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Characterizing the strong maximum principle for constant coefficient subequations

F. REESE HARVEY – H. BLAINE LAWSON, JR.

Abstract: *In this paper we characterize the degenerate elliptic equations* $\mathbf{F}(D^2u) = 0$ *whose subsolutions* $(\mathbf{F}(D^2u) \geq 0)$ *satisfy the strong maximum principle. We introduce an easily computed function* f *on* $(0, \infty)$ *which is determined by* $\mathbf{F}$*, and we show that the strong maximum principle holds depending on whether* $\int_{0+} \frac{dy}{f(y)}$ *is infinite or finite. This is in the spirit of previous work characterizing the ordinary maximum principle in terms of the geometry of the set of symmetric matrices* $F = \{\mathbf{F} \geq 0\}$*. Along the way, radial subsolutions are characterized, and, as an application, a sufficient condition for strong comparison is established. A number of examples, important for the theory of such equations, are examined.*

1 – Introduction

This paper is concerned with differential equations of the form $\mathbf{F}(D^2u) = 0$ where $\mathbf{F}$ is degenerate elliptic, and attention is focused on the set $F(X)$ of viscosity subsolutions $(\mathbf{F}(D^2u) \geq 0)$ on an open set $X \subset \mathbf{R}^n$ as defined in the seminal papers [7] or [8]. Our interest here is in the theory of such equations, rather than concern with specific cases. The main point of the paper is to address the following.

Question. When do the subsolutions satisfy the strong maximum principle?

By the **maximum principle** and the **strong maximum principle** for $\mathbf{F}$ we mean the following. Given a bounded domain $\Omega \subset \mathbf{R}^n$, let $F(\overline{\Omega})$ denote the space upper semi-continuous functions on $\overline{\Omega}$ which are subsolutions on Ω. Consider the

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implications:

$$u \in F(\overline{\Omega}) \qquad \Rightarrow \qquad \sup_{\overline{\Omega}} u \ \le \ \sup_{\partial\Omega} u \tag{MP}$$

$$u \in F(\overline{\Omega}) \ \text{ has an interior maximum point } \qquad \Rightarrow \qquad u \ \text{ is constant.} \tag{SMP}$$

We say that the (MP)/(SMP) holds for F if it is true for all such Ω and u. Of course, (SMP) $\Rightarrow$ (MP).

There is, of course, a huge literature concerned with the (SMP) for viscosity subsolutions of nonlinear equations (for just a few examples see [2, 3, 5, 18, 20] and the references therein. We note in particular the landmark paper [3] and refer the reader to Remark 8.10 for a discussion of its relationship to the results here.) However, these authors typically make structural assumptions such as uniform ellipticity or homogeneity.

Here we confine our attention to the special case of constant coefficient, pure second-order (degenerate elliptic) equations in $\mathbf{R}^n$, sometimes having an invariance property. No other structural assumptions are made. The point of this paper is that in this special but important setting one can give a complete and somewhat unexpected characterization of exactly when the (SMP) holds.

This work is a natural outgrowth of the results in [11] where the ordinary (MP) is characterized, in several equivalent ways, in terms of the geometry of the set $F \equiv \{A : \mathbf{F}(A) \ge 0\}$. They are given in Theorem 2.1 below. We have reviewed these (MP) results in Section 2 for two reasons. First, they are scattered in various remarks throughout [11]. Secondly, these geometrically elegant and complete characterizations available for the (MP) provide the motivation for our discussion of the (SMP) in this paper.

Our discussion of the (SMP) divides into three cases. The first case is relatively simple and classical, and the (SMP) always holds:

$$\mathbf{F}(0) \ < \ 0 \qquad \Rightarrow \qquad \text{The (SMP) holds.} \tag{1.1}$$

The second case is also simple and classical, and the (SMP) always fails. To describe this case we set some notation. Given a non-zero vector $e \in \mathbf{R}^n$, let P_e and $P_{e^\perp}$ denote orthogonal projection onto the line spanned by e and the hyperplane perpendicular to e respectively, so that $P_e + P_{e^\perp} = I$. Our second case is the following.

$$\mathbf{F}(-\mu P_e) \ge 0 \ \text{ for some } \mu > 0, e \ne 0 \quad \Rightarrow \quad \text{(SMP) fails.} \tag{1.2}$$

Consequently, we concentrate on the remaining case, which will be referred to as the **borderline case**.

In this paper a key role is played by the increasing radial subsolutions. They are determined by a "characteristic function" f of $\mathbf{F}$, which is defined as follows. For simplicity we first assume the following weak form of **invariance**: For all $\lambda, \mu \in \mathbf{R}$,

$$\begin{aligned} & \mathbf{F}(\lambda P_{e^\perp} - \mu P_e) \ \geq \ 0 \quad \text{for one} \quad e \neq 0 \\ \Rightarrow \quad & \mathbf{F}(\lambda P_{e^\perp} - \mu P_e) \ \geq \ 0 \quad \text{for all} \quad e \neq 0. \end{aligned} \tag{1.3}$$

This holds, for example, if $\mathbf{F}$ is invariant under a group such as O_n or $\mathrm{SU}_{n/2}$ acting transitively on the $n-1$ sphere in $\mathbf{R}^n$ (a condition called *ST-invariance* in [15]). Given an invariant $\mathbf{F}$, the **characteristic function** f associated to $\mathbf{F}$ for $0 \leq \lambda < \infty$ is defined by

$$f(\lambda) \ \equiv \ \sup \{\mu : \mathbf{F}\left(\lambda P_{e^\perp} - \mu P_e\right) \ \geq \ 0\}. \tag{1.4}$$

The **borderline** cases are exactly the cases where $f(0) = 0$ (see Lemma 3.4).

Now we can state our main result, simplified by assuming invariance.

THEOREM A. *Suppose* $\mathbf{F}$ *is invariant and borderline. Then*

$$\textit{The (SMP) holds for } \mathbf{F} \qquad \iff \qquad \int_{0^+} \frac{dy}{f(y)} \ = \ \infty.$$

The general (non-invariant) version of this result is given below in Theorem A′.

The characteristic function f determines the following one-dimensional variable coefficient operator

$$\left(R_f^{\uparrow}\psi\right)(t) \ \equiv \ \min\left\{\psi'(t), \ \psi''(t) + f\left(\frac{\psi'(t)}{t}\right)\right\}. \tag{1.5}$$

The next result is of general interest, and probably classical in the C^2-case.

PROPOSITION B. *A radial function* $u(x) = \psi(|x|)$ *with* ψ *increasing, is an* $\mathbf{F}$*-subsolution if and only if* ψ *is an* $R_f^{\uparrow}$*-subsolution.*

The "only if" part of this result requires a technical lemma for general upper semi-continuous functons, which is given in Appendix A.

These two results lead to the following.

QUESTION. Given an upper semi-continuous, increasing function $f : [0, \infty) \to [0, \infty]$ with $f(0) = 0$, is there a way to describe all the equations $\mathbf{F}$ which have f as their characteristic function, or equivalently (by Proposition B) have the same set of increasing radial subsolutions.

This question is answered here. First, such equations $\mathbf{F}$ always exist for any such f (as above). Here are two crucial examples. Let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ denote the ordered eigenvalues of a symmetric matrix A so that $\lambda_{\min} = \lambda_1$ and $\lambda_{\max} = \lambda_n$. Define

$$\begin{aligned} \mathbf{F}_f^{\min/\max}\left(D^2u\right) \ &\equiv \ \min\left\{\lambda_{\max}\left(D^2u\right), \lambda_{\min}\left(D^2u\right) + f\left(\lambda_{\max}\left(D^2u\right)\right)\right\}, \quad \text{and} \\ \mathbf{F}_f^{\min/2}\left(D^2u\right) \ &\equiv \ \min\left\{\lambda_2\left(D^2u\right), \lambda_{\min}\left(D^2u\right) + f\left(\lambda_2\left(D^2u\right)\right)\right\}. \end{aligned}$$

Both have associated characteristic function f (see Lemma 8.3). In fact, they are the largest and the smallest such examples. (A priori it is not clear that there is a largest or a smallest.)

THEOREM C. *If* $\mathbf{F}$ *is invariant and borderline with characteristic function* f, *then its subsolutions satisfy*

$$F_f^{\min/2}(X) \;\subset\; F(X) \;\subset\; F_f^{\min/\max}(X).$$

Conversely, these containments imply that $\mathbf{F}$ *must have characteristic function* f.

Our first main result, Theorem A above, extends to $\mathbf{F}$'s which are not necessarily invariant as follows. We define the *upper and lower characteristic functions* $\overline{f}$ *and* $\underline{f}$ *for* $\mathbf{F}$ by:

$$\begin{aligned} \overline{f}(\lambda) \;&\equiv\; \sup\{\mu : \mathbf{F}\,(\lambda P_{e^\perp} - \mu P_e) \geq 0 \ \text{ for some } e \neq 0\} \\ \underline{f}(\lambda) \;&\equiv\; \sup\{\mu : \mathbf{F}\,(\lambda P_{e^\perp} - \mu P_e) \geq 0 \ \text{ for all } e \neq 0\}\,. \end{aligned}$$

When $\mathbf{F}(0) = 0$, we have $\underline{f}(0) \;=\; \overline{f}(0) \;=\; 0$.

THEOREM A′. *Suppose that* F *is borderline and has upper and lower characteristic functions* $\overline{f}$ *and* $\underline{f}$.

$$\begin{aligned} &\text{(a)} \quad \textit{If} \int_{0^+} \frac{dy}{\overline{f}(y)} \;=\; \infty, \quad \textit{then the} \ \text{(SMP)} \ \textit{holds for } \mathbf{F}. \\ &\text{(b)} \quad \textit{If} \int_{0^+} \frac{dy}{\underline{f}(y)} \;<\; \infty, \quad \textit{then the} \ \text{(SMP)} \ \textit{fails for } \mathbf{F}. \end{aligned}$$

Now that our main result has been stated in the traditional manner using nonlinear operators $\mathbf{F}$, we switch to the geometric point of view (pioneered by Krylov [19]) which replaces $\mathbf{F}$ with the subset $F = \{\mathbf{F} \geq 0\}$ of $\mathrm{Sym}^2(\mathbf{R}^n)$, the space of $n \times n$ symmetric matrices. This is particularly appropriate for discussing questions such as ours concerning the (MP) and (SMP) since they only depend on the space of subsolutions $F(X)$ which in turn only depends on the geometry of the subset F and not on the operator $\mathbf{F}$ used to define it. (The situation is analogous to studying submanifolds independently of any implicit defining function.) Let

$$\mathcal{P} \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0\}.$$

Instead of "operators" we consider **subequations** which by definition are closed subsets $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ satisfying the weakest form of ellipticity, namely:

$$F + \mathcal{P} \;\subset\; F, \tag{P}$$

called **positivity**. Subsolutions are defined in the usual manner, except that one requires $D^2_x\varphi \in F$, rather than $\mathbf{F}(D^2_x\varphi) \geq 0$, for test functions φ at x. To emphasize the parallels with potential theory in several complex variables, we will use the terminology F**-subharmonic** rather than $\mathbf{F}$-*subsolution*. The key topological property of F is that:

$$F \;=\; \overline{\mathrm{Int} F}. \tag{T}$$

This follows easily from (P) and the assumption that F is closed.

Some Technical Points: With the operator $\mathbf{F}$ replaced by the set $F \equiv \{\mathbf{F} \geq 0\}$, the positivity condition for F is weaker than degenerate ellipticity for the operator $\mathbf{F}$. Positivity is equivalent to requiring that: $\mathbf{F}(A) \geq 0 \quad \Rightarrow \quad \mathbf{F}(A+P) \geq 0$ for all $A \in \mathrm{Sym}^2(\mathbf{R}^n), P \in \mathcal{P}$. (Weak ellipticity is the requirement that $\mathbf{F}(A+P) \geq \mathbf{F}(A)$ for all such A and P.)

Our notion of a supersolution v is (for some $\mathbf{F}$) more restrictive than the classical notion $\mathbf{F}(D^2 v) \leq 0$. We require $-v$ to be subharmonic for the dual subequation $\widetilde{F} = -(\sim \mathrm{Int} F)$. This has an advantage over the standard notion of supersolution. For example, we were able to prove that comparison always holds for any subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ ([11, Theorem 6.4]). For degenerate elliptic operators $\mathbf{F}$, the statement becomes: comparison holds if and only if $\{\mathbf{F} \leq 0\}$ is the complement of the interior of $\{\mathbf{F} \geq 0\}$. The reader is referred to the "Pocket Dictionary" in Appendix A of [13] for a more complete translation of concepts.

By *comparison* holding for a subequation F we mean

$$u + w \ \text{ satisfies the (MP) for all } \ u \in F(\overline{\Omega}),\ w \in \widetilde{F}(\overline{\Omega}). \tag{C}$$

By *strong comparison* holding for a subequation F we mean

$$u + w \ \text{ satisfies the (SMP) for all } \ u \in F(\overline{\Omega}),\ w \in \widetilde{F}(\overline{\Omega}). \tag{SC}$$

Unlike (C), strong comparison (SC) does not always hold for pure second-order constant coefficient subequations (for instance $F = \mathcal{P}$). In Section 9 we establish a sufficient condition for (SC) utilizing a "monotonicity subequation" M_F associated to F.

Theorem D. *If the dual $\widetilde{M_F}$ satisfies the strong maximum principle, then the strong comparison principle holds for F.*

We leave as an open question: When does the strong comparison principle (SC) for F imply the (SMP) for $\widetilde{M_F}$?

Surprisingly, not all monotonicity subequations M_F are convex cones. In Section 10, utilizing such M_F, we construct many new examples of borderline equations for which strong comparison holds. Specifically, for each decreasing continuous function $g : [0, \infty) \to \mathbf{R}$ with $g(0) = 0$ and $g(x) < 0$ for $x > 0$, we construct two equations

M^g and $\widetilde{M}^g$, with $\widetilde{M}^g$ borderline, and compute the characteristic function f of $\widetilde{M}^g$ in terms of g.

THEOREM E. *If g is subadditive and $\int_{0^+} \frac{dy}{f(y)} = \infty$, where f is the characteristic function associated with $\widetilde{M}^g$, then the strong comparison principle holds for M^g and $\widetilde{M}^g$.*

Many such functions g exist. For further examples see (10.4) and [4], and see Example 10.7 for a specific example related to the Hopf function (10.10).

We use "increasing" to mean non-decreasing throughout the paper.

2 – Characterizing the maximum principle

In this section we review and amplify the (MP) results in [11].

Let $\widetilde{\mathcal{P}}$ denote the subset of $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ with at least one non-negative eigenvalue, *i.e.*, with $\lambda_{\max}(A) \geq 0$. For the maximum principle we only need to consider subequations $F \subset \widetilde{\mathcal{P}}$, since if $A \notin \widetilde{\mathcal{P}}$, then A is negative definite and $\langle Ax, x\rangle$ violates (MP). Note that $\widetilde{\mathcal{P}}$ is a subequation, that is, it is a closed set which satisfies (P). In fact, $\widetilde{\mathcal{P}}$ is universal for (MP) in the following sense.

THEOREM 2.1 (Part I). *Suppose that F is a subequation.*

(a)
$$\text{The (MP) holds for } F \qquad \iff \qquad F \subset \widetilde{\mathcal{P}}. \tag{2.1}$$

PROOF. It remains to show that (MP) holds for $\widetilde{\mathcal{P}}$, which follows from Proposition 2.3. ☐

DEFINITION 2.2. A function u is **subaffine on** X if it is upper semi-continuous on X and

$$\text{for all compact sets } K \subset X \text{ and affine functions } a(x) \equiv \langle p, x\rangle + c,$$
$$u \leq a \text{ on } \partial K \qquad \Rightarrow \qquad u \leq a \text{ on } K.$$

Subaffine functions clearly satisfy (MP) (take $a(x) = c =$ constant in Definition 2.2).

Furthermore, for any pure second-order subequation F, *the functions $u \in F(X)$ satisfy* (MP) *if and only if they are subaffine*, since the sum $u + a$ of a function u in $F(X)$ and an affine function a is again in $F(X)$. The subaffine fundtions have an advantage over the larger class of functions satisfying the (MP) in that they are determined by a local property.

PROPOSITION 2.3.

$$u \in \widetilde{\mathcal{P}}(X) \qquad \iff \qquad u \quad \textit{is subaffine on } \ X.$$

Proof. Suppose u is not subaffine. Then there exists a compact set $K \subset X$ and an affine function a so that (MP) fails for $w \equiv u - a$ on K, *i.e.*, w has a strict interior maximum point on K. This also holds for $w + \epsilon \frac{|x|^2}{2}$ with $\epsilon > 0$ sufficiently small. Then $\varphi = -\epsilon \frac{|x|^2}{2}$ is a test function for w at the maximum point $\bar{x} \in \mathrm{Int} K$. Since $D^2_{\bar{x}} \varphi = -\epsilon I < 0$, we conclude that $w \notin \widetilde{\mathcal{P}}(X)$ and so $u \notin \widetilde{\mathcal{P}}(X)$.

If $u \notin \widetilde{\mathcal{P}}(X)$, then there exists a test function φ for u at a point $\bar{x} \in X$ with $D^2_{\bar{x}} \varphi \notin \widetilde{\mathcal{P}}$, *i.e.*, $A \equiv D^2_{\bar{x}} \varphi < 0$. Set $a(x) \equiv \langle D_{\bar{x}} \varphi, x - \bar{x} \rangle + \varphi(\bar{x})$. Then $u(x) \leq a(x) + \frac{1}{4} \langle A(x - \bar{x}), x - \bar{x} \rangle$ near $\bar{x}$, showing that u is not subaffine on a small ball K about $\bar{x}$. □

In particular, we have, as advertised, that if $u \in \mathrm{USC}(X)$ is locally subaffine, then u is subaffine. We will refer to $\widetilde{\mathcal{P}}$ as the *subaffine subequation.*

Note that in addition to Theorem 2.1(a) we have established the following additional characterizations of the maximum principle. (The condition in (2.3) implies $0 \in \mathrm{Int} F$ by positivity.)

Theorem 2.1 (Continued).

(b) *The* (MP) *holds for* $F \iff 0 \notin \mathrm{Int} F$. (2.2)

(c) *The* (MP) *fails for* $F \iff -\epsilon \dfrac{|x|^2}{2}$ is F subharmonic for some $\epsilon > 0$. (2.3)

Remark 2.4. Part (a) of Theorem 2.1 states that $\widetilde{\mathcal{P}}$ is the "universal" subequation for the maximum principle. Part (b) provides the simplest test for the (MP) to hold for F. Part (c) states that the function $-\epsilon \frac{|x|^2}{2}$ is a "universal" counterexample to the maximum principle.

An obvious corollary of Theorem 2.1(b) is the following.

Corollary 2.5 (Localization).

$$\begin{gathered} \textit{If two subequations } F \textit{ and } G \textit{ agree} \\ \textit{in a neighborhood of the origin in } \mathrm{Sym}^2(\mathbf{R}^n), \textit{ then} \\ (\mathrm{MP}) \textit{ holds for } F \iff (\mathrm{MP}) \textit{ holds for } G. \end{gathered} \tag{2.4}$$

A discussion of the subaffine subequation is not complete without mentioning its duality with the convex subequation $\mathcal{P}$.

2.1 – Duality

For any subset F of $\mathrm{Sym}^2(\mathbf{R}^n)$, the **Dirichlet dual** $\widetilde{F}$ is defined to be:

$$\widetilde{F} = -(\sim \mathrm{Int} F) = \sim (-\mathrm{Int} F). \tag{2.5}$$

One can calculate the key property that

$$\widetilde{F+A} \;=\; \widetilde{F} - A \quad \text{for each} \quad A \in \mathrm{Sym}^2(\mathbf{R}^n). \tag{2.6}$$

This can be used to show that

$$F \;\text{ satisfies (P)} \quad \Rightarrow \quad \widetilde{F} \;\text{ satisfies (P)}. \tag{2.7}$$

Other properties of the subequations and their dual subequations include:

$$F_1 \;\subset\; F_2 \;\;\Rightarrow\;\; \widetilde{F}_2 \;\subset\; \widetilde{F}_1, \qquad \widetilde{F_1 \cap F_2} \;=\; \widetilde{F}_1 \cup \widetilde{F}_2, \tag{2.8}$$

$$\widetilde{\widetilde{F}} \;=\; F \qquad\qquad \mathrm{Int}\widetilde{F} \;=\; -(\sim F) \qquad\qquad \partial\widetilde{F} \;=\; -\partial F. \tag{2.9}$$

The first assertion in (2.9) follows from $\mathrm{Int}F \subset \widetilde{\widetilde{F}} \subset F$ combined with condition (T) for F. The second assertion in (2.9) is a restatement of the first.

The Dirichlet dual of $\mathcal{P}$ can be computed as follows. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of $A \in \mathrm{Sym}^2(\mathbf{R}^n)$. By definition

$$\mathcal{P} \;=\; \{A : \lambda_{\min}(A) \geq 0\}. \tag{2.10}$$

Since $\lambda_{\min}(-A) = -\lambda_{\max}(A)$ it is easy to see that the dual of $\mathcal{P}$ is

$$\widetilde{\mathcal{P}} \;=\; \{A : \lambda_{\max}(A) \geq 0\}, \tag{2.11}$$

justifying the notation $\widetilde{\mathcal{P}}$ for the subaffine subequation.

3 – Characterizing the strong maximum principle – three cases

Given a subequation F, we consider the following three mutually exclusive cases.

The Stable Case. $\quad F \cap (-\mathcal{P}) \;=\; \emptyset.$

The Algebraic Counterexample Case. $\quad (F - \{0\}) \cap (-\mathcal{P}) \;\neq\; \emptyset.$

The Borderline Case. $\quad F \cap (-\mathcal{P}) \;=\; \{0\}.$

The first case is stable or generic among subequations where the (SMP) holds, while in the second case the (SMP) fails via a quadratic counterexample. They are both very easy to analyze.

THEOREM 3.1.

(a) *If F is stable, then the* (SMP) *holds for F* (*and for all subequations in a small distance neighborhood of F*).
(b) *If F falls into the algebraic counterexample case, then the* (SMP) *fails for F.*
(c) *If F is borderline, then the* (MP) *holds but the* (SMP) *may or may not hold.*

Proof. (a) For completeness first note the equivalent ways of saying that F is stable.

$$F \cap (-\mathcal{P}) \;=\; \emptyset \qquad \iff \qquad F \;\subset\; \mathrm{Int}\widetilde{\mathcal{P}} \qquad \iff \qquad 0 \notin F \tag{3.1}$$

(by positivity, if $A \in F$, $A \leq 0$, then $0 \in F$). Suppose the (SMP) fails for F. Then for some domain Ω there exists $u \in F(\overline{\Omega})$ non-constant, but with an interior maximum point x_0. The constant function $M \equiv \sup_{\overline{\Omega}} u$ is a test function for u at x_0. Hence, $0 = D^2_{x_0} M \in F$, so F is not stable. The second claim in (a) follows from the last part of (3.1).

(b) By positivity,

$$\begin{aligned} (F - \{0\}) \cap (-\mathcal{P}) \;\neq\; \emptyset \quad &\iff \quad \exists\, A \leq 0,\ A \neq 0 \quad \text{and} \quad A \in F, \\ &\iff \quad -\mu P_e \in F \quad \text{for some} \quad \mu > 0 \quad \text{and} \quad e \neq 0, \end{aligned} \tag{3.2}$$

in which case the functions $u(x) = \frac{1}{2}\langle Ax, x\rangle$ and $-\frac{\mu}{2}\langle e, x\rangle^2$ are counterexamples to the (SMP) on any domain Ω containing the origin.

(c) The (MP) follows from Theorem 2.1(b). Borderline examples where (SMP) holds and where (SMP) fails will be given in Section 8 after we prove our main result. $\square$

The rest of this section is devoted to further discussion of the borderline case.

3.1 – Borderline subequations

There are several equivalent ways of describing the borderline subequations.

Lemma 3.2. *A subequation F is* **borderline** *if and only if any (or all) of the following equivalent conditions holds for F.*

$$\begin{aligned} &(1) \quad 0 \in \partial F \ \textit{and}\ F - \{0\} \subset \mathrm{Int}\widetilde{\mathcal{P}}. \\ &(1)' \quad 0 \in \partial \widetilde{F} \ \textit{and}\ \mathcal{P} - \{0\} \subset \mathrm{Int}\widetilde{F}. \\ &(2) \quad 0 \in \partial F \ \textit{and}\ -\mu P_e \notin F\ \forall\, \mu > 0, e \neq 0. \\ &(2)' \quad 0 \in \partial \widetilde{F} \ \textit{and}\ \mu P_e \in \mathrm{Int}\widetilde{F}\ \forall\, \mu > 0, e \neq 0. \end{aligned}$$

Proof. Since $\mathrm{Int}\widetilde{\mathcal{P}} = \sim(-\mathcal{P})$, (1) is a rephrasing of the definition of borderline. The equivalences (1) $\iff$ (1)$'$ and (2) $\iff$ (2)$'$ follow from (2.8) and (2.9). Condition (1) implies Condition (2) because $-\mu P_e \notin \mathrm{Int}\widetilde{\mathcal{P}}$ for $\mu > 0$. Condition (2)$'$ implies Condition (1)$'$ since, by (P), $\mathrm{Int}\widetilde{F} + \mathcal{P} \subset \mathrm{Int}\widetilde{F}$, and $\mathcal{P} - \{0\}$ is the convex hull of the elements μP_e for $\mu > 0$ and $e \in \mathbf{R}^n$. $\square$

3.2 – The characteristic function of a subequation

In order to further analyze (not necessarily borderline or invariant) subequations we associate two functions $\underline{f} \leq \overline{f}$ with F. We begin by considering a general subequation F. The motivation and more details will be provided later in Section 5. First we associate the following two closed sets in $\mathbf{R}^2$ with F, called the **upper (larger) and lower (smaller) radial profiles of** F:

$$\begin{aligned} \overline{\Lambda} &\equiv \{(\lambda, \mu) : \lambda P_{e^\perp} + \mu P_e \in F \ \text{ for some } e \neq 0\}, \\ \underline{\Lambda} &\equiv \{(\lambda, \mu) : \lambda P_{e^\perp} + \mu P_e \in F \ \text{ for all } e \neq 0\}. \end{aligned} \tag{3.3}$$

Since F is $\mathcal{P}$-monotone,

$$\overline{\Lambda} \ \text{ and } \ \underline{\Lambda} \ \text{ are } \ \mathbf{R}_+ \times \mathbf{R}_+ \ \text{monotone.} \tag{3.4}$$

Closed subsets $\Lambda \subset \mathbf{R}^2$ which are $\mathbf{R}_+ \times \mathbf{R}_+$-monotone can be classified in several ways. The classification we need is in the following lemma.

Lemma 3.3. *A set $\Lambda \subset \mathbf{R}^2$ is closed and $\mathbf{R}^2_+$-monotone $\iff$ there exists a lower semi-continuous, decreasing function $h : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ such that $\Lambda = \{(\lambda, \mu) : \mu \geq h(\lambda)\}$.*

Proof. Given Λ, for each $\lambda \in \mathbf{R}$, define $h(\lambda) = \inf\{\mu : (\lambda, \mu) \in \Lambda\}$, with $h(\lambda) = -\infty$ if this set is all of $\mathbf{R}$ and $h(\lambda) = \infty$ if this set is empty. The $\mathbf{R}^2_+$-monotonicity implies that h is decreasing. Now Λ is closed if and only if h is lower semi-continuous. The remainder of the proof is left to the reader. $\square$

It is more convenient to replace h by the function $f \equiv -h$ so that $f : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ is upper semi-continuous, increasing and

$$\Lambda \equiv \{(\lambda, \mu) : \mu + f(\lambda) \geq 0\}. \tag{3.5}$$

Thus the radial profiles $\overline{\Lambda}$ and $\underline{\Lambda}$ of F can be used interchangeably with the following associated functions $\overline{f}$ and $\underline{f}$ describing them.

Definition 3.4. Suppose that F is a subequation. The **upper (larger) and lower (smaller) characteristic functions $\overline{f}$ and $\underline{f}$ associated with** F are defined by:

$$\begin{aligned} \overline{f}(\lambda) &\equiv \sup\{\mu : \lambda P_{e^\perp} - \mu P_e \in F \ \text{ for some } e \neq 0\} \\ \underline{f}(\lambda) &\equiv \sup\{\mu : \lambda P_{e^\perp} - \mu P_e \in F \ \text{ for all } e \neq 0\}. \end{aligned}$$

Summarizing, we have

$$\lambda P_{e^\perp} + \mu P_e \in F \quad \text{for some } \ e \neq 0 \qquad \iff \qquad \mu + \overline{f}(\lambda) \geq 0. \tag{3.6}$$

$$\lambda P_{e^\perp} + \mu P_e \in F \quad \text{for all } \ e \neq 0 \qquad \iff \qquad \mu + \underline{f}(\lambda) \geq 0. \tag{3.7}$$

We will use the following fact to further analyze the borderline case.

LEMMA 3.5.

$$F \quad \textit{is borderline} \qquad \Longleftrightarrow \qquad \underline{f}(0) \;=\; \overline{f}(0) \;=\; 0. \tag{3.8}$$

PROOF. Use Definition 3.4 and condition (2) in Lemma 3.2. □

The asymptotic structure of F near 0 is reflected in the asymptotic behavior of $\underline{f}$ and $\overline{f}$ near 0. Now we can state the main result of this paper. Note that only the behavior of $\underline{f}(\lambda)$ and $\overline{f}(\lambda)$ for λ positive (and small) affects the outcomes.

THEOREM 3.6. *Suppose that F is a borderline subequation with upper and lower characteristic functions $\overline{f}$ and $\underline{f}$.*

(a) $$\textit{If} \quad \int_{0+} \frac{dy}{\overline{f}(y)} \;=\; \infty, \quad \textit{then the} \quad \text{(SMP)} \ \textit{holds for } F.$$

(b) $$\textit{If} \quad \int_{0+} \frac{dy}{\underline{f}(y)} \;<\; \infty, \quad \textit{then the} \quad \text{(SMP)} \ \textit{fails for } F.$$

The only case not covered is when $\int_{0+} \frac{dy}{\overline{f}(y)} < \infty$ and $\int_{0+} \frac{dy}{\underline{f}(y)} = \infty$. In this case F will be referred to as a **gap subequation**.

DEFINITION 3.7 (Weakly invariant subequations). For most equations of interest, $\underline{f} = \overline{f}$, and in this case Theorem 3.6 gives a necessary and sufficient condition for F to satisfy the (SMP). First note that $\underline{f} = \overline{f}$ if and only if $\underline{\Lambda} = \overline{\Lambda}$, or equivalently, for all λ, μ:

$$\text{If } \lambda P_{e^\perp} + \mu P_e \in F \ \text{ for some } \ e \neq 0, \ \text{ then } \lambda P_{e^\perp} + \mu P_e \in F \ \text{ for all } \ e \neq 0. \tag{3.9}$$

We take this as the definition of F being **weakly invariant**, and for simplicity we shall refer to it by just saying that F is **invariant**. Note also that $P_{e^\perp}, P_e$ have the same span as I, P_e, and therefore, for any subequation F which is invariant under the action of a group G acting transitively on the unit sphere $S^{n-1} \subset \mathbf{R}^n$, the characteristic functions $\underline{f}$ and $\overline{f}$ are equal. Among possibilities for G are SO_n acting on $\mathbf{R}^n$, SU_n acting on $\mathbf{R}^{2n} = \mathbf{C}^n$, Sp_n acting on $\mathbf{R}^{4n} = \mathbf{H}^n$, G_2 acting on $\mathbf{R}^7$ and Spin_7 acting on $\mathbf{R}^8$.

SOME EXAMPLES 3.8. Suppose $\chi : \mathbf{R} \to \mathbf{R}$ is odd ($\chi(-t) = -\chi(t)$) and strictly increasing. Fix $1 \le k \le n$ and define $F = F_{\chi,k}$ to be the set of $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ such that

$$\sigma_\ell(\chi(A)) \;\ge\; 0 \qquad \text{for } \ell = 1, ..., k,$$

where σ_ℓ denotes the ℓ^{th} elementary symmetric function. That is, $A \in F$ if and only if

$$\sigma_\ell\left(\chi(\lambda_1(A)), ..., \chi(\lambda_n(A))\right) \;\ge\; 0 \qquad \text{for } \ell = 1, ..., k,$$

where $\lambda_1(A), ..., \lambda_n(A)$ are the eigenvalues of A. One checks that F satisfies Condition (P) and is therefore a subequation. Direct calculation shows that the characteristic function $f = \overline{f} = \underline{f}$ is

$$f(\lambda) \;=\; \chi^{-1}\left\{\left(\frac{n}{k}-1\right)\chi(\lambda)\right\}. \tag{3.10}$$

One concludes that*if χ is smooth, or just Lipschitz, in a neighborhood of 0, then the* (SMP) *holds.*

A basic case is where $\chi(t) = \mathrm{sign}(t)|t|^\beta$ for $\beta > 0$. For example, if $k = 1$ and $\beta = \frac{1}{3}$, then $F = \{A : \mathrm{tr}(A^{\frac{1}{3}}) \geq 0\}$. In these cases $f(\lambda) = (\frac{n}{k}-1)^{\frac{1}{\beta}}\lambda$, and so the (SMP) holds.

In fact, by Theorem 8.5 below, this basic example is contained in the cone subequation $\mathcal{P}_\alpha^{\min/\max}$ with $\alpha \equiv (\frac{n}{k}-1)^{\frac{1}{\beta}}$. Now subequations which are cones can be treated more classically. Nevertheless, they are useful in understanding our main result in the non-invariant case, so we examine them next.

3.3 – Local cone subequations

Perhaps the simplest examples where Theorem 3.6 applies are the cone subequations. The results in this case are not really new, but they give a nice illustration of our geometric point of view and parallel the characterizations obtained for the (MP) as described in Remark 2.4.

We say that a subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a **cone** if $tF \subset F$ for all $t > 0$, and a **local cone** if for some $\delta > 0$ we have that $F \cap B_\delta(0)$ is the cone on the (non-empty) link $F \cap \partial B_\delta(0)$.

THEOREM 3.9 ($n \geq 2$). *Suppose that F is a local cone subequation.*

(a) *The* (SMP) *holds for F* $\iff$ $F - \{0\} \subset \mathrm{Int}\widetilde{\mathcal{P}}$.

(b) *The* (SMP) *holds for F* $\iff$ $-\,\epsilon P_e \notin F \;\; \forall\, \epsilon > 0$ and $|e| = 1$.

(c) *The* (SMP) *fails for F* $\iff$ $-\,\frac{\epsilon}{2}\langle e, x\rangle^2$ *is F-subharmonic for some $\epsilon > 0$ and some $|e| = 1$.*

Moreover,

$$\int_{0^+} \frac{dy}{\overline{f}(y)} = \infty \quad \text{is both necessary and sufficient for the (SMP)} \tag{3.11}$$

(*cf. Theorem* 3.6(a)), *and is equivalent to F being borderline.*

Proof. For a local cone subequation we have $0 \in \partial F$. Hence by Lemma 3.2, part (2)

$$F \text{ is borderline} \qquad \iff \qquad -\epsilon P_e \notin F \ \ \forall\, \epsilon > 0 \ \text{ and } \ e \neq 0. \tag{3.12}$$

Said differently, F is not borderline $\iff$ $-\epsilon P_e \in F$ for some $\epsilon > 0$ and $e \neq 0$. This proves that

$$\text{either } F \text{ is borderline or } h(x) \equiv -\tfrac{\epsilon}{2}\langle e, x\rangle^2 = -\tfrac{\epsilon}{2}\langle P_e x, x\rangle \text{ is } F \text{ subharmonic,} \tag{3.13}$$

in which case the (SMP) fails for F.

Define $0 \leq \overline{\alpha} \leq \infty$ by

$$\overline{\alpha} \ \equiv \ \sup\left\{\alpha : \tfrac{\delta}{\sqrt{n-1+\alpha^2}}(P_{e^\perp} - \alpha P_e) \in \partial B_\delta(0) \cap F \ \text{ for some } |e| = 1\right\}. \tag{3.14}$$

The local cone condition implies that if $\overline{\alpha} < \infty$, then

$$\overline{f}(\lambda) \ = \ \overline{\alpha}\lambda \quad \text{for} \quad \lambda \ \leq \ \lambda_0 \tag{3.15}$$

where $\lambda_0 \ = \ (\frac{1}{1+\alpha^2})^{\frac{1}{2}}$. Hence, $\int_{0+} \frac{1}{\overline{f}} \ = \ \frac{1}{\overline{\alpha}} \int_{0+} \frac{d\lambda}{\lambda} \ = \infty$, in which case by Theorem 3.6(a), F satisfies the (SMP). On the other hand, if $\overline{\alpha} = \infty$, then since $F \cap \partial B_\delta(0)$ is closed, one has that $-\delta P_e \in F$ for some $|e| = 1$, and hence the (SMP) fails.

Finally, note that conditions in (a) and (b) are just two ways of saying that F is borderline (*cf.* Theorem 3.6(a)). In addition, this proves (3.11). $\square$

The Remark 2.4 describing Parts (a), (b) and (c) of Theorem 2.1 has the following parallel describing parts (a), (b) and (c) of Theorem 3.9.

Remark 3.10 (The (SMP) for local cones)). Part (a) states that $\mathrm{Int}\widetilde{\mathcal{P}}$ is a "universal" set for the (SMP), while part (b) gives the simplest test for the (SMP). Part (c) says that one-variable quadratic functions such as $-\frac{1}{2}x_1^2$ provide a "universal" set of counterexamples to the (SMP).

Remark 3.11. The hard half of Theorem 3.9 (a) or (b) says that borderline cone subequations satisfy the (SMP). This is easy to prove using the classical "Hopf Lemma" construction, and so, in this sense, Theorem 3.9 is not new.

Example 3.12 (Gap cone subequations)). These are cone subequations where $\int_{0+}(1/\overline{f}) \ = \ 0$ (equivalently, $\overline{\alpha} \ = \ \infty$) and $\int_{0+}(1/\underline{f}) \ = \ \infty$ (equivalently, $\underline{\alpha} \ < \ \infty$). Neither Part (a) nor Part (b) of Theorem 3.6 applies. However, Theorem 3.9 does apply (see (3.11)) to show that the (SMP) fails for all gap cone subequations.

Such subequations are easy to construct. For example, on $\mathbf{R}^2$ define F by: $u_{xx} \geq 0$ or, if $u_{xx} < 0$, then $u_{yy} + \underline{\alpha} u_{xx} \geq 0$.

4 – The radial subequation associated to F

Supppose ψ is of class C^2 on an interval contained in the positive real numbers. Also consider $\psi(|x|)$ as a function of x on the corresponding annular region in $\mathbf{R}^n$.

LEMMA 4.1.

$$D_x^2\psi \;=\; \frac{\psi'(|x|)}{|x|}P_{x^\perp} + \psi''(|x|)P_x.$$

PROOF. First note that $D(|x|) = \frac{x}{|x|}$, and therefore $D^2(|x|) = D(\frac{x}{|x|}) = \frac{1}{|x|}I - \frac{x}{|x|^2} \circ \frac{x}{|x|} = \frac{1}{|x|}(I - P_x) = \frac{1}{|x|}P_{x^\perp}$. Hence,

$$D_x\psi \;=\; \psi'(|x|)\frac{x}{|x|} \qquad \text{and}$$

$$D_x^2\psi \;=\; \psi'(|x|)D\left(\frac{x}{|x|}\right) + \psi''(|x|)\frac{x}{|x|} \circ \frac{x}{|x|} \;=\; \frac{\psi'(|x|)}{|x|}P_{x^\perp} + \psi''(|x|)P_x. \quad \square$$

COROLLARY 4.2. *The second derivative $D_x^2\psi$ has eigenvalues $\frac{\psi'(|x|)}{|x|}$ with multiplicity $n-1$ and $\psi''(|x|)$ with multiplicity* 1.

For simplicity we shall now assume that F is invariant as in Definition 3.7 and let $f = \overline{f} = \underline{f}$ denote its characteristic function. Recall from (3.6), or (3.7), that

$$\lambda P_{e^\perp} + \mu P_e \;\in\; F \;\; \forall\, e \neq 0 \qquad \iff \qquad \mu + f(\lambda) \geq 0. \tag{4.1}$$

With motivation from Lemma 4.1 this leads to a subequation R_f on $(0, \infty)$. Let $p = \psi'(t)$ and $a = \psi''(t)$ denote jet coordinates.

DEFINITION 4.3. The **radial subequation R_f associated to F** is defined by

$$R_f \;:\; a + f\left(\frac{p}{t}\right) \;\geq\; 0 \qquad 0 < t < \infty \tag{4.2}$$

where f is the characteristic function associated with the subequation F.

It follows immediately from these definitions and Lemma 4.1 that if $\psi(t)$ is a C^2-function defined on a subinterval of $(0, \infty)$, with $u(x) \equiv \psi(|x|)$ defined on the corresponding annular region in $\mathbf{R}^n$, then

$$u(x) \equiv \psi(|x|) \;\text{ is }\; F \text{ subharmonic} \qquad \iff \qquad \psi(t) \;\text{ is }\; R_f \text{ subharmonic.} \tag{4.3}$$

This is extended to upper semi-continuous functions in Appendix A (Theorem A.1). The proof of the implication $\Rightarrow$ is elementary, whereas the proof of $\Leftarrow$ requires some details. However, note that $u(x) = \psi(|x|)$ is upper semicontinuous $\iff$ $\psi(t)$ is upper semicontinuous.

REMARK 4.4. The radial subequation R_f associated to F satisfies the topological conditions (T) in [12]. Namely,

$$\text{(i)}\ \ R \ = \ \overline{\text{Int}R}, \qquad \text{(ii)}\ \ R_t \ = \ \overline{\text{Int}R_t}, \qquad \text{(iii)}\ \ \text{Int}_t R_t \ = \ (\text{Int}R)_t,$$

where R_t is the fibre of R above t and Int_t denotes the interior in R_t. Note that $\text{Int}R$ is not defined by $a + f(\frac{p}{t}) > 0$ but by $a + f_-(\frac{p}{t}) > 0$ where $f_-(y) \equiv \lim_{z \to y^-} f(z)$ is lower semi-continuous. The proof is left to the reader.

REMARK 4.5 (Radial Harmonics). If $\psi(t)$ is R_f-harmonic on an interval $I \subset (0, \infty)$, then for any constants $r > 0$ and $k \in \mathbf{R}$, the 2-parameter family of functions $\psi_r(t) \equiv r^2\psi(t/r) + k$ consists of R_f-harmonics on rI. This follows since φ is a test function for $\pm\psi$ if and only if φ_r is a test function for $\pm\psi_r$, and the assertion is true for C^2-functions.

5 – Increasing radial subharmonics for borderline subequations

As in the last section, we assume for simplicity that F is invariant, *i.e.*, $f = \underline{f} = \overline{f}$. Because of the next result we focus on radial subharmonics which are increasing.

LEMMA 5.1. *Suppose that F is borderline and $u(x) = \psi(|x|)$ is a radial F-subharmonic function. There is only one way that $u(x)$ can violate the* (SMP). *Namely, for some r, $\psi(t)$ must satisfy:*

$$\psi(t) \ < \ M \quad \text{for} \ \ t < r \qquad \text{and} \qquad \psi(t) \ \equiv \ M \quad \text{for} \ \ t \geq r. \tag{5.1}$$

Moreover,

$$\psi \ \text{must be increasing on} \ (\bar{a}, r), \ \text{for some} \ \bar{a} < r. \tag{5.2}$$

PROOF. By the borderline hypothesis the (MP) holds for F. Since u satisfies the (MP), so does $\psi(t)$. If ψ has an interior maximum point at t_0 on an interval $[a, b]$, then either ψ is equal to the maximum value M on $[a, t_0]$ or on $[t_0, b]$ since otherwise ψ violates the (MP) on an interval about t_0. If ψ equals this maximum value on $[a, t_0]$, we can extend $u(x)$ to the ball of radius b (to be constant on the ball of radius a) as an F-subharmonic function which violates the (MP). This proves (5.1).

Pick $\bar{a}$ to be a minimum point for ψ on $[a, b]$. Then ψ must be increasing on $[\bar{a}, b]$. Otherwise, there exist α, β with $\bar{a} < \alpha < \beta < r$ and $\psi(\alpha) > \psi(\beta)$. In this case $\psi(\alpha) > \psi(\bar{a})$ also (since $\psi(\beta) \geq \psi(\bar{a})$), and this violates the maximum principle on $[\bar{a}, \beta]$. $\square$

DEFINITION 5.2. Suppose that F is borderline. The **increasing radial subharmonic equation** $R_f^{\uparrow}$ on $(0,\infty)$ is defined by

$$R_f^{\uparrow} \;:\; a + f\left(\frac{p}{t}\right) \;\geq\; 0 \quad \text{and} \quad p \geq 0, \tag{5.3}$$

where f is the characteristic function of F.

For C^2-functions $\psi(t)$ it is obvious that:

$$\psi(t) \;\text{ is } R_f^{\uparrow} \text{ subharmonic} \qquad \iff \qquad \psi(|x|) \;\text{ is } F \cap \{x \cdot p \geq 0\} \text{ subharmonic} \tag{5.4}$$

where $F \cap \{x \cdot p \geq 0\}$ is a variable coefficient subequation on $\mathbf{R}^n$ depending on both the first and second derivatives. The equivalence (5.4) is extended using Theorem A.1.

THEOREM 5.3 (Increasing radial subharmonics). *Suppose that F is borderline. The function $u(x) = \psi(|x|)$ is F-subharmonic and radially increasing on an annular region in $\mathbf{R}^n$ if and only if $\psi(t)$ is $R_f^{\uparrow}$-subharmonic on the corresponding subinterval of* $(0,\infty)$.

PROOF. Theorem A.1 states that: u is F-subharmonic $\iff$ ψ is R_f-subharmonic. By definition $u(x)$ is radially increasing if u satisfies the first-order variable coefficient subequation $\{p \cdot x \geq 0\}$. It remains to show that

$$u \;\text{ satisfies the subequation } \{p \cdot x \geq 0\} \qquad \iff \qquad \psi \;\text{ satisfies } \psi'(t) \geq 0. \tag{5.5}$$

Suppose $\psi(|x|)$ is $\{x \cdot p \geq 0\}$-subharmonic and that $\varphi(t)$ is a test function for $\psi(t)$ at a point t_0. Then $\varphi(|x|)$ is a test function for $u(x)$ at x_0 if $|x_0| = t_0$. Now

$$D_{x_0}\varphi \;=\; \varphi'(|x_0|)\frac{x_0}{|x_0|} \quad \text{and hence} \quad x_0 \cdot D_{x_0}\varphi \;=\; |x_0|\varphi'(|x_0|). \tag{5.6}$$

Thus $\varphi'(t_0) \geq 0$ proving that $\psi(t)$ is increasing. Conversely, if $\psi(t)$ is increasing and $\varphi(x)$ is a test function for $u(x)$ at x_0, then $\overline{\varphi}(t) \equiv \varphi(\frac{tx_0}{|x_0|})$ is a test function for $\psi(t)$ at $t_0 = |x_0|$. Hence, $\overline{\varphi}'(t_0) \geq 0$. However, $\overline{\varphi}'(t_0) = (D_{x_0}\varphi) \cdot x_0$. $\square$

REMARK 5.4 (Decreasing radial subharmonics). For borderline F we define the **decreasing radial subharmonic equation** $R_f^{\downarrow}$ on $(0,\infty)$ by

$$R_f^{\downarrow} \;:\; a + f\left(\frac{p}{t}\right) \;\geq\; 0 \quad \text{and} \quad p \leq 0, \tag{5.7}$$

where again f is the characteristic function of F. We leave it to the reader to show the following. For ψ upper semi-continuous,

$$\psi(t) \;\text{ is } R_f^{\downarrow} \text{ subharmonic} \qquad \iff \qquad \psi(|x|) \;\text{ is } F \cap \{x \cdot p \leq 0\} \text{ subharmonic.} \tag{5.8}$$

6 – Proof of the (SMP)

In this section we prove Part (a) of Theorem 3.6. The subequation F is assumed to be borderline, and we can assume that it is O_n-invariant because of the following construction. Set

$$F^{\#} \ \equiv \ \bigcup_{g\in \mathrm{O}_n} g(F). \tag{6.1}$$

First note that $F^{\#}$ is also a subequation. Now from Definition 3.4 of the characteristic function $\overline{f}$ of F and the fact that $P_{e^\perp}$, P_e have the same span as I, P_e, it is easy to see that the characteristic function for $F^{\#}$ is $\overline{f}$. Moreover, $F^{\#}$ is an O_n-invariant subequation which contains F so that it suffices to prove Theorem 3.6(a) for $F^{\#}$.

From now on we assume that F is an O_n-invariant borderline subequation, and we let f denote the restriction of $\overline{f} = \underline{f}$ to $[0, \infty)$. Hence $f(0) = 0$ and f is increasing. Furthermore, let both $R_f^{\uparrow}$ and $R_F^{\uparrow}$ denote the subequation defined by (5.4).

Part (a) of Theorem 3.6 follows from two implications.

Lemma 6.1. *Suppose F is an O_n-invariant borderline subequation. Then*

$$\int_{0^+} \frac{dy}{f(y)} \ = \ \infty \qquad \Rightarrow \qquad \text{(SMP) for } R_f^{\uparrow}, \ \text{ and} \tag{6.2}$$

$$\text{(SMP) for } R_F^{\uparrow} \qquad \Rightarrow \qquad \text{(SMP) for } F. \tag{6.3}$$

Proof. We prove the second implication (6.3) first. Suppose u is a counterexample to the (SMP) for F on a bounded domain Ω. We will show this leads to a counterexample to the (SMP) for $R_F^{\uparrow}$. $\square$

First, for all sufficiently small $\overline{r} > 0$, there exists a ball $B_{\overline{r}}(x_0) \subset \Omega$ of radius $\overline{r}$ such that the maximum $M \equiv \sup_{\overline{\Omega}} u$ satisfies

$$\begin{array}{llll} \text{(a)} & u(x) \ < \ M & \text{for all} \ \ x \in B_{\overline{r}}(x_0) & \text{and} \\ \text{(b)} & u(\overline{x}) \ = \ M & \text{for some} \ \ \overline{x} \in \partial B_{\overline{r}}(x_0). & \end{array} \tag{6.4}$$

This can be seen as follows. Since (SMP) is false, there exist points in Ω which are not in the maximum set $\{u = M\}$. Pick such a point x_0 closer to $\{u = M\}$ than to $\partial\Omega$ and set $\overline{r} \equiv \mathrm{dist}(x_0, \{u = M\})$. Let $B_t \equiv B_t(x_0)$ and $M(t) \equiv \sup_{\partial B_t} u$.

Second, choose an annulus

$$A \ = \ A(r, R) \ \equiv \ \{x : r \ \le \ |x - x_0| \ \le \ R\} \ \subset \ \Omega \tag{6.5}$$

containing $\partial B_{\overline{r}}$ in its interior. *i.e.*, with $r < \overline{r} < R$. Then

$$u(\overline{x}) \ = \ M \ \text{ at } \ \overline{x} \in \mathrm{Int} A, \ \text{ while on } \partial A: \ u\big|_{\partial B_r} \ < \ M \ \text{ and } \ u\big|_{\partial B_R} \ \le \ M. \tag{6.6}$$

Since F is borderline, $0 \in \partial F$, and hence by Theorem 2.1 the (MP) holds for u on B_t since u is F-subharmonic on Ω. Therefore $M(t)$ must be increasing for $r < t < R$. Hence, by (6.4) and (6.6)

$$M(t) \ < \ M \quad \text{for} \quad r < t < \bar{r} \qquad \text{and} \qquad M(t) = M \ \ \text{for} \ \ \bar{r} \le t < R. \tag{6.7}$$

That is, the (SMP) for $M(t)$ on $r \le t \le R$ fails. It remains to show that $M(t)$ is $R_F^{\uparrow}$-subharmonic.

Lemma 6.2. *For any upper semi-continuous function u, the function $M(t) \equiv \sup_{\partial B_t} u$ is upper semi-continuous.*

Proof. Assume the balls B_t are centered at the origin. Given $\delta > 0$,

$$N_\delta \ \equiv \ \{x : u(x) < M(t) + \delta\}$$

is an open set containing $\partial B_t = \{x : |x| = t\}$. Hence the annulus $\{x : t - \epsilon \le |x| \le t + \epsilon\}$ is contained in N_δ for $\epsilon > 0$ small. Thus $M(r) < M(t) + \delta$ if $t - \epsilon \le r \le t + \epsilon$. This proves that $M(t)$ is upper semi-continuous. $\square$

Since $M(t)$ satisfies the subequation $\{p \ge 0\}$ it remains to show that $M(t)$ satisfies the subequation R_F. By Theorem A.1 it suffices to show that $M(|x|)$ is F-subharmonic on $r < |x| < R$. The next result completes the proof of (6.3).

Lemma 6.3. *If u is F-subharmonic on an annulus, then $M(|x|)$ is also F-subharmonic on the same annulus where $M(t) \equiv \sup_{|x|=t} u$.*

Proof. By Lemma 6.2 $M(t)$ is upper semi-continuous, and hence $M(|x|)$ is upper semi-continuous. Let $u_g(x) \equiv u(gx)$ with $g \in \mathrm{O}_n$. Each u_g is F-subharmonic since F is O_n-invariant. Thus

$$M(|x|) \ = \ \sup_{g \in \mathrm{O}_n} u_g(x) \tag{6.8}$$

is F-subharmonic by the standard "families locally bounded above" property for F. $\square$

6.1 – A one-variable result

The point of this subsection is to prove the one-variable result (6.2) which completes the proof of Theorem 3.6 part (a). We assume throughout that $f : [0, \infty) \to [0, \infty]$ is an upper semi-continuous, increasing function with $f(0) = 0$, and we define the subequation $R_f^{\uparrow}$ on $(0, \infty)$ by (5.4).

PROPOSITION 6.4.

$$\int_{0+} \frac{dy}{f(y)} = \infty \qquad \Rightarrow \qquad \textit{The (SMP) holds for } R_f^{\uparrow}.$$

To prove this we first consider the following one-variable constant coefficient subequation E defined by

$$E \ : \ a + f(p) \ \geq \ 0 \quad \text{and} \quad p \geq 0. \tag{6.9}$$

PROPOSITION 6.5.

$$\int_{0+} \frac{dy}{f(y)} = \infty \qquad \Rightarrow \qquad \textit{The (SMP) holds for } E.$$

PROOF THAT 6.5 $\Rightarrow$ 6.4. Suppose that the (SMP) fails for $R_f^{\uparrow}$ on $[r_1, r_2] \subset (0, \infty)$. Choose r with $0 < r < r_1$. Consider the constant coefficient subequation E_r defined by

$$E_r \ : \ a + f\left(\frac{p}{r}\right) \ \geq \ 0 \qquad \text{and} \qquad p \geq 0. \tag{6.10}$$

If $t > r$, then $a + f\left(\frac{p}{t}\right) \geq 0$ implies that $a + f\left(\frac{p}{r}\right) \geq 0$ since f is increasing. That is, each fibre $(R_f^{\uparrow})_t \subset E_r$ if $t > r$, so that on a neighborhood of $[r_1, r_2]$, if ψ is $R_f^{\uparrow}$-subharmonic, then ψ is E_r-subharmonic. Therefore the (SMP) fails for E_r. The function $f(\frac{y}{r})$ satisfies the same conditions as the function f. Hence, by Proposition 6.5, $\int_{0+} \frac{dy}{f(y)} = \frac{1}{r} \int_{0+} \frac{dy}{f(\frac{y}{r})} < \infty$. $\square$

PROOF OF PROPOSITION 6.5. Suppose that ψ is a counterexample to the (SMP) for E. Since ψ is upper semi-continuous and increasing, there exists a point r_0 such that

$$\psi(t) \ < \ M \quad \text{for} \ \ t < r_0, \qquad \text{and} \qquad \psi(t) \ \equiv \ M \quad \text{for} \ \ r_0 \leq t. \tag{6.11}$$

By sup-convolution we may assume that ψ is quasi-convex and still satisfies E with a new r_0 slightly smaller than the old one. Since f is increasing we have the following.

LEMMA 6.6. *The derivative ψ' can be assumed to be absolutely continuous.*

PROOF. Since $\psi(t) + \frac{1}{2}\lambda t^2$ is convex for some $\lambda > 0$, the second distributional derivative $\psi'' = \mu - \lambda$ where $\mu \geq 0$ is a non-negative measure. Consider the Lebesgue decomposition $\mu = \alpha + \nu$ of μ into its L^1_{loc}-part α and its singular part ν. Since ν is supported on $t \leq r_0$, there exists a unique convex function β with $\beta'' = \nu$ and $\beta \equiv 0$ on $r_0 \leq t$. It follows easily that $\beta(t) \geq 0$ and β is decreasing. Therefore $\bar{\psi}(t) \equiv \psi(t) - \beta(t) \leq \psi(t)$ and $\bar{\psi}(t)$ is increasing. Hence $\bar{\psi}$ also satisfies (6.11). Now $\bar{\psi}'' = \alpha - \lambda$, and therefore $\bar{\psi}'$ is absolutely continuous. Since ν is singular, $\beta''(t) = 0$

a.e., and since β is decreasing, $\bar{\psi}'(t) = \psi'(t) - \beta'(t) \geq \psi'(t)$ a.e.. Therefore, since f is increasing and ψ is E-subharmonic,

$$\bar{\psi}''(t) + f(\bar{\psi}'(t)) \ \geq \ 0 \qquad \text{a.e.} \tag{6.12}$$

This almost-everywhere inequality is all that will be used to complete the proof of Proposition 6.5. However, in general, if a quasi-convex function satisfies a subequation F a.e., then it must be F-subharmonic (see Corollary 7.5 in [11] for pure second-order case and (7.3) below for the general case). $\square$

Now let $\varphi(t) \equiv \psi'(t)$. This function φ is absolutely continuous since $\varphi'(t) \equiv \alpha(t) - \lambda$. The properties that ψ is increasing and $\psi(t) \equiv M$ for $t \geq r_0$ translate into the properties:

$$\varphi(t) \ \geq \ 0 \qquad \text{and} \qquad \varphi(t) \ = \ 0 \ \text{ if } \ t \geq r_0. \tag{6.13}$$

The inequality (6.12) states that

$$\varphi'(t) + f(\varphi(t)) \ \geq \ 0 \qquad \text{a.e.} \tag{6.14}$$

Note that at a point t where φ is differentiable, if $\varphi(t) = 0$, then this implies that $\varphi'(t) \geq 0$. Thus (6.14) can be rewritten as

$$\frac{-\varphi'(t)}{f(\varphi(t))} \ \leq \ 1 \qquad \text{a.e.} \tag{6.15}$$

where the LHS equals $-\infty$ at points where $\varphi(t) = 0$. Therefore, for any measurable set B we have

$$-\int_B \frac{\varphi'(t)}{f(\varphi(t))} \ \leq \ |B|. \tag{6.16}$$

On the set B^- where φ is differentiable and $\varphi'(t) < 0$, the inequality (6.16) has content. Otherwise the integrand $\frac{-\varphi'(t)}{f(\varphi(t))} \leq 0$.

Choose s_1 and s_0 so that $r_1 < s_1 < s_0 < r_0$ and $0 < \varphi(s_0) < \varphi(s_1)$. We will show that

$$\int_{\varphi(s_0)}^{\varphi(s_1)} \frac{dy}{f(y)} \ \leq \ r_0 - r_1 \qquad \text{for all such } \ s_0 > s_1. \tag{6.17}$$

Because of (6.13) the point s_0 with $\varphi(s_0) > 0$ can be chosen arbitrarily close to r_0. Then taking the limit as s_0 increases to r_0 proves that

$$\int_0^{\varphi(s_1)} \frac{dy}{f(y)} \ \leq \ r_0 - r_1 \ < \ \infty.$$

It remains to prove (6.17). Let $N(\varphi\big|_A, y)$ denote the cardinality of $\{t \in A : \varphi(t) = y\}$. Set $A = [s_1, s_0]$, and let $V_A(\varphi)$ denote the total variation of φ on A. Since φ is absolutely continuous, we have, by [10, Theorem 2.10.13, page 177], that

$$V_A(\varphi) \text{ is finite, and } V_A(\varphi) \;=\; \int N\left(\varphi\big|_A, y\right)\, dy. \tag{6.18}$$

Now set

$$f_\epsilon(y) \;\equiv\; \max\{f(y), \epsilon\} \quad \text{where } \epsilon > 0.$$

Then

$$\int \frac{1}{f_\epsilon(y)} N\left(\varphi\big|_A, y\right)\, dy \;\leq\; \frac{1}{\epsilon} V_A(\varphi) \;<\; \infty.$$

Hence, the second half of [10, Theorem 3.2.6 (p. 245)] applies to yield

$$\int_{\varphi(s_1)}^{\varphi(s_0)} \frac{1}{f_\epsilon(y)}\, dy = -\int_{s_1}^{s_0} \frac{\varphi'(t)}{f_\epsilon(\varphi(t))}\, dt. \tag{6.19}$$

Since $\frac{1}{f_\epsilon(y)} \leq \frac{1}{f(y)}$ on the set B^- where φ is differentiable and $\varphi'(t) < 0$, we have

$$\int_{B^-} \frac{-\varphi'(t)\, dt}{f_\epsilon(\varphi(t))} \;\leq \int_{B^-} \frac{-\varphi'(t)\, dt}{f(\varphi(t))} \;\leq\; |B^-| \;\leq\; r_0 - r_1 \tag{6.20}$$

by (6.16). Combining (6.19) and (6.20) proves that

$$\int_{\varphi(s_1)}^{\varphi(s_0)} \frac{dy}{f_\epsilon(y)} \;\leq\; r_0 - r_1,$$

since $\int_{\sim B^-} \frac{-\varphi'(t)\, dt}{f_\epsilon(\varphi(t))} \;\leq\; 0$. By the Monotone Convergence Theorem this proves (6.17). $\square$

Remark 6.7. In the proof of Proposition 6.5, the fact that f is increasing was only used in Lemma 6.6. Therefore, if a subequation E is defined by an upper semi-continuous function $f : [0, \infty) \to [0, \infty]$ with $f(0) = 0$ and $f(y) > 0$ for $y > 0$, then we have that:

$$\int_{0^+} \frac{dy}{f(y)} = \infty \;\Rightarrow\; \text{the (SMP) holds}$$

for all E-subharmonic functions ψ for which ψ' is absolutely continuous.

7 – Radial (harmonic) counterexamples to the (SMP)

In this section we give the proof of Part (b) of Theorem 3.6 by constructing a radial counterexample to the (SMP) for F. Let f denote the restriction of $\underline{f}$ to $[0,\infty)$, where $\underline{f}$ is the (smaller) characteristic function (Definition 3.4) of the given borderline subequation F. Then

$$f : [0,\infty) \to [0,\infty] \ \text{ is upper semicontinuous, increasing and } f(0) = 0. \tag{7.1}$$

More precisely we prove the following.

Theorem 7.1. *Suppose that F is a borderline subequation with f as described above. If $\int_{0+} \frac{dy}{f(y)} < \infty$, then there exists a radially increasing F-subharmonic function $u(x) = \psi(|x|)$ on $|x| > 1$ where ψ is of class $C^{1,1}$ on $(1,\infty)$ and satisfies*

$$\psi(t) \ < \ m \quad \textit{for} \ \ 1 < t < t_0 \quad \textit{and} \quad \psi(t) \ = \ m \quad \textit{for} \ \ t \geq t_0. \tag{7.2}$$

By Theorem 5.3, it suffices to construct an increasing $C^{1,1}$-function which is $R_f^{\uparrow}$-subharmonic and satisfies (7.2).

In order to explicate the proof we will use the "almost-everywhere theorem" for quasi-convex functions, which holds for the most general possible subequations F. This AE Theorem states that for a quasi-convex function u

$$\text{If } u \text{ has its } 2-\text{jet in } F \text{ a.e., then } u \text{ is } F \,\text{subharmonic}, \tag{7.3}$$

and was established in [16]. We will also make use of the fact (*cf.* [17, 9], or [16]) that

$$u \text{ is of class } C^{1,1} \qquad \Longleftrightarrow \qquad u \text{ and } -u \text{ are quasiconvex}. \tag{7.4}$$

Proof Theorem 7.1 We start by solving the constant coefficient subequation E on $\mathbf{R}$ defined by

$$E \ : \quad a + f(p) \ \geq \ 0 \qquad \text{and} \qquad p \ \geq \ 0, \tag{7.5}$$

which is simpler than $R_f^{\uparrow}$. □

Lemma 7.2. *If $\int_{0+} \frac{dy}{f(y)} < \infty$, then there exists an E-subharmonic function $\varphi(s)$ of class $C^{1,1}$ on $(0,\infty)$ with*

$$\varphi(s) < m \ \text{ strictly increasing on } (0, s_0) \qquad \text{and} \qquad \varphi(s) \equiv m \ \text{ on } [s_0, \infty).$$

Proof. Set $s(y) = \int_0^y \frac{dy}{f(y)}$ for $y \geq 0$. For $0 \leq y_1 < y_2 \leq y_0$ we have

$$\frac{y_2 - y_1}{f(y_2)} \quad \leq \quad \int_{y_1}^{y_2} \frac{dt}{f(t)} \quad = \quad s_2 - s_1. \tag{7.6}$$

Therefore, this function $s(y)$ is strictly increasing until f equals $+\infty$ (and is constant afterwards). In particular, it is a homeomorphism from $[0, y_0]$ to $[0, s_0]$ for some $y_0 > 0$ with $s_0 = s(y_0) < \infty$. Let $y(s)$ denote the inverse, which is also strictly increasing with $y(0) = 0$. The inequality (7.6) implies that $y(s)$ is Lipschitz on $[0, s_0]$ with Lipschitz constant $f(y_0)$, since $f(y_2) \leq f(y_0)$ if $y_2 \leq y_0$.

Taking $y_1 = 0$, $y_2 = y(s)$ yields $y(s) \leq sf(y(s))$ which implies that y is differentiable from the right at $s = 0$ with $y'(0) = 0$. Moreover, since $y(s)$ is Lipschitz, it is differentiable a.e. and

$$y'(s) = f(y(s)) \quad \text{a.e.} \tag{7.7}$$

Fix m and consider the function $\varphi(s)$ defined on $(0, \infty)$ by $\varphi(s_0) = m$ and

$$\varphi'(s) \;\equiv\; \begin{cases} y(s_0 - s) & \text{if } 0 < s \leq s_0 \\ 0 & \text{if } s \geq s_0. \end{cases}$$

Since $\varphi'(s)$ is continuous and strictly decreasing to zero on $(0, s_0]$, $\varphi(s)$ must be strictly increasing to m on $(0, s_0]$ and identically equal to m afterwards.

Since φ is twice differentiable at $s = s_0$, with $\varphi'(s_0) = \varphi''(s_0) = 0$, the function φ is class $C^{1,1}$ on all of $(0, \infty)$. Moreover, (7.7) implies that

$$\varphi''(s) + f(\varphi'(s)) \;=\; 0 \quad \text{a.e. on} \quad (0, \infty). \tag{7.8}$$

By (7.4) and (7.3) this implies that φ is E-subharmonic on $(0, \infty)$. ☐

We will use Lemma 7.2 applied to the subequation E' defined by

$$E' \;:\quad a + p + f(p) \;\geq\; 0 \qquad \text{and} \qquad p \;\geq\; 0, \tag{7.9}$$

rather than E. Now consider the radial subequation $R_f^{\uparrow}$ on $(0, \infty)$ defined by

$$R_f^{\uparrow} \;:\quad a + f\left(\frac{p}{t}\right) \;\geq\; 0, \qquad \text{and} \qquad p \;\geq\; 0 \tag{7.10}$$

which depends on the variable $t \in (0, \infty)$.

Proposition 7.3. *Suppose $\varphi(s)$ is the E'-subharmonic function given by Lemma 7.2 applied to E' rather than E. Then the function $\psi(t)$ defined on $(1, \infty)$ by*

$$\psi'(t) \;=\; t\varphi'(\log t) \qquad \text{and} \qquad \psi(t_0) = m, \tag{7.11}$$

where $t_0 = e^{s_0}$, is a $C^{1,1}$ subharmonic for $R_f^{\uparrow}$. Moreover,

$$\begin{gathered} \psi(t) \;\text{ is strictly increasing with} \\ \psi(t) \;<\; m \;\text{ on }\; 1 < t < t_0 \qquad \text{and} \qquad \psi(t) \;\equiv\; m \;\text{ on }\; t_0 \leq t. \end{gathered} \tag{7.12}$$

Proof. That φ' is Lipschitz implies that ψ' is Lipschitz. Therefore ψ is class $C^{1,1}$. At a point of differentiability we have $\psi''(t) = \varphi'(\log t) + \varphi''(\log t)$, and hence $\psi''(t) + f(\frac{\psi'(t)}{t}) = \varphi''(\log t) + \varphi'(\log t) + f(\varphi'(\log t)) = 0$. Therefore $\psi(t)$ satisfies (7.10) a.e. (Since $\varphi'(s)$ is continuous and > 0 on $(0, s_0)$, $\psi'(t)$ is also continuous and > 0 on $(1, t_0)$. Thus ψ is strictly increasing on $(1, t_0)$.) Thus by (7.4) and (7.3), ψ is $R_f^{\uparrow}$-subharmonic. The properties (7.12) are straightforward. $\square$

Remark 7.4 (F-Harmonicity). The F-subharmonic function $u(x) = \psi(|x|)$ constructed in this section is, in fact, F-harmonic if F is invariant as in Definition 3.7. We leave it to the reader to show that $-\psi$ is $\widetilde{R}_f^{\uparrow}$-subharmonic and hence $-u$ is $\widetilde{F}$-subharmonic. One can show that

$$a + f(p) \;=\; 0, \quad p \;\geq\; 0 \qquad \Rightarrow \qquad (p, a) \;\in\; \partial E, \tag{7.13}$$

but the converse is not true if f has a jump.

Example 7.5. One of the simpler examples where Theorem 7.1 applies is the subequation F defined by $\lambda_{\max}(A) \geq 0$ and $\lambda_{\min}(A) + \sqrt{\lambda_{\max}(A)} \geq 0$ (See Example 8.1 below). The characteristic function is $f(\lambda) = \sqrt{\lambda}$. However, carrying out the construction of the F-harmonic counterexample provided in the proof of Theorem 7.1 involves taking a complicated inverse. To obtain more explicit harmonics consider the subequation F defined by $\lambda_{\max}(A) \geq 0$ and $\lambda_{\min}(A) + f(\lambda_{\max}(A)) \geq 0$ where

$$f(\lambda) \;\equiv\; \sqrt{\lambda}\left(\frac{4R}{\sqrt{4R+\lambda}+\sqrt{\lambda}}\right) \;=\; \sqrt{\lambda^2 + 4R\lambda} - \lambda.$$

The characteristic function is $f(\lambda)$, and $\lim_{\lambda\to 0} f(\lambda)/\sqrt{\lambda} = 2\sqrt{R}$ so that $\int_{0^+} 1/f < \infty$, and hence again Theorem 7.1 applies.

The increasing radial harmonics for this subequation F on $\mathbf{R}^n - \{0\}$ are very explicit:

$$h(x) \;\equiv\; \begin{cases} -\frac{R}{3r}(r - |x|)^3 + k & \text{for } \ |x| \leq r \\ 0 & \text{for } \ |x| \geq r. \end{cases} \tag{7.14}$$

A general version of this example is provided in 8.13 in the next section.

8 – Subequations with the same increasing radial subharmonics

In order to begin to understand examples and applications of the main Theorem A, it is helpful to describe all the borderline invariant subequations with a given characteristic function f.

REMARK. By Theorem 5.3, this problem is equivalent to describing all borderline invariant subequation with the same set of increasing radial subharmonics (or, equivalently, the same set of increasing radial harmonics) satisfying

$$R_f^{\uparrow} \quad : \quad \psi'(t) \;\geq\; 0 \qquad \text{and} \qquad \psi''(t) + f\left(\frac{\psi'(t)}{t}\right) \;\geq\; 0 \tag{8.1}$$

on $(\alpha, \beta) \subset (0, \infty)$.

We assume that an increasing upper semi-continuous function

$$f : [0, \infty) \;\to\; [0, \infty) \qquad \text{with} \;\; f(0) \;=\; 0 \tag{8.2}$$

is given. The problem is to determine all subequations (if any) with this characteristic function f.

We start with the two examples that play a central role. Given $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ denote the ordered eigenvalues of A. In particular, the minimum and maximum eigenvalues are $\lambda_{\min}(A) = \lambda_1(A)$ and $\lambda_{\max}(A) = \lambda_n(A)$ respectively. Recall the monotonicity $\lambda_k(A + P) \geq \lambda_k(A)$ for $P \in \mathcal{P}$.

EXAMPLE 8.1 (The f-min/max subequation).

$$F_f^{\min/\max} \;\equiv\; \{A : \lambda_{\max}(A) \geq 0 \;\;\text{and}\;\; \lambda_{\min}(A) + f(\lambda_{\max}(A)) \geq 0\}.$$

EXAMPLE 8.2 (The f-min/2 subequation).

$$F_f^{\min/2} \;\equiv\; \{A : \lambda_2(A) \geq 0 \;\;\text{and}\;\; \lambda_{\min}(A) + f(\lambda_2(A)) \geq 0\}.$$

PROPOSITION 8.3. *The sets $F_f^{\min/\max}$ and $F_f^{\min/2}$ are subequations which are borderline and O_n-invariant. Moreover, for both subequations, the characteristic function restricted to $[0, \infty)$ equals f.*

PROOF. Since f is upper semi-continuous, both sets are closed. Since f is increasing, positivity follows from the $\mathcal{P}$-monotonicity of the ordered eigenvalues. To prove these subequations are borderline, suppose A lies in the larger subequation $F_f^{\min/\max}$ and $A \in -\mathcal{P}$, *i.e.*, $\lambda_{\max}(A) \leq 0$. Then $\lambda_{\max}(A) = 0$ and since $f(0) = 0$, $\lambda_{\min}(A) = 0$. Hence, $A = 0$. Invariance follows because the ordered eigenvalues themselves are O_n-invariant.

To complete the proof we compute the full radial profiles (not just the increasing part).

The subequation $F_f^{\min/\max}$ has radial profile

$$\{(\lambda, \mu) : \lambda \geq 0 \;\;\text{and}\;\; \mu + f(\lambda) \geq 0\} \cup \{(\lambda, \mu) : \mu \geq 0 \;\;\text{and}\;\; \lambda + f(\mu) \geq 0\}. \tag{8.3}$$

For $n \geq 3$, the subequation $F_f^{\min/2}$ has radial profile

$$\{(\lambda, \mu) : \lambda \geq 0 \ \text{ and } \ \mu + f(\lambda) \geq 0\}. \tag{8.4}$$

We see this as follows. Note that the radial profile of $F_f^{\min/\max}$ is symmetric about the diagonal. Recall that if $A \equiv \lambda P_{e^\perp} + \mu P_e$ belongs to any borderline subequation, then either $\lambda \geq 0$ or $\mu \geq 0$.

For (8.3), suppose $A \equiv \lambda P_{e^\perp} + \mu P_e \in F_f^{\min/\max}$. If $\lambda \geq \mu$, then $\lambda = \lambda_{\max} \geq 0$ and $\mu = \lambda_{\min}$ satisfies $\lambda_{\min} + f(\lambda_{\max}) \geq 0$. If $\mu \geq \lambda$, then $\mu = \lambda_{\max} \geq 0$ and $\lambda = \lambda_{\min}$ satisfies $\lambda_{\min} + f(\lambda_{\max}) \geq 0$.

For (8.4), suppose $A \equiv \lambda P_{e^\perp} + \mu P_e \in F_f^{\min/2}$. Since $n \geq 3$, $\lambda = \lambda_2 \geq 0$ and either $\mu = \lambda_1$ or $\mu > \lambda$. In either case $\lambda \geq 0$ and $\mu + f(\lambda) \geq 0$. □

COROLLARY 8.4. *Both $F_f^{\min/\max}$ and $F_f^{\min/2}$ have their increasing radial subharmonics $u(x) = \psi(|x|)$ determined by the subequation $R_f^{\uparrow}$ defined in* (8.1).

The subequations $F_f^{\min/\max}$ and $F_f^{\min/2}$ are of central importance because they are the largest and smallest possible under our invariance hypothesis (3.9) on F:

$$\lambda P_{e^\perp} + \mu P_e \in F \ \text{ for some } e \neq 0 \qquad \Rightarrow \qquad \lambda P_{e^\perp} + \mu P_e \in F \ \text{ for all } e \neq 0. \tag{3.9}$$

THEOREM 8.5. *Suppose F is invariant. Then F has characteristic function f, or equivalently, the radial increasing subharmonics $u(x) = \psi(|x|)$ for F are determined by $R_f^{\uparrow}$ as in* (8.1), *if and only if*

$$F_f^{\min/2} \ \subset \ F \ \subset \ F_f^{\min/\max}. \tag{8.5}$$

PROOF. Each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ can be written as a sum $A = \lambda_1 P_{e_1} + \cdots + \lambda_n P_{e_n}$ using the ordered eigenvalues of A. Set $B_0 \equiv \lambda_1 P_{e_1} + \lambda_2 P_{e_1^\perp}$ and $B_1 \equiv \lambda_1 P_{e_1} + \lambda_n P_{e_1^\perp}$, and note that $B_0 \leq A \leq B_1$.

If $A \in F_f^{\min/2}$, then $\lambda_2 \geq 0$ and $\lambda_1 + f(\lambda_2) \geq 0$. Thus $B_0 \equiv \lambda_1 P_{e_1} + \lambda_2 P_{e_1^\perp} \in F_f^{\min/2}$. Since $F_f^{\min/2}$ and F have the same radial profile in the half-plane $\{\lambda \geq 0\}$ by (8.4), we conclude that $B_0 \in F$. However, $B_0 \leq A$ proving that $A \in F$.

For the other inclusion, pick $A \in F$. Since $F \subset \widetilde{\mathcal{P}}$ we have $\lambda_{\max} \geq 0$. Now $A \leq B_1$ implies $B_1 \in F$. By the invariance hypothesis and (8.3), F and $F_f^{\min/\max}$ have the same same radial profile in the half-plane $\{\lambda \geq 0\}$. Therefore, $B_1 \in F_f^{\min/\max}$, *i.e.*, $\lambda_n \geq 0$ and $\lambda_1 + f(\lambda_n) \geq 0$. This implies by definition that $A \in F_f^{\min/\max}$. □

Remark 8.6. Theorem 8.5 can be used to construct vast numbers of invariant borderline subequations which satisfy the (SMP), or, if one prefers, which do not satisfy the (SMP).

Remark 8.7. Dropping the invariance assumption (3.9), the proof of Theorem 8.5 shows that for any borderline subequation F with characteristic functions $\underline{f}$ and $\overline{f}$

$$F_{\underline{f}}^{\min/2} \subset F \subset F_{\overline{f}}^{\min/\max}. \tag{8.5$'$}$$

The subequation $F^{\#}$ defined by (6.1) as the O_n-orbit of F, has characteristic function $\overline{f}$. It satisfies

$$F \subset F^{\#} \subset F_{\overline{f}}^{\min/\max}, \tag{8.6}$$

and is the smallest O_n-invariant subequation containing F.

8.1 – Explicit borderline examples

It is natural to look for the largest possible subequations which satisfy the (SMP). Because of Theorem 3.6(a) and Theorem 8.5 these are max-min subequations F whose characteristic function f is as large as possible subject to the condition $\int_{0^+} \frac{dy}{f(y)} = \infty$. Since this only depends on the behavior of the germ at 0^+ of f, we can also localize F at the origin in $\mathrm{Sym}^2(\mathbf{R}^n)$ by replacing f on $[\epsilon, \infty)$ by the function $\equiv +\infty$, which yields a larger subequation. We present three examples where the (SMP) holds, and two where the (SMP) fails.

If F is a cone, then by (3.15) $\overline{f}(\lambda) = \overline{\alpha}\lambda$, and the corresponding min/max-subequation $F_{\overline{f}}^{\min/\max}$ containing F is given as follows.

Example 8.8 (Min/Max cones). $(0 < \alpha < \infty)$

$$f(y) = \alpha y \qquad \left(\text{and hence } \int_{0^+} \frac{1}{f} = \infty\right). \tag{a}$$

The borderline O_n-invariant cone subequation

$$\mathcal{P}_\alpha^{\min/\max} : \quad \lambda_{\min}(A) + \alpha\lambda_{\max}(A) \ \ge\ 0 \qquad \text{satisfies the (SMP).} \tag{b}$$

Thus all subequations F which are contained in $\mathcal{P}_\alpha^{\min/\max}$ for some $\alpha > 0$ satisfy the (SMP). The increasing radial harmonics $\psi(|x|)$ are important classical functions given (with $p \equiv \alpha + 1$) by:

$$\begin{aligned}
\psi(t) &= at^{2-p} + b \quad \text{with } a > 0 \text{ if } 1 \le p < 2 \\
\psi(t) &= a\log t + b \quad \text{with } a > 0 \text{ if } p = 2 \\
\psi(t) &= -\frac{a}{t^{p-2}} + b \quad \text{with } a > 0 \text{ if } p > 2.
\end{aligned} \tag{8.7}$$

We leave the computation of M_F for this F as an open problem.

This example can be localized.

EXAMPLE 8.9 (8.9. (Localizing the Min/Max cone). ($0 < \alpha < \infty$ and $\epsilon > 0$)

(a)
$$f(y) \equiv \begin{cases} \alpha y & \text{if } \ 0 \le y < \epsilon \\ +\infty & \text{if } \ \epsilon \le y. \end{cases}$$

(b) The borderline O_n-invariant cone subequation

$$\mathcal{P}^{\min/\max}_{\alpha,\mathrm{loc}} \quad : \quad \text{Either } \ \lambda_{\min}(A) \ \ge \ \epsilon \quad \text{or} \quad \lambda_{\min}(A) + \alpha\lambda_{\max}(A) \ \ge \ 0$$

satisfies the (SMP). Thus

$$\begin{array}{l}\text{All subequations } F \text{ which are contained in } \mathcal{P}^{\min/\max}_{\alpha,\mathrm{loc}} \\ \text{for some } \alpha \text{ satisfy the (SMP).}\end{array} \tag{8.8}$$

REMARK 8.10 (The Barles and Busca Hopf Lemma 3.2 [3]). Under their hypothesis "(F3b)" they prove that the (SMP) holds. This landmark paper on comparison covers a wide range of subequations. For the constant coefficient, pure second-order subequations considered here their hypothesis can be restated as follows:

$$\begin{array}{c} \forall\, \lambda > 0, \ \ \exists\, \mu, \delta \ > \ 0 \quad \text{such that} \\ E \ \equiv \ \{t\,(\lambda P_{e^\perp} - \mu P_e) : 0 < t < \delta \ \text{ and } \ |e| = 1\} \\ \text{is contained in the complement of the subequation } F. \end{array} \tag{F3b}$$

Now the assertion that $E \subset (\sim F)$ is equivalent to saying that $\overline{f}(y) < \frac{\mu}{\lambda} y$ for all $0 < y < \lambda\delta$. By Theorem 8.5 and Remark 8.7 this proves that the condition (F3b) is equivalent to

$$F \ \subset \ \mathcal{P}^{\min/\max}_{\alpha,\mathrm{loc}} \qquad \text{for some } \ \alpha$$

(take $\alpha = \frac{\mu}{\lambda}$). Thus, the Hopf Lemma (3.2) in [3], when restricted to subequations of the type considered here, is equivalent to the corollary (8.8) of Theorem 3.6(a).

EXAMPLE 8.11 (A localized Hopf subequation). ($0 < k < \alpha < \infty$ and $0 < \epsilon \le 1$)

(a)
$$f(y) = \begin{cases} y\left(\alpha + k \log \frac{1}{y}\right) & \text{if } \ 0 \le y < \epsilon \\ +\infty & \text{if } \ \epsilon \le y \end{cases}$$

is an upper semi-continuous increasing function with associated min/max subequation

(b) $H(\alpha)$: Either $\lambda_{\min}(A) \ \ge \epsilon$ or $\lambda_{\min}(A) + \lambda_{\max}(A)\,(\alpha - k \log \lambda_{\max}(A)) \ \ge \ 0.$

This subequation satisfies the (SMP) since

$$\int \frac{dy}{y(\alpha - k\log y)} \;=\; -\frac{1}{k}\log(\alpha - k\log y)$$

which implies $\int_{0+} \frac{du}{f(y)} = \infty$. The increasing radial harmonics satisfy

$$\psi'(t) \;=\; \beta t e^{ct^k} \quad \text{where} \quad \log\beta \;=\; \frac{1+\alpha}{k} \tag{8.9}$$

and c is the constant of integration. For example, if we take $k = 2$ and set $c = -\beta/2$, we see that $\psi'(t) = \beta t e^{\frac{-\beta t^2}{2}}$ integrates to

$$\psi(t) \;=\; e^{\frac{-\beta R^2}{2}} - e^{\frac{-\beta t^2}{2}} \tag{8.10}$$

which is the standard Hopf function (*cf.* [6]). Here $f(y) = y(2\log(\frac{\beta}{y}) - 1)$ for y small.

Obviously,

$$\mathcal{P}_\alpha^{\max/\min} \;\subset\; \mathcal{P}_\alpha^{\rm loc} \;\subset\; H_\alpha, \tag{8.11}$$

and for larger α each subequation is larger. Our notation suppresses the dependence of $\mathcal{P}_\alpha^{\rm loc}$ on ϵ and of H_α on ϵ and k.

Example 8.12 (The (SMP) Fails)). Let $f : [0,\infty) \to [0,\infty)$ be defined by

$$f(y) \;=\; Ny^{\frac{N-1}{N}} \qquad (N \;>\; 1). \tag{8.12}$$

Since $\int \frac{dy}{f(y)} \;=\; y^{\frac{1}{N}}$, we have $\int_{0+} \frac{dy}{f(y)} \;<\; \infty$. Therefore the (SMP) fails for the corresponding min/max subequation

$$F \;: \lambda_{\min}(A) + N\lambda_{\max}(A)^{\frac{N-1}{N}} \;\geq\; 0 \quad \text{and} \quad \lambda_{\max}(A) > 0. \tag{8.13}$$

The associated constant coefficient subequation

$$E \quad : \quad a \;+\; Np^{\frac{N-1}{N}} \qquad \text{and} \qquad p \;\geq\; 0$$

has radial harmonics

$$\psi(t) \equiv -\frac{1}{N+1}(R-t)^{N+1} + k \text{ for } t \leq R \qquad \text{and} \qquad \psi(t) \;\equiv\; 0 \quad \text{for} \;\; t \geq R, \tag{8.14}$$

but the radial harmonics for $R_f^{\uparrow}$ are more complicated. Note that a better example (*i.e.*, f is smaller) where (SMP) fails is $f(y) \equiv y(\log y)^2 < y^\beta$ $(0 < \beta < 1$ and y small), since $\int \frac{dy}{f(y)} = 1 + \frac{1}{\log y} < 1$ for $y > 0$ small.

EXAMPLE 8.13 (The (SMP) fails with explicit harmonics). However, fixing $R > 0$ there exists a modification of (8.13) of the form

$$F \; : \; \lambda_{\min}(A) + N(\lambda_{\max}(A) g(\lambda_{\max}(A)))^{\frac{N-1}{N}} \; \geq \; 0 \qquad \text{and} \qquad \lambda_{\max}(A) \; \geq \; 0 \tag{8.15}$$

with g defined below so that F has the simple/explicit radial harmonics:

$$\psi(|x|) \equiv \frac{-r^2}{(N+1)R^2}(r - |x|)^{N+1} + k \text{ for } |x| \leq r \quad \text{and} \quad \psi(|x|) \equiv 0 \text{ for } |x| \geq r. \tag{8.16}$$

The proof is omitted.

The characteristic function for F is

$$f(y) \; = \; N(yg(y))^{\frac{N-1}{N}}. \tag{8.17}$$

The function $g(y)$ is defined to be the inverse of $y(t) \equiv (R-t)^N/t = \psi'(t)/t$ for $0 \leq t \leq R$. Since $y(t)$ is strictly decreasing from ∞ to 0 on $[0, R]$, the function $g(y)$ is strictly decreasing from R to 0 on $[0, \infty]$. One can show that $x = \frac{1}{N} f(y)$ has inverse

$$y(x) = \frac{x^{\frac{N}{N-1}}}{R - x^{\frac{N}{N-1}}},$$

and hence is strictly increasing on $[0, \infty]$ from 0 to NR^{N-1} ensuring that F is a subequation.

9 – Strong comparison and monotonicity

By the **strong comparison principle** for a **subequation** F we mean the following.

$$\text{If } u \in F(\overline{\Omega}) \text{ and } v \in \widetilde{F}(\overline{\Omega}), \text{ then the (SMP) holds for } u + v \text{ on } \overline{\Omega}. \tag{SC}$$

This is, of course, immediate if $u + v$ is G-subharmonic for some subequation G for which the strong maximum principle holds. In this section we address the question of when such a G exists. The geometric point of view is, we think, an advantage here. This question is reduced to algebra by the following.

THEOREM 9.1 (Addition). *If three subequations satisfy*

$$F + H \; \subset \; G,$$

then

$$F(\overline{\Omega}) + H(\overline{\Omega}) \quad \subset \quad G(\overline{\Omega}).$$

REMARK. This result is immediate from sup-convolution and either of the classical Jensen or Slodkowsky Lemmas (which are in a strong sense equivalent, *cf.* [16]). It is referred to as "Transitivity of inequalities in the viscosity sense" on [1, page 745], and is proved in the book [3] of Caffarelli-Cabré (Proposition 2.9) in the case where F and H are uniformly elliptic. See also classic works of Crandall and Crandall, Ishii and Lions [7, 8].

Thus the (SC) question is reduced to asking when is $F + \widetilde{F}$ contained in G, where G satisfies the (SMP).

Using the fact (2.6) that $\widetilde{F + A} = \widetilde{F} - A$ one can show that for any two subequations F and G

$$F + \widetilde{F} \subset G \qquad \iff \qquad F + \widetilde{G} \subset F.$$

Rewriting this with $\widetilde{G}$ replaced by M gives

$$F + M \subset F \qquad \iff \qquad F + \widetilde{F} \subset \widetilde{M}. \tag{9.1}$$

A subequation M satisfying $F + M \subset F$ will be called a **monotonicity subequation for** F. It is easy to show that M is a monotonicity subequation for F if and only if M is monotonicity subequation for $\widetilde{F}$. (See (5) below.)

THEOREM 9.2 (Strong comparison). *Suppose that M is a monotonicity subequation for F. Then*

$$\text{(SMP)} \ \ \text{for} \ \ \widetilde{M} \qquad \Rightarrow \qquad \text{(SC)} \ \ \text{for} \ \ F. \tag{9.2}$$

PROOF. By (9.1) and Theorem 9.1, $F + M \subset F \ \Rightarrow \ F + \widetilde{F} \subset \widetilde{M} \ \Rightarrow$ $F(\overline{\Omega}) + \widetilde{F}(\overline{\Omega}) \subset \widetilde{M}(\overline{\Omega})$. ☐

9.1 – The largest monotonicity subequation for F

Increasing the size of a subequation M satisfying $F + M \subset F$ decreases the size of $G = \widetilde{M}$, thereby increasing the liklyhood that $G = \widetilde{M}$ satisfies the (SMP). Hence, it is natural to look for the largest subequation M satisfying $F + M \subset F$. It is somewhat surprising that there is such a subequation. We define the **monotonicity subequation for** F to be the set

$$M_F \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : F + A \subset F\}. \tag{9.3}$$

We leave the following facts as an exercise.

(1) M_F is a subequation, (2) $0 \in \partial M_F$ and $\widetilde{M}_F \subset \widetilde{\mathcal{P}}$,
(3) M_F is its own monotonicity subequation, in particular M_F is additive,
(4) If M_F is a cone, then M_F is a convex cone subequation,
(5) $M_{\widetilde{F}} = M_F$, in fact, for any A, $F + A \subset F \iff \widetilde{F} + A \subset \widetilde{F}$,
(6) $\mathrm{Int}\widetilde{M}_F \subset F + \mathrm{Int}\widetilde{F} \subset F + \widetilde{F} \subset \widetilde{M}_F$, and hence $\widetilde{M}_F = \mathrm{Cl}(F + \widetilde{F})$. (9.4)

Definition 9.3. A subequation M such that $0 \in M$ and M is additive, *i.e.*, $M + M \subset M$, will be called a **monotonicity subequation**.

$$\begin{aligned} &M \text{ is a monotonicity subequation} \iff \\ &M = M_F \text{ for some subequation } F. \end{aligned} \tag{9.5}$$

Proof. In fact M is its own monotonicity subequation, because if $M + A \subset M$, then $0 \in M \Rightarrow A \in M$. □

Note that M_F is maximal, that is, it contains every monotonicity subequation for F. Consequently, Theorem 9.2 could be restated equivalently as follows.

Theorem 9.2′

$$\widetilde{M}_F \text{ satisfies the (SMP)} \quad \Rightarrow \quad F \text{ satisfies (SC).} \tag{9.2′}$$

For most subequations F, even when F is not a cone, M_F is a cone. These subequations F will be referred to as **normal subequations**. If F is normal, then in fact, by (4) above, M_F is a convex cone. The verious criteria in Theorem 3.9 apply to $\widetilde{M}_F$. In addition, uniform ellipticity can be added to the list since M_F is a convex cone.

Proposition 9.4. *Suppose F is a normal subequation (i.e., M_F is a cone). Then*

$$\begin{aligned} \text{(SMP) } \textit{holds for } \widetilde{M}_F &\iff -P_e \notin \widetilde{M}_F \ \forall\, e \neq 0 \iff P_e \in \mathrm{Int} M_F \ \forall\, e \neq 0 \\ &\iff M_F \ \textit{is a convex conical neighborhood of } \mathcal{P} \\ &\iff F \ \textit{is uniformly elliptic.} \end{aligned}$$

Proof. The first equivalence is just part (b) of Theorem 3.9. The second follows from the definition of the dual of M_F. The third follows since M_F is a convex cone. The last follows from Lemma B.1 in Appendix B. □

We note that since M_F is maximal, there is the possibility that the reverse implication in $(9.2)'$ holds. We leave this as an open question even in the case where F is a cone. However, the following is a partial answer in this case.

PROPOSITION 9.5. *Suppose that F is a normal subequation and $F + \widetilde{F} = \widetilde{M_F}$. Then*

$$\text{(SC) } \textit{holds for } F \qquad \iff \qquad \textit{the } \text{(SMP) } \textit{holds for } \widetilde{M_F}.$$

PROOF. Suppose that the (SMP) fails for $\widetilde{M_F}$. Then by Proposition 9.4 we have $-P_e \in \widetilde{M_F}$ for some e. By the hypothesis that $F + \widetilde{F} = \widetilde{M_F}$, we have

$$-P_e \;=\; Q + \widetilde{Q} \qquad \text{with } \; Q \in F \;\text{ and }\; \widetilde{Q} \in \widetilde{F}. \tag{9.6}$$

Let $w(x) \equiv \frac{1}{2}\langle e, x\rangle^2$, $u(x) \equiv \frac{1}{2}\langle Qx, x\rangle$, and $v(x) \equiv \frac{1}{2}\langle \widetilde{Q}x, x\rangle$ denote the corresponding quadratic functions. Then $w = u + v$, $u \in F(\mathbf{R}^n), v \in \widetilde{F}(\mathbf{R}^n)$ but the (SMP) fails for w. Hence, u and v provide a counterexample to (SC) for F. $\square$

The following corollary probably comes as no surprise.

COROLLARY 9.6. *Suppose that F is a convex cone subequation. Then*

$$\text{(SC) } \textit{holds for } F \qquad \iff \qquad F \textit{ is uniformly elliptic.}$$

PROOF. If $A, B \in F$, then $\frac{1}{2}(A+B) \in F$ by convexity, and since F is a cone, this proves $F+F \subset F$ which implies $F \subset M_F$. Since $0 \in F$, $M_F = 0 + M_F \subset F + M_F \subset F$. Thus, $M_F = F$, and so $\widetilde{M_F} = \widetilde{F}$. By (5) above, $\widetilde{F} + F \subset \widetilde{F} = \widetilde{M_F}$, while $\widetilde{M_F} = \widetilde{F} \subset \widetilde{F} + 0 \subset \widetilde{F} + F$. This proves that $\widetilde{F} + F = \widetilde{M_F} = \widetilde{F}$ so that Proposition 9.5 applies. Finally, by Proposition 9.4 the (SMP) holds for $\widetilde{F} = \widetilde{M_F} \iff F$ is uniformly elliptic. $\square$

The hypothesis $F + \widetilde{F} = \widetilde{M_F}$ in Proposition 9.5 can be analyzed further because the set $F + \widetilde{F}$ can be explicitly computed. For this we introduce the **strict monotonicity set S_F for F**

$$S_F \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : F + A \subset \mathrm{Int}F\}, \tag{9.7}$$

along with a secondary notion for the dual, this time for an arbitrary subset G, namely,

$$G^* \;\equiv\; -(\sim G) \;=\, \sim(-G). \tag{9.8}$$

REMARK 9.7. Although we restrict attention in this paper to subequations $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$, it is worth noting that the next result holds for an arbitrary subequation $F \subset J^2(X)$ on a manifold X.

LEMMA 9.8. *For any subequation F*

$$F + \widetilde{F} \;=\; S_F^*.$$

PROOF. Note that $E \in S_F^* \iff -E \notin S_F \iff \exists A \in F$ such that $-B = A - E \notin \mathrm{Int}F \iff E = A + B$ for some $A \in F$ and $B \in \widetilde{F}$. $\square$

Remark 9.9 (Reformulating $F+\widetilde{F} = \widetilde{M}_F$). For any subequation F the following are equivalent statements.

$$\widetilde{M}_F \ \subset\ F+\widetilde{F} \tag{1}$$
$$\widetilde{M}_F \ \subset\ S_F^* \tag{2}$$
$$S_F \ \subset\ \mathrm{Int}M_F \tag{3}$$
$$S_F \cap \partial M_F \ =\ \emptyset. \tag{4}$$

The reverse containments in (1), (2) and (3) are always true. Thus if any of (1) through (4) are true, then equality holds in (1) through (3).

Proof. Assertions (1) and (2) are equivalent since $S_F^* = F+\widetilde{F}$ Assertions (2) and (3) are equivalent since $(\widetilde{M}_F)^* = \mathrm{Int}M_F$, $(S_F^*)^* = S_F$, and taking the secondary dual $(\cdot)^*$ reverses containments. Assertions (1) and (2) are equivalent since $S_F \subset M_F$. $\square$

Example 9.10 ($F+\widetilde{F} = \widetilde{M}_F$ is true). Let F be a convex cone subequation. We saw, in the proof of Corollary 9.6, that $F+\widetilde{F} = \widetilde{M}_F$. Here we give a second proof involving S_F. Note that $S_F = \mathrm{Int}M_F = \mathrm{Int}F$, since any $B \in \partial M_F = \partial F$ cannot be in S_F (because $B \in S_F$ would imply $0+B \in \mathrm{Int}F$ by definition). Therefore $S_F^* = \widetilde{F} = \widetilde{M}_F$. Now apply Lemma 9.8.

Example 9.11 ($F+\widetilde{F} = \widetilde{M}_F$ is false). The simplest example is the Monge-Ampère equation $F : \det A \geq 1, A > 0$. Here $M_F = \mathcal{P}$, but $S_F = \mathcal{P} - \{0\}$ is larger than $\mathrm{Int}M_F$. Thus, S_F^* is smaller than $\widetilde{M}_F$. In fact, $S_F^* = (\mathrm{Int}\widetilde{\mathcal{P}}) \cup \{0\} = F+\widetilde{F}$ does not contain $-P_e$ for any $e \neq 0$, but $-P_e \in \widetilde{M}_F$.

This example does not contradict the equivalence of (SC) for F and the (SMP) for $\widetilde{M}_F$ since both conditions fail in this case. To see that (SC) fails for the Monge-Ampère equation F, one employs the classical Pogorelov harmonics

$$h_\alpha(t,x) \ \equiv\ \frac{|x|^{2-\frac{2}{n}}}{f_\alpha(t)^{1-\frac{2}{n}}} \qquad \text{where} \quad f''_\alpha + f_\alpha^{n-1} \ =\ 0, \quad \text{and} \quad f_\alpha(0) = \alpha > 0.$$

Then near $t=0$, $h_{2\alpha} - h_\alpha \leq 0$ attains the maximum value zero on the t-axis.

This section leads to the question: are there examples where M_F is not a cone (*i.e.*, F is not normal)? The answer is yes. This involves some intriguing new subequations discussed in the next section.

10 – Examples of exotic monotonicity subequations which are not cones

The examples will be constructed as follows.

DEFINITION 10.1. Suppose $g : [0, \infty) \to \mathbf{R}$ is a continuous decreasing function with $g(0) = 0$ and $g(x) < 0$ for $x > 0$. Set

$$M^g \;\equiv\; \{A : \; \mathrm{tr}A \geq 0 \;\text{ and }\; \lambda_{\min}(A) \geq g(\mathrm{tr}A)\}. \tag{10.1}$$

PROPOSITION 10.2. *M^g is a subequation which is orthogonally invariant with*

$$M^g \cap \{\mathrm{tr}A = 0\} \;=\; \{0\} \qquad \text{and} \qquad \mathcal{P} - \{0\} \;\subset\; \mathrm{Int}M^g. \tag{10.2}$$

PROOF. Since g is continuous, M^g is a closed set. Recall that

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B). \tag{10.3}$$

This combined with the fact that g is decreasing easily implies that positivity (P) holds for M^g. Obviously M^g is O_n-invariant.

If $A \in M^g$ and $\mathrm{tr}A = 0$, then since $g(0) = 0$, the minimum eigenvalue $\lambda_{\min}(A) \geq g(0) = 0$. But then $\mathrm{tr}A = 0$ implies $A = 0$.

If $P \geq 0$ and $P \neq 0$, then $\mathrm{tr}P > 0$. Since $x > 0$ implies $g(x) < 0$, we have $g(\mathrm{tr}P) < 0$. Thus $\lambda_{\min}(P) \geq 0 > g(\mathrm{tr}P)$ which implies that $P \in \mathrm{Int}M^g$, since g is continuous. $\square$

COROLLARY 10.3. *The dual subequation $\widetilde{M^g}$ is borderline.*

PROOF. The first part of (10.2) implies that $0 \in \partial M^g = -\partial\widetilde{M^g}$. Combined with the second part of (10.2), this is condition $(1)'$ in Lemma 3.2 for the subequation $F = \widetilde{M^g}$, which proves that $\widetilde{M^g}$ is borderline. $\square$

PROPOSITION 10.4. *The subequation M^g is additive, i.e., $M^g + M^g \subset M^g$, if and only if g is subadditive, i.e., $g(x + y) \leq g(x) + g(y)$.*

PROOF. Use (10.3) and $\mathrm{tr}(A + B) = \mathrm{tr}A + \mathrm{tr}B$. $\square$

If $g(x) \equiv -\delta x$ ($\delta > 0$), then $M^g \equiv \mathcal{P}(\delta)$ is the convex cone subequation discussed in Appendix B. However, there are plenty of other subadditive decreasing functions g.

Suppose g is concave on $[0, a]$ with $g(0) = 0$. Then, as noted in the introduction to [4], the extension of $g(x)$ from $[0, a]$ to $[0, \infty)$ defined by

$$g(x) \;\equiv\; jg(a) + g(x - ja), \qquad ja \leq x \leq (j + 1)a, \qquad j = 1, 2, ... \tag{10.4}$$

is subadditive on $[0, \infty)$ and has the property that $g \geq h$ for any other subadditive function h on $[0, \infty)$ which agrees with g on $[0, a]$. The elementary proof is omitted. Summarizing, we have the following.

Theorem 10.5. *Suppose that $g : [0,\infty) \to \mathbf{R}$ is the extension of a decreasing concave function on $[0,a]$ defined by (10.4) with $g(0) = 0$. Then M^g is a monotonicity subequation (orthogonally invariant), and its dual $\widetilde{M^g}$ is borderline.*

Lemma 10.6. *The dual subequation $\widetilde{M^g}$ is defined by*

$$\widetilde{M^g} \quad : \quad \mathrm{tr}A \ge 0 \quad \text{or} \quad \lambda_{\max}(A) \ \ge \ -g\,(-\mathrm{tr}A) \quad (\text{with} \ \ \mathrm{tr}A \le 0).$$

Proof. Note that

$$\begin{aligned} A \in \widetilde{M^g} \iff -A \notin \mathrm{Int}M^g &\iff \lambda_{\min}(-A) \ \le \ g\,(-\mathrm{tr}A) \quad \text{or} \quad \mathrm{tr}(-A) \le 0 \\ &\iff \lambda_{\max}(A) \ \ge \ -g\,(-\mathrm{tr}A) \quad \text{or} \quad \mathrm{tr}(A) \ge 0 \end{aligned}$$

since $\lambda_{\max}(A) = -\lambda_{\min}(-A)$. □

Proposition 10.7. *The characteristic function f for the dual subequation $\widetilde{M^g}$ on $\mathbf{R}^n$ is $f(\lambda) = g^{-1}(-\lambda) + (n-1)\lambda$ for $\lambda \ge 0$.*

Proof. The increasing radial profile of $\widetilde{M^g}$ is by definition

$$\Lambda \equiv \ \{(\lambda,\mu) : \lambda P_{e^\perp} + \mu P_e \in \widetilde{M^g} \ \ \text{and} \ \ \lambda \ge 0\}.$$

Note that $\mathrm{tr}A = (n-1)\lambda + \mu$ if $A \equiv \lambda P_{e^\perp} + \mu P_e$. If $\lambda \ge 0$ and $A \in \widetilde{M^g}$ with $\mathrm{tr}A \le 0$, then

$$\lambda \ \equiv \ \lambda_{\max} \ \ge \ 0, \quad \mu \ \le \ 0, \qquad \text{and hence} \qquad \lambda \ge -g\,(-(n-1)\lambda - \mu)\,.$$

Set $x \equiv -(n-1)\lambda - \mu \ge 0$ and $y \equiv -\lambda \le 0$. Then $y \le g(x)$ is equivalent to $x \le g^{-1}(y)$ since g is decreasing and $g(0) = 0$. Thus $-(n-1)\lambda - \mu \le g^{-1}(-\lambda)$, or $\mu + g^{-1}(-\lambda) + (n-1)\lambda \ge 0$. Since f is defined by $\mu + f(\lambda) \ge 0$ for such pairs (λ,μ),this completes the proof. □

Example 10.8 (An explicit example where (SC) holds but the subequation is not contained in a uniformly Elliptic subequation). Define $g : [0,a] \to [-b,0]$ via its inverse by

$$g^{-1}(-\lambda) \ \equiv \ \lambda(\alpha - 2\log\lambda) \qquad 0 \le \lambda \le a. \tag{10.5}$$

Here α is a constant chosen first, and then a is chosen small enough so that $h(\lambda) \equiv g^{-1}(-\lambda)$ is strictly increasing on $[0,a]$, and finally we set $-b = g(a)$. Note that $h'(\lambda) = \alpha - 2 - 2\log\lambda$. Also, $h''(\lambda) = -\frac{2}{\lambda} < 0$. Therefore g is concave and strictly decreasing on $[0,a]$ with $g(0) = 0$. Applying Theorem 10.4 we see that

$$M^g \ \ \text{is a monotonicity subequation whose dual } \widetilde{M^g} \text{ is borderline.} \tag{10.6}$$

By Proposition 10.7

$$\text{The dual } \widetilde{M^g} \text{ has characteristic function } f(\lambda) = \lambda(\alpha + n - 1 - 2\log\lambda) \text{ on } [0,a]. \tag{10.7}$$

Recall the subequation $H(\alpha')$ discussed in Example 8.11. If we take $y = \lambda$, $k = 2$, and the α there to be the $\alpha' \equiv \alpha + n - 1$ for (10.7), then the characteristic function $f(\lambda)$ for $\widetilde{M^g}$ is the same as the characteristic function for $H(\alpha')$, for λ small. Since $\int_{0+} \frac{1}{f} = \infty$, as shown there, this proves

Proposition 10.9. *The* (SMP) *holds for this dual subequation* $\widetilde{M^g}$*, and so the* (SC) *holds for* M^g*.*

It is easy to see that $\widetilde{M^g}$ is not contained in a uniformly elliptic subequation since $f(\lambda)/\lambda = \alpha + n - 1 - 2\log\lambda \to \infty$ as $\lambda \to \infty$.

Finally we remark that, as in Example 8.11, if $\beta \equiv e^{\frac{1}{2}(1+\alpha)}$, then the Hopf function

$$\psi(|x|) \;\equiv\; e^{-\beta R^2/2} - e^{-\beta|x|^2/2} \quad \text{is } \widetilde{M^g} \text{ harmonic for } |x| \text{ small.} \tag{10.8}$$

(This function $\psi(|x|)$ is also a harmonic for the subequations $F_f^{\min/2} \subset \widetilde{M^g} \subset F_f^{\min/\max}$ described in Theorem 8.5.)

11 – Another application – product subequations

In this section we apply our main result to study the (SMP) for product subequations. Let $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ and $G \subset \mathrm{Sym}^2(\mathbf{R}^m)$ be invariant pure second-order subequations, and consider the *product subequation* $H \equiv$ “$F \times G$” $\subset \mathrm{Sym}^2(\mathbf{R}^{n+m})$ defined, for coordinates $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, by requiring that $D_x^2 u \in F$ and $D_y^2 u \in G$. In other words, u is separately F-subharmonic in x and G-subharmonic in y.

One easily checks that if either F or G is stable, then H is stable. On the other hand, if either F or G has a counterexample to the (SMP), so does H (take the same counterexample considered as a function of all the variables). For the remaining case we have the following. The proofs are omitted.

Theorem 11.1. *Let F and G be invariant borderline subequations for which the* (SMP) *holds. Let f and g denote their respective characteristic functions. Suppose one of these, say g, satisfies*

$$\frac{g(y) - g(x)}{y - x} \quad \text{is bounded for} \quad 0 < x < y \quad \text{small.} \tag{11.1}$$

Then the (SMP) *holds for the product subequation $H \equiv$ “$F \times G$”.*

Outline of Proof. The first step is the following.

Proposition 11.2. *Let $\overline{h}$ and $\underline{h}$ be the upper and lower characteristic functions of H. Then*

$$\overline{h}(\lambda) \;=\; f(\lambda) + g(\lambda) + \lambda \qquad \text{and} \qquad \underline{h}(\lambda) \;=\; \min\{f(\lambda), g(\lambda)\}.$$

The next step is that

$$\int_{0^+} \frac{1}{f} = \infty \quad \text{and} \quad \int_{0^+} \frac{1}{g} = \infty \quad \text{and} \quad (11.1) \quad \Rightarrow \quad \int_{0^+} \frac{1}{f+g} \ = \ \infty$$
$$\text{and therefore} \quad \Rightarrow \quad \int_{0^+} \frac{1}{f+g+\lambda} \ = \ \infty.$$

Theorem 11.1 now follows from Theorem A′. □

REMARK 11.3. There exist functions $f, g > 0$ with $\int_{0^+} \frac{1}{f} = \int_{0^+} \frac{1}{g} = \infty$ but $\int_{0^+} \frac{1}{f+g} < \infty$.

Appendix

A – Radial subharmonics

Since our characterization of radial subharmonics is useful for many purposes, it is separated out in this appendix. Recall the characteristic lower function $\underline{f}$ associated with a subequation F and the radial subequation $R_{\underline{f}}$ defined by

$$\psi'' + \underline{f}\left(\frac{\psi'}{t}\right) \ \geq \ 0 \qquad \text{on } 0 < t < \infty.$$

In the following we drop the bar, letting f denote $\underline{f}$.

THEOREM A.1 (Radial subharmonics). *The function $u(x) \equiv \psi(|x|)$ is F-subharmonic on an annular region in $\mathbf{R}^n$ if and only if $\psi(t)$ is R_f-subharmonic on the corresponding sub-interval of* $(0, \infty)$.

PROOF. ($\Rightarrow$): Suppose $u(x) \equiv \psi(|x|)$ is F-subharmonic. If $\varphi(t)$ is a test function for $\psi(t)$ at t_0, then $\varphi(|x|)$ is a test function for $\psi(|x|)$ at any point on the t_0-sphere in $\mathbf{R}^n$. Therefore $D^2_{x_0}\varphi \in F$. Applying the formula (Lemma 4.1) for $D^2_{x_0}\varphi$ in terms of $\varphi'(t_0)$ and $\varphi''(t_0)$, the equivalence (4.3), and the definition of $(R_F)_{t_0}$, we have $J^2_{t_0}\varphi \in R_F$. This proves that $\psi(t)$ is R_F-subharmonic.

($\Leftarrow$): Suppose that $\psi(t)$ is R_F-subharmonic. We must show that $u(x) \equiv \psi(|x|)$ is F-subharmonic. That is, given a test function $\varphi(x)$ for $u(x)$ at a point x_0, we must show that $D^2_{x_0}\varphi \in F$.

Suppose that there exists a *smooth* function $\overline{\psi}(t)$, defined near $t_0 = |x_0|$, such that $\overline{\varphi}(x) \equiv \overline{\psi}(|x|)$ satisfies

$$u(x) \ \leq \ \overline{\varphi}(x) \ \leq \varphi(x) \tag{A.1}$$

near x_0. Then $\overline{\psi}(t)$ is a test function for $\psi(t)$ at t_0. Hence, the 2-jet of $\overline{\psi}$ at t_0 belongs to R_F. By Lemma 4.1 and the discussion above, this implies that $D^2_{x_0}\overline{\varphi} \in F$. The inequality $\overline{\varphi}(x) \leq \varphi(x)$ (with equality at x_0) implies that $D^2_{x_0}\varphi = D^2_{x_0}\overline{\varphi} + P$ for some $P \geq 0$, which proves that $D^2_{x_0}\varphi \in F$ as desired.

To complete this argument by finding $\overline{\psi}(t)$ there is some flexibility given by [12, Lemma 2.4] so that not all test functions $\varphi(x)$ need be considered. First we may choose new coordinates $z = (t, y)$ near x_0 so that $t \equiv |x|$. (Thus t = constant defines the sphere of radius t near x_0.) Furthermore, we may assume that $\varphi(z)$ is a polynomial of degree ≤ 2 in $z = (t, y)$ and that it is a *strict* local test function, *i.e.*, $u(z) < \varphi(z)$ for $z \neq z_0$. Now Lemma A.2 below ensures the existence of $\overline{\varphi}(x) = \overline{\psi}(|x|)$ satisfying (A.1). $\square$

Let $z = (t, y)$ denote standard coordinates on $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^\ell$. Fix a point $z_0 = (t_0, y_0)$ and let $u(t)$ be an upper semi-continuous function (of t alone) and $\varphi(z)$ a C^2-function, both defined in a neighborhood of z_0.

Lemma A.2. *Suppose $u(t) < \varphi(z)$ for $z \neq z_0$ with equality at z_0. If $\varphi(z)$ is a polynomial of degree ≤ 2, then there exists a polynomial $\overline{\varphi}(t)$ of degree ≤ 2 with*

$$u(t) \ \leq\ \overline{\varphi}(t) \ \leq\ \varphi(z) \qquad \text{near} \ \ z_0. \tag{A.2}$$

Proof. We may assume $z_0 = 0$ and $u(0) = \varphi(0) = 0$. Then

$$\varphi(z) \ = \ \langle p, t\rangle + \langle q, y\rangle + \langle At, t\rangle + 2\langle Bt, y\rangle + \langle Cy, y\rangle.$$

We assume $u(t) < \varphi(t, y)$ for $|t| \leq \epsilon$ and $|y| \leq \delta$ with $(t, y) \neq (0, 0)$.

Setting $t = 0$, we have $0 = u(0) < \langle q, y\rangle + \langle Cy, y\rangle$ for $y \neq 0$ sufficiently small. Therefore, $q = 0$ and $C > 0$ (positive definite). Now define

$$\overline{\varphi}(t) \ \equiv\ \langle p, t\rangle + \langle (A - B^t C^{-1} B)t, t\rangle. \tag{A.3}$$

The inequalities in (A.2) follow from the fact that for t sufficiently small,

$$\overline{\varphi}(t) \ = \ \inf_{|y|\leq\delta} \varphi(z) \ = \ \langle p, t\rangle + \langle At, t\rangle + \inf_{|y|\leq\delta}\{2\langle Bt, y\rangle + \langle Cy, y\rangle\}. \tag{A.4}$$

To prove (A.4) fix t and consider the function $2\langle Bt, y\rangle + \langle Cy, y\rangle$. Since $C > 0$, it has a unique minimum point at the critical point $y = -C^{-1}Bt$. The minimum value is $-\langle B^t C^{-1} Bt, t\rangle$. If t is sufficiently small, the critical point y satisfies $|y| < \delta$, which proves (A.4). $\square$

B – Uniform ellipticity

This is a geometric discussion of uniform ellipticity. A family of convex cone subequations $\{M_\delta\}$ is said to be a *fundamental neighborhood system* for $\mathcal{P}$ if given any conical neighborhood G of $\mathcal{P}$ (this means that $\mathcal{P}-\{0\} \subset \mathrm{Int}G$ and G is a cone), there exists δ with $M_\delta \subset G$. Given such a family $\{M_\delta\}$, a subequation F is **uniformly elliptic** if one of the M_δ is a monotonicity subequation for F. That is,

$$F + M_\delta \ \subset \ F \qquad \text{for some} \ \ \delta. \tag{B.1}$$

This definition is easily seen to be independent of the choice of the neighborhood system $\{M_\delta\}$ for $\mathcal{P}$. (The monotonicity condition (B.1) can always be rephrased classically, in terms of the operator defining M_δ, as two inequalities – see, for example, $(4.5.1)'$ in [14]).

The standard choice made in the literature consists of the **Pucci cones**

$$\mathcal{P}_{\lambda,\Lambda} \ \equiv \ \{A : \lambda \mathrm{tr}\, A^+ + \Lambda \mathrm{tr}\, A^- \geq 0\}$$

with $0 < \lambda < \Lambda$, where $A = A^+ + A^-$ is the decomposition of A into positive and negative parts. Another good choice is the **δ-uniformly elliptic regularization** $\mathcal{P}(\delta)$ of $\mathcal{P}$

$$\mathcal{P}(\delta) \ \equiv \ \{A : A + \delta(\mathrm{tr}\, A)I \geq 0\} \qquad (\delta > 0).$$

Both $\mathcal{P}_{\lambda,\Lambda}$ and $\mathcal{P}(\delta)$ are convex cone subequations as required. See [13, Section 4.5] for more details regarding $\mathcal{P}_{\lambda,\Lambda}$ and $\mathcal{P}(\delta)$ (The Riesz characteristics are computed in Example 6.2.5.)

Since there is a largest monotonicity subequation M_F for F, uniform ellipticity can be defined equivalently as

$$M_F \ \ \text{contains a convex conical neighborhood of } \mathcal{P}, \ \ \text{or as} \tag{B.1$'$}$$

$$\widetilde{M}_F \ \subset \ \widetilde{\mathcal{P}(\delta)} \quad \text{for some} \ \ \delta > 0. \tag{B.1$''$}$$

Lemma B.1. *If F is normal, i.e., M_F is a cone (and hence a convex cone), then F is uniformly elliptic $\iff$ M_F is a conical neighborhood of $\mathcal{P}$ $\iff$ $P_e \in \mathrm{Int}M_F$ for all $e \neq 0$ $\iff$ $-P_e \notin \widetilde{M}_F$ for all $e \neq 0$ $\iff$ $\widetilde{M}_F$ is borderline.*

The next remark is to be applied to $F = M_G$ where G is normal.

Remark B.2 (Cone subequations and the Riesz characteristic). For simplicity suppose that $f = \underline{f} = \overline{f}$ is the characteristic function for a cone subequation F. Then $f(t\lambda) = tf(\lambda)$ for $t > 0$, and hence the characteristic function reduces to two numerical invariants

$$\alpha \ \equiv \ f(1) \qquad \text{and} \qquad \alpha^* \ \equiv \ -f(-1), \qquad 0 \leq \alpha, \alpha^* \leq \infty \tag{B.2}$$

where we have

$$f(\lambda) \ = \ \alpha\lambda \quad \text{for } \lambda > 0 \qquad \text{and} \qquad f(\lambda) \ = \ \alpha^*\lambda \quad \text{for } \lambda < 0. \tag{B.3}$$

The radial profile Λ is defined by

$$\mu + \alpha\lambda \ \geq \ 0 \quad \text{if } \lambda \geq 0 \qquad \text{and} \qquad \lambda + \alpha^*\mu \ \geq \ 0 \ \text{ if } \lambda \leq 0. \tag{B.4}$$

Note that $\alpha = \infty \iff P_{e^\perp} - \mu P_e \in F$ for all $\mu \iff -P_e \in F \iff F$ is not borderline. That is,

$$F \ \text{ satisfies the (SMP)} \qquad \iff \qquad \alpha \equiv \alpha_F \ < \ \infty. \tag{B.5}$$

The invariant $p_F \equiv \alpha_F + 1$ is called the **Riesz characteristic of** F because of its connection with Riesz kernels. See [14], [15] for applications, examples and a fuller discussion, where it is proved, in particular, that $\alpha\alpha^* \geq 1$.

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Indirizzi degli Autori:
F. Reese Harvey – Department of Mathematics – Rice University – Houston – TX 77251-1892, USA
Email address: harvey@rice.edu

H. Blaine Lawson Jr. – Department of Mathematics – Stony Brook University – Stony Brook – NY 11794-3651, USA
Email address: blaine@math.sunysb.edu