attains its maximum at (C,p,q), it is constant along $(\mathcal{L} \cap Y)_{(C,p,q)}$ and so $(\mathcal{L} \cap Y)_{(C,p,q)}$ is contained inside Y_1.

Then one considers $t_2|_{Y_1}$ and one can check that, either it is already constant, or it never attains the value μ_1. In the former case, we are done; in the latter case, the restriction of t_2 to $\mathcal{L} \cap \{t_1 > t_2\}$ is locally the maximum of finitely many harmonic functions (after a possible base-change, as usual) for every leaf $\mathcal{L}$. So one proceeds in the same way as above.

3.6 – Results in low genera

For $g \leq 5$, Fontanari-Looijenga [30] exhibit a stratification of $\mathcal{M}_g$ with $g-1$ affine locally closed layers. More explicitly,

$$\begin{aligned}
&\mathcal{M}_2 = \mathcal{S}_2^2 \\
&\mathcal{M}_3 = \mathcal{S}_3^3 \supset \mathcal{S}_3^2 = \mathcal{H}yp_3 \\
&\mathcal{M}_4 = \mathcal{S}_4^4 \supset \mathcal{S}_4^3 = \mathcal{N}_4 \supset \mathcal{S}_4^2 = \mathcal{H}yp_4 \\
&\mathcal{M}_5 = \mathcal{S}_5^5 \supset \mathcal{S}_5^4 = \mathcal{T}_5 \supset \mathcal{S}_5^3 = \mathcal{T}_5 \cap \mathcal{N}_5 \supset \mathcal{S}_5^2 = \mathcal{H}yp_5
\end{aligned}$$

where $\mathcal{H}yp_g$ and $\mathcal{T}_g$ are the loci respectively of hyperelliptic and trigonal curves of genus g and $\mathcal{N}_g$ is the locus of curves $C \in \mathcal{M}_g$ that admit an even effective theta-characteristic, *i.e.* a line bundle L such that $L^{\otimes 2} \cong K_C$ and $h^0(C,L)$ is positive and even.

The result was later improved by Fontanari-Pascolutti [31], who exhibited an affine open cover made of $g-1$ layers for $g \leq 5$. They take a direct approach as described in Section 3.4 but they work in the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties of dimension g, in which $\mathcal{M}_g$ embeds via the Torelli map $j : \mathcal{M}_g \to \mathcal{A}_g$ defined as $j(C) := (J(C), \Theta_C)$, where $J(C)$ is the Jacobian variety of C and Θ_C is its Theta divisor. The advantage of working in $\mathcal{A}_g$ is that one can use the theory of modular forms, whose vanishing loci are ample divisors. Thus, in genus 3 it is enough to find a modular form that does not vanish on the hyperelliptic locus. In genus 4 (resp. 5) the authors succeed in finding modular forms $F_{\mathrm{null}}, F_1, F_H$ (resp. $F_{\mathrm{null}}, F_1, F_H, F_T$) that do not simultaneously vanish at $(J(C), \Theta_C)$ of a smooth curve C. As the combinatorial complexity increases very rapidly with g, it seems hard to push this strategy much further.

3.7 – Non-affineness of Arbarello's stratification

Fontanari [29] observed that Diaz's computation [22] of the class of $\overline{\mathcal{W}}_{g-1}$ inside $\mathcal{M}_g$ together with the ampleness criterion by Cornalba-Harris [16] showed that Arbarello's open stratum $\mathcal{W}_g$ is always affine. Clearly, the hyperelliptic locus $\mathcal{W}_2$ is affine too. But whether Arbarello's (locally closed) strata were affine remained unknown for long time, though some people suspected they were not in general.

The main difficulties in studying Arbarello's strata are that they are not complete intersections in the whole space (with the obvious exception of the divisorial one) and that in general $\overline{\mathcal{W}}_{k-1}$ is not Cartier inside $\overline{\mathcal{W}}_k$.

The negative suspects were confirmed by Arbarello-Mondello [9], who showed that most strata $\mathcal{W}_k$ and $\mathcal{W}_k^*$ are not affine, by embedding them as open subsets inside smooth spaces in such a way that the complement is not purely divisorial.

The case of the strata $\mathcal{W}_k$ essentially relies on computations by Diaz [21]. For strata $\mathcal{W}_k^*$ in $\mathcal{M}_{g,1}$, the idea is to use the desingularization

$$\mathcal{G}^1_{d,*} = \left\{ (C,p,V) \,\middle|\, (C,p) \in \mathcal{M}_{g,1} \text{ and } 1 \in V \subset H^0(C,kp),\ \dim_{\mathbb{C}}(V) = 2 \right\}$$

of $\overline{\mathcal{W}}^*_k$, which resolves $\mathcal{W}^*_{k-1}$ as the union of $\mathcal{G}^1_{d-1,*}$ and

$$\mathcal{Z} = \left\{ (C,p,V) \in \mathcal{G}^1_{d,*} \,\middle|\, h^0(kp) \geq 3 \right\}.$$

The authors show that in most cases $\mathcal{Z}$ is not contained inside $\mathcal{G}^1_{d,*}$ and is not purely divisorial.

3.8 – Stratification of the Hodge bundle

The theory of translation surfaces naturally suggests a stratification of the Hodge bundle $\pi_*(\omega_\pi)$ over $\mathcal{M}_g$, whose total space will be denoted by $\mathcal{H}_g$.

Indeed, if the space $\mathcal{H}(m)$ of translation surfaces with fixed singularities is defined as in Section 2.7, then strata of $\mathcal{H}_g$ can be identified to $\mathcal{H}(m)/\mathrm{Aut}(m)$. By regrouping strata by codimension, we obtain a stratification of $\mathcal{H}_g$ (and so of its projectivization $\mathbb{P}\mathcal{H}_g$) by $2g-2$ layers.

One can rephrase the stratification problem of $\mathcal{M}_g$ for the $\mathbb{P}^{g-1}$-bundle $\mathbb{P}\mathcal{H}_g$: the hope is to find an affine stratification of $\mathbb{P}\mathcal{H}_g$ with $(g-1)+(g-1) = 2g-2$ layers.

Problem 3.7. Are strata $\mathbb{P}\mathcal{H}(m)$ affine?

Affirmative answer to this question would easily allow to conclude that $\text{coh-dim}_{\text{alg}}(\mathcal{M}_g) \leq g-2$.

One can observe that all hyperelliptic strata are affine, as well as the strata $\mathbb{P}\mathcal{H}(4)$ and $\mathbb{P}\mathcal{H}(3,1)$ in genus 3 (see [65]). Moreover, in private conversation with Looijenga, it became apparent that [84, 88] and [95] imply that $\mathbb{P}\mathcal{H}(6)$ and $\mathbb{P}\mathcal{H}(8)^{\text{odd}}$ are affine too (see Kontsevich-Zorich [59] for a classification of connected components of $\mathbb{P}\mathcal{H}(m)$).

Really, to bound the algebraic cohomological dimension from above it is enough to prove the assertion below, which is weaker than the statement of Problem 3.7.

Problem 3.8. Is $\text{coh-dim}_{\text{alg}}(\mathbb{P}\mathcal{H}(m_1,\dots,m_n)) \leq n-1$?

A less ambitious question is the following.

Problem 3.9. Are smallest strata $\mathbb{P}\mathcal{H}(2g-2)$ affine?

Recently, it was proven [67] that $\text{coh-dim}_{\text{Dol}}(\mathbb{P}\mathcal{H}(m)) \leq g$ and that $\text{coh-dim}_{\text{Dol}}(\mathcal{M}_g) \leq 2g-2$, which is certainly not optimal, though it is the first non-trivial result valid for all g. This only mildly connects to Problem 3.8, as Dolbeault and algebraic cohomology of quasi-projective varieties need not be easily related. Moreover, passing from the cohomological dimension of strata to the cohomological dimension of the whole space is something formal for de Rham or algebraic cohomology, but it can be considerably more involved for Dolbeault cohomology (because non-algebraic holomorphic functions on a locally closed stratum need not leave a "trace" on the boundary of such stratum).

The strategy for proving such a bound goes through finding an exhaustion function ξ with controlled complex Hessian (see Section 3.4.3). The function $(C,P,[\varphi]) \mapsto \xi(C,P,[\varphi])$ is cooked up using the area of φ, namely

$$A(\varphi) = \frac{i}{2}\int_C \varphi \wedge \overline{\varphi}$$

and the φ-lengths along short smooth φ-geodesics γ joining zeroes of φ, namely

$$\ell_\gamma(\varphi) = \left|\int_\gamma \varphi\right|.$$

The complex Hessian is then explicitly analyzed using local period coordinates.

It is not unlikely that a similar strategy might also work for a partial compactification of $\mathcal{M}_g$ contained inside $\overline{\mathcal{M}}_g$.

A – Cohomological dimensions

A.1 – De Rham cohomological dimension

A.1.1 – Definition

Given a smooth connected manifold M of (real) dimension n, we can define the *de Rham cohomological dimension* of M as

$$\text{coh-dim}_{dR}(M) := \sup\left\{k \in \mathbb{N} \,\middle|\, H^k_{dR}(M;\mathbb{L}) \neq 0,\ \text{for some } \mathbb{C}\text{-local system } \mathbb{L} \text{ on } M\right\}$$

and it has the following properties:

(a) $\text{coh-dim}_{dR}(M) \in [0,n]$ and $\text{coh-dim}_{dR}(M) = n$ if and only if M is compact;
(b) if $\widetilde{M} \to M$ is a finite unramified cover, then $\text{coh-dim}_{dR}(\widetilde{M}) = \text{coh-dim}_{dR}(M)$;
(c) if $\pi : \widetilde{M} \to M$ locally looks like $U \to U/G$ with G a finite group, then $\text{coh-dim}_{dR}(\widetilde{M}) = \text{coh-dim}_{dR}(M)$;
(d) $\text{coh-dim}_{dR}(M \times N) = \text{coh-dim}_{dR}(M) + \text{coh-dim}_{dR}(N)$;
(e) if $M \to N$ is a fibration with fiber of (real) dimension r, then $\text{coh-dim}_{dR}(M) \leq \text{coh-dim}_{dR}(N) + r$ and equality holds if the map is proper;
(f) if $f : M \to \mathbb{R}$ is a smooth proper function, bounded from below, such that $n_+(\text{Hess}(f)_p) + n_0(\text{Hess}(f)_p) \leq q$ for all critical points $p \in M$ of f, then $\text{coh-dim}_{dR}(M) \leq q$.

Clearly, given a good open cover $\mathcal{U} = \{U_i\}$ of M, *i.e.* such that all finite intersections $U_I := \bigcap_{i\in I} U_i$ are either empty or contractible, one can compute $H^k_{dR}(M;\mathbb{L})$ using the Cech complex associated to $\mathcal{U}$. This way one can easily prove (c). Property (e) is a consequence of Leray-Serre spectral sequence and (f) follows from Morse theory.

A.1.2 – Orbifolds

An orbifold M of (real) dimension n is a topological spaces modelled on U/G, where U is a contractible open subset of $\mathbb{R}^n$ and G is a finite group. A subchart V/H of U/G is the datum of an injective homomorphism of groups $H \hookrightarrow G$ and an H-equivariant diffeomorphism $V \to U$ onto its image, such that the induced map $V/H \to U/G$ is an inclusion of open subsets of M. A differential form (in the orbifold sense) in U/G is a G-invariant differential form on U and its restriction to V/H is by the definition the H-invariant restriction to V.

Thus, one can extends the notion of global (smooth) differential forms, of exterior derivative and of de Rham cohomology to orbifolds. Thus, the definition of de Rham cohomological dimension still makes sense and the same properties as above hold for orbifolds.

A.2 – Dolbeault cohomological dimension

An analogous quantity can be produced in the context of complex analytic geometry. Namely, given a complex manifold X of (complex) dimension d, we define the *Dolbeault cohomological dimension* of X as

$$\text{coh-dim}_{\text{Dol}}(X) := \sup\left\{ q \in \mathbb{N} \,\middle|\, H^{0,q}_{\overline{\partial}}(X; E) \neq 0, \text{ for some hol.vector bundle } E \text{ on } X \right\}$$

which has the following properties:

(a) $\text{coh-dim}_{\text{Dol}}(X) \in [0, d]$ and $\text{coh-dim}_{\text{Dol}}(X) = d$ if and only if X has a compact d-dimensional component;
(b) if $\widetilde{X} \to X$ is a finite (unramified) cover, then $\text{coh-dim}_{\text{Dol}}(\widetilde{X}) = \text{coh-dim}_{\text{Dol}}(X)$;
(c) if $\pi : \widetilde{X} \to X$ locally looks like $U \to U/G$ with G a finite group, then $\text{coh-dim}_{\text{Dol}}(\widetilde{X}) = \text{coh-dim}_{\text{Dol}}(X)$;
(d) $\text{coh-dim}_{\text{Dol}}(X \times Y) = \text{coh-dim}_{\text{Dol}}(X) + \text{coh-dim}_{\text{Dol}}(Y)$;
(e) if $X \to Y$ has fibers of (complex) dimension s, then $\text{coh-dim}_{\text{Dol}}(X) \leq \text{coh-dim}_{\text{Dol}}(Y) + s$ and equality holds if the map is a proper submersion;
(f) if $\xi : X \to \mathbb{R}$ is a smooth proper function, bounded from below, such that $n_+(i\partial\overline{\partial}\xi)_x + n_0(i\partial\overline{\partial}\xi)_x \leq q$ for all $x \in X$, then $\text{coh-dim}_{\text{Dol}}(X) \leq q$.

Property (f) is due to Andreotti-Grauert [3] and uses Bochner technique instead of Morse theory.

De Rham and Dolbeault cohomological dimensions are related as follows. If $\mathbb{L}$ is a $\mathbb{C}$-linear system on a complex manifold X, then $\mathbb{L} \otimes_{\mathbb{C}} \Omega^{p,0}_X$ is a holomorphic vector bundle and there is a spectral sequence

$$E_2^{p,q} = H^{0,q}_{\overline{\partial}}(X; \mathbb{L} \otimes_{\mathbb{C}} \Omega^{p,0}_X) \implies H^{p+q}_{dR}(X; \mathbb{L})$$

so that the two cohomological dimensions satisfy

(dD) $\text{coh-dim}_{dR}(X) \leq \text{coh-dim}_{\text{Dol}}(X) + \dim_{\mathbb{C}}(X)$.

Clearly every definition and remark can be formulated in the category of complex analytic orbifolds.

A.3 – Algebraic cohomological dimension

Let X be a complex algebraic variety X of dimension d and assume that X is Cohen-Macaulay. We can define the *algebraic cohomological dimension* of X as

$$\text{coh-dim}_{\text{alg}}(X) := \sup\left\{ q \in \mathbb{N} \,\middle|\, H^q(X; \mathcal{E}) \neq 0, \text{ for some alg.coherent sheaf } \mathcal{E} \text{ on } X \right\}$$

which has the following properties:

(a) $\text{coh-dim}_{\text{alg}}(X) \in [0, d]$ and $\text{coh-dim}_{\text{alg}}(X) = d$ if and only if X has a complete irreducible component of dimension d;
(b) if $\widetilde{X} \to X$ is a finite (unramified) cover, then $\text{coh-dim}_{\text{alg}}(\widetilde{X}) = \text{coh-dim}_{\text{alg}}(X)$;
(c) if $\pi : \widetilde{X} \to X$ locally looks like $U \to U/G$ with G a finite group, then $\text{coh-dim}_{\text{alg}}(\widetilde{X}) = \text{coh-dim}_{\text{alg}}(X)$;
(d) $\text{coh-dim}_{\text{alg}}(X \times Y) = \text{coh-dim}_{\text{alg}}(X) + \text{coh-dim}_{\text{alg}}(Y)$;
(e) if $X \to Y$ is of relative dimension s, then $\text{coh-dim}_{\text{alg}}(X) \leq \text{coh-dim}_{\text{alg}}(Y) + s$ and equality holds if the map is proper, flat and relatively Gorenstein.
(f) Let $X = \overline{X} \setminus D$ with $\overline{X}$ smooth and complete and $D \subset \overline{X}$ a divisor. Suppose that $L = \mathcal{O}_{\overline{X}}(D)$ carries a Hermitean metric h such that the curvature $i\Theta(L, h)$ has at most q non-positive eigenvalues at every $x \in \overline{X}$. Then $\text{coh-dim}_{\text{alg}}(X) \leq q$.
(g) If X is a smooth complex affine variety, then it is Stein and so $\text{coh-dim}_{\text{Dol}}(X) = \text{coh-dim}_{\text{alg}}(X) = 0$.

Property (f) is again essentially due to Andreotti-Grauert [3] (see also [19]).

Algebraic and Dolbeault cohomologies are related by Serre's GAGA theorems [90]: if X is a smooth complex projective variety and E is an algebraic vector bundle on X, then $H^q(X; E) = H^{0,q}_{\overline{\partial}}(X; E)$, where X and E on the right hand side are considered as a complex manifold and a holomorphic vector bundle. Unfortunately, these cohomological dimensions are not interesting invariants for smooth projective varieties.

All the considerations can be extended to the category of complex Deligne-Mumford stacks, which are indeed locally of the form U/G, where U is a complex affine variety and G is a finite group.

A.4 – A naive look at the cases of $\mathcal{M}_{g,n}$ for $g = 0, 1, 2$

Let $g, n \geq 0$ such that $2g - 2 + n > 0$. The map $f : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ that forgets the $(n+1)$-th point has fiber $C \setminus P$, where C is a complex projective curve of genus g and P is a collection of n distinct points of C.

If $n = 0$, then f is proper of relative (complex) dimension 1. Hence,

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{g,1}) = \text{coh-dim}_{\text{alg}}(\mathcal{M}_g) + 1, \ \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{g,1}) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_g) + 1.$$

If $n \geq 1$, then f is algebraic affine of relative (complex) dimension 1, and so

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{g,n+1}) \leq \text{coh-dim}_{\text{alg}}(\mathcal{M}_{g,n}), \ \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{g,n+1}) \leq \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{g,n}).$$

A.4.1 – Genus zero

A point in $\mathcal{M}_{0,n}$ is represented by $(\mathbb{P}^1, 0, 1, \infty, x_1, \dots, x_{n-3})$ with $x_i \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $x_i \neq x_j$ for $i \neq j$. Thus, $\mathcal{M}_{0,n} \cong (\mathbb{C} \setminus \{0,1\})^{n-3} \setminus \Delta$, where Δ is the big diagonal (and so a Cartier divisor). Hence, $\mathcal{M}_{0,n}$ is a smooth complex affine variety of dimension $n-3$. It follows that

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{0,n}) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{0,n}) = 0.$$

A.4.2 – Genus one

A point in $\mathcal{M}_{1,1}$ is represented by an elliptic curve $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\tau \in \mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$. Indeed, if $\{\alpha, \beta\}$ is a positively oriented basis of $H_1(E;\mathbb{Z})$ and φ is a nonzero holomorphic $(1,0)$-form on E (which is unique up to rescaling), then

$$\tau = \left(\int_\beta \varphi\right) \Big/ \left(\int_\alpha \varphi\right).$$

Getting rid of the choice of the basis amount to quotienting $\mathbb{H}$ by the action of $\mathrm{SL}_2(\mathbb{Z})$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Hence, $\mathcal{M}_{1,1} \cong \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ and in fact $\mathfrak{T}_{1,1} \cong \mathbb{H}$ and $\mathrm{MCG}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})$. As $\mathcal{M}_{1,1}$ is irreducible, not complete and of dimension 1, it is affine. Hence, so is $\mathcal{M}_{1,n}$ and we can conclude that

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{1,n}) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{1,n}) = 0\,.$$

A.4.3 – Genus 2

Every curve of genus 2 is hyperelliptic and so it canonically maps onto $\mathbb{P}^1$ with degree 2 branching over 6 distinct points. Hence, there is a natural map $Br : \mathcal{M}_2 \to \mathcal{M}_{0,6}/\mathfrak{S}_6$, that takes a curve C to the quotient $(C/\iota, P)$, where ι is its hyperelliptic involution and P is the branching divisor of $C \to C/\iota$. Clearly, such a map Br is a $B(\mathbb{Z}/2\mathbb{Z})$-fibration, because of the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \cong \langle \iota \rangle \to \mathrm{Aut}(C) \to \mathrm{Aut}(C/\iota, P) \to 0$$

and so $\mathcal{M}_2$ has the same cohomological dimensions as $\mathcal{M}_{0,6}$, that is

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_2) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_2) = 0.$$

As a consequence,

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{2,1}) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{2,1}) = 1$$

and for $n \geq 2$ we have

$$\text{coh-dim}_{\text{alg}}(\mathcal{M}_{2,n}) = \text{coh-dim}_{\text{Dol}}(\mathcal{M}_{2,n}) \leq 1\,.$$

Acknowledgements

I am indebted to Enrico Arbarello and Eduard Looijenga for fruitful exchange of ideas over many years.

REFERENCES

[1] W. Abikoff: *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics, Springer, Berlin, Vol. 820, 1980, MR 590044 (82a:32028).
[2] D. Abramovich – A. Corti – A. Vistoli: *Twisted bundles and admissible covers*, Comm. Algebra, **31** (2003), no. 8, 3547–3618, Special issue in honor of Steven L. Kleiman, MR 2007376 (2005b:14049).
[3] A. Andreotti – H. Grauert: *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, **90** (1962), 193–259, MR 0150342 (27 #343).
[4] S. Ju. Arakelov: *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk SSSR Ser. Mat., **35** (1971), 1269–1293, MR 0321933 (48 #298).
[5] E. Arbarello: *Weierstrass points and moduli of curves*, Compositio Math., **29** (1974), 325–342, MR 0360601 (50 #13048).
[6] E. Arbarello – M. Cornalba: *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom., **5** (1996), no. 4, 705–749, MR 1486986 (99c:14033).
[7] E. Arbarello – M. Cornalba: *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 88, 97–127 (1999), MR 1733327 (2001h:14030).
[8] E. Arbarello – M. Cornalba – P. A. Griffiths: *Geometry of algebraic curves. Vol. II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris, MR 2807457 (2012e:14059)
[9] E. Arbarello – G. Mondello: *Two remarks on the Weierstrass flag*, Compact moduli spaces and vector bundles, Contemp. Math., Vol. 564, Amer. Math. Soc., Providence, RI, (2012), 137–144, MR 2895187.
[10] S. J. Bigelow: *Braid groups are linear*, J. Amer. Math. Soc., **14** (2001), no. 2, 471–486 (electronic), MR 1815219 (2002a:20043).
[11] M. Boggi – E. Looijenga: preprint.
[12] M. Boggi – M. Pikaart: *Galois covers of moduli of curves*, Compositio Math., **120** (2000), no. 2, 171–191, MR 1739177 (2002a:14025).
[13] S. K. Boldsen: *Improved homological stability for the mapping class group with integral or twisted coefficients*, Math. Z., **270** (2012), no. 1-2, 297–329, MR 2875835 (2012k:57026).

[14] B. H. Bowditch – D. B. A. Epstein: *Natural triangulations associated to a surface*, Topology, **27** (1988), no. 1, 91–117, MR 935529 (89e:57004).

[15] A. Buryak – S. Shadrin: *A new proof of Faber's intersection number conjecture*, Adv. Math., **228** (2011), no. 1, 22–42, MR 2822226 (2012j:14040).

[16] M. Cornalba – J. Harris: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. École Norm. Sup., (4) **21** (1988), no. 3, 455–475, MR 974412 (89j:14019).

[17] M. Dehn: *Die Gruppe der Abbildungsklassen*, Acta Math., **69** (1938), no. 1, 135–206, Das arithmetische Feld auf Flächen, MR 1555438.

[18] P. Deligne – D. Mumford: *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math., (1969), no. 36, 75–109, MR 0262240 (41 #6850).

[19] J. P. Demailly: *Complex analytic and differential geometry*, OpenContent Book, `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[20] S. Diaz: *A bound on the dimensions of complete subvarieties of $\mathcal{M}_g$*, Duke Math. J., **51** (1984), no. 2, 405–408, MR 747872 (85j:14042).

[21] S. Diaz: *Moduli of curves with two exceptional Weierstrass points*, J. Differential Geom., **20** (1984), no. 2, 471–478, MR 788291 (86f:14012).

[22] S. Diaz: *Exceptional Weierstrass points and the divisor on moduli space that they define*, Mem. Amer. Math. Soc., **56** (1985), no. 327, iv+69, MR 791679 (86j:14022).

[23] C. Faber: *Chow rings of moduli spaces of curves. I. The Chow ring of $\overline{\mathcal{M}}_3$*, Ann. of Math., (2) **132** (1990), no. 2, 331–419, MR 1070600 (91h:14009a).

[24] C. Faber: *Chow rings of moduli spaces of curves. II. Some results on the Chow ring of $\overline{\mathcal{M}}_4$*, Ann. of Math., (2) **132** (1990), no. 3, 421–449, MR 1078265 (91h:14009b).

[25] C. Faber: *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, Aspects Math., E33, Vieweg, Braunschweig, (1999), 109–129, MR 1722541 (2000j:14044).

[26] C. Faber: *Tautological algebras of moduli spaces of curves*, Moduli spaces of Riemann surfaces, IAS/Park City Math. Ser., Vol. 20, Amer. Math. Soc., Providence, RI, (2013), 197–219, MR 3114686.

[27] C. Faber – S. Shadrin – D. Zvonkine: *Tautological relations and the r-spin Witten conjecture*, Ann. Sci. Éc. Norm. Supér., (4) **43** (2010), no. 4, 621–658, MR 2722511 (2011k:14065).

[28] B. Farb – D. Margalit: *A primer on mapping class groups*, Princeton Mathematical Series, Vol. 49, Princeton University Press, Princeton, NJ, 2012, MR 2850125 (2012h:57032).

[29] C. Fontanari: *Moduli of curves via algebraic geometry*, Rend. Sem. Mat. Univ. Politec. Torino, **59** (2001), no. 2, 137–139 (2003), Liaison and related topics (Turin, 2001), MR 1977098 (2004d:14027).

[30] C. Fontanari – E. Looijenga: *A perfect stratification of $\mathcal{M}_g$ for $g \leq 5$*, Geom. Dedicata, **136** (2008), 133–143, MR 2443348 (2009f:14050).

[31] C. Fontanari – S. Pascolutti: *An affine open covering of $\mathcal{M}_g$ for $g \leq 5$*, Geom. Dedicata, **158** (2012), 61–68, MR 2922703.

[32] W. Fulton – R. Pandharipande: *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., Vol. 62, Amer. Math. Soc., Providence, RI, (1997), 45–96, MR 1492534 (98m:14025).

[33] F. P. Gardiner – N. Lakic: *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs, Vol. 76, American Mathematical Society, Providence, RI, 2000, MR 1730906 (2001d:32016).

[34] E. Getzler: *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The moduli space of curves (Texel Island, 1994), Progr. Math., Vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 199–230, MR 1363058 (96k:18008).

[35] E. Getzler – E. Looijenga: *The Hodge polynomial of* $\overline{\mathcal{M}}_{3,1}$, e-print `http://arxiv.org/abs/math/9910174`.

[36] E. Getzler – R. Pandharipande: *Virasoro constraints and the Chern classes of the Hodge bundle*, Nuclear Phys. B, **530** (1998), no. 3, 701–714, MR 1653492 (2000b:14073).

[37] A. B. Givental: *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J., **1** (2001), no. 4, 551–568, 645, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary, MR 1901075 (2003j:53138).

[38] G. Gonzalez Diez – W. J. Harvey: *On complete curves in moduli space. I, II*, Math. Proc. Cambridge Philos. Soc., **110** (1991), no. 3, 461–466, 467–472, MR 1120481 (92j:14028).

[39] A. G. Gorinov: *Rational cohomology of the moduli space of pointes genus* 1 *curves*, e-print `http://arxiv.org/abs/1303.5693`.

[40] I. P. Goulden – D. M. Jackson – R. Vakil: *The moduli space of curves, double Hurwitz numbers, and Faber's intersection number conjecture*, Ann. Comb., **15** (2011), no. 3, 381–436, MR 2836449 (2012h:14129).

[41] T. Graber – R. Vakil: *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J., **130** (2005), no. 1, 1–37, MR 2176546 (2006j:14035).

[42] A. Grothendieck: *Techniques de construction en géométrie analytique X: construction de l'espace de Teichmüller*, Vol. 13, Séminaire Henri Cartan, no. 2, ch. 17, pp. 1–20, Secr'etariat Mathématique, Paris, 1960-1961.

[43] S. Grushevsky – I. Krichever: *The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces*, Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, Surv. Differ. Geom., Vol. 14, Int. Press, Somerville, MA, (2009), 111–129, MR 2655324 (2011f:14058).

[44] J. Harer – D. Zagier: *The Euler characteristic of the moduli space of curves*, Invent. Math., **85** (1986), no. 3, 457–485, MR 848681 (87i:32031).

[45] J. Harer: *The second homology group of the mapping class group of an orientable surface*, Invent. Math., **72** (1983), no. 2, 221–239, MR 700769 (84g:57006).

[46] J. Harer: *The third homology group of the moduli space of curves*, Duke Math. J., **63** (1991), no. 1, 25–55, MR 1106936 (92d:57012).

[47] J. L. Harer: *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math., (2) **121** (1985), no. 2, 215–249, MR 786348 (87f:57009).

[48] J. L. Harer: *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math., **84** (1986), no. 1, 157–176, MR 830043 (87c:32030).

[49] J. Harris – D. Mumford: *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67** (1982), no. 1, 23–88, with an appendix by William Fulton, MR 664324 (83i:14018).

[50] J. H. Hubbard: *Teichmüller theory and applications to geometry, topology, and dynamics*. Vol. 1, Matrix Editions, Ithaca, NY, 2006, *Teichmüller theory*, with contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle, MR 2245223 (2008k:30055).

[51] K. Igusa: *Combinatorial Miller-Morita-Mumford classes and Witten cycles*, Algebr. Geom. Topol., **4** (2004), 473–520, MR 2077674 (2005g:57041).

[52] E.-N. Ionel: *Topological recursive relations in* $H^{2g}(\mathcal{M}_{g,n})$, Invent. Math., **148** (2002), no. 3, 627–658, MR 1908062 (2003d:14065).

[53] E.-N. Ionel: *Relations in the tautological ring of* $\mathcal{M}_g$, Duke Math. J., **129** (2005), no. 1, 157–186, MR 2155060 (2006c:14040).

[54] N. V. Ivanov: *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., Vol. 150, Amer. Math. Soc., Providence, RI, (1993), 149–194, MR 1234264 (94h:57022).

[55] E. Izadi: *The Chow ring of the moduli space of curves of genus* 5, The moduli space of curves (Texel Island, 1994), Progr. Math., Vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 267–304, MR 1363060 (97b:14029).

[56] Y. Kida: *Measure equivalence rigidity of the mapping class group*, Ann. of Math., (2) **171** (2010), no. 3, 1851–1901, MR 2680399 (2011e:37009).

[57] K. Kodaira: *A certain type of irregular algebraic surfaces*, J. Analyse Math., **19** (1967), 207–215, MR 0216521 (35 #7354).

[58] M. Kontsevich: *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys., **147** (1992), no. 1, 1–23, MR 1171758 (93e:32027).

[59] M. Kontsevich – A. Zorich: *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math., **153** (2003), no. 3, 631–678, MR 2000471 (2005b:32030).

[60] D. Krammer: *Braid groups are linear*, Ann. of Math., (2) **155** (2002), no. 1, 131–156, MR 1888796 (2003c:20040).

[61] K. Liu – H. Xu: *A proof of the Faber intersection number conjecture*, J. Differential Geom., **83** (2009), no. 2, 313–335, MR 2577471 (2011d:14051).

[62] E. Looijenga: *Cohomology of* $\mathcal{M}_3$ *and* $\mathcal{M}_3^1$, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., Vol. 150, Amer. Math. Soc., Providence, RI, 1993, pp. 205–228, MR 1234266 (94i:14032).

[63] E. Looijenga: *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geom., **3** (1994), no. 2, 283–293, MR 1257324 (94m:14029).

[64] E. Looijenga: *On the tautological ring of* $\mathcal{M}_g$, Invent. Math., **121** (1995), no. 2, 411–419, MR 1346214 (96g:14021).

[65] E. Looijenga – G. Mondello: *The fine structure of the moduli space of Abelian differentials in genus 3*, Geom. Dedicata, **169** (2014), 109–128, MR 3175239.

[66] Ib Madsen – M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math., (2) **165** (2007), no. 3, 843–941, MR 2335797 (2009b:14051).

[67] G. Mondello: *On the cohomological dimension of the moduli space of Riemann surfaces*, e-print `http://arxiv.org/abs/1405.2608`.

[68] G. Mondello: *Combinatorial classes on* $\overline{\mathcal{M}}_{g,n}$ *are tautological*, Int. Math. Res. Not. (2004), no. 44, 2329–2390, MR 2078260 (2005g:14056).

[69] G. Mondello: *A remark on the homotopical dimension of some moduli spaces of stable Riemann surfaces*, J. Eur. Math. Soc. (JEMS), **10** (2008), no. 1, 231–241, MR 2349902 (2008h:14028).

[70] G. Mondello: *Riemann surfaces, ribbon graphs and combinatorial classes*, Handbook of Teichmüller theory. Vol. II, IRMA Lect. Math. Theor. Phys., Vol. 13, Eur. Math. Soc., Zürich, 2009, pp. 151–215, MR 2497787 (2010f:32012).

[71] G. Mondello: *Triangulated Riemann surfaces with boundary and the Weil-Petersson Poisson structure*, J. Differential Geom., **81** (2009), no. 2, 391–436, MR 2472178 (2010a:32027).

[72] G. Mondello: *Poisson structures on the Teichmüller space of hyperbolic surfaces with conical points*, In the tradition of Ahlfors-Bers. V, Contemp. Math., Vol. 510, Amer. Math. Soc., Providence, RI, (2010), 307–329, MR 2581842 (2010m:53129).

[73] G. Mondello: *Riemann surfaces with boundary and natural triangulations of the Teichmüller space*, J. Eur. Math. Soc. (JEMS), **13** (2011), no. 3, 635–684, MR 2781928 (2012i:32020).

[74] S. Morita: *Generators for the tautological algebra of the moduli space of curves*, Topology, **42** (2003), no. 4, 787–819, MR 1958529 (2004g:14029).

[75] D. Mumford: *Abelian quotients of the Teichmüller modular group*, J. Analyse Math., **18** (1967), 227–244, MR 0219543 (36 #2623).

[76] D. Mumford: *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., Vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328, MR 717614 (85j:14046).

[77] R. Pandharipande – A. Pixton: *Relations in the tautological ring of the moduli space of curves*, e-print `http://arxiv.org/abs/1301.4561`.

[78] R. Pandharipande – A. Pixton – Zvonkine D.: *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*, e-print `http://arxiv.org/abs/1303.1043`.

[79] R. C. Penner: *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys., **113** (1987), no. 2, 299–339, MR 919235 (89h:32044).

[80] R. C. Penner: *Perturbative series and the moduli space of Riemann surfaces*, J. Differential Geom., **27** (1988), no. 1, 35–53, MR 918455 (89h:32045).

[81] R. C. Penner: *Weil-Petersson volumes*, J. Differential Geom., **35** (1992), no. 3, 559–608, MR 1163449 (93d:32029).

[82] R. C. Penner: *The Poincaré dual of the Weil-Petersson Kähler two-form*, Comm. Anal. Geom., **1** (1993), no. 1, 43–69, MR 1230273 (94i:32025)¿

[83] D. Petersen – O. Tommasi: *The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$*, Invent. Math., **196** (2014), no. 1, 139–161, MR 3179574.

[84] H. C. Pinkham: *Deformations of algebraic varieties with $\mathbb{G}_m$-action*, Société Mathématique de France, Paris, 1974, Astérisque, no. 20, MR 0376672 (51 #12847).

[85] A. Pixton: *Conjectural relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$*, e-print `http://arxiv.org/abs/1207.1918`.

[86] A. Putman: *A note on the abelianizations of finite-index subgroups of the mapping class group*, Proc. Amer. Math. Soc., **138** (2010), no. 2, 753–758, MR 2557192 (2011c:57004).

[87] A. Putman: *The second rational homology group of the moduli space of curves with level structures*, Adv. Math., **229** (2012), no. 2, 1205–1234, MR 2855091 (2012i:57032).

[88] D. Sang Rim – M. A. Vitulli: *Weierstrass points and monomial curves*, J. Algebra, **48** (1977), no. 2, 454–476, MR 0457447 (56 #15652).

[89] H. L. Royden: *Intrinsic metrics on Teichmüller space*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2, Canad. Math. Congress, Montreal, Que., 1975, pp. 217–221, MR 0447636 (56 #5946).

[90] J.-P. Serre: *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble, **6** (1955–1956), 1–42, MR 0082175 (18,511a).

[91] K. Strebel: *On quadratic differentials with closed trajectories and second order poles*, J. Analyse Math., **19** (1967), 373–382, MR 0224808 (37 #407).

[92] O. Tommasi: *Rational cohomology of the moduli space of genus 4 curves*, Compos. Math., **141** (2005), no. 2, 359–384, MR 2134272 (2006c:14043).

[93] O. Tommasi: *Rational cohomology of $\mathcal{M}_{3,2}$*, Compos. Math., **143** (2007), no. 4, 986–1002, MR 2339836 (2008d:14043).

[94] A. J. Tromba: *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992, Lecture notes prepared by Jochen Denzler, MR 1164870 (93h:32028).

[95] M. A. Vitulli: *The obstruction of the formal moduli space in the negatively graded case*, Pacific J. Math., **82** (1979), no. 1, 281–294, MR 549851 (82c:14006).

[96] E. Witten: *On the Kontsevich model and other models of two-dimensional gravity*, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), World Sci. Publ., River Edge, NJ, 1992, pp. 176–216, MR 1225112 (94g:32029).

[97] E. Witten: *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 235–269, MR 1215968 (94c:32012).

[98] C. G. Zaal: *A complete surface in M_6 in characteristic* > 2, Compositio Math., **119** (1999), no. 2, 209–212, MR 1723129 (2000i:14031).

[99] C. Zaal: *Explicit complete curves in the moduli space of curves of genus three*, Geom. Dedicata, **56** (1995), no. 2, 185–196, MR 1338958 (96k:14020).

Lavoro pervenuto alla redazione il 24 settembre 2014
ed accettato per la pubblicazione il 1 ottobre 2014
bozze licenziate il 16 dicembre 2014

Indirizzo dell'Autore:
Gabriele Mondello – Dipartimento di Matematica – "Sapienza" Università di Roma – P.le Aldo Moro 5 – 00185 – Roma – Italy
E-mail: mondello@mat.uniroma1.it

The author's research was partially supported by the MIUR grant FIRB 2010 "Low-dimensional geometry and topology" (RBFR10GHHH_003).