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On the theorem of Walsh-Runge

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Abstract: *A version of the classical holomorphic and meromorphic approximation therems by Behnke-Stein and Walsh is proved in the case of a general, not necessarily smooth, one-dimensional complex space. It contains all the known results, which are briefly reviewed.*

1 – Runge's approximation theorem

The condition that allows uniform approximation of holomorphic functions by rational functions on the compact subsets of a domain has been originally given by Runge's in his classical Approximationssatz of 1885:

Theorem 1.1 (Runge, [1]). *Every holomorphic function defined in an open subset U of the Riemann sphere can be uniformly approximated on the compact subsets of U by rational functions.*

Later, in 1948, Behnke and Stein have characterized the domains in a general open Riemann surface in which holomorphic functions can be approximated by global ones:

Theorem 1.2 (Behnke-Stein, [3]). *Every holomorphic function defined in an open set U of the open Riemann surface X can be approximated by holomorphic functions defined in X if and only if $X \setminus U$ has no relatively (to X) compact connected component.*

The one-dimensional particular case of the main result in [6] gives the extension of Behnke-Stein theorem in the presence of singularities:

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Theorem 1.3 ([6]). *Let X be a Stein analytic space of pure dimension* 1*, Y an open space of X. If for each open neighbourhood U of $X \setminus Y$ and for each irreducible component C of U, $C \cap (X \setminus Y)$ is non compact, then every holomorphic function on Y can uniformly approximated on the compact subsets of Y, by means of restrictions to Y of holomorphic functions on X.*

This result of course reduces to the sufficient condition of the Behnke-Stein theorem when X is smooth.

Mihalache, in 1988, has found a condition of topological nature, which is necessary and sufficient for approximation to hold:

Theorem 1.4 ([5]). *Let X be a Stein analytic space of pure dimension* 1*, Y an open space of X. Every holomorphic function on Y can uniformly approximated on the compact subsets of Y, by means of restrictions to Y of holomorphic functions on X if an only if the homomorphism induced by the inclusion between the singular homology groups*

$$H_1(Y) \to H_1(X)$$

is injective.

Let us remark that compactness of the ambient space has no longer been considered in the above mentioned string of results and that not all non-compact one-dimensional analytic spaces are Stein spaces.

In the vein of the first of these remarks, J.L. Walsh has given in 1935 what is the sharpest result, to the author's knowledge, generalizing Runge's theorem in the classical setting [7]:

Theorem 1.5 (Walsh). *Let U be an open subset and P be a subset given on the Riemann sphere. Let us suppose that P meets only a finite number of connected components of the complement of U. In order that a holomorphic function defined on U be uniformly approximable on compact subsets by means of rational functions with poles contained in P is necessary and sufficient that each connected component of the complement of U contains a point of P.*

Walsh's result has been "rediscovered" by Auderset, [2], and generalized to the case of holomorphic sections of line bundles on a general Riemann surface.

We wish to prove here the following statement, which we shall call the Walsh-Runge theorem and which contains both Mihalache's and Auderset's results, generalizes Walsh's theorem to the 1-dimensional singular case and shows that the condition in Theorem 1.3 is also necessary:

Theorem 1.6. *Let X be a reduced analytic space of dimension* 1 (*with countable topology*) *and $Y \subset X$ an open subspace. Let $P \subset X \setminus Y$. If X has irreducible compact components C_α, let us suppose that $P \cap C_\alpha \neq \emptyset$ but for a finite number of α's. Let $\mathbb{L}$ be a holomorphic line bundle on X.*

The following statements are equivalent:

(*i*) *Every holomorphic section of* $\mathbb{L}$ *on* Y *can be uniformly approximated on the compact subsets of* Y, *by means of meromorphic sections of* $\mathbb{L}$ *on* X *with poles contained in* P.

(*ii*) *Every holomorphic section of* $\mathbb{L}$ *on* Y *can be uniformly approximated on the compact subsets of* Y, *by means of restrictions to* Y *of holomorphic sections of* $\mathbb{L}$ *on* $X \setminus \overline{P}$.

(*iii*) *The homomorphism induced by the inclusion between the singular homology groups*

$$H_1(Y) \to H_1(X \setminus \overline{P})$$

is injective.

(*iv*) *For every open neighbourhood* U *of* $(X \setminus \overline{P}) \setminus Y$ *and for every irreducible component* C *of* U,

$$C \cap ((X \setminus \overline{P}) \setminus Y)$$

is non-compact.

2 – A proof of Theorem 1.6

The following preliminary results are needed:

Proposition 2.1. *Let* X' *be a non compact holomorphically convex analytic space of dimension* 1; $Y \subset X'$ *be an open subspace. Let* $\pi : \tilde{X} \to X'$ *be the normalization map and* $\tilde{Y} = \pi^{-1}(Y)$.

The following statements are equivalent:

(*a*) *For every locally free sheaf* Ω *on* X', *the restriction map*

$$\Gamma(X', \Omega) \to \Gamma(Y, \Omega)$$

has dense image.

(*b*) *For every locally free sheaf* $\tilde{\Omega}$ *on* $\tilde{X}$, *the restriction map*

$$\Gamma(\tilde{X}, \tilde{\Omega}) \to \Gamma(\tilde{Y}, \tilde{\Omega})$$

has dense image.

(*c*) *The homomorphism induced by the inclusion between singular homology groups*

$$H_1(Y) \to H_1(X')$$

is injective.

(*d*) *For every open neighbourhood U of $(X'\setminus Y)$ and for every irreducible component C of U,*

$$C \cap (X' \setminus Y)$$

is non-compact.

(*e*) *The homomorphism induced by the inclusion between singular homology groups*

$$H_1(\tilde{Y}) \to H_1(\tilde{X})$$

is injective.

(*f*) *$\tilde{X} \setminus \tilde{Y}$ has no compact connected components.*

Proof.

$(a) \iff (b) \iff (f) \implies (d) \implies (c) \implies (e) \iff (f)$

$(b) \iff (f)$ is the Thorem of Behnke-Stein, Theorem 1.2.

$(e) \iff (f)$ is in Auderset, [2].

$(f) \implies (d)$: Let C be an irreducible component of U such that $A = C \cap (X' \setminus Y)$ is compact and let $\tilde{A} = \pi^{-1}(A)$. $\tilde{A}$ is compact.

$\tilde{C} = \pi^{-1}(C)$ is a connected component of $\pi^{-1}(U)$ and $\tilde{A} = \tilde{C} \cap (\tilde{X} \setminus \tilde{Y})$, hence $\tilde{A}$ is open; it follows that $\tilde{X} \setminus \tilde{Y}$ contains an open compact subset hence it has a compact connected component.

$(d) \implies (c)$: Let $\mathbb{Z}$ be the sheaf of germs of locally constant integer valued functions on X'. If $U \subset X'$ is open, then the support of an element in $H^0_c(U, \mathbb{Z})$ is a compact irreducible component of U. Hence, $H^0_c(U, \mathbb{Z}) = 0$ if and only if U has no compact irreducible component.

Let us take now an open neighbourhood U of $(X' \setminus Y)$. Each irreducible component of U is non compact.

From the exact sequence :

$$\ldots \to {H_c}^0(U, \mathbb{Z}) \to {H_c}^1(Y, \mathbb{Z}) \to {H_c}^1(X', \mathbb{Z}) \to \ldots,$$

and from ${H_c}^0(U, \mathbb{Z}) = 0$, the Alexander-Pontryagin duality gives (c).

$(c) \implies (e)$: Let S be the singular locus of Y. The normalization map $\pi : \tilde{Y} \to Y$ gives a biholomorphism:

$$\pi \restriction_{\tilde{Y} \setminus \pi^{-1}(S)} : \tilde{Y} \setminus \pi^{-1}(S) \to Y \setminus S.$$

The singular homology of $Y \setminus S$, respectively $\tilde{Y} \setminus \pi^{-1}(S)$, is the homology of the pair (Y, S), $(\tilde{Y}, \pi^{-1}(S))$, respectively, hence we have :

$$H_1(Y, S) \cong H_1(\tilde{Y}, \pi^{-1}(S))$$

and since $\dim S = \dim \pi^{-1}(S) = 0$, the exact sequences of relative homology give the commutative diagram:

$$\begin{array}{ccccc}
0 = H_1(\pi^{-1}(S)) & \longrightarrow & H_1(\tilde{Y}) & \longrightarrow & H_1(\tilde{Y} \setminus \pi^{-1}(S)) \\
\downarrow & & \pi_* \downarrow & & \downarrow \\
0 = H_1(S) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(Y \setminus S)
\end{array}$$

whence $\pi_* : H_1(\tilde{Y}) \to H_1(Y)$ is injective.

The commutative diagram:

$$\begin{array}{ccc}
H_1(\tilde{Y}) & \xrightarrow{\pi_*} & H_1(Y) \\
\tilde{i} \downarrow & & i \downarrow \\
H_1(\tilde{Y}) & \xrightarrow{\pi_*} & H_1(X')
\end{array}$$

gives the injectivity of $\tilde{i}$.

$(a) \Longleftrightarrow (b)$ It is enough to show this for $\Omega = \mathcal{O}'_X$ and $\tilde{\Omega} = \mathcal{O}_{\tilde{X}}$.

It is clear that $(a) \Longrightarrow (b)$.

Conversely, since both $\tilde{X}$ and $\tilde{Y}$ are holomorphically convex, holomorphic convexity being a property that carries over to normalization, we have that $H^1(\tilde{X}, \mathcal{I})$ is Hausdorff for every coherent ideal sheaf $\mathcal{I}$ on $\tilde{X}$ and the same holds for $\tilde{Y}$.

Let $\mathcal{D}$ the sheaf of universal denominators on X',[4], and $\tilde{\mathcal{O}}_{X'}$ be the sheaf of germs of locally bounded holomorphic functions on X'. $\mathcal{D}\tilde{\mathcal{O}}_{X'}$ is a subsheaf of $\mathcal{O}_{X'}$, and $\operatorname{supp}(\mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'})$ is contained in the singular locus S of X'.

Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(X', \mathcal{D}\tilde{\mathcal{O}}_{X'}) & \longrightarrow & \Gamma(X', \mathcal{O}_{X'}) & \xrightarrow{\rho} & \Gamma(X', \mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'}) \\
& & r_1 \downarrow & & r_2 \downarrow & & r_3 \downarrow \\
0 & \longrightarrow & \Gamma(Y, \mathcal{D}\tilde{\mathcal{O}}_{X'}) & \longrightarrow & \Gamma(Y, \mathcal{O}_{X'}) & \xrightarrow{\rho_Y} & \Gamma(Y, \mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'})
\end{array}$$

If we show that r_1 and r_3 have dense images and that ρ and ρ_Y are surjective, a standard argument of topological vector spaces applied to the above diagram will give us the density of r_2, hence our thesis.

Since $\pi^*(\mathcal{D}\tilde{\mathcal{O}}_{X'}) = \pi^*(\mathcal{D})\mathcal{O}_{\tilde{X}}$ and $\tilde{\mathcal{O}}_{X'} = \pi_* \mathcal{O}_{\tilde{X}}$,

$$\Gamma(X', \mathcal{D}\tilde{\mathcal{O}}_{X'}) = \Gamma(\tilde{X}, \pi^*(\mathcal{D})\mathcal{O}_{\tilde{X}})$$

and

$$H^1(X', \mathcal{D}\tilde{\mathcal{O}}_{X'}) = H^1(\tilde{X}, \pi^*(\mathcal{D})\mathcal{O}_{\tilde{X}})$$

and the same holds for Y and $\tilde{Y}$, we have that r_1 has dense image and that $H^1(X', \mathcal{D}\tilde{\mathcal{O}}_{X'})$ is Hausdorff and the same holds for Y.

Furthermore, since $\dim \operatorname{supp}(\mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'}) = 0$, r_3 has dense image as well.

Let $S = \{x_i\}_{i\in\mathbb{N}}$. Since X' is holomorphically convex, for every choice of $\{x_{i_1}\dots x_{i_n}\} \subset S$ and $\{z_1\dots z_n\} \subset \mathbb{C}$, there exists $f \in \Gamma(X', \mathcal{O}_{X'})$ such that $\rho(f) = f(x_{i_j}) = z_j$. Identifying the Frèchet space $\Gamma(X', \mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'})$ with $\mathbb{C}^{\{x_1\cdots\}}$, we have that ρ has dense image. But

$$\operatorname{Im}\rho = \ker\left(\delta : \Gamma(\mathrm{X}', \mathcal{O}_{\mathrm{X}'}/\mathcal{D}\tilde{\mathcal{O}}_{\mathrm{X}'}) \to \mathrm{H}^1(\mathrm{X}', \mathcal{D}\tilde{\mathcal{O}}_{\mathrm{X}'})\right)$$

Since the latter space is Hausdorff, $\ker\delta$ and therefore $\operatorname{Im}\rho$ are closed. It follows that $\operatorname{Im}\rho$ is a closed dense subset of $\Gamma(X', \mathcal{O}_{X'}/\mathcal{D}\tilde{\mathcal{O}}_{X'})$ so they have to coincide .

Hence ρ, and by the same reasoning, ρ_Y are both surjective. □

We will need also to prove the following remark:

LEMMA 2.2. *Let X be a reduced complex analytic space of dimension 1 with countable topology.*

If X has only finitely many compact irreducible components of dimension 1, X is strongly 1-pseudoconvex (hence holomorphically convex).

PROOF. Let X' be the union of the non compact irreducible components of X. Then there exists a 1-convex $\mathcal{C}^\infty$ exhausting function ψ on X'.

Let A be the union of all compact irreducible components of X, and

$$\beta = \sup_{A\cap X'} \psi + 1.$$

If we set:

$$\phi = \begin{cases} \exp\left(-\dfrac{1}{(\psi-\beta)^2} + \psi - \beta\right) & \text{on } X' \\ 0 & \text{on } A. \end{cases}$$

we obtain that ϕ is a $\mathcal{C}^\infty$ exhausting function on X which is strongly 1-convex outside a compact subset of X. □

In order to prove the Walsh-Runge theorem, we observe that the assumption of holomorphic convexity being warranted by Lemma 2.2, the equivalences between (ii), (iii) and (iv) are a consequence of Proposition 2.1, if we set $X \setminus \overline{P} = X'$, $\Omega = \mathcal{O}(\mathbb{L})$.

Now, $(i) \implies (ii)$ is immediate. To conclude let us recall that if $\mathbf{D}$ is the set of effective divisors with support contained in P, a meromorphic section of $\mathbb{L}$ with poles contained in P is a holomorphic section of the locally free sheaf $\mathcal{O}(\mathbb{L}\otimes[D])$, where $[D]$ is the line bundle associated with $D \in \mathbf{D}$, and D is such that if f is a meromorphic section of $\mathbb{L}$ with poles contained in P, then $-D \leq div\,(f)$, which proves (i). □

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