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Landau-Kolmogorov type inequalities in several variables for the Jacobi measure

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Abstract: *This paper is devoted to Landau-Kolmogorov type inequalities in several variables in L^2 norm for Jacobi measures. These measures are chosen in such a way that the partial derivatives of the Jacobi orthogonal polynomials are also orthogonal. These orthogonal polynomials in several variables are built by tensor product of the orthogonal polynomials in one variable. These inequalities are obtained by using a variational method and they involve the square norms of a polynomial p and at most those of the partial derivatives of order 2 of p.*

1 – Introduction

These inequalities appear for the first time in the papers of E. Landau [13] and A. Kolmogorov [12] who used the maximum norm. The use of L^2 norms involves another form of these inequalities which generally is

$$\| p' \|_{(1)}^2 \leqslant C_1(n) \| p \|_{(0)}^2 + C_2(n) \| p'' \|_{(2)}^2$$

for any polynomial p, in one variable, of degree at most n. $C_1(n)$ and $C_2(n)$ are two positive constants depending on n. The different L^2 norms $\| \,.\, \|_{(0)}$, $\| \,.\, \|_{(1)}$ and $\| \,.\, \|_{(2)}$ depend on the used measures (see [3, 4, 6, 7, 16]; see also [14] from the page 614 for a general presentation of the different types of inequalities). These inequalities are called Landau-Kolmogorov type inequalities. More general Landau-Kolmogorov type inequalities involving more than three L^2 norms can be found in [1].

Key Words and Phrases: *Markov-Bernstein inequalities – Landau-Kolmogorov type inequalities – L^2 norm – Jacobi measure – Orthogonal polynomials – Several variables – Variational method – Sobolev spaces*
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The study of such inequalities in several variables is new. The first results are given for the Hermite and Laguerre-Sonin measures in [2]. This paper is devoted to Landau-Kolmogorov type inequalities in several variables for Jacobi measures.

We will use the following notations. Let $x = (x_1, \dots, x_s)$ be a point of $\mathbb{R}^s$, for $s \geqslant 2$. For any element $n = (n_1, \dots, n_s)$ of $\mathbb{N}^s$, $|n|$ will denote the sum of its components.

$$|n| = \sum_{j=1}^{s} n_j.$$

Let $i = (i_1, \dots, i_s)$ be an element of $\mathbb{N}^s$. Let $L^2(\Omega; \mu_{i_j,j})$, $j = 1, \dots, s$, be the Hilbert space of square integrable real functions in the variable x_j on the open set $\Omega \subset \mathbb{R}$. $\mu_{i_j,j}$ is a Jacobi measure supported on $\Omega =]-1, 1[$.

$$\mu_{i_j,j} = (1-x)^{\alpha_j + i_j}(1+x)^{\beta_j + i_j}, \quad j = 1, \dots, s, \text{ and } \forall i_j \in \mathbb{N}, \\ \text{with } \alpha_j > -1 \text{ and } \beta_j > -1.$$

We will put $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\beta = (\beta_1, \dots, \beta_s)$.

The inner product on every Hilbert space $L^2(\Omega; \mu_{i_j,j})$ is defined by

$$(f(x_j), g(x_j))_{L^2(\Omega;\mu_{i_j,j})} = \int_\Omega f(x_j) g(x_j) d\mu_{i_j,j}(x_j), \qquad \forall f, g \in L^2(\Omega; \mu_{i_j,j}),$$

and the norm is

$$\|f\|_{L^2(\Omega;\mu_{i_j,j})} = ((f,f)_{L^2(\Omega;\mu_{i_j,j})})^{\frac{1}{2}}.$$

From the different measures $\mu_{i_j,j}$, we define the measure μ_i by tensor product

$$\mu_i = \prod_{j=1}^{s} \mu_{i_j,j}.$$

Note that the element $(0, \dots, 0)$ of $\mathbb{N}^s$ will simply be denoted by 0. Therefore, $\mu_0 = \prod_{j=1}^{s} \mu_{0,j}$.

Let $r = (r_1, \dots, r_s)$ be an element of $\mathbb{N}^s$. Let Q_r be the space of real polynomials in s variables of degree at most r_j with respect to the variable x_j, $j = 1, \dots, s$. Let us define a notation for the partial derivatives. Let p be a polynomial of Q_r. Let $\nu = (\nu_1, \dots, \nu_s)$ be an element of $\mathbb{N}^s$. $\partial^\nu p$ will denote the following partial derivative of p:

$$\partial^\nu p = \frac{\partial^{|\nu|} p}{\partial x_1^{\nu_1} \dots \partial x_s^{\nu_s}}.$$

Let $\ell = (\ell_1, \dots, \ell_s)$ be an element of $\mathbb{N}^s$. Let $P_\ell^{(\alpha,\beta)}$ be the monic Jacobi polynomial orthogonal with respect to the measure μ_0. Then, $P_\ell^{(\alpha,\beta)}$ will be equal to

$P_{\ell_1}^{(\alpha_1,\beta_1)}(x_1)\dots P_{\ell_s}^{(\alpha_s,\beta_s)}(x_s)$ where $P_{\ell_j}^{(\alpha_j,\beta_j)}(x_j)$ is the monic Jacobi polynomial, of degree j in one variable x_j, orthogonal with respect to $\mu_{0,j}$ (see [11]).
Let us denote by $k_{\ell_j}^{(\nu_j)}$ the following square norm:

$$k_{\ell_j}^{(\nu_j)} = \left(\frac{d^{\nu_j}}{dx_j^{\nu_j}} P_{\ell_j}^{(\alpha_j,\beta_j)}(x_j), \frac{d^{\nu_j}}{dx_j^{\nu_j}} P_{\ell_j}^{(\alpha_j,\beta_j)}(x_j) \right)_{L^2(\Omega;\mu_{\nu_j,j})}$$

and by $k_\ell^{(\nu)}$ the tensor square norm

$$k_\ell^{(\nu)} = \prod_{j=1}^{s} k_{\ell_j}^{(\nu_j)}.$$

Of course $k_\ell^{(\nu)} = 0$ if at least one of the ℓ_j's is such that $\ell_j < \nu_j$. We recall (see [8]) that $\frac{d^{\nu_j}}{dx_j^{\nu_j}} P_{\ell_j}^{(\alpha_j,\beta_j)}(x_j) = (\ell_j - \nu_j + 1)_{\nu_j} P_{\ell_j-\nu_j}^{(\alpha_j+\nu_j,\beta_j+\nu_j)}(x_j)$.
$(a)_m$ is the Pochhammer symbol: $(a)_m = a(a+1)\dots(a+m-1)$. By definition $(a)_0 = 1$.

For the Jacobi measure we have (see [8, 15])

$$k_{\ell_j}^{(\nu_j)} = \frac{2^{\delta_{\ell_j,j}}(\ell_j)!\Gamma(\alpha_{\ell_j,j})\Gamma(\beta_{\ell_j,j})\Gamma(\delta_{\ell_j,j} - \ell_j + \nu_j)}{\delta_{\ell_j,j}(\Gamma(\delta_{\ell_j,j}))^2} (\ell_j - \nu_j + 1)_{\nu_j}$$

with

$$\begin{aligned} \alpha_{\ell_j,j} &= \alpha_j + \ell_j + 1, \\ \beta_{\ell_j,j} &= \beta_j + \ell_j + 1, \\ \delta_{\ell_j,j} &= \alpha_{\ell_j,j} + \beta_{\ell_j,j} - 1. \end{aligned}$$

Γ is the gamma function.

Therefore, all these norms satisfy the following ratio

$$C_{\ell_j}^{(\nu_j)} = \frac{k_{\ell_j}^{(\nu_j)}}{k_{\ell_j}^{(0)}} = (\ell_j + \alpha_j + \beta_j + 1)_{\nu_j} (\ell_j - \nu_j + 1)_{\nu_j}.$$

It will be convenient to put

$$\begin{aligned} C_\ell^{(\nu)} &= \prod_{j=1}^{s} C_{\ell_j}^{(\nu_j)} = \prod_{j=1}^{s} \frac{k_{\ell_j}^{(\nu_j)}}{k_{\ell_j}^{(0)}} = \frac{k_\ell^{(\nu)}}{k_\ell^{(0)}} \\ &= (\ell + \alpha + \beta + 1)_\nu (\ell - \nu + 1)_\nu \end{aligned}$$

where $(\ell+\alpha+\beta+1)_\nu = \prod_{j=1}^{s}(\ell_j+\alpha_j+\beta_j+1)_{\nu_j}$ and $(\ell-\nu+1)_\nu = \prod_{j=1}^{s}(\ell_j-\nu_j+1)_{\nu_j}$.

Note that, if $(\ell_j - \nu_j + 1)_{\nu_j} = 0$ (*i.e.* for $\nu_j \geqslant 1$ and $(\ell_j - \nu_j + 1) \leqslant 0$), then $(\ell - \nu + 1)_\nu = 0$ and $C_\ell^{(\nu)}$ too. For the sake of simplicity with the writing of all the relations, for example like $\sum_{|\nu|} C_\ell^{(\nu)}$ for a fixed $|\nu|$, we will keep all the indexes, even those for which $C_\ell^{(\nu)} = 0$.

Our study of Landau-Kolmogorov type inequalities for the Jacobi measure will be limited to the case where the total order of the derivatives $|\nu|$ is at most 2.

Like in the previous papers (see [1, 9, 10]), the best tool for studying such inequalities is to use a bilinear functional a_λ defined for functions belonging to a Sobolev space. Let $H^2(\Omega^s; \mu_\nu, |\nu| \leqslant 2)$ be the Sobolev space defined by

$$H^2(\Omega^s; \mu_\nu, |\nu| \leqslant 2) = \left\{ f | \partial^\nu f \in L^2(\Omega^s; \mu_\nu), \forall |\nu| \leqslant 2 \right\}.$$

$$L^2(\Omega^s; \mu_\nu) = L^2(\Omega; \mu_{\nu_1,1}) \times \ldots \times L^2(\Omega; \mu_{\nu_s,s}).$$

The inner product on this space is given by

$$(f,g)_{H^2(\Omega^s;\mu_\nu,|\nu|\leqslant 2)} = \sum_{|\nu|\leqslant 2} (\partial^\nu f, \partial^\nu g)_{L^2(\Omega^s;\mu_\nu)}.$$

The corresponding norm is

$$\|f\|_{H^2(\Omega^s;\mu_\nu,|\nu|\leqslant 2)} = ((f,f)_{H^2(\Omega^s;\mu_\nu,|\nu|\leqslant 2)})^{\frac{1}{2}}.$$

The bilinear functional a_λ

$$a_\lambda : H^2(\Omega^s; \mu_\nu, |\nu| \leqslant 2) \times H^2(\Omega^s; \mu_\nu, |\nu| \leqslant 2) \to \mathbb{R}$$

is defined by

$$a_\lambda(f,g) = \sum_{m=0}^{2} \lambda_m \sum_{|\nu|=m} (\partial^\nu f, \partial^\nu g)_{L^2(\Omega^s;\mu_\nu)}, \qquad \forall f,g \in H^2(\Omega^s; \mu_\nu, |\nu| \leqslant 2), \quad (1)$$

λ_m, $m = 0,1,2$, are three fixed real numbers such that $\lambda_0 = 1$ and $\lambda_2 \neq 0$. We will put $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Now we define some domains of $\mathbb{R}^2$.

$$\mathcal{D}_r = \left\{ \lambda \in \mathbb{R}^2 | \quad a_\lambda(p,p) > 0, \quad \forall p \in \mathcal{Q}_r - \{0\} \right\},$$
$$\bar{\mathcal{D}}_r = \left\{ \lambda \in \mathbb{R}^2 | \quad a_\lambda(p,p) \geqslant 0, \quad \forall p \in \mathcal{Q}_r \right\}.$$

If we have to compare two elements $u = (u_1, \ldots, u_s)$ and $v = (v_1, \ldots, v_s)$ of $\mathbb{N}^s$ or $\mathbb{R}^s$, we will use the classical inequalities: $u < v$ (resp. $u \leqslant v$) $\Leftrightarrow u_j < v_j$ (resp. $u_j \leqslant v_j$), $j = 1, \ldots, s$.

The previous domains can also be obtained from the monic Jacobi polynomials. Indeed, we have the following obvious property:

Theorem 1.1.

$$a_\lambda(p,p) > 0, \quad \forall p \in \mathcal{Q}_r - \{0\} \quad \Longleftrightarrow \quad a_\lambda(P_\ell^{(\alpha,\beta)}(x), P_\ell^{(\alpha,\beta)}(x)) > 0, \quad \forall \ell \leqslant r.$$

Proof.
$\Rightarrow$ Obvious
$\Leftarrow$ If p is written in the basis of the monic Jacobi polynomials $P_\ell^{(\alpha,\beta)}(x)$

$$p(x) = \sum_{\ell \leqslant r} \beta_\ell P_\ell^{(\alpha,\beta)}(x),$$

then $a_\lambda(p,p) = \sum_{\ell \leqslant r} \beta_\ell^2 a_\lambda(P_\ell^{(\alpha,\beta)}(x), P_\ell^{(\alpha,\beta)}(x)) > 0$. □

2 – Landau-Kolmogorov type inequalities in Ω^s for Jacobi measures.

We use Theorem 1.1 to define some points belonging to the boundary of $\bar{\mathcal{D}}_r$. First, we give the expression of $a_\lambda(P_\ell^{(\alpha,\beta)}(x), P_\ell^{(\alpha,\beta)}(x))$ when $\ell \leqslant r$.

$$\begin{aligned} a_\lambda(P_\ell^{(\alpha,\beta)}, P_\ell^{(\alpha,\beta)}) &= \left\| P_\ell^{(\alpha,\beta)} \right\|^2_{L^2(\Omega^s;\mu_0)} + \lambda_1 \sum_{|\nu|=1} \left\| \partial^\nu P_\ell^{(\alpha,\beta)} \right\|^2_{L^2(\Omega^s;\mu_\nu)} \\ &\quad + \lambda_2 \sum_{|\nu|=2} \left\| \partial^\nu P_\ell^{(\alpha,\beta)} \right\|^2_{L^2(\Omega^s;\mu_\nu)} \\ &= k_\ell^{(0)} + \lambda_1 \sum_{|\nu|=1} k_\ell^{(\nu)} + \lambda_2 \sum_{|\nu|=2} k_\ell^{(\nu)} \\ &= k_\ell^{(0)}(1 + \lambda_1 \sum_{|\nu|=1} C_\ell^{(\nu)} + \lambda_2 \sum_{|\nu|=2} C_\ell^{(\nu)}). \end{aligned}$$

For the sake of simplicity we will put

$$\begin{aligned} \varphi_1(\ell) &= \sum_{|\nu|=1} C_\ell^{(\nu)} = \sum_{j=1}^{s} \ell_j(\ell_j + \gamma_j + 1) \\ \varphi_2(\ell) &= \sum_{|\nu|=2} C_\ell^{(\nu)} = \sum_{j=1}^{s} \ell_j(\ell_j - 1)(\ell_j + \gamma_j + 1)(\ell_j + \gamma_j + 2) \\ &\qquad + \sum_{1 \leqslant i < j \leqslant s} \ell_i \ell_j (\ell_i + \gamma_i + 1)(\ell_j + \gamma_j + 1) \end{aligned}$$

where $\gamma_j = \alpha_j + \beta_j$, $j = 1, \ldots, s$.

$a_\lambda(P_\ell^{(\alpha,\beta)}, P_\ell^{(\alpha,\beta)}) = 0$ corresponds to a straight line $\mathcal{H}_\ell$.

$$\mathcal{H}_\ell = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | 1 + \lambda_1\varphi_1(\ell) + \lambda_2\varphi_2(\ell) = 0\right\}.$$

$a_\lambda(P_\ell^{(\alpha,\beta)}, P_\ell^{(\alpha,\beta)}) > 0$ corresponds to a positive open half space $\mathcal{O}_\ell$.

$$\mathcal{O}_\ell = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | 1 + \lambda_1\varphi_1(\ell) + \lambda_2\varphi_2(\ell) > 0\right\}.$$

Therefore, $\mathcal{D}_r = \bigcap\limits_{\ell \leqslant r} \mathcal{O}_\ell$.

Of course $\mathcal{D}_r$ (resp. $\bar{\mathcal{D}}_r$) is a convex domain as shown in the following property.

Property 2.1. $\mathcal{D}_r$ (resp. $\bar{\mathcal{D}}_r$) is a convex domain.

Proof. Let λ and λ^* be two points of $\mathcal{D}_r$ (resp. $\bar{\mathcal{D}}_r$), and a_λ and a_{λ^*} be the corresponding bilinear functionals defined by (1).
For any $\theta \in [0, 1]$, we have

$$a_{\theta\lambda+(1-\theta)\lambda^*} = \theta a_\lambda + (1-\theta)a_{\lambda^*}.$$

Therefore, if a_λ and a_{λ^*} are positive (resp. non negative), then $a_{\theta\lambda+(1-\theta)\lambda^*}$ is positive (resp. non negative). □

The method to give Landau-Kolmogorov type inequalities for Jacobi measures is different enough from the method used for Hermite and Laguerre measures in [2]. Indeed, in these two cases all the straight lines $\mathcal{H}_\ell$, for $|\ell|$ fixed, have a common point which is very useful to obtain some points in $\bar{\mathcal{D}}_r$.

Without loss of generality we can assume that $r_j \geqslant 1$, $\forall j = 1, \ldots, s$. Indeed, if one of the r_j's is equal to 0, the space $\mathbb{R}^s$ can be reduced to $\mathbb{R}^{s-1}$.

Our aim is to prove that a segment on the straight line $\mathcal{H}_r$ belongs to the boundary of $\bar{\mathcal{D}}_r$.

Property 2.2. The point $A = (-\frac{1}{\varphi_1(r)}, 0)$ belongs to the boundary of $\bar{\mathcal{D}}_r$.

Proof. A is on $\mathcal{H}_r$. Let us show that $A \in \mathcal{O}_\ell$, $\forall \ell \leqslant r$, $\ell \neq r$. Indeed, we have

$$\begin{aligned} 1 + \lambda_1(A)\varphi_1(\ell) &= \frac{\varphi_1(r) - \varphi_1(\ell)}{\varphi_1(r)} \\ &= \frac{\sum\limits_{j=1}^{s}(r_j - \ell_j)(r_j + \ell_j + \gamma_j + 1)}{\varphi_1(r)} > 0. \end{aligned}$$

Therefore, A belongs to the boundary of $\bar{\mathcal{D}}_r$. □

Let us denote by B_ℓ the point of intersection of $\mathcal{H}_r$ and $\mathcal{H}_\ell$, $\forall \ell \leqslant r$, $\ell \neq r$. The coordinates of such a point are

$$\lambda_1(B_\ell) = -\frac{\varphi_2(r) - \varphi_2(\ell)}{\varphi_2(r)\varphi_1(\ell) - \varphi_2(\ell)\varphi_1(r)}, \tag{2}$$

$$\lambda_2(B_\ell) = \frac{\varphi_1(r) - \varphi_1(\ell)}{\varphi_2(r)\varphi_1(\ell) - \varphi_2(\ell)\varphi_1(r)}. \tag{3}$$

PROPERTY 2.3. If $\ell \leqslant r$ with $\ell \neq r$,

$$\varphi_1(r) - \varphi_1(\ell) > 0,$$
$$\varphi_2(r) - \varphi_2(\ell) > 0.$$

PROOF. Property 2.2 already gives us the result $\varphi_1(r) - \varphi_1(\ell) > 0$.

$$\begin{aligned}\varphi_2(r) - \varphi_2(\ell) = &\sum_{j=1}^{s}(r_j(r_j-1)(r_j+\gamma_j+1)(r_j+\gamma_j+2)\\ &- \ell_j(\ell_j-1)(\ell_j+\gamma_j+1)(\ell_j+\gamma_j+2))\\ &+ \sum_{1\leqslant i<j\leqslant s}(r_ir_j(r_i+\gamma_i+1)(r_j+\gamma_j+1)\\ &- \ell_i\ell_j(\ell_i+\gamma_i+1)(\ell_j+\gamma_j+1)).\end{aligned}$$

The result holds by comparing every factor with similar forms in the different sums and by using the fact that there exists at least one index such that $r_j > \ell_j$. □

Now, we give another expression of $\varphi_2(\ell)$.

PROPERTY 2.4.

$$\begin{aligned}\varphi_2(\ell) = &\frac{1}{2}\left(\sum_{j=1}^{s}\ell_j(\ell_j+\gamma_j+1)\right)^2 + \frac{1}{2}\sum_{j=1}^{s}\ell_j^2(\ell_j+\gamma_j+1)^2\\ &- \sum_{j=1}^{s}(2+\gamma_j)\ell_j(\ell_j+\gamma_j+1).\end{aligned} \tag{4}$$

PROOF.

$$\begin{aligned}\varphi_2(\ell) &= \sum_{j=1}^{s}\ell_j(\ell_j+\gamma_j+1)(\ell_j-1)(\ell_j+\gamma_j+1+1)\\ &\quad+ \sum_{1\leqslant i<j\leqslant s}\ell_i\ell_j(\ell_i+\gamma_i+1)(\ell_j+\gamma_j+1)\\ &= \sum_{j=1}^{s}\ell_j^2(\ell_j+\gamma_j+1)^2 - \sum_{j=1}^{s}(2+\gamma_j)\ell_j(\ell_j+\gamma_j+1)\\ &\quad+ \sum_{1\leqslant i<j\leqslant s}\ell_i\ell_j(\ell_i+\gamma_i+1)(\ell_j+\gamma_j+1).\end{aligned} \tag{5}$$

Hence

$$\begin{aligned}\varphi_2(\ell) + \sum_{1\leqslant i<j\leqslant s} \ell_i\ell_j(\ell_i+\gamma_i+1)(\ell_j+\gamma_j+1) \\ = \sum_{j=1}^{s} \ell_j^2(\ell_j+\gamma_j+1)^2 + 2\sum_{1\leqslant i<j\leqslant s} \ell_i\ell_j(\ell_i+\gamma_i+1)(\ell_j+\gamma_j+1) \\ - \sum_{j=1}^{s}(2+\gamma_j)\ell_j(\ell_j+\gamma_j+1) \\ = \left(\sum_{j=1}^{s} \ell_j(\ell_j+\gamma_j+1)\right)^2 - \sum_{j=1}^{s}(2+\gamma_j)\ell_j(\ell_j+\gamma_j+1). \qquad (6)\end{aligned}$$

The result is obtained by addition of (5) and (6). □

We already know that A belongs to the boundary of $\bar{\mathcal{D}}_r$. We want to prove that one of the points B_ℓ also belongs to the boundary of $\bar{\mathcal{D}}_r$. This point is taken among the points B_ℓ for which $\ell \leqslant r$ with $|\ell| = |r| - 1$. We begin to prove that $\lambda_1(B_\ell) < 0$ and $\lambda_2(B_\ell) > 0$ for such points.

THEOREM 2.5. *$\lambda_1(B_\ell) < 0$ and $\lambda_2(B_\ell) > 0$ for $\ell \leqslant r$ with $|\ell| = |r| - 1$.*

PROOF. If $\ell \leqslant r$ with $|\ell| = |r| - 1$, then there exists an index i, $1 \leqslant i \leqslant s$, for which $\ell_i = r_i - 1$ and $\ell_j = r_j$, $\forall j \neq i$. $\varphi_1(\ell)$ and $\varphi_2(\ell)$ can be expressed in function of $\varphi_1(r)$ and $\varphi_2(r)$

$$\varphi_1(\ell) = \varphi_1(r) - (2r_i + \gamma_i),$$

$$\varphi_2(\ell) = \varphi_2(r) - (2r_i + \gamma_i)(\varphi_1(r) + (r_i - 2)(r_i + \gamma_i + 1)).$$

We put

$$\Delta_1 = \varphi_1(r) - \varphi_1(\ell),$$

$$\Delta_2 = \varphi_2(r) - \varphi_2(\ell).$$

From Property 2.3, $\Delta_1 > 0$ and $\Delta_2 > 0$.

Let us prove that the common denominator of $\lambda_1(B_\ell)$ and $\lambda_2(B_\ell)$ is positive.

$$\varphi_2(r)\varphi_1(\ell) - \varphi_2(\ell)\varphi_1(r) = \varphi_1(r)\Delta_2 - \varphi_2(r)\Delta_1.$$

$$\Delta_2 = \Delta_1(\varphi_1(r) + (r_i - 2)(r_i + \gamma_i + 1)). \qquad (7)$$

Thus, by using (4) we have

$$\begin{aligned}
&\varphi_1(r)\Delta_2 - \varphi_2(r)\Delta_1 \\
&= \Delta_1 \Bigg(\varphi_1(r)(\varphi_1(r) + r_i^2 - r_i + r_i\gamma_i - 2\gamma_i - 2) - \frac{1}{2}\sum_{j=1}^{s} r_j^2(r_j+\gamma_j+1)^2 \\
&\quad - (\varphi_1(r))^2 + 2\varphi_1(r) + \sum_{j=1}^{s} \gamma_j r_j (r_j+\gamma_j+1) \Bigg) \\
&= \Delta_1 \Bigg(\frac{1}{2}(\varphi_1(r))^2 - \frac{1}{2}\sum_{j=1}^{s} r_j^2(r_j+\gamma_j+1)^2 + \varphi_1(r)(r_i^2 - r_i + r_i\gamma_i - 2\gamma_i) \\
&\quad + \sum_{j=1}^{s} \gamma_j r_j (r_j+\gamma_j+1) \Bigg) \qquad (8) \\
&= \Delta_1 \sum_{j=1}^{s} r_j(r_j+\gamma_j+1) \Bigg(\frac{1}{2} \sum_{m=1, m\neq j}^{s} r_m(r_m+\gamma_m+1) \\
&\quad + (r_i - 2)(r_i+\gamma_i+1) + 2 + \gamma_j \Bigg). \qquad (9)
\end{aligned}$$

If $r_i \geqslant 2$, then (9) is positive.

If $r_i = 1$, then (8) becomes.

$$\Delta_1 \left(\frac{1}{2}(\varphi_1(r))^2 - \frac{1}{2}\sum_{j=1}^{s} r_j^2(r_j+\gamma_j+1)^2 + \sum_{j=1, j\neq i}^{s} (\gamma_j - \gamma_i) r_j (r_j+\gamma_j+1) \right). \qquad (10)$$

$\frac{1}{2}(\varphi_1(r))^2 - \frac{1}{2}\sum\limits_{j=1}^{s} r_j^2(r_j+\gamma_j+1)^2$ can be written as

$$\sum_{1\leqslant j<m\leqslant s} r_j(r_j+\gamma_j+1) r_m(r_m+\gamma_m+1) = r_i(r_i+\gamma_i+1) \sum_{j=1, j\neq i}^{s} r_j(r_j+\gamma_j+1) + R$$

where R is positive. Hence, the new expression of (10)

$$\begin{aligned}
&\Delta_1 \left(R + (2+\gamma_i) \sum_{j=1, j\neq i}^{s} r_j(r_j+\gamma_j+1) + \sum_{j=1, j\neq i}^{s} (\gamma_j - \gamma_i) r_j(r_j+\gamma_j+1) \right) \\
&= \Delta_1 \left(R + \sum_{j=1, j\neq i}^{s} (2+\gamma_j) r_j(r_j+\gamma_j+1) \right) > 0.
\end{aligned}$$

Hence, the result holds. □

Now we choose a point B_{ℓ^*} among the points B_ℓ as follows. We take B_{ℓ^*} such that

$$\lambda_2(B_{\ell^*}) = \min_{\ell\in\mathcal{L}} \lambda_2(B_\ell)$$

where $\mathcal{L} = \{\ell \quad | \quad \ell \leqslant r \text{ and } |\ell| = |r| - 1 \geqslant 1\}$.

Let us prove that such a point B_{ℓ^*} belongs to the boundary of $\bar{\mathcal{D}}_r$.

THEOREM 2.6. *The point B_{ℓ^*} belongs to the boundary of $\bar{\mathcal{D}}_r$.*

PROOF. All the slopes $-\frac{\varphi_1(\ell)}{\varphi_2(\ell)}$ of the straight lines $\mathcal{H}_\ell$ are negative. Moreover, their slopes are smaller than the slope of $\mathcal{H}_r$ when $\ell \leqslant r$ with $|\ell| = |r| - 1$. We still have $\ell_i = r_i - 1$ and $\ell_j = r_j$, $\forall j \neq i$. Indeed, by using Theorem 2.5,

$$-\frac{\varphi_1(r)}{\varphi_2(r)} + \frac{\varphi_1(\ell)}{\varphi_2(\ell)} = \frac{\varphi_2(r)\varphi_1(\ell) - \varphi_2(\ell)\varphi_1(r)}{\varphi_2(r)\varphi_2(\ell)} > 0.$$

A straight line $\mathcal{H}_\ell$ contains the point $(-\frac{1}{\varphi_1(\ell)}, 0)$ which is on the left of the point A on the axis of the λ_1's. Therefore, $B_{\ell^*} \in \mathcal{O}_\ell$, $\forall \ell \leqslant r$ with $|\ell| = |r| - 1$.
Since

$$\lambda_2(B_\ell) = \frac{1}{\sum_{j=1}^{s} r_j(r_j + \gamma_j + 1)\left(\frac{1}{2}\sum_{m=1,m\neq j}^{s} r_m(r_m + \gamma_m + 1) + 2 + \gamma_j\right) + (r_i - 2)(r_i + \gamma_i + 1)\varphi_1(r)}, \tag{11}$$

to minimize $\lambda_2(B_\ell)$, we have to maximize $(r_i - 2)(r_i + \gamma_i + 1)\varphi_1(r)$.

Let us denote by (r_{i^*}, γ_{i^*}) a couple which maximizes $(r_i - 2)(r_i + \gamma_i + 1)$.

Let us show that $B_{\ell^*} \in \mathcal{O}_\ell$, $\forall \ell \leqslant r$ with $|\ell| = |r| - 2$, that is to say,

$$1 + \varphi_1(\ell)\lambda_1(B_{\ell^*}) + \varphi_2(\ell)\lambda_2(B_{\ell^*}) > 0. \tag{12}$$

By using (2) and (3) the left hand side of (12) gives us

$$(\varphi_1(r) - \varphi_1(\ell^*))(\varphi_2(\ell) - \varphi_2(r)) + (\varphi_2(r) - \varphi_2(\ell^*))(\varphi_1(r) - \varphi_1(\ell)). \tag{13}$$

$$\varphi_1(r) - \varphi_1(\ell) = \sum_{j=1}^{s}(r_j - \ell_j)(r_j + \ell_j + \gamma_j + 1).$$

$$\begin{aligned}\varphi_2(r) - \varphi_2(\ell) &= \frac{1}{2}(\varphi_1(r) - \varphi_1(\ell))(\varphi_1(r) + \varphi_1(\ell)) \\ &\quad + \frac{1}{2}\sum_{j=1}^{s}(r_j^2(r_j + \gamma_j + 1)^2 - \ell_j^2(\ell_j + \gamma_j + 1)^2) \\ &\quad - \sum_{j=1}^{s}(2 + \gamma_j)(r_j(r_j + \gamma_j + 1) - \ell_j(\ell_j + \gamma_j + 1)).\end{aligned}$$

Moreover, by using (7), (13) can be written as

$$
\begin{aligned}
&(\varphi_1(r)-\varphi_1(\ell^*))\Big((\varphi_1(r)-\varphi_1(\ell))(\varphi_1(r)+r_{i^*}^2-r_{i^*}+r_{i^*}\gamma_{i^*}-2\gamma_{i^*}-2)-(\varphi_2(r)-\varphi_2(\ell))\Big)\\
&=(\varphi_1(r)-\varphi_1(\ell^*))\sum_{j=1}^{s}(r_j(r_j+\gamma_j+1)-\ell_j(\ell_j+\gamma_j+1))\left(\frac{1}{2}(\varphi_1(r)-\varphi_1(\ell))+r_{i^*}^2-r_{i^*}\right.\\
&\quad\left.+r_{i^*}\gamma_{i^*}-2\gamma_{i^*}-2-\frac{1}{2}(r_j(r_j+\gamma_j+1)-\ell_j(\ell_j+\gamma_j+1))+2+\gamma_j\right)\\
&=(\varphi_1(r)-\varphi_1(\ell^*))\sum_{j=1}^{s}(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)\Big(\frac{1}{2}\sum_{m=1}^{s}(r_m-\ell_m)(r_m+\ell_m+\gamma_m+1)\\
&\quad+((r_{i^*}-2)(r_{i^*}+\gamma_{i^*}+1)-(r_j-2)(r_j+\gamma_j+1))+\frac{1}{2}(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)-2r_j-\gamma_j\Big)\\
&=(\varphi_1(r)-\varphi_1(\ell^*))\sum_{j=1}^{s}(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)\Big(\frac{1}{2}\sum_{m=1,m\neq j}^{s}(r_m-\ell_m)(r_m+\ell_m+\gamma_m+1)\\
&\quad+((r_{i^*}-2)(r_{i^*}+\gamma_{i^*}+1)-(r_j-2)(r_j+\gamma_j+1))\\
&\quad+(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)-2r_j-\gamma_j\Big).
\end{aligned}
\tag{14}
$$

All the factors of (14) are positive, except possibly

$$
(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)-2r_j-\gamma_j. \tag{15}
$$

If $r_j-\ell_j=1$, then $(15)=0$.

If $r_j=\ell_j+\delta$ with $\delta\geqslant 2$, then $(15)=(\delta-1)(r_j+\ell_j+\gamma_j)>0$, since $r_j\geqslant 2$.

If $r_j-\ell_j=0$, then the factor $(r_j-\ell_j)$ in the sum $\sum\limits_{j=1}^{s}$ involves that

$$
\begin{aligned}
&(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)\Big(\frac{1}{2}\sum_{m=1,m\neq j}^{s}(r_m-\ell_m)(r_m+\ell_m+\gamma_m+1)\\
&+\Big((r_{i^*}-2)(r_{i^*}+\gamma_{i^*}+1)-(r_j-2)(r_j+\gamma_j+2)\Big)+(r_j-\ell_j)(r_j+\ell_j+\gamma_j+1)-2r_j-\gamma_j\Big)=0.
\end{aligned}
$$

Therefore, (14) is always positive.

Hence, B_{ℓ^*} is on $\mathcal{H}_r$ and belongs to all the $\mathcal{O}_\ell$'s for $\ell\leqslant r$, $\ell\neq r$. Thus, B_{ℓ^*} is on the boundary of $\bar{\mathcal{D}}_r$. □

In the particular case where $|r| = 2$, the straight lines $\mathcal{H}_\ell$, for $|\ell| = 1$ with $\ell \leqslant r$ are given by the following equations:

$$\lambda_1 = -\frac{1}{\varphi_1(\ell)} = -\frac{1}{\ell_j(\ell_j + \gamma_j + 1)} = -\frac{1}{2 + \gamma_j}.$$

The set of positive half spaces $\mathcal{O}_\ell$, when $|\ell| = 1$ with $\ell \leqslant r$, gives a global positive half space $\mathcal{O}_{\ell^*}$ defined by

$$\lambda_1 > -\frac{1}{2 + \max\limits_j \gamma_j} = -\frac{1}{2 + \gamma_{i^*}}.$$

Thus, $\lambda_1(B_{\ell^*}) = -\frac{1}{2+\gamma_{i^*}}$. $\lambda_2(B_{\ell^*}))$ is given by (11).

Corollary 2.7. *Any point $M = (1 - \theta)A + \theta B_{\ell^*}$, for $0 \leqslant \theta \leqslant 1$ belongs to the boundary of $\bar{\mathcal{D}}_r$.*

Proof. A and B_{ℓ^*} belong to the boundary of $\bar{\mathcal{D}}_r$. $\bar{\mathcal{D}}_r$ is a convex domain. Thus, all the points of the interval $[B_{\ell^*}, A]$ belong to $\bar{\mathcal{D}}_r$. But they lie on $\mathcal{H}_r$. Therefore, they belong to the boundary of $\bar{\mathcal{D}}_r$. □

For such points M we have Landau-Kolmogorov type inequalities.

Corollary 2.8. *For any point $M = (1 - \theta)A + \theta B_{\ell^*}$, with $0 < \theta \leqslant 1$, we have the following Landau-Kolmogorov type inequality*

$$\begin{aligned}\sum_{|\nu|=1} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)} \leqslant -\frac{1}{\lambda_1(M)} ||p||^2_{L^2(\Omega^s;\mu_0)} \\ - \frac{\lambda_2(M)}{\lambda_1(M)} \sum_{|\nu|=2} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)}, \quad \forall p \in \mathcal{Q}_r.\end{aligned} \tag{16}$$

When $0 < \theta < 1$, this inequality is an equality if and only if $p = cP_r^{(\alpha,\beta)}$ with $c \in \mathbb{R}$. If $\theta = 1$, then this inequality is an equality if and only if $p = cP_r^{(\alpha,\beta)} + \sum\limits_{\ell \in \mathcal{L}^*} c_\ell P_\ell^{(\alpha,\beta)}$. $\mathcal{L}^*$ is the subset of $\mathcal{L}$ such that for any element of $\mathcal{L}^*$ we have the maximum of $(r_i - 2)(r_i + \gamma_i + 1)$. c and the c_ℓ's belong to $\mathbb{R}$.

Proof. $\lambda_1(M) < 0$ and $\lambda_2(M) > 0$. Thus (16) holds.
If $0 < \theta < 1$, then M, belonging to $\mathcal{H}_r$, gives us

$$1 + \lambda_1(M)\varphi_1(r) + \lambda_2(M)\varphi_2(r) = 0.$$

Hence (16) is an equality for $cP_r^{(\alpha,\beta)}$.

If $\theta = 1$, then M belongs to $\mathcal{H}_r$ and the different $\mathcal{H}_\ell$ such that $\ell \in \mathcal{L}^*$. Hence (16) is an equality for any Jacobi polynomial $P_r^{(\alpha,\beta)}$ and $P_\ell^{(\alpha,\beta)}$ for $\ell \in \mathcal{L}^*$. $\square$

Note that if $\theta = 0$, we have a Markov-Bernstein inequality

$$\sum_{|\nu|=1} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)} \leqslant -\frac{1}{\lambda_1(A)} ||p||^2_{L^2(\Omega^s;\mu_0)}, \quad \forall p \in \mathcal{Q}_r,$$

that is to say,

$$\sum_{|\nu|=1} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)} \leqslant \sum_{j=1}^{s} r_j(r_j + \gamma_j + 1) ||p||^2_{L^2(\Omega^s;\mu_0)}, \quad \forall p \in \mathcal{Q}_r.$$

This inequality is an equality if and only if $p = cP_r^{(\alpha,\beta)}$ with $c \in \mathbb{R}$.

To finish, we want to propose an optimal point M on the segment $[B_{\ell^*}, A[$ which minimizes the right hand side of the inequality (16) when p is any fixed polynomial of $\mathcal{Q}_r$.

THEOREM 2.9. *The best point M on the segment $[B_{\ell^*}, A[$ which minimizes the right hand side of the inequality* (16) *for any fixed polynomial $p \in \mathcal{Q}_r$, is B_ℓ^*.*

PROOF. $M = (1-\theta)A + \theta B_{\ell^*}$ for $0 < \theta \leqslant 1$. Thus

$$\lambda_1(M) = -\frac{1}{\varphi_1(r)} + \theta \left(\frac{1}{\varphi_1(r)} + \lambda_1(B_\ell^*) \right),$$

$$\lambda_2(M) = \theta \lambda_2(B_\ell^*).$$

Let $G(\theta)$ be the right hand side of (16). Its derivative $G'(\theta)$ is

$$G'(\theta) = \frac{\dfrac{1}{\varphi_1(r)} + \lambda_1(B_\ell^*)}{(\lambda_1(M))^2} ||p||^2_{L^2(\Omega^s;\mu_0)} + \frac{\lambda_2(B_\ell^*)}{\varphi_1(r)} \frac{1}{(\lambda_1(M))^2} \sum_{|\nu|=2} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)}$$

$$= \frac{\hat{G}}{(\lambda_1(M))^2}.$$

We replace $\lambda_1(B_\ell^*)$ and $\lambda_2(B_\ell^*)$ with the relations (2) and (3). We obtain

$$\begin{aligned}\hat{G} &= \frac{\varphi_1(\ell^*) - \varphi_1(r)}{\varphi_1(r)(\varphi_2(r)\varphi_1(\ell^*) - \varphi_2(\ell^*)\varphi_1(r))}\left(\varphi_2(r)||p||^2_{L^2(\Omega^s;\mu_0)} - \sum_{|\nu|=2} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)}\right)\\ &= \frac{\varphi_1(\ell^*) - \varphi_1(r)}{\varphi_1(r)(\varphi_2(r)\varphi_1(\ell^*) - \varphi_2(\ell^*)\varphi_1(r))}G^*.\end{aligned}$$

The factor of G^* is negative.

Let p be a polynomial of $\mathcal{Q}_r$. Thus, p can be written in the basis of monic Jacobi polynomials $P_\rho^{(\alpha,\beta)}$, $\forall \rho \leqslant r$.

$$p = \sum_{\rho \leqslant r} \beta_\rho P_\rho^{(\alpha,\beta)}.$$

We have

$$\begin{aligned}||p||^2_{L^2(\Omega^s;\mu_0)} &= \sum_{\rho\leqslant r} \beta_\rho^2 k_\rho^{(0)},\\ ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)} &= \sum_{\rho\leqslant r} \beta_\rho^2 k_\rho^{(\nu)} = \sum_{\rho\leqslant r} C_\rho^{(\nu)} \beta_\rho^2 k_\rho^{(0)},\\ \sum_{|\nu|=2} ||\partial^\nu p||^2_{L^2(\Omega^s;\mu_\nu)} &= \sum_{|\nu|=2}\sum_{\rho\leqslant r} C_\rho^{(\nu)} \beta_\rho^2 k_\rho^{(0)}\\ &= \sum_{\rho\leqslant r}\sum_{|\nu|=2} C_\rho^{(\nu)} \beta_\rho^2 k_\rho^{(0)}\\ &= \sum_{\rho\leqslant r} \beta_\rho^2 \varphi_2(\rho) k_\rho^{(0)}.\end{aligned}$$

Thus, $G^* = \sum_{\rho\leqslant r} k_\rho^{(0)} \beta_\rho^2(\varphi_2(r) - \varphi_2(\rho)) > 0$, by using Property 2.3.

Therefore, $G' < 0$ and $G(\theta)$ is a strictly decreasing function with respect to $\theta \in]0, 1]$. The minimum is attained for $\theta = 1$, that is to say, when $M = B_{\ell^*}$. $\square$

Remark 2.10. If p is a polynomial in one variable x_j, by using $\mu_{i_j,j} = (1 - x_j)^{\alpha_j+i_j}(1 + x_j)^{\beta_j+i_j} = (1 - x_j^2)^{i_j}\mu_{0,j}$, $j = 1, \ldots, s$, we get

$$(p^{(\nu_j)}(x_j), p^{(\nu_j)}(x_j))_{L^2(\Omega;\mu_{\nu_j,j})} = ((1-x_j^2)^{\nu_j/2} p^{(\nu_j)}(x_j), (1-x_j^2)^{\nu_j/2} p^{(\nu_j)}(x_j))_{L^2(\Omega;\mu_{0,j})}.$$

Therefore,

$$\sum_{|\nu|=1} ||\partial^{\nu} p||^2_{L^2(\Omega^s;\mu_\nu)} = \sum_{j=1}^{s} \| \, (1-x_j^2)^{1/2} \partial^{\nu(j)} p \, \|^2_{L^2(\Omega^s;\mu_0)}, \tag{17}$$

$$\begin{aligned} \sum_{|\nu|=2} ||\partial^{\nu} p||^2_{L^2(\Omega^s;\mu_\nu)} &= \sum_{j=1}^{s} ||(1-x_j^2)\partial^{\hat{\nu}(j)} p||^2_{L^2(\Omega^s;\mu_0)} \\ &+ \sum_{1\leqslant i<j\leqslant s} ||((1-x_i^2)(1-x_j^2))^{1/2} \partial^{\nu(i,j)} p||^2_{L^2(\Omega^s;\mu_0)} \end{aligned} \tag{18}$$

where $\nu(j) = (\nu_1, \ldots, \nu_s) \in \mathbb{N}^s$. $\nu(j)$ is such that $\nu_j = 1$ and $\nu_i = 0$, $\forall i \neq j$. $\hat{\nu}(j) = (\hat{\nu}_1, \ldots, \hat{\nu}_s) \in \mathbb{N}^s$ and $\nu(i,j) = (\nu_1^*, \ldots, \nu_s^*) \in \mathbb{N}^s$. $\hat{\nu}(j)$ is such that $\hat{\nu}_j = 2$ and $\hat{\nu}_i = 0$, $\forall i \neq j$. $\nu(i,j)$ is such that $\nu_i^* = \nu_j^* = 1$ $(i < j)$ and $\nu_m^* = 0$, $\forall m \neq i, j$.
We can use (17-18) in (16) to give new expressions of Landau-Kolmogorov type inequalities only by using the norm defined on the space $L^2(\Omega^s; \mu_0)$.
The special forms of these inequalities are similar to those given in [5] (from the page 614) in the case of polynomials in one variable.

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