

Ekland sequences compact in L^∞

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ABSTRACT: *In this paper we use Ekeland's ε -variational principle and we prove that, for some integral functionals, there exists a minimizing sequence compact in L^∞ .*

1 – Introduction

Let us recall Ekeland's ε -variational principle (see [8, 9]).

LEMMA 1.1. *Let (V, d) be a complete metric space, and let $\mathcal{F} : V \rightarrow (-\infty, +\infty]$ be a lower semicontinuous function such that $\inf_V \mathcal{F}$ is finite. Let $\varepsilon > 0$ and $u \in V$ be such that*

$$\mathcal{F}(u) \leq \inf_{v \in V} \mathcal{F}(v) + \varepsilon.$$

Then there exists $v \in V$ such that

- (i) $d(u, v) \leq \sqrt{\varepsilon}$;
- (ii) $\mathcal{F}(v) \leq \mathcal{F}(u)$;
- (iii) v minimizes the functional $\mathcal{G}(w) = \mathcal{F}(w) + \sqrt{\varepsilon} d(v, w)$.

Many papers used the above ε -variational principle in many different frameworks: it is impossible to quote all of them; we only recall [1, 10, 11, 12, 17, 18, 20] (and the references therein) and the papers referred below.

In this paper we carry on the study of additional properties of the Ekeland sequences (begun in [5] and developed in [7]), in the case of functionals defined through multiple integrals.

In [5], we considered some properties of the minimizing sequences for integral functionals J . Thanks to the Ekeland Lemma, we proved the existence of a minimizing sequence compact in $L^s(\Omega)$ or in $C^{0,\alpha}$ for functionals which do not need to have a minimum, without using the integral representation of the relaxed functional J^* .

In [7], we improved the study done in the paper [5], under the assumption that the functional J has a minimum belonging to $L^\infty(\Omega)$. Using again Ekeland's ε -variational principle, we proved that there exists a minimizing sequence uniformly converging to a minimum u .

In this paper, we prove similar results if the coefficients satisfy a control assumption (quite natural and introduced in [2]; see (2.3) below).

2 – Setting and statement of the result

Let Ω be an open, bounded subset of \mathbb{R}^N , $N \geq 2$, and let p be a real number, with $2 \leq p < N$. We will denote by p^* the Sobolev exponent of p : $p^* = \frac{Np}{N-p}$.

Let $j : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function (i.e., measurable with respect to x for every $\xi \in \mathbb{R}^N$, and continuous with respect to ξ for almost every $x \in \Omega$) convex with respect to ξ , and such that

$$\alpha |\xi|^p \leq j(x, \xi) \leq \beta |\xi|^p, \quad (2.1)$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where α, β are positive real numbers. Let $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be defined by

$$J(v) = \int_{\Omega} j(x, \nabla v) + \frac{1}{r} \int_{\Omega} b(x) |v|^r - \int_{\Omega} f(x) v, \quad v \in W_0^{1,p}(\Omega),$$

where

$$r > 1, \quad (2.2)$$

and the coefficient $b(x)$ and the datum $f(x)$, belonging to $L^1(\Omega)$, satisfy the dominated assumption

$$\text{there exists } Q \in \mathbb{R}^+ \text{ such that } |f(x)| \leq Q b(x). \quad (2.3)$$

Note that (2.1) implies

$$J(v) \geq \alpha \int_{\Omega} |\nabla v|^p + \frac{1}{r} \int_{\Omega} b(x) |v|^r - \int_{\Omega} f(x) v(x).$$

First of all, we point out that the assumptions (2.3) and (2.2) imply

$$\begin{aligned} & \left[\frac{1}{r} \int_{\Omega} b(x) |v|^r - \int_{\Omega} f(x) v \geq \frac{1}{r} \int_{\Omega} b(x) |v|^r - \int_{\Omega} Q b(x) |v| \right. \\ & \left. = \int_{|v|^{r-1} \geq Q} b(x) [|v|^{r-1} - Q] |v| + \int_{|v|^{r-1} < Q} b(x) [|v|^{r-1} - Q] |v|, \right. \end{aligned} \quad (2.4)$$

where $b(x)[|v|^{r-1} - Q]|v|$ is positive on the set $\{|v|^{r-1} \geq Q\}$ and $b(x)[|v|^{r-1} - Q]|v|$ belongs to L^1 on the set $\{|v|^{r-1} < Q\}$.

Thus, under the above assumptions J is well defined on $W_0^{1,p}(\Omega)$ even though f is only in $L^1(\Omega)$; possibly with value $+\infty$.

Since J is both weakly lower semicontinuous and coercive on $W_0^{1,p}(\Omega)$, there exists a minimum u of J .

Moreover the assumption (2.1) and the inequality (2.4) say that the functional J is strongly lower semicontinuous in $W_0^{1,1}(\Omega)$, so that it is possible to apply the Ekeland Lemma 1.1.

Also we have the following results on the summability of minima of J .

PROPOSITION 2.1. *Let u be a minimum of J on $W_0^{1,p}(\Omega)$. Then*

- (i) *in [19] (see also [14]) is proved that, if $b(x) = 0$, $f \in L^m(\Omega)$, $m > \frac{N}{p}$, then u belongs to $L^\infty(\Omega)$;*
- (ii) *in [4] is proved that, if $b(x) = 0$, $f \in L^m(\Omega)$, $1 < m < \frac{N}{p}$, then u belongs to $L^\sigma(\Omega)$, $\sigma = \frac{(pm)^*}{p'}$;*
- (iii) *in [2] is proved that, under the assumptions (2.2) and (2.3), $u \in L^\infty(\Omega)$, with the explicit bound*

$$|u(x)| \leq Q^{\frac{1}{r-1}}. \quad (2.5)$$

REMARK 2.2. Let $\{\bar{u}_n\}$ be a minimizing sequence. Recall the definitions ($k > 0$)

$$G_k(s) = (|s| - k)^+ \frac{s}{|s|}, \quad T_k(s) = s - G_k(s).$$

Before proving Theorem 2.3, note that, under the assumptions (2.1), (2.2), (2.3), since we have estimate (2.5), the sequence $\{u_n\}$, with $u_n = T_M(\bar{u}_n)$, $M = Q^{\frac{1}{r-1}}$, $\varepsilon_n \rightarrow 0$, satisfies

$$\begin{aligned} & \int_{\Omega} j(x, \nabla T_M(\bar{u}_n)) + \frac{1}{r} \int_{\Omega} b(x) |T_M(\bar{u}_n)|^r - \int_{\Omega} f(x) T_M(\bar{u}_n) \\ & \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + \frac{1}{r} \int_{\Omega} b(x) [|T_M(\bar{u}_n)|^r - |\bar{u}_n|^r] + \int_{\Omega} f(x) G_M(\bar{u}_n), \\ & \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + \int_{\Omega} f(x) G_M(\bar{u}_n), \end{aligned}$$

and since

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f(x) G_M(\bar{u}_n) = \int_{\Omega} f(x) G_M(u) = 0,$$

we have that

$$\int_{\Omega} j(x, \nabla u_n) + \frac{1}{r} \int_{\Omega} b(x) |u_n|^r - \int_{\Omega} f(x) u_n \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \bar{\varepsilon}_n.$$

That is, the sequence $\{u_n\}$ is a minimizing sequence for J , and it is bounded in $L^\infty(\Omega)$: $|u_n(x)| \leq M$.

We assume that

there exists $a(x, \xi) = j_\xi(x, \xi)$, which satisfies the Leray-Lions assumptions (2.6)

(see [16]); that is $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that the following holds for almost every $x \in \Omega$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x, \xi)\xi \geq \alpha |\xi|^p, \\ |a(x, \xi)| \leq \beta |\xi|^{p-1}, \\ [a(x, \xi) - a(x, \eta)](\xi - \eta) > 0, \end{cases} \tag{2.7}$$

where α, β are positive constants. Then the minimum $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of the Euler-Lagrange equation

$$\int_\Omega a(x, \nabla u) \nabla \varphi + \int_\Omega b(x) u |u|^{r-2} \varphi = \int_\Omega f(x) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{2.8}$$

With respect to the above Remark 2.2, if we assume (2.6), in Theorem 2.3, we prove more (as in [7], even with different assumptions): thanks to the ε -variational principle, it is possible to build a minimizing sequence not only bounded in $L^\infty(\Omega)$, but also strongly convergent to u in the same space.

Our main result is the following.

THEOREM 2.3. *We assume (2.1), (2.2), (2.3). Let u be a minimum of J on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let $\{\bar{u}_n\}$ be any minimizing sequence for J ,*

$$\|\bar{u}_n\| \leq M = Q^{\frac{1}{r-1}}, \text{ thanks to Remark 2.2.} \tag{2.9}$$

Then the minimizing sequence $\{u_n\}$ built after $\{\bar{u}_n\}$ using the ε -variational principle satisfies

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{W_0^{1,p}(\Omega)} = 0, \tag{2.10}$$

and

$$\text{the sequence } \{u_n\} \text{ is compact in } L^\infty(\Omega). \tag{2.11}$$

PROOF.

STEP 1. Let ε_n be a sequence of positive real numbers, converging to zero, and let \bar{u}_n be such that, for every $n \in \mathbb{N}$,

$$J(\bar{u}_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n.$$

Let us now consider the complete metric space $W_0^{1,1}(\Omega)$, endowed with the distance

$$d_n(w, v) = \frac{1}{\sqrt{\varepsilon_n}} \int_{\Omega} |\nabla w - \nabla v|.$$

Thanks to Fatou lemma, to the fact that $j(x, \xi) \geq 0$ and to (2.4), we have that J is strongly lower semicontinuous on $W_0^{1,1}(\Omega)$. Thus, in view of Lemma 1.1, there exists a sequence $\{u_n\}$ in $W_0^{1,1}(\Omega)$ such that

$$\int_{\Omega} |\nabla u_n - \nabla \bar{u}_n| \leq \sqrt{\varepsilon_n},$$

and

$$J(u_n) \leq J(\bar{u}_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n, \quad (2.12)$$

$$J(u_n) \leq J(w) + \sqrt{\varepsilon_n} \int_{\Omega} |\nabla u_n - \nabla w|, \quad \forall w \in W_0^{1,1}(\Omega). \quad (2.13)$$

Using the growth properties of J we now prove that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By (2.1), we have (recall (2.4))

$$\begin{aligned} & \int_{\Omega} j(x, \nabla u_n) + \int_{\{|u_n|^{r-1} < Q\}} \left[\frac{1}{r} b(x) |u_n|^r - f(x) u_n \right] \\ & \leq \int_{\Omega} j(x, \nabla u_n) + \frac{1}{r} \int_{\Omega} b(x) |u_n|^r - \int_{\Omega} f(x) u_n. \end{aligned}$$

Thus

$$\alpha \int_{\Omega} |\nabla u_n|^p \leq J(u_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + \int_{\{|u_n|^{r-1} < Q\}} \left[f(x) u_n - \frac{1}{r} b(x) |u_n|^r \right].$$

and

$$\alpha \int_{\Omega} |\nabla u_n|^p \leq J(u_n) \leq \inf_{v \in W_0^{1,p}(\Omega)} J(v) + \varepsilon_n + C_Q,$$

which implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus $\|u_n\|_{W_0^{1,p}(\Omega)} \leq R$ and, up to subsequences, still denoted by $\{u_n\}$, there exists a function u in $W_0^{1,p}(\Omega)$ such that

$$u_n \rightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and almost everywhere in } \Omega. \quad (2.14)$$

By the weak lower semicontinuity of J on $W_0^{1,p}(\Omega)$, and by (2.12), u is a minimum of J on this space.

Now we follow a classic method by I. Ekeland and we use (2.6). Choosing $w = u_n - t\psi$ in (2.13), where t is a positive real number and ψ is a function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we obtain

$$J(u_n - t\psi) - J(u_n) + \sqrt{\varepsilon_n} t \int_{\Omega} |\nabla\psi| \geq 0.$$

Dividing by t , and letting t tend to zero, we get, since J is differentiable,

$$-\langle J'(u_n), \psi \rangle + \sqrt{\varepsilon_n} \int_{\Omega} |\nabla\psi| \geq 0,$$

so that

$$\langle J'(u_n), \psi \rangle \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla\psi|. \quad (2.15)$$

Recalling that $J'(u) = 0$ since u is a minimum, we have

$$\langle J'(u_n) - J'(u), \psi \rangle \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla\psi|,$$

for every ψ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Observe that

$$\langle J'(u_n), \psi \rangle = \int_{\Omega} a(x, \nabla u_n) \nabla\psi - \int_{\Omega} f(x)\psi, \quad (2.16)$$

that is

$$\langle J'(u_n) - J'(u), \psi \rangle = \int_{\Omega} [a(x, \nabla u_n) - a(x, \nabla u)] \nabla\psi + \int_{\Omega} b(x)(u_n|u_n|^{r-2} - u|u|^{r-2})\psi.$$

Choosing $\psi = T_k(u_n - u)$, and using the fact that $s \mapsto s|s|^{r-2}$ is monotone, we have

$$\left[\begin{aligned} & \int_{|u_n - u| \leq k} [a(x, \nabla u_n) - a(x, \nabla u)] \nabla(u_n - u) \\ & \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla T_k(u_n - u)| \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla(u_n - u)| \leq C_R \sqrt{\varepsilon_n}. \end{aligned} \right.$$

Here we use the Fatou lemma, as $k \rightarrow \infty$, and we obtain

$$\int_{\Omega} [a(x, \nabla u_n) - a(x, \nabla u)] \nabla(u_n - u) \leq C_R \sqrt{\varepsilon_n}.$$

A result by J. Leray and J.-L. Lions says that the above limit implies that

$$\nabla u_n(x) \text{ converges a.e. to } \nabla u(x). \quad (2.17)$$

Then in [6] is proved that, under our assumption on the function $a(x, \xi)$, this a.e. convergence implies (2.10).

STEP 2. Let $0 \leq \zeta \leq 1$ be a smooth function and let $\varphi_k(t) = T_1[G_k(t)]$, with $k \geq M = Q^{\frac{1}{r-1}}$. Choose $\psi = \varphi_k(u_n) \zeta^p$ in (2.15); then we have

$$\left[\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla [\varphi_k(u_n) \zeta^p] + \int_{\Omega} b(x) u_n |u_n|^{r-2} \varphi_k(u_n) \zeta^p - \int_{\Omega} f(x) \varphi_k(u_n) \zeta^p \\ & \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla [\varphi_k(u_n) \zeta^p]| \end{aligned} \right.$$

and, recalling (2.3), (2.4), (2.9),

$$\left[\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla [\varphi_k(u_n) \zeta^p] \\ & \leq \sqrt{\varepsilon_n} \int_{\Omega} |\nabla [\varphi_k(u_n) \zeta^p]| + \int_{\{|u_n|^{r-1} < Q\}} b(x) [Q - |u_n|^{r-1}] \varphi_k(u_n) \zeta^p \\ & = \sqrt{\varepsilon_n} \int_{\Omega} |\nabla [\varphi_k(u_n) \zeta^p]|, \end{aligned} \right.$$

which implies (with $B_{n,k} = \{x \in \Omega : k \leq |u_n(x)| < k+1\}$, $A_{n,k} = \{x \in \Omega : k \leq |u_n(x)|\}$).

$$\left[\begin{aligned} & \alpha \int_{B_{n,k}} |\nabla u_n|^p \zeta^p \\ & \leq \sqrt{\varepsilon_n} \int_{B_{n,k}} |\nabla u_n| |\zeta^p| + p \int_{\Omega} |\nabla \zeta| |\varphi_k(u_n)| \zeta^{p-1} \\ & + p\beta \int_{\Omega} (|\nabla u_n| \zeta)^{p-1} |\nabla \zeta| |\varphi_k(u_n)| \end{aligned} \right.$$

and

$$\left[\begin{aligned} & \alpha \sum_{k=j}^{\infty} \int_{B_{n,k}} |\nabla u_n|^p \zeta^p \\ & \leq \sqrt{\varepsilon_n} \sum_{k=j}^{\infty} \int_{B_{n,k}} |\nabla u_n| \zeta^p + p \sum_{k=j}^{\infty} \int_{\Omega} |\nabla \zeta| |\varphi_k(u_n)| \zeta^{p-1} \\ & + p\beta \sum_{k=j}^{\infty} \int_{\Omega} (|\nabla u_n| \zeta)^{p-1} |\nabla \zeta| |\varphi_k(u_n)|. \end{aligned} \right.$$

Since

$$\sum_{k=j}^{+\infty} \varphi_k(t) = G_j(t),$$

we have (note that $\nabla u_n G_j(u_n) = \nabla G_j(u_n) G_j(u_n)$ and use the Young inequality with $0 < B < \frac{\alpha}{2}$)

$$\left[\begin{array}{l} \alpha \int_{\Omega} |\nabla G_j(u_n)|^p \zeta^p \\ \leq \sqrt{\varepsilon_n} \int_{\Omega} [|\nabla G_j(u_n)| \zeta] \zeta^{p-1} + p \int_{\Omega} |\nabla \zeta| |G_j(u_n)| \zeta^{p-1} \\ \quad + p\beta \int_{\Omega} (|\nabla G_j(u_n)| \zeta)^{p-1} |\nabla \zeta| |G_j(u_n)|, \\ \leq B \int_{\Omega} |\nabla G_j(u_n)|^p \zeta^p + C_B (\sqrt{\varepsilon_n})^{p'} \int_{A_{n,j}} \zeta^p \\ \quad + C_1 \int_{\Omega} |\nabla \zeta|^p |G_j(u_n)|^p + C_2 \int_{A_{n,j}} \zeta^p \\ \quad + B \int_{\Omega} |\nabla G_j(u_n)|^p \zeta^p + C_B \int_{\Omega} |\nabla \zeta|^p |G_j(u_n)|^p. \end{array} \right.$$

Thus we have

$$(\alpha - 2B) \int_{\Omega} |\nabla G_j(u_n)|^p \zeta^p \leq (C_1 + C_B) \int_{\Omega} |\nabla \zeta|^p |G_j(u_n)|^p + (C_2 + C_B (\sqrt{\varepsilon_n})^{p'}) \int_{A_{n,j}} \zeta^p.$$

This estimate implies (see [13, 15]) that the sequence $\{u_n\}$ is bounded in the De Giorgi class $\mathcal{B}_2(\Omega, M)$; that is the functions u_n are equi-Hölder continuous in Ω and we proved the statement (2.11). \square

REMARK 2.4. In [3] we presented a different method to prove the compactness in $W_0^{1,p}(\Omega)$ of minimizing sequences proved in (2.10).

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*Lavoro pervenuto alla redazione il 15 marzo 2016
ed accettato per la pubblicazione il 12 aprile 2016*

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