Rend. Mat. Appl. (7).
Volume 38, (2017), 1 – 169

RENDICONTI DI MATEMATICA
E DELLE SUE APPLICAZIONI

Lectures on algebraic stacks

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Abstract. *These lectures give a detailed and almost self-contained introduction to algebraic stacks. A great part of the paper is devoted to preliminary technical topics, both from category theory (like Grothendieck topologies, fibred categories and stacks) and algebraic geometry (like faithfully flat descent). All this machinery is finally used to present the definition and some basic properties first of algebraic spaces and then of algebraic stacks.*

Contents

2010 Mathematics Subject Classification: 14A20, 14B25, 14D23, 18D30, 18F10.
Keywords: Grothendieck topologies, fibred categories, stacks, faithfully flat descent, algebraic spaces, algebraic stacks.

1 Introduction

These notes are based on a Ph.D. course taught at the Università di Roma "La Sapienza" during the academic year 2002/2003. The aim of the course was to give an introduction to algebraic stacks by trying to explain all the foundational material (both from category theory and algebraic geometry) which is at the base of the subject. It turned out that the about 30 hours of lectures were barely sufficient to the purpose, as the definition of algebraic stack was given during the last one! Moreover, many details were necessarily skipped, and these written notes (a draft of which was already available in 2004) try to fill those gaps, providing a more organic and neat presentation at the same time. The reader must be warned that, although a little expanded, this paper essentially covers the same topics of the original lectures. This means that, beyond the definition, only some of the first properties of algebraic stacks are treated here. So this is certainly not the right place for those looking for a comprehensive reference or one that gets to the point quickly.

Algebraic stacks were introduced in the 1960's: even if not yet defined, in practice they were already used in [18]. The term first appeared in [4] to denote what are now called Deligne-Mumford stacks, and in the more general current meaning in [1]. Despite their importance in algebraic geometry, in particular for moduli problems, for a long time they had a reputation of an esoteric and almost intractable subject. This was at least in part justified by the fact that knowledge of (algebraic) stacks requires some familiarity with a number of technical topics (like Grothendieck topologies, fibred categories, descent theory, ...), for which accessible references were hard to find. In fact, this was still true at the time of the course, even if very good introductory articles to algebraic stacks (like [5], [6] and the appendix of [24]) and the important but difficult book [15] were already available. Indeed, one of the ambitions of these notes was (and still is) to provide a detailed and reasonably self-contained exposition of the preliminary material needed to make [15] readable. It must be said that since then, in parallel with a more widespread use of algebraic stacks, other important references have appeared, some of which take good care also of the foundational aspects of the theory. Among them, the following certainly deserve a citation: [25] (which actually does not even define algebraic stacks), [19], the excellent recent book [20] and, of course, the encyclopedic ongoing project [22] (already over 6000 pages long). Moreover it should be mentioned that, with the development of derived algebraic geometry, other more abstract and general objects entered the scene, like higher[1] and derived stacks, which are not considered here (see, for instance, [23] for a survey on them).

Also as a guide to the reading, it can be useful to describe briefly the contents of the paper. Section 2 collects, without any proof or reference, what the reader is assumed to know. It should not represent a problem for those with a basic knowledge of category theory and algebraic geometry. However, also the expert is advised to have a look at it, since the notation and results introduced here are freely used without further mention in the following sections. In the rest of the paper, every time a full proof is not provided, either a precise reference is given (this happens for some algebro-geometric results) or it is left to the reader (this is the case of many categorical statements, requiring a possibly long and boring, but really not difficult check). Section 3 deals with various aspects of scheme theory (often not adequately treated in standard references like [11]), which are needed later mainly for faithfully flat descent. The categorical part of the paper begins in Section 4, and those who prefer can start reading here without any serious trouble, as in this part schemes appear only in the examples. First presheaves on a category (i.e., contravariant functors to the category of sets) are studied. Then, after introducing (Grothendieck pre)topologies, the focus is restricted to sheaves, namely presheaves satisfying natural gluing conditions with respect to a fixed topology. Equivalence relations and their quotients in the category of sheaves are also analyzed. Beyond being needed later, in particular for algebraic spaces, this section is pedagogically important because it represents a simplified model of the contents of the following two sections. Indeed, Section 5 deals with a natural generalization of the notion of presheaf, which can be defined either as a fibred category, or, equivalently, as a contravariant (lax) 2-functor to the 2-

[1]There is a notion of n-stack for every n and even of ∞-stack. Ordinary stacks (as in this paper) are just 1-stacks in this wider sense.

category of categories. What was proved for presheaves is then extended to fibred categories in a conceptually clear way (hopefully), but with inevitable technical complications coming from the 2-categorical setting. Similar considerations hold for Section 6, where the theory of sheaves is generalized to the one of stacks (of categories). In particular, groupoids in the category of sheaves are investigated, together with their quotients, which are stacks of groupoids. Section 7 returns to algebraic geometry, presenting the main elements of faithfully flat descent theory: from now on the base category is taken to be the category of schemes (or some of its variants) and it is proved that every scheme is a sheaf and quasi-coherent sheaves form a stack for a suitable faithfully flat topology. These two important results allow to deduce descent or local properties for many classes of morphisms of schemes. Section 8 is devoted to algebraic spaces, which, loosely speaking, are spaces (namely sheaves for the étale topology) which can be approximated by a scheme via an étale cover. After proving that algebraic spaces can be characterized as quotient spaces of étale equivalence relations in the category of schemes, the first steps are taken to illustrate how most aspects of scheme theory can be extended to algebraic spaces (a more thorough treatment can be found in [14] or [20]). The notion of algebraic space is actually necessary in order to give the precise definition of algebraic stack, but, similarly as before, the usefulness of this section is mainly due to the fact that it anticipates, in a simpler form, many features of the theory of algebraic stacks. It is also worth pointing out that this section (with the exception of a couple of minor points) can be read without any prior knowledge of fibred categories and stacks, provided one is willing to assume without proof some of the needed statements from descent theory. The only caveat in doing so is that one has to adopt as working definition of property which "satisfies effective descent" the characterization of Corollary 6.33 instead of Definition 6.32. Eventually Section 9 introduces algebraic stacks: they are stacks of groupoids (always for the étale topology) which can be approximated by an algebraic space via a smooth (étale in the Deligne-Mumford case) cover. In analogy with the previous section, it is proved that algebraic stacks can be characterized as quotient stacks of smooth groupoids in the category of algebraic spaces. Then it is briefly explained how some properties of schemes or algebraic spaces can be extended to algebraic stacks. Finally, the contents of Appendix A are used in the paper, whereas Appendix B can be safely ignored by the uninterested reader.

2 Prerequisites

2.1 Category theory

We will systematically ignore all logical problems of category theory, usually solved (for instance in [2]) with the use of notions like that of universe.

If $\mathbf{C}$ is a category, $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ will denote, respectively, the set of objects and the set of morphisms of $\mathbf{C}$. If U is an object of $\mathbf{C}$, by abuse of notation we will usually write $U \in \mathbf{C}$ instead of $U \in \mathrm{Ob}(\mathbf{C})$, and id_U (or simply id, if there can be no doubt about U) will be the identity morphism of U. Given $U, V \in \mathbf{C}$, the elements of $\mathrm{Mor}(\mathbf{C})$ with source U and target V will be denoted by $\mathrm{Hom}_{\mathbf{C}}(U, V)$ (or simply by $\mathrm{Hom}(U, V)$), and its subset consisting of isomorphisms by $\mathrm{Isom}_{\mathbf{C}}(U, V)$; we will usually write $\mathrm{Aut}_{\mathbf{C}}(U)$ instead of $\mathrm{Isom}_{\mathbf{C}}(U, U)$ (notice that it is a group under composition, the group of automorphisms of U). The composition of two morphisms $f \colon U \to V$ and $g \colon V \to W$ will be denoted by $g \circ f \colon U \to W$. The symbol $f \colon U \xrightarrow{\sim} V$ will be used to indicate that f is an isomorphism; we will write $U \cong V$ if U and V are isomorphic (i.e., $\mathrm{Isom}_{\mathbf{C}}(U, V) \neq \emptyset$).

Set, **Grp**, **Ab** and **Rng** will be, respectively, the categories of sets, of groups, of abelian groups and of rings (always assumed to be commutative with unit, with morphisms preserving the unit).

Definition 2.1. A morphism $f \colon U \to V$ in a category $\mathbf{C}$ is a *monomorphism* (respectively an *epimorphism*) if the map $\mathrm{Hom}_{\mathbf{C}}(W, U) \xrightarrow{f\circ} \mathrm{Hom}_{\mathbf{C}}(W, V)$ (respectively $\mathrm{Hom}_{\mathbf{C}}(V, W) \xrightarrow{\circ f} \mathrm{Hom}_{\mathbf{C}}(U, W)$) is injective for every $W \in \mathbf{C}$.

We will write $f \colon U \hookrightarrow V$ (respectively $f \colon U \twoheadrightarrow V$) to mean that f is a monomorphism (respectively an epimorphism).

Example 2.2. A morphism in **Set**, **Grp** or **Ab** is a monomorphism (respectively an epimorphism) if and only if it is injective (respectively surjective). Also in **Rng** it is true that a morphism is a monomorphism if and only if it is injective, but, while every surjective morphism of rings is an epimorphism, the converse is not true (consider, for instance, $\mathbb{Z} \to \mathbb{Q}$).

Definition 2.3. A morphism $f \colon U \to V$ in a category is *left* (respectively *right*) *invertible* if there exists a morphism $g \colon V \to U$ such that $g \circ f = \mathrm{id}_U$ (respectively $f \circ g = \mathrm{id}_V$).

Remark 2.4. It is easy to see that a morphism is an isomorphism if and only if it is left and right invertible. It is also clear that every left (respectively right) invertible morphism is a monomorphism (respectively an epimorphism), but the converse is not true in general ($\mathbb{Z} \to \mathbb{Q}$ is a monomorphism and an epimorphism in **Rng**, but it is neither left nor right invertible).

Given a category $\mathbf{C}$, we will denote by $\mathbf{C}^\circ$ the opposite category (it has the same objects as $\mathbf{C}$ and, if $U, V \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}^\circ}(U, V) := \mathrm{Hom}_{\mathbf{C}}(V, U)$; the composition of morphisms $g \circ f$ in $\mathbf{C}^\circ$ is of course given by $f \circ g$ in $\mathbf{C}$).

Remark 2.5. Many categorical properties admit "dual versions", obtained by passing to the opposite category. For instance, a morphism of $\mathbf{C}$ is a monomorphism if and only if, as a morphism of $\mathbf{C}^\circ$, it is an epimorphism.

Definition 2.6. Let $f_1, f_2 \colon U \to V$ be two morphisms with same source and target in a category $\mathbf{C}$. A morphism $g \colon U' \to U$ is a *kernel* or *equalizer* of $U \underset{f_2}{\overset{f_1}{\rightrightarrows}} V$ if for every $W \in \mathbf{C}$ the sequence of sets

$$\mathrm{Hom}_{\mathbf{C}}(W,U') \xrightarrow{g\circ} \mathrm{Hom}_{\mathbf{C}}(W,U) \overset{f_1\circ}{\underset{f_2\circ}{\rightrightarrows}} \mathrm{Hom}_{\mathbf{C}}(W,V)$$

is exact (by definition, this means that the map on the left is injective and that its image consists precisely of the elements of the middle set which are mapped to the same element by the two maps on the right).

Remark 2.7. It is clear from the definition that if a kernel $g\colon U' \to U$ of $U \overset{f_1}{\underset{f_2}{\rightrightarrows}} V$ exists, then it is unique up to isomorphism, and it is a monomorphism; we will write $g \cong \ker(U \overset{f_1}{\underset{f_2}{\rightrightarrows}} V)$ (or $U' \cong \ker(U \overset{f_1}{\underset{f_2}{\rightrightarrows}} V)$, by abuse of notation).

Given objects $\{U_i\}_{i\in I}$ in a category $\mathbf{C}$, their *product* is an object $U = \prod_{i\in I} U_i$ together with *projection* morphisms $pr_i\colon U \to U_i$ for $i \in I$ such that the following universal property is satisfied: given morphisms $f_i\colon V \to U_i$ for $i \in I$, there exists a unique $f\colon V \to U$ such that $pr_i \circ f = f_i$ for every $i \in I$; such a morphism f will be usually denoted by $(f_i)_{i\in I}$. Again, it is clear that if such a product exists, then it is unique up to isomorphism (every object $V \cong U$ satisfies the same universal property, with projections obtained composing the pr_i with the given isomorphism). We will say that $\mathbf{C}$ has (finite) products if all (finite, i.e. those indexed by a finite set) products exist in $\mathbf{C}$. The product indexed by the empty set (if it exists) is a *terminal object*: by definition, it is an object $*$ such that for every $U \in \mathbf{C}$ there is a unique morphism from U to $*$ (which will be denoted by $*_U$). Notice that $\mathbf{C}$ has finite products if and only if it has a terminal object and for every couple of objects $U, V \in \mathbf{C}$, their product (denoted, of course, by $U \times V$) exists.

Given two morphisms $f\colon U \to T$ and $g\colon V \to T$ in $\mathbf{C}$, their *fibred product* is an object $U\,{}_f{\times}_g\,V$ (often denoted by $U \times_T V$, if there can be no doubt about f and g), together with *projection* morphisms $pr_1\colon U\,{}_f{\times}_g\,V \to U$ and $pr_2\colon U\,{}_f{\times}_g\,V \to V$ such that $f \circ pr_1 = g \circ pr_2$ and the following universal property is satisfied: given a commutative diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "k"'] & U \arrow[d, "f"] \\ V \arrow[r, "g"'] & T \end{tikzcd} \tag{2.1}$$

there exists a unique $s\colon W \to U\,{}_f{\times}_g\,V$ (usually denoted by (h,k)) such that the diagram

$$\begin{tikzcd}
W \arrow[dr, "s"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& U {}_f\times_g V \arrow[r, "pr_1"'] \arrow[d, "pr_2"] & U \arrow[d, "f"] \\
& V \arrow[r, "g"'] & T
\end{tikzcd}$$

is also commutative. In case s is an isomorphism, we will say that the diagram (2.1) is *cartesian*; this will be usually expressed with the following standard notation:

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\
V \arrow[r, "g"'] & T.
\end{tikzcd}$$

Notice that, if $\mathbf{C}$ has a terminal object $*$, then $U \times V \cong U \times_* V$; therefore, a category with fibred products and terminal object has also finite products.

By passing to $\mathbf{C}^\circ$, from the notions of kernel or equalizer, (fibred) product, terminal object and cartesian diagram, one gets the notions of *cokernel* or *coequalizer* (denoted by coker), *(fibred) coproduct* (for which the symbol $\coprod$ will be used in place of $\prod$ or $\times$),[2] *initial* object and *cocartesian* diagram: so, for instance the diagram (2.1) is cocartesian in $\mathbf{C}$ if and only if the corresponding diagram is cartesian in $\mathbf{C}^\circ$ (and this happens if and only if the induced morphism $U \coprod_W V \to T$ is an isomorphism).

Example 2.8. The category **Set** has all (fibred) products and (fibred) coproducts, as well as kernels and cokernels. For instance, products are the usual ones (in particular, a terminal object is a set with one element, usually denoted by $\{*\}$), coproducts are disjoint unions and, given maps $f\colon X \to Z$ and $g\colon Y \to Z$, $X {}_f\times_g Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ (with the obvious projections).

Remark 2.9. (Fibred) products and coproducts and kernels and cokernels are particular instances of a general notion of limit (the interested reader can see Section B.1), which includes also the well known case of (direct and inverse) limits over filtered[3] sets, which will be used occasionally and with which the reader is assumed to have some familiarity.

[2] Coproducts in an additive category (like **Ab**) are usually called *direct sums* and are denoted by $\oplus$

[3] A *filtered* or (*directed*) set is a preordered set $(X, \leq)$ (by definition, this means that $\leq$ is a reflexive and transitive relation on X) with the property that for all $x, y \in X$ there exists $z \in X$ such that $x \leq z$ and $y \leq z$.

If $f\colon U \to V$ is a morphism (in some category) such that $U\,{}_f\!\times_f U$ exists, then the diagonal morphism $(\mathrm{id}_U, \mathrm{id}_U)\colon U \to U\,{}_f\!\times_f U$ will be usually denoted by $\Delta_f\colon U \to U \times_V U$.

Similarly, if U is an object such that $U \times U$ exists, the diagonal morphism $(\mathrm{id}_U, \mathrm{id}_U)\colon U \to U \times U$ will be denoted by $\Delta_U\colon U \to U \times U$. Given morphisms $f\colon U \to V$ and $f'\colon U' \to V'$, we will denote by $f \times f'\colon U \times U' \to V \times V'$ the morphism $(f \circ pr_1, f' \circ pr_2)$. The same notation will be used for morphisms between two fibred products.

Definition 2.10. A property P of morphisms of $\mathbf{C}$ is *stable under base change* if the following condition is satisfied for every morphism $f\colon U \to V$ of $\mathbf{C}$ which satisfies P: for every morphism $g\colon V' \to V$ of $\mathbf{C}$ there exists a cartesian diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\ V' \arrow[r, "g"'] & V \end{tikzcd}$$

and in every such diagram f' satisfies P, too.

Remark 2.11. If P is stable under base change and $h\colon \tilde{U} \to U$ is an isomorphism, then it follows from the definition that $f\colon U \to V$ satisfies P if and only if $f \circ h$ does.

Example 2.12. It is easy to see that in a category with fibred products the property of being a monomorphism is stable under base change, whereas the property of being an epimorphism is not, in general.

When P is a property (not stable under base change) of morphisms of $\mathbf{C}$ and $f\colon U \to V$ is a morphism of $\mathbf{C}$ such that for every cartesian diagram as in the above definition the morphism $f'\colon U' \to V'$ satisfies P, we will say that f satisfies P *universally* (so for instance, we can speak of universally closed continuous maps).

If $\mathbf{C}$ and $\mathbf{D}$ are two categories, $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ will be the category having as objects functors from $\mathbf{C}$ to $\mathbf{D}$ and as morphisms natural transformations of functors.

Definition 2.13. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *faithful* (respectively *full*) if the induced map $\mathrm{Hom}_{\mathbf{C}}(U, V) \to \mathrm{Hom}_{\mathbf{D}}(F(U), F(V))$ is injective (respectively surjective) for all $U, V \in \mathbf{C}$. If F is faithful and full, it is usually said to be *fully faithful.* F is *essentially surjective* if every object V of $\mathbf{D}$ is isomorphic to $F(U)$ for some object U of $\mathbf{C}$.

Remark 2.14. One could define a functor $F\colon \mathbf{C} \to \mathbf{D}$ to be essentially injective if, given objects U and U' of $\mathbf{C}$, $F(U) \cong F(U')$ implies $U \cong U'$. Notice, however, that a fully faithful functor is always essentially injective.

Definition 2.15. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories* if there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ such that $G \circ F \cong \mathrm{id}_{\mathbf{C}}$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{C})$ and $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ in $\mathbf{Fun}(\mathbf{D}, \mathbf{D})$; such a functor G is called a *quasi-inverse* of F.

Using choice axiom it is easy to prove the following important result.

Proposition 2.16. *A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Definition 2.17. A subcategory $\mathbf{C}'$ of a category $\mathbf{C}$ is *full* if $\mathrm{Hom}_{\mathbf{C}'}(U,V) = \mathrm{Hom}_{\mathbf{C}}(U,V)$ for all $U,V \in \mathbf{C}'$. $\mathbf{C}'$ is a *strictly full* subcategory of $\mathbf{C}$ if it is full and every object of $\mathbf{C}$ isomorphic to an object of $\mathbf{C}'$ is in $\mathbf{C}'$.

If S is an object of a category $\mathbf{C}$, then $\mathbf{C}_{/S}$ will denote the category whose objects are the morphisms of $\mathbf{C}$ having target S and whose morphisms are defined as follows: if $f\colon U \to S$ and $g\colon V \to S$ are two objects of $\mathbf{C}_{/S}$, then

$$\mathrm{Hom}_{\mathbf{C}_{/S}}(f,g) := \{h \in \mathrm{Hom}_{\mathbf{C}}(U,V) \mid g \circ h = f\}$$

(composition of morphisms is defined as in $\mathbf{C}$); clearly id_S is a terminal object of $\mathbf{C}_{/S}$. More generally, if $\mathbf{C}'$ is a subcategory of $\mathbf{C}$, even if S is not an object of $\mathbf{C}'$, we can still define the category $\mathbf{C}'_{/S}$ (having as objects the morphisms of $\mathbf{C}$ with source an object of $\mathbf{C}'$ and with target S, and as morphisms the morphisms of $\mathbf{C}_{/S}$ which are in $\mathbf{C}'$). Notice that $\mathbf{C}'_{/S}$ is a subcategory of $\mathbf{C}_{/S}$, and that it is full if $\mathbf{C}'$ is a full subcategory of $\mathbf{C}$. There is a natural functor $\mathbf{C}'_{/S} \to \mathbf{C}'$, which sends an object of $\mathbf{C}'_{/S}$ (which is a morphism of $\mathbf{C}$) to its source and which is the identity on morphisms; this functor is obviously faithful, and it is an equivalence if S is a terminal object of $\mathbf{C}$ or $\mathbf{C}'$. Observe also that, if $\mathbf{C}'$ has fibred products, the same is true for $\mathbf{C}'_{/S}$, and $\mathbf{C}'_{/S} \to \mathbf{C}'$ preserves fibred products.

Definition 2.18. Let $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ be two functors. Then F is a *left adjoint* of G and G is a *right adjoint* of F if for every $U \in \mathbf{C}$ and every $V \in \mathbf{D}$ there is a natural bijection $\mathrm{Hom}_{\mathbf{D}}(F(U),V) \cong \mathrm{Hom}_{\mathbf{C}}(U,G(V))$ (i.e., there must be an isomorphism $\mathrm{Hom}_{\mathbf{D}}(F(-),-) \cong \mathrm{Hom}_{\mathbf{C}}(-,G(-))$ of functors $\mathbf{C}^\circ \times \mathbf{D} \to \mathbf{Set}$).

Remark 2.19. It is plain that a (left or right) adjoint of a functor, if it exists, is unique up to isomorphism. Moreover, if the functor $F\colon \mathbf{C} \to \mathbf{D}$ is left adjoint of $G\colon \mathbf{D} \to \mathbf{C}$, then for every $U \in \mathbf{C}$ the images of $\mathrm{id}_{F(U)}$ under the natural isomorphisms $\mathrm{Hom}_{\mathbf{D}}(F(U),F(U)) \cong \mathrm{Hom}_{\mathbf{C}}(U,G(F(U)))$ define a natural transformation $\mathrm{id}_{\mathbf{C}} \to G \circ F$; similarly, there is an induced natural transformation $F \circ G \to \mathrm{id}_{\mathbf{D}}$.

Every set X naturally determines a category, having as objects the elements of X and as morphisms only the identities. Conversely, every category with only the identities as morphisms comes from a set in this way; therefore, by abuse of notation, we will call such a category a set. Notice that a functor between two sets is just a usual map of sets.

An equivalence relation on a set X is a subset R of $X \times X$ satisfying the following well known properties: $(x,x) \in R$ for every $x \in X$ (reflexive); if $(x,y) \in R$, then $(y,x) \in R$ (symmetric); if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

(transitive). To an equivalence relation $R \subseteq X \times X$ we can associate a category $[X, R]$, with $\mathrm{Ob}([X, R]) := X$ and $\mathrm{Mor}([X, R]) := R$; more precisely, $(x, y) \in R$ is viewed as a morphism from y to x, and the composition $(x, y) \circ (y, z)$ is defined to be (x, z). Notice that the reflexive and transitive properties of the relation imply, respectively, that every object of $[X, R]$ has an identity and that the composition of morphisms is well defined, so that $[X, R]$ is indeed a category. Moreover, (since R is a subset of $X \times X$) there is at most one morphism between two objects of $[X, R]$, whereas the symmetric property implies that every morphism is an isomorphism. Conversely, it is easy to see that every category satisfying these two additional properties is induced by an equivalence relation; again, we will say freely that such a category is an equivalence relation.

If $R \subseteq X \times X$ is an equivalence relation, we will denote by X/R the set of equivalence classes in X under R. It is clear that the natural functor $[X, R] \to X/R$ (defined by the natural map $X \to X/R$ on objects, and in the unique possible way on morphisms) is fully faithful and essentially surjective (hence an equivalence of categories). In fact, it is immediate to see that equivalence relations can be characterized as follows.

Proposition 2.20. *A category is an equivalence relation if and only if it is equivalent to a set.*

Definition 2.21. A *groupoid* is a category in which every morphism is an isomorphism.

Of course, every equivalence relation (in particular, every set) is a groupoid. Moreover, it is clear that a groupoid $\mathbf{C}$ is an equivalence relation if and only if there is at most one (iso)morphism between any two objects of $\mathbf{C}$ (which is true if and only if $\mathrm{Aut}_{\mathbf{C}}(U) = \{\mathrm{id}_U\}$ for every $U \in \mathbf{C}$).

Remark 2.22. A category equivalent to a groupoid is a groupoid. The analogous statement is true also for equivalence relations (but not for sets, of course).

A groupoid with only one object is completely described by the group of automorphisms of the object, and conversely, every group determines a groupoid with one object having that group as group of automorphisms; notice also that a functor between two groups (considered as groupoids) is just given by an ordinary group homomorphism.

Remark 2.23. It is easy to prove that every groupoid is equivalent to a groupoid $\mathbf{C}$ with the property that $\mathrm{Hom}_{\mathbf{C}}(U, V) = \emptyset$ if $U \neq V$. Therefore, giving a groupoid up to equivalence amounts to giving a collection of groups (the automorphism groups of every object in such a groupoid).

Example 2.24. To every action (on the right) $\varrho \colon X \times G \to X$ of a group G on a set X (i.e., ϱ is a map such that $\varrho(\varrho(x, g), h) = \varrho(x, gh)$ and $\varrho(x, \mathrm{id}) = x$ for every $x \in X$ and for all $g, h \in G$) we can naturally associate a groupoid $\mathbf{C}(\varrho)$

with $\mathrm{Ob}(\mathbf{C}(\varrho)) := X$ and $\mathrm{Mor}(\mathbf{C}(\varrho)) := X \times G$ as follows. $(x, g) \in X \times G$ is a morphism with source $\varrho(x, g)$ and target x, and the composition $(x, g) \circ (\varrho(x, g), h)$ is defined to be (x, gh). The axioms of action (together with group axioms for G) imply that composition is well defined and associative, that every object $x \in X$ has an identity $\mathrm{id}_x = (x, \mathrm{id})$ and that every morphism $(x, g) \in X \times G$ has inverse $(x, g)^{-1} = (\varrho(x, g), g^{-1})$, so that $\mathbf{C}(\varrho)$ is indeed a groupoid. Note that ϱ is a free action (i.e., $\varrho(x, g) = x$ implies $g = \mathrm{id}$) if and only if $\mathbf{C}(\varrho)$ is an equivalence relation, whereas ϱ is transitive (i.e., for all $x, y \in X$ there exists $g \in G$ such that $\varrho(x, g) = y$) if and only if $\mathbf{C}(\varrho)$ is equivalent to a group.

2.2 Commutative algebra

If A is a ring, A-**Mod** will be the category of A-modules (it is an abelian category with direct sums); we will usually write Hom_A instead of $\mathrm{Hom}_{A\text{-}\mathbf{Mod}}$. Similarly, A-**Alg** will be the category of A-algebras: its objects are morphisms of rings with source A (by abuse of notation, an A-algebra $A \to B$ will be usually denoted simply by B) and its morphisms are morphisms of rings commuting with the morphisms from A. Note that, since $\mathbb{Z}$ is an initial object of **Rng**, $\mathbb{Z}$-**Alg** can be identified with **Rng**.

Given a ring A and morphisms $\phi_i \colon B \to C_i$ (for $i = 1, 2$) of A-**Alg**, the tensor product

$$C_1 \otimes_B C_2 = C_1 \, {}_{\phi_1}\!\otimes_{\phi_2} C_2$$

is in a natural way an A-algebra, which, together with the natural morphisms $C_i \to C_1 \otimes_B C_2$, gives the fibred coproduct $C_1 \coprod_B C_2$ in A-**Alg**. Also (fibred) products exist in A-**Alg** (they are defined as in **Set**, with operations componentwise).

Definition 2.25. Let A be a ring. An A-module M is *finitely generated* or *of finite type* (respectively *finitely presented* or *of finite presentation*) if there is an exact sequence in A-**Mod** of the form $A^n \to M \to 0$ (respectively $A^m \to A^n \to M \to 0$) for some $n, m \in \mathbb{N}$.

Definition 2.26. A morphism of rings $A \to B$ is *of finite type* (or B is a *finitely generated* A-algebra) if there exist $n \in \mathbb{N}$ and a surjective morphism of A-algebras $\pi \colon A[t_1, \dots, t_n] \twoheadrightarrow B$. $A \to B$ is *of finite presentation* (or B is a *finitely presented* A-algebra) if it is of finite type and π as above can be chosen so that $\ker \pi$ is a finitely generated ideal of $A[t_1, \dots, t_n]$.

Remark 2.27. If A is a noetherian ring, then a finitely generated A-module is also finitely presented. Similarly, if $\phi \colon A \to B$ is a morphism of finite type of rings with A noetherian, then by Hilbert's basis theorem also B is noetherian and ϕ is actually of finite presentation.

Definition 2.28. A morphism of rings $A \to B$ is *finite* if B is a finitely generated A-module.

Remark 2.29. It is easy to see that, if $A \to B \to C$ are morphisms of finite type (respectively of finite presentation, respectively finite) and $A \to A'$ is an arbitrary morphism of rings, then the induced morphisms $A \to C$ and $A' \to B \otimes_A A'$ are of finite type (respectively of finite presentation, respectively finite).

If A is a ring and $S \subset A$ is a multiplicative system, $S^{-1}A$ will be the corresponding localized ring; similarly, for every $M \in A$-**Mod**, $S^{-1}M \cong S^{-1}A \otimes_A M$ will be the corresponding localized $S^{-1}A$-module. When $S = \{a^n \mid n \in \mathbb{N}\}$ for some $a \in A$ (respectively $S = A \setminus \mathtt{p}$ for some prime ideal $\mathtt{p} \subset A$), $S^{-1}A$ and $S^{-1}M$ will be denoted by A_a and M_a (respectively by $A_{\mathtt{p}}$ and $M_{\mathtt{p}}$).

Definition 2.30. Let A be a ring. An A-module M is A-*flat* (or simply *flat*) if the functor $M \otimes_A -\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ (which is always right exact) is exact.

A morphism of rings $A \to B$ is *flat* if B is A-flat.

Remark 2.31. If $\phi\colon A \to B$ is a morphism of rings, $B \otimes_A -\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ factors (up to canonical isomorphism) as the composition of a functor $\phi_*\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ (often denoted also by $B \otimes_A -$) and of the forgetful functor $D\colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ (it is easy to see that D is right adjoint of ϕ_*, namely for every $M \in A$-**Mod** and every $N \in B$-**Mod** there is a natural isomorphism of B-modules $\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N)$). Since D is exact and faithful, it is clear that $\phi_*\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ is always right exact, and it is exact if and only if B is a flat A-module.

Remark 2.32. It is clear that, if $A \to B \to C$ are flat morphisms and $A \to A'$ is an arbitrary morphism of rings, then the induced morphisms $A \to C$ and $A' \to B \otimes_A A'$ are flat.

Proposition 2.33.

1. *If A is a ring and $S \subset A$ is a multiplicative system, then the natural morphism of rings $A \to S^{-1}A$ is flat.*
2. *A morphism of rings $\phi\colon A \to B$ is flat if and only if the induced morphism of local rings $\phi_{\mathtt{q}}\colon A_{\phi^{-1}(\mathtt{q})} \to B_{\mathtt{q}}$ is flat for every prime ideal $\mathtt{q} \subset B$.*

Given $B \in A$-**Alg** and $M \in B$-**Mod**, we denote by $\mathrm{Der}_A(B, M)$ the set of A-derivations from B to M (i.e., those maps $d \in \mathrm{Hom}_A(B, M)$ such that $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$), which is in a natural way a B-module. If $N \in B$-**Mod** and $d\colon B \to N$ is an A-derivation, the map $\circ d\colon \mathrm{Hom}_B(N, M) \to \mathrm{Der}_A(B, M)$, is a morphism of B-**Mod**. It can be proved that there exists (necessarily unique up to unique isomorphism) a B-module $\Omega_{B/A}$ (called the *module of relative differentials*) together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$ (called the *universal derivation*) such that $\circ d_{B/A}\colon \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$ is an isomorphism for every $M \in B$-**Mod**. Indeed, denoting by $\mu\colon B \otimes_A B \to B$ the (surjective) morphism of rings defined by $b \otimes b' \mapsto bb'$ and setting $\mathtt{J} := \ker \mu$, we can take $\Omega_{B/A} := \mathtt{J}/\mathtt{J}^2$ (note that $\mathtt{J}/\mathtt{J}^2$ is in a natural way a $(B \otimes_A B)/\mathtt{J} \cong B$-module) and define $d_{B/A}(b)$ to be the class in $\mathtt{J}/\mathtt{J}^2$ of $1 \otimes b - b \otimes 1 \in J$. Notice that $\Omega_{B/A}$ is generated (as a B-module) by $\mathrm{im}\, d_{B/A}$.

Example 2.34. If $A \to A' := A[t_i]_{i\in I}$ is the natural morphism, then $\Omega_{A'/A}$ is a free A'-module with base $\{d_{A'/A}(t_i)\}_{i\in I}$.

Proposition 2.35. *Let $A \to B$ be a morphism of rings and let $S \subset B$ be a multiplicative system. Then there is a natural isomorphism of $S^{-1}B$-modules $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

Proposition 2.36. *Let $A \to B$ and $A \to A'$ be morphisms of rings. Setting $B' := B\otimes_A A'$, there is a natural isomorphism of B'-modules $\Omega_{B'/A'} \cong \Omega_{B/A}\otimes_B B'$.*

Proposition 2.37. *Given morphisms of rings $A \to B \to C$ there is a naturally induced exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

If moreover $B \to C$ is surjective (so that obviously $\Omega_{C/B} = 0$) with kernel $\mathtt{I}$, then the above sequence extends on the left to an exact sequence of C-modules

$$\mathtt{I}/\mathtt{I}^2 \xrightarrow{d^{\mathtt{I}}_{B/A}} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where $d^{\mathtt{I}}_{B/A} := (d_{B/A}|_{\mathtt{I}}) \otimes_B C$.

GRng will be the category whose objects are positively graded rings and whose morphisms are morphisms of rings preserving degree. If $R = \bigoplus_{n\geq 0} R_n \in$ **GRng**, we will denote by R-**GMod** the category whose objects are graded R-modules $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and whose morphisms are morphisms of R-modules preserving degree. If $M \in R$-**GMod** and $n \in \mathbb{Z}$, $M(n) \in R$-**GMod** is defined by $M(n)_m := M_{n+m}$ for every $m \in \mathbb{Z}$. Given a homogeneous element $r \in R_+ := \bigoplus_{n>0} R_n$, we define $R_{(r)}$ to be the ring $(R_r)_0$ and $M_{(r)}$ to be the $R_{(r)}$-module $(M_r)_0$. Similarly, if $\mathtt{p} \subset R$ is a homogeneous prime ideal, we set $R_{(\mathtt{p})} := (S^{-1}R)_0$ and $M_{(\mathtt{p})} := (S^{-1}M)_0 \in R_{(\mathtt{p})}$-**Mod**, where S is the multiplicative system of homogeneous elements of $R \setminus \mathtt{p}$. Finally, if A is a ring, A-**GAlg** will be the category of positively graded A-algebras (its objects are morphisms of **GRng** with source $A = A_0 \in$ **GRng**).

Nakayama's lemma. Let A be a ring and $\mathtt{J} \subset A$ the intersection of all maximal ideals of A. If M is a finitely generated A-module such that $\mathtt{J}M = M$, then $M = 0$.

For a ring A we will denote by $\dim A$ its Krull dimension (the maximal length of a chain of prime ideals). If A is a noetherian local ring with maximal ideal $\mathtt{m}$ and residue field $\mathbb{K} = A/\mathtt{m}$, then $\dim A \leq \dim_{\mathbb{K}} \mathtt{m}/\mathtt{m}^2 < \infty$ (note that $\dim_{\mathbb{K}} \mathtt{m}/\mathtt{m}^2$ is the minimal number of generators of $\mathtt{m}$ by Nakayama's lemma); A is *regular* if $\dim A = \dim_{\mathbb{K}} \mathtt{m}/\mathtt{m}^2$.

If $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ will denote its algebraic closure.

2.3 Ringed spaces

If X is a ringed space, $|X|$ (or simply X, if no confusion is possible) will denote the underlying topological space and $\mathcal{O}_X$ the structure sheaf of rings. Similarly, for a morphism of ringed spaces $f\colon X \to Y$, $|f|\colon |X| \to |Y|$ (or simply f, if no confusion is possible) will denote the (continuous) map between the underlying topological spaces and $f^{\#}\colon \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ the morphism of sheaves of rings (on Y). For every open subset U of a ringed space X we will always consider U as a ringed space with structure sheaf $\mathcal{O}_U := \mathcal{O}_X|_U$. We will say that a ringed space X is connected, or irreducible, or quasi-compact if the same is true for $|X|$,[4] and that a morphism of ringed spaces f is injective, or surjective, or open, or closed if the same is true for $|f|$. Moreover, for a ringed space X we will denote by $\dim X$ the dimension of $|X|$ (the maximal length of a chain of irreducible closed subsets of $|X|$) and for every $x \in X$ we define $\dim_x X$ to be the minimum of $\dim U$ for U open neighbourhood of x; if $f\colon X \to Y$ is a morphism of ringed spaces, we also set $\dim_x f := \dim_x |f|^{-1}(f(x))$.

The category of (sheaves of) $\mathcal{O}_X$-modules (respectively $\mathcal{O}_X$-algebras, respectively positively graded $\mathcal{O}_X$-algebras) on a ringed space X will be denoted by $\mathbf{Mod}(X)$ (respectively $\mathbf{Alg}(X)$, respectively $\mathbf{GAlg}(X)$). $\mathbf{Mod}(X)$ is an abelian category with direct sums: if $\mathcal{F}_i$ ($i \in I$) are $\mathcal{O}_X$-modules the direct sum $\bigoplus_{i\in I} \mathcal{F}_i$ is the sheaf associated to the presheaf (which is a sheaf if I is finite, but not in general) defined by $U \mapsto \bigoplus_{i\in I} \mathcal{F}_i(U)$ ($U \subseteq X$ open); if $\mathcal{F} \in \mathbf{Mod}(X)$, we will write $\mathcal{F}^I$ for $\bigoplus_{i\in I} \mathcal{F}$. Given $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(X)$, we will usually write $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ instead of $\mathrm{Hom}_{\mathbf{Mod}(X)}(\mathcal{F}, \mathcal{G})$, whereas $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ will be the sheaf defined for every $U \subseteq X$ open by $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U)$; $\mathcal{F} \otimes_X \mathcal{G} \in \mathbf{Mod}(X)$ (or simply $\mathcal{F} \otimes \mathcal{G}$) will denote the sheaf associated to the presheaf defined by $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Definition 2.38. Let X be a ringed space. $\mathcal{F} \in \mathbf{Mod}(X)$ is *of finite type* (respectively *of finite presentation*) if for every $x \in X$ there is an open neighbourhood U of x and an exact sequence in $\mathbf{Mod}(U)$ of the form $\mathcal{O}_U^n \to \mathcal{F}|_U \to 0$ (respectively $\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0$) for some $n, m \in \mathbb{N}$.

Definition 2.39. Let X be a ringed space. $\mathcal{F} \in \mathbf{Mod}(X)$ is *quasi-coherent* if for every $x \in X$ there is an open neighbourhood U of x and an exact sequence in $\mathbf{Mod}(U)$ of the form $\mathcal{O}_U^I \to \mathcal{O}_U^J \to \mathcal{F}|_U \to 0$ for some sets I, J. $\mathcal{F}$ is *coherent* if it is of finite type and for every $U \subseteq X$ open the kernel of every morphism $\mathcal{O}_U^n \to \mathcal{F}|_U$ of $\mathbf{Mod}(U)$ (with $n \in \mathbb{N}$) is of finite type.

We will denote by $\mathbf{QCoh}(X)$ (respectively $\mathbf{Coh}(X)$) the full subcategory of $\mathbf{Mod}(X)$ whose objects are quasi-coherent (respectively coherent) $\mathcal{O}_X$-modules. Similarly, $\mathbf{QCohAlg}(X)$ (respectively $\mathbf{QCohGAlg}(X)$) will be the full subcategory of $\mathbf{Alg}(X)$ (respectively $\mathbf{GAlg}(X)$) whose objects are quasi-coherent as

[4] Following the French terminology, we say that a topological space T is *quasi-compact* if every open cover of T admits a finite subcover (then T is compact if it is quasi-compact and Hausdorff).

$\mathcal{O}_X$-modules (note that $\mathcal{R} = \bigoplus_{n\in\mathbb{N}} \mathcal{R}_n \in \mathbf{GAlg}(X)$ is quasi-coherent if and only if $\mathcal{R}_n$ is quasi-coherent for every $n \in \mathbb{N}$).

Remark 2.40. An $\mathcal{O}_X$-module of finite presentation is also of finite type and quasi-coherent. Every coherent $\mathcal{O}_X$-module is of finite presentation, whereas the converse is true if and only if $\mathcal{O}_X$ is coherent: this follows from the fact $\mathbf{Coh}(X)$ is an abelian subcategory of $\mathbf{Mod}(X)$ (if $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathbf{Coh}(X)$, then $\ker\alpha$ and $\operatorname{coker}\alpha$ are also coherent).

We also recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if for every $x \in X$ there is an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U^n$ for some $n \in \mathbb{N}$ (then $\mathcal{F}$ is obviously of finite presentation); if the integer n is the same for every $x \in X$, $\mathcal{F}$ is said to be of rank n. A locally free $\mathcal{O}_X$-module $\mathcal{L}$ of rank 1 is also called *invertible*, since in this case there exists another invertible $\mathcal{O}_X$-module $\mathcal{M}$ (namely $\mathcal{M} = \mathcal{H}om_X(\mathcal{L}, \mathcal{O}_X)$) such that $\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_X$.

A morphism of ringed spaces $f\colon X \to Y$ induces a (left exact) functor

$$f_*\colon \mathbf{Mod}(X) \to \mathbf{Mod}(Y)$$

and its left adjoint $f^*\colon \mathbf{Mod}(Y) \to \mathbf{Mod}(X)$, which is right exact and which preserves tensor products and direct sums. Then it is clear that if $\mathcal{F} \in \mathbf{Mod}(Y)$ is quasi-coherent, (but not coherent, in general) or of finite type, or of finite presentation, or locally free (of some rank n), then $f^*(\mathcal{F}) \in \mathbf{Mod}(X)$ has the same property. In particular, f induces a functor $f^*\colon \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$, hence also functors $f^*\colon \mathbf{QCohAlg}(Y) \to \mathbf{QCohAlg}(X)$ and $f^*\colon \mathbf{QCohGAlg}(Y) \to \mathbf{QCohGAlg}(X)$.

If X is a locally ringed space (i.e., the stalk $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$) we will denote by $\mathtt{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x)$ the residue field $\mathcal{O}_{X,x}/\mathtt{m}_x$. $\mathbf{LRngSp}$ will be the subcategory of the category of ringed spaces whose objects are locally ringed spaces and whose morphisms are those morphisms of ringed spaces $f\colon X \to Y$ such that for every $x \in X$ the morphism of local rings $f_x^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local, which means that $(f_x^\#)^{-1}(\mathtt{m}_x) = \mathtt{m}_{f(x)}$ (so that there is an induced extension of fields $\kappa(f(x)) \hookrightarrow \kappa(x)$).

If A is a ring, $\operatorname{Spec} A$ is a locally ringed space, where $|\operatorname{Spec} A|$ is the set of prime ideals of A whose closed subsets are those of the form $V(\mathtt{I}) := \{\mathtt{p} \in \operatorname{Spec} A \mid \mathtt{p} \supseteq \mathtt{I}\}$ (for $\mathtt{I} \subseteq A$ an ideal) and the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is such that $\mathcal{O}_{\operatorname{Spec} A,\mathtt{p}} \cong A_{\mathtt{p}}$ for every $\mathtt{p} \in \operatorname{Spec} A$. Every morphism of rings $\phi\colon A \to B$ induces a morphism of locally ringed spaces $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$, so that $\operatorname{Spec}\colon \mathbf{Rng}^\circ \to \mathbf{LRngSp}$ is a functor.

2.4 Schemes

We will denote by **AffSch** (respectively **Sch**) the category of affine schemes (respectively of schemes): by definition, they are (strictly) full subcategories of **LRngSp** and a locally ringed space X is an affine scheme if and only if there is a ring A such that $X \cong \operatorname{Spec} A$, whereas X is a scheme if and only if every point of X has an open neighbourhood which is an affine scheme. If A is a ring, the open subsets of $\operatorname{Spec} A$ of the form $D(a) := \operatorname{Spec} A \setminus V(a)$ ($a \in A$) form a base of the topology of $\operatorname{Spec} A$; moreover, $D(a) \cong \operatorname{Spec} A_a$ (and, denoting by $\iota_a \colon A \to A_a$ the localization map, the morphism of affine schemes $\operatorname{Spec} \iota_a \colon \operatorname{Spec} A_a \cong D(a) \to \operatorname{Spec} A$ is the natural inclusion). It follows that every open subset of a scheme is also a scheme and that open affine subsets form a base of the topology of every scheme.

Remark 2.41. Every affine scheme is quasi-compact.

Proposition 2.42. *The functor* $\Gamma \colon \mathbf{Sch} \to \mathbf{Rng}^\circ$, *defined on objects by setting* $\Gamma(X) := \mathcal{O}_X(X)$ *and on morphisms by*

$$\Gamma(X \xrightarrow{f} Y) := f^{\#}(Y) \colon \mathcal{O}_Y(Y) \to f_*(\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$$

is left adjoint of $\operatorname{Spec} \colon \mathbf{Rng}^\circ \to \mathbf{Sch}$ *(i.e., there is a natural bijection*

$$\operatorname{Hom}_{\mathbf{Rng}}(A, \mathcal{O}_X(X)) \cong \operatorname{Hom}_{\mathbf{Sch}}(X, \operatorname{Spec} A)$$

for every $X \in \mathbf{Sch}$ *and every* $A \in \mathbf{Rng}$*) and* $\Gamma \circ \operatorname{Spec} \cong \mathrm{id}_{\mathbf{Rng}^\circ}$. *In particular,* $\operatorname{Spec} \colon \mathbf{Rng}^\circ \to \mathbf{AffSch}$ *is an equivalence, with quasi-inverse the restriction of* Γ.

If A is a ring, we will usually write $\mathbf{AffSch}_{/A}$ instead of $\mathbf{AffSch}_{/\operatorname{Spec} A}$ and $\mathbf{Sch}_{/A}$ instead of $\mathbf{Sch}_{/\operatorname{Spec} A}$. Note that the above result implies that $\operatorname{Spec} \mathbb{Z}$ is a terminal object of $\mathbf{Sch}$, so that $\mathbf{Sch}_{/\mathbb{Z}}$ can be identified with $\mathbf{Sch}$ (and $\mathbf{AffSch}_{/\mathbb{Z}}$ with $\mathbf{AffSch}$). We denote by $\mathbb{A}^n_A$ the affine scheme $\operatorname{Spec} A[t_1, \dots t_n]$.

Proposition 2.43. *Let A be a ring. The exact functor* A-$\mathbf{Mod} \to \mathbf{Mod}(\operatorname{Spec} A)$ *defined on objects by* $M \mapsto \widetilde{M}$ *(the associated sheaf* $\widetilde{M}$ *is such that* $\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ *for every* $\mathfrak{p} \in \operatorname{Spec} A$*) induces an equivalence of categories between* A-$\mathbf{Mod}$ *and* $\mathbf{QCoh}(\operatorname{Spec} A)$, *whose quasi-inverse is the functor defined on objects by* $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec} A)$. *The same functors induce equivalences between* A-$\mathbf{Alg}$ *(respectively* A-$\mathbf{GAlg}$*) and* $\mathbf{QCohAlg}(\operatorname{Spec} A)$ *(respectively* $\mathbf{QCohGAlg}(\operatorname{Spec} A)$*).*

Remark 2.44. It follows from the above result that for a scheme X the category $\mathbf{QCoh}(X)$ is abelian and the general definition of quasi-coherent $\mathcal{O}_X$-module coincides with that of [11] (namely, $\mathcal{F} \in \mathbf{Mod}(X)$ is quasi-coherent if and only if for every $x \in X$ there is an open affine neighbourhood U of x such that $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_U(U)$-module M). On the other hand, the definition of coherent $\mathcal{O}_X$-module given in [11] (the same as that of quasi-coherent, with the additional requirement that M be a finitely generated $\mathcal{O}_U(U)$-module) is not the right one in general (for instance, when $X = \operatorname{Spec} A$ with A a non noetherian ring), but it is equivalent to that of quasi-coherent $\mathcal{O}_X$-module of finite type.

If X is a scheme, the (left exact) functor $\mathbf{Mod}(X) \to \mathcal{O}_X(X)$-$\mathbf{Mod}$ defined on objects by $\mathcal{F} \mapsto \mathcal{F}(X)$ is often denoted by $H^0(X, -)$ and its right derived functors by $H^i(X, -)$ ($i \in \mathbb{N}$).

Proposition 2.45. *If X is an affine scheme, then $H^i(X, \mathcal{F}) = 0$ for every $i > 0$ and every $\mathcal{F} \in \mathbf{QCoh}(X)$.*

Sch has fibred products (hence also finite products, since it has a terminal object) and the inclusion functor **AffSch** $\subset$ **Sch** preserves fibred products (of course, fibred products in **AffSch** correspond to tensor products of rings under the functor Spec). Given morphisms of schemes $X \to Z$ and $Y \to Z$, the projection $pr_1 \colon X \times_Z Y \to X$ is such that $pr_1^{-1}(U) \cong U \times_Z Y$ for every open subset U of X. The natural continuous map $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ (the latter denoting fibred product in the category of topological spaces) is in general not injective, but it is always surjective. This implies that for morphisms of schemes the property of being surjective is stable under base change; the same is not true for the property of being injective (consider for instance morphisms of the form $\operatorname{Spec} \phi$, where ϕ is a separable extension of fields, e.g. $\mathbb{R} \subset \mathbb{C}$). If $f \colon X \to Y$ is a morphism of schemes, the *fibre* of f over a point $y \in Y$ is the scheme $f^{-1}(y) := X \times_Y \operatorname{Spec} \kappa(y)$: in this case the map $|f^{-1}(y)| \to |X| \times_{|Y|} |\operatorname{Spec} \kappa(y)| = |f|^{-1}(y)$ (the latter being endowed with the induced topology as a subspace of $|X|$) is a homeomorphism.

Remark 2.46. It is immediate to see that every monomorphism of schemes is injective, hence universally injective. On the other hand, a universally injective morphism of schemes is not a monomorphism, in general (just consider $\operatorname{Spec} \mathbb{Z}[t]/(t^2) \to \operatorname{Spec} \mathbb{Z}$).

Definition 2.47. A morphism of schemes $f \colon Y \to X$ is an *immersion* if $\operatorname{im}|f|$ is a locally closed subspace (i.e., the intersection of a closed and an open subset) of $|X|$, $|f|$ induces a homeomorphism between $|Y|$ and $\operatorname{im}|f|$ and $f^\#_y \colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective for every $y \in Y$. f is a *closed* (respectively an *open*) *immersion* if moreover $\operatorname{im}|f|$ is closed in $|X|$ (respectively if $\operatorname{im}|f|$ is open in $|X|$ and $f^\#_y$ is an isomorphism for every $y \in Y$).

Remark 2.48. An immersion of schemes is clearly a monomorphism, but the converse is not true (e.g., consider $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$).

A (*closed*, respectively *open*) *subscheme* of a scheme X is an equivalence class of (closed, respectively open) immersions with target X, where two immersions $f \colon Y \hookrightarrow X$ and $f' \colon Y' \hookrightarrow X$ are equivalent if $f \cong f'$ in $\mathbf{Sch}_{/X}$. If $f \colon Y \hookrightarrow X$ is an immersion, by abuse of notation we will also say that f (or simply Y) is a subscheme of X, and we will often write $Y \subseteq X$ (or $Y \subset X$ is f is not an isomorphism). It is clear that open subschemes of X are in natural bijection with open subsets of $|X|$, and as for closed subschemes we have the following result.

Proposition 2.49. *If $f \colon Y \hookrightarrow X$ is a closed immersion, then $f^\# \colon \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective and its kernel $\mathcal{I}_{Y/X}$ (or simply $\mathcal{I}_Y$) is a quasi-coherent ideal of $\mathcal{O}_X$, called the ideal (sheaf) of Y in X. The map $(Y \subseteq X) \mapsto \mathcal{I}_{Y/X}$ gives a bijection between closed subschemes of X and quasi-coherent ideals of $\mathcal{O}_X$.*

Remark 2.50. If $X = \operatorname{Spec} A$ (A a ring), then the associated sheaf functor gives a bijection between ideals of A and quasi-coherent ideals of $\mathcal{O}_{\operatorname{Spec} A}$, and the closed subscheme corresponding to an ideal $\mathtt{I} \subset A$ is $\operatorname{Spec} A/\mathtt{I} \subseteq \operatorname{Spec} A$. In particular, a closed subscheme of an affine scheme is also affine.

Remark 2.51. If Y is a subscheme of X, then Y is a closed subscheme of some open subscheme U of X, and there is a largest open subscheme U of X with this property (namely, $|U| = |X| \setminus (\overline{|Y|} \setminus |Y|)$, where $\overline{|Y|}$ is the closure of $|Y|$ in $|X|$).

Proposition 2.52. *For every morphism of schemes $f\colon X \to Y$ the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is an immersion.*

So for every morphism $X \to Y$ we can regard X as a closed subscheme of an open subscheme W of $X \times_Y X$. Setting $\mathcal{J} := \mathcal{I}_{X/W}$, the $\mathcal{O}_W$-module $\mathcal{J}/\mathcal{J}^2$ is in a natural way an $\mathcal{O}_W/\mathcal{J} \cong \mathcal{O}_X$-module, which is called the *sheaf of relative differentials* and is denoted by $\Omega_{X/Y}$: clearly, for every open affine subsets $\operatorname{Spec} B \cong U \subseteq X$ and $\operatorname{Spec} A \cong V \subseteq Y$ such that $f(U) \subseteq V$, there is a natural isomorphism $\Omega_{X/Y}|_U \cong \Omega_{U/Y} \cong \widetilde{\Omega_{B/A}}$ (in particular, $\Omega_{X/Y} \in \mathbf{QCoh}(X)$).

Remark 2.53. If $f\colon X \to Y$ is a morphism of schemes, it is easy to see that for every $x \in X$ there is a natural isomorphism of $\mathcal{O}_{X,x}$-modules $(\Omega_{X/Y})_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$.

Proposition 2.54. *Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes. Setting $X' := X \times_Y Y'$, there is a natural isomorphism of $\mathcal{O}_{X'}$-modules $\Omega_{X'/Y'} \cong pr_1^*(\Omega_{X/Y})$.*

Proposition 2.55. *Given morphisms of schemes $X \xrightarrow{f} Y \to Z$ there is a naturally induced exact sequence in $\mathbf{QCoh}(X)$*

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

If moreover f is a closed immersion (so that obviously $\Omega_{X/Y} = 0$), then the above sequence extends on the left to an exact sequence in $\mathbf{QCoh}(X)$

$$\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2 \to f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to 0.$$

Morphisms of schemes can be glued: this means that given an open cover $\{U_i\}_{i \in I}$ of a scheme X and $f_i \in \operatorname{Hom}_{\mathbf{Sch}}(U_i, Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $f \in \operatorname{Hom}_{\mathbf{Sch}}(X, Y)$ such that $f|_{U_i} = f_i$ for every $i \in I$. Also schemes can be glued, in the following sense. Assume that $\{X_i\}_{i \in I}$ is a collection of schemes and that, for all $i, j \in I$ with $i \neq j$, $U_{i,j} \subseteq X_i$ is an open subscheme and $f_{i,j}\colon U_{j,i} \xrightarrow{\sim} U_{i,j}$ is an isomorphism such that $f_{j,i} = f_{i,j}^{-1}$ and, if $i, j, k \in I$ are distinct, the following cocycle condition is satisfied: $f_{i,j}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k}$ and

$$f_{i,k}|_{U_{k,i} \cap U_{k,j}} = f_{i,j}|_{U_{j,i} \cap U_{j,k}} \circ f_{j,k}|_{U_{k,i} \cap U_{k,j}} \colon U_{k,i} \cap U_{k,j} \to U_{i,j} \cap U_{i,k}.$$

Then there exists (unique up to isomorphism) a scheme X together with an open cover $\{U_i\}_{i\in I}$ of X and isomorphisms $g_i\colon U_i \xrightarrow{\sim} X_i$ such that $g_i(U_i \cap U_j) = U_{i,j}$ and $f_{i,j} = g_i|_{U_i\cap U_j} \circ (g_j|_{U_i\cap U_j})^{-1}$ for all $i, j \in I$. In the particular case in which all the $U_{i,j}$ are empty, the scheme X is the coproduct $\coprod_{i\in I} X_i$ (it is also called the *disjoint union* of the X_i), whereas when $I = \{1, 2\}$ (then to give the $U_{i,j}$ and the $f_{i,j}$ is equivalent to giving a scheme U and two open immersions $U \hookrightarrow X_i$) the scheme X is the fibred coproduct $X_1 \coprod_U X_2$.

Given a positively graded ring R, $\operatorname{Proj} R$ is a scheme with $|\operatorname{Proj} R|$ the set of homogeneous prime ideals of R not containing R_+ whose closed subsets are those of the form $V_+(\mathtt{I}) := \{\mathtt{p} \in \operatorname{Proj} R \mid \mathtt{p} \supseteq \mathtt{I}\}$ (for $\mathtt{I} \subseteq A$ a homogeneous ideal) and with structure sheaf $\mathcal{O}_{\operatorname{Proj} R}$ such that $\mathcal{O}_{\operatorname{Proj} R,\mathtt{p}} \cong R_{(\mathtt{p})}$ for every $\mathtt{p} \in \operatorname{Proj} R$. The open subsets of the form $D_+(r) := \operatorname{Proj} R \setminus V_+(r)$ (where $r \in R_+$ is a homogeneous element) form a base of the topology of $\operatorname{Proj} R$ and $D_+(r) \cong \operatorname{Spec} R_{(r)}$. The morphisms of schemes $D_+(r) \to \operatorname{Spec} R_0$ (induced by the natural morphisms of rings $R_0 \to R_{(r)}$) glue to a structure morphism $\operatorname{Proj} R \to \operatorname{Spec} R_0$; it follows that every morphism of rings $A \to R_0$ induces a morphism $\operatorname{Proj} R \to \operatorname{Spec} A$. For every ring A the scheme $\operatorname{Proj} A[t_0, \dots, t_n]$ (where each t_i has degree 1) is denoted by $\mathbb{P}^n_A$. More generally, for every scheme X we set $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X$ (obviously $\mathbb{P}^n_{\operatorname{Spec} A} \cong \mathbb{P}^n_A$ in $\mathbf{Sch}_{/A}$);[5] note that $\mathbb{P}^0_X \cong X$.

If $R \in \mathbf{GRng}$, there is a natural exact functor $R\text{-}\mathbf{GMod} \to \mathbf{QCoh}(\operatorname{Proj} R)$ defined on objects by $M \mapsto \widetilde{M}$ (the associated sheaf $\widetilde{M}$ is such that $\widetilde{M}_{\mathtt{p}} \cong M_{(\mathtt{p})}$ for every $\mathtt{p} \in \operatorname{Proj} R$ and $\widetilde{M}|_{D_+(r)} \cong \widetilde{M_{(r)}}$ under the natural isomorphism $D_+(r) \cong \operatorname{Spec} R_{(r)}$ for every homogeneous $r \in R_+$). The sheaves $\widetilde{R(n)}$ (for $n \in \mathbb{Z}$) are denoted by $\mathcal{O}_{\operatorname{Proj} R}(n)$; they are invertible if R is generated by R_1 as R_0-algebra. Every morphism $\phi\colon R \to R'$ of $\mathbf{GRng}$ induces a morphism of schemes

$$\operatorname{Proj} \phi\colon P(\phi) \to \operatorname{Proj} R$$

(where $P(\phi) \subseteq \operatorname{Proj} R'$ is the open subset $\operatorname{Proj} R' \setminus V_+(\phi(R_+))$) such that, for every homogeneous $r \in R_+$, $(\operatorname{Proj} \phi)^{-1}(D_+(r)) = D_+(\phi(r))$ and

$$(\operatorname{Proj} \phi)|_{D_+(\phi(r))}\colon D_+(\phi(r)) \to D_+(r)$$

is induced by the natural morphism of rings $R_{(r)} \to R'_{(\phi(r))}$.

Definition 2.56. A scheme X is *locally noetherian* if for every $x \in X$ there is an open affine neighbourhood U of x such that the ring $\mathcal{O}_U(U)$ is noetherian.

A scheme is *noetherian* if it is locally noetherian and quasi-compact.

Proposition 2.57. *If a scheme X is locally noetherian, then for every open affine subset $U \subseteq X$ the ring $\mathcal{O}_U(U)$ is noetherian. In particular, a ring A is noetherian if and only if* $\operatorname{Spec} A$ *is a noetherian scheme.*

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[5] Similarly, we define $\mathbb{A}^n_X := \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X$, and then $\mathbb{A}^n_{\operatorname{Spec} A} \cong \mathbb{A}^n_A$ in $\mathbf{Sch}_{/A}$.

Remark 2.58. If a scheme X is (locally) noetherian then $|X|$ is (locally) noetherian (a topological space is noetherian if it satisfies the descending chain condition for closed subsets, and it is locally noetherian if every point has a noetherian neighbourhood). It is easy to see that a topological space is noetherian if and only if it is locally notherian and quasi-compact if and only if every open subset is quasi-compact.

Remark 2.59. On a locally noetherian scheme X every quasi-coherent $\mathcal{O}_X$-module of finite type is coherent (so, in this case, the three notions "coherent", "of finite presentation" and "quasi-coherent of finite type" coincide).

Definition 2.60. A scheme X is *reduced* if $\mathcal{O}_{X,x}$ is a reduced ring (i.e., with no nilpotent elements) for every $x \in X$ (or, equivalently, if $\mathcal{O}_X(U)$ is a reduced ring for every $U \subseteq X$ open).

Definition 2.61. A scheme X is *normal* if $\mathcal{O}_{X,x}$ is an integrally closed domain for every $x \in X$.

Definition 2.62. A locally noetherian scheme X is *regular at* $x \in X$ if the noetherian local ring $\mathcal{O}_{X,x}$ is regular. X is *regular* if it is regular at every point.

Definition 2.63. A morphism of schemes $f\colon X \to Y$ is a *local isomorphism at* $x \in X$ if there exist open neighbourhoods U of x and V of $f(x)$ such that $f|_U\colon U \xrightarrow{\sim} V$ is an isomorphism. f is a *local isomorphism* if it is a local isomorphism at every point of X.

Definition 2.64. A property P of schemes is *local* if the following holds: given an open cover $\{U_i\}_{i\in I}$ of a scheme X, X satisfies P if and only if U_i satisfies P for every $i \in I$.

Example 2.65. The following properties of schemes are local: locally noetherian, reduced, normal, regular.

Definition 2.66. A property P of morphisms of schemes is *local on the domain* (respectively *local on the codomain*) if the following holds: given a morphism of schemes $f\colon X \to Y$ and an open cover $\{U_i\}_{i\in I}$ of X (respectively $\{V_i\}_{i\in I}$ of Y), f satisfies P if and only if $f|_{U_i}$ (respectively $f|_{f^{-1}(V_i)}\colon f^{-1}(V_i) \to V_i$) satisfies P for every $i \in I$.

Example 2.67. It is easy to see that the following properties of morphisms of schemes are local on the codomain (and stable under composition): universally injective, surjective, universally open, universally closed, open immersion, closed immersion, immersion, local isomorphism (all these are also stable under base change), injective, open, closed. The properties of being open, universally open and a local isomorphism are also local on the domain.

Remark 2.68. When working in $\mathbf{Sch}_{/S}$ (S a scheme) it will be customary to denote an object $X \to S$ simply by X, and consequently properties of schemes (respectively of morphisms of schemes) will be extended to objects (respectively morphisms) of $\mathbf{Sch}_{/S}$.

Definition 2.69. Let $\mathbb{K}$ be a field. $X \in \mathbf{Sch}_{/\mathbb{K}}$ is *geometrically connected* (respectively *geometrically irreducible*, respectively *geometrically reduced*) if the scheme $X \times_{\operatorname{Spec}\mathbb{K}} \operatorname{Spec}\overline{\mathbb{K}}$ is connected (respectively irreducible, respectively reduced).

3 Some properties of morphisms of schemes

3.1 Quasi-compact and (quasi)separated morphisms

Definition 3.1. A morphism of schemes $f\colon X \to Y$ is *quasi-compact* if $f^{-1}(U)$ is quasi-compact for every $U \subseteq Y$ open and quasi-compact.

Remark 3.2. If $\{U_i\}_{i\in I}$ is a base of the topology of Y with each U_i quasi-compact, then $f\colon X \to Y$ is quasi-compact if and only if $f^{-1}(U_i)$ is quasi-compact for every $i \in I$: this follows immediately from the fact that a finite union of quasi-compact subspaces is again quasi-compact.

Example 3.3. Every closed immersion is quasi-compact, whereas an open immersion need not be quasi-compact.

Proposition 3.4. *For morphisms of schemes the property of being quasi-compact is stable under composition and base change and is local on the codomain.*

Proof. As for base change, by Remark 3.2 it is enough to prove that $X \times_Z Y$ is quasi-compact if Y and Z are affine and X is quasi-compact. Now, this is clear, since X can be covered by a finite number of open affine subsets and since the fibred product of affine schemes is again affine (hence quasi-compact). The rest of the proof is straightforward. □

Lemma 3.5. *If $f\colon X \to Y$ is a morphism of schemes with X quasi-compact and Y affine, then f is quasi-compact. In particular, a scheme X is quasi-compact if and only if $X \to \operatorname{Spec}\mathbb{Z}$ is quasi-compact.*

Proof. Since X can be covered by a finite number of open affine subsets, we can assume that also X is affine. Now, for a morphism of rings $\phi\colon A \to B$, we have $(\operatorname{Spec}\phi)^{-1}(D(a)) = D(\phi(a))$ for every $a \in A$, and the conclusion follows from Remark 3.2 □

Definition 3.6. A morphism of schemes $f\colon X \to Y$ is *separated* (respectively *quasi-separated*) if $\Delta_f\colon X \to X \times_Y X$ is a closed (respectively quasi-compact) immersion. A scheme X is *(quasi)separated* if the morphism $X \to \operatorname{Spec}\mathbb{Z}$ is (quasi)separated.

Remark 3.7. A separated morphism is quasi-separated.

Example 3.8. A morphism of affine schemes is separated; in particular, an affine scheme is separated.

Example 3.9. Every monomorphism (in particular, every immersion, and, more in particular, every diagonal morphism) of schemes is separated. Indeed, if $f\colon X \to Y$ is a monomorphism, then $\Delta_f\colon X \to X \times_Y X$ is an isomorphism by Lemma A.2.

Proposition 3.10. *For morphisms of schemes the properties of being separated and quasi-separated are stable under composition and base change and are local on the codomain.*

Proof. By Proposition 3.4 and remembering that also the property of being a closed immersion is stable under composition and base change and is local on the codomain, the statement about composition and base change follows from Lemma A.4, whereas the statement about locality on the codomain can be similarly proved using Lemma A.3 (this last fact will be proved in greater generality in Lemma 4.50). □

Proposition 3.11. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes.*

1. *If $g \circ f$ is (quasi)separated, then f is (quasi)separated.*
2. *If $g \circ f$ is quasi-compact and g is quasi-separated, then f is quasi-compact.*

Proof. Just remember that Δ_g is in any case separated, and apply Lemma A.5. □

Corollary 3.12. *Let $f\colon X \to Y$ be a morphism of schemes. If X is (quasi)separated, then f is (quasi)separated, and the viceversa holds if Y is (quasi)separated.*

We denote by **QSch** the full subcategory of **Sch** whose objects are quasi-separated schemes.

Remark 3.13. The category **QSch** has fibred products and the inclusion functor **QSch** $\subset$ **Sch** preserves them: if $f\colon X \to Y$ and $Y' \to Y$ are morphisms of **QSch**, then $X' := X \times_Y Y' \in$ **QSch**. Indeed, f is quasi-separated by Corollary 3.12, whence the induced morphism $f'\colon X' \to Y'$ is quasi-separated (by Proposition 3.10), and then X' is quasi-separated again by Corollary 3.12. It is also clear that, if X_i $(i \in I)$ are quasi-separated schemes, then $\coprod_{i\in I} X_i$ is quasi-separated, too.

Proposition 3.14. *A scheme X is quasi-separated if and only if for all $U, V \subseteq X$ open and quasi-compact, $U \cap V$ is also quasi-compact.*

Proof. Clearly $U \cap V = \Delta_X^{-1}(U \times V)$ if U and V are open subsets of X. It is then enough to note that $U \times V$ is quasi-compact if U and V are, and that the open subsets of the form $U \times V$ (with $U, V \subseteq X$ open and quasi-compact) form a base of the topology of $X \times X$. □

Corollary 3.15. *If X is a scheme such that $|X|$ is locally noetherian (in particular, if X is locally noetherian), then X is quasi-separated.*

Example 3.16. Let $Y := \operatorname{Spec} \mathbb{K}[t_i]_{i\in I}$ ($\mathbb{K}$ a field), $U := Y \setminus \{(t_i)_{i\in I}\}$ and $X := Y \coprod_U Y$. If I is finite it is well known that X is not separated (but it is quasi-separated, since it is noetherian). If I is infinite, then X is not even quasi-separated: indeed, denoting by $j_1, j_2 \colon Y \hookrightarrow X$ the natural morphisms, $j_1(Y), j_2(Y) \subset X$ are open and quasi-compact, whereas $j_1(Y) \cap j_2(Y) \cong U$ is not quasi-compact.

Proposition 3.17. *Let $f \colon X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If $\mathcal{F} \in \mathbf{QCoh}(X)$, then $f_*(\mathcal{F}) \in \mathbf{QCoh}(Y)$.*

Proof. Using Proposition 3.14 it is easy to adapt the proof of [11, II, Prop. 5.8] (otherwise, see [9, 1.7.4]). □

3.2 Morphisms (locally) of finite type and presentation

Definition 3.18. A morphism of schemes $f \colon X \to Y$ is *of finite type* (respectively *of finite presentation*) *at* $x \in X$ if there exist open affine neighbourhoods U of x and V of $f(x)$ such that $f(U) \subseteq V$ and the induced morphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite type (respectively of finite presentation). $f \colon X \to Y$ is *locally of finite type* (respectively *locally of finite presentation*) if it is of finite type (respectively of finite presentation) at every point of X.

Definition 3.19. A morphism of schemes is *of finite type* (respectively *of finite presentation*) if it is locally of finite type and quasi-compact (respectively locally of finite presentation, quasi-compact and quasi-separated).

Remark 3.20. If $f \colon X \to Y$ is (locally) of finite type and Y is (locally) noetherian, then X is also (locally) noetherian and f is (locally) of finite presentation.

Remark 3.21. If $f \colon X \to Y$ is of finite type at x, then $\dim_x f < \infty$.

Example 3.22. An open immersion (and more generally a local isomorphism) is obviously locally of finite presentation (hence locally of finite type), but it need not be of finite type. On the other hand, a closed immersion is of finite type, but it need not be locally of finite presentation. In any case, we see that every immersion is locally of finite type.

Example 3.23. Let X and Y be as in Example 3.16 with I infinite. Then the natural morphism $X \to Y$ is locally of finite presentation (it is a local isomorphism) and quasi-compact (by Lemma 3.5), but not quasi-separated (since Y is quasi-separated and X is not), hence not of finite presentation.

Proposition 3.24. *If a morphism of schemes $f \colon X \to Y$ is locally of finite type (respectively presentation) and $U \subseteq X$ and $V \subseteq Y$ are open affine subsets such that $f(U) \subseteq V$, then the induced morphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite type (respectively presentation).*

Proof. One can easily reduce to prove that a morphism of rings $\phi\colon A \to B$ is of finite type (respectively presentation) if $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is (locally) of finite type (respectively presentation), which is done in [7, Prop. 6.3.3] (respectively [9, Prop. 1.4.6]). □

Taking into account Proposition 3.4 and Proposition 3.10, the following result is then straightforward.

Corollary 3.25. *For morphisms of schemes, the properties of being of finite type and of finite presentation are stable under composition and base change and are local on the codomain; the same is true for the properties of being locally of finite type and locally of finite presentation, which are also local on the domain.*

Lemma 3.26. *If $f\colon X \to Y$ is a morphism locally of finite type of schemes, then $\Delta_f\colon X \to X \times_Y X$ is locally of finite presentation.*

Proof. It is enough to see that if $A \to B$ is a morphism of finite type of rings, then $\mathtt{J} := \ker(B \otimes_A B \twoheadrightarrow B)$ is a finitely generated ideal of $B \otimes_A B$. Indeed, it is easy to prove that, if $b_1, \dots, b_n \in B$ generate B as an A-algebra, then $\mathtt{J} = (b_1 \otimes 1 - 1 \otimes b_1, \dots, b_n \otimes 1 - 1 \otimes b_n)$. □

Proposition 3.27. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes.*

1. *If $g \circ f$ is of locally of finite type, then f is locally of finite type.*
2. *If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Since Δ_g is in any case locally of finite type (by Example 3.22), it follows from Lemma 3.26 and Lemma A.5. □

Lemma 3.28. *If $X \to Y$ is a morphism locally of finite type (respectively presentation) of schemes, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite type (respectively presentation).*

Proof. The question being local, it is enough to prove that, if $A \to B$ is a morphism of finite type (respectively presentation) of rings, then $\Omega_{B/A}$ is a B-module of finite type (respectively presentation). Since $B \cong A'/\mathtt{I}$ as an A-algebra, where $A' = A[t_1, \dots, t_n]$ for some $n \in \mathbb{N}$ and $\mathtt{I} \subset A'$ is an ideal (respectively a finitely generated ideal), this follows from the exact sequence

$$\mathtt{I}/\mathtt{I}^2 \to \Omega_{A'/A} \otimes_{A'} B \to \Omega_{B/A} \to 0,$$

as $\Omega_{A'/A} \otimes_{A'} B \cong B^n$ and $\mathtt{I}/\mathtt{I}^2$ is a finitely generated B-module if $\mathtt{I}$ is a finitely generated ideal of A'. □

Proposition 3.29. *If a morphism of schemes $f\colon X \to Y$ is open, then the morphism $\operatorname{Spec} f_x^{\#}\colon \operatorname{Spec}\mathcal{O}_{X,x} \to \operatorname{Spec}\mathcal{O}_{Y,f(x)}$ is surjective for every $x \in X$, and the viceversa is also true if f is locally of finite presentation.*

Proof. [9, Cor. 1.10.4]. □

Definition 3.30. A morphism of schemes is *proper* if it is separated, of finite type and universally closed.

Example 3.31. Every closed immersion is proper.

Taking into account Proposition 3.10 and Corollary 3.25, it is easy to prove the following result.

Proposition 3.32. *For morphisms of schemes the property of being proper is stable under composition and base change and is local on the codomain.*

3.3 (Very) ample sheaves, (quasi)affine and (quasi)projective morphisms

Definition 3.33. A morphism of schemes $f\colon X \to Y$ is *affine* (respectively *finite*) if there is an open affine cover $\{V_i\}_{i\in I}$ of Y such that $U_i := f^{-1}(V_i) \subseteq X$ is affine (respectively U_i is affine and the induced morphism of rings $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ is finite) for every $i \in I$.

Remark 3.34. An affine morphism is separated and quasi-compact.

Example 3.35. A closed immersion is a finite morphism.

Proposition 3.36. *If a morphism of schemes $f\colon X \to Y$ is affine (respectively finite) and $V \subseteq Y$ is an open affine subset, then $U := f^{-1}(V) \subseteq X$ is affine (respectively U is affine and the induced morphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is finite).*

Proof. [8, Cor. 1.3.2] (respectively [8, Prop. 6.1.4]). □

Corollary 3.37. *For morphisms of schemes the properties of being affine and finite are stable under composition and base change and are local on the codomain.*

Given a scheme Y and $\mathcal{A} \in \mathbf{QCohAlg}(Y)$, there is a natural way to define a scheme $\mathcal{S}pec\,\mathcal{A}$ together with a structure morphism $f\colon \mathcal{S}pec\,\mathcal{A} \to Y$ such that, for every open affine subset $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec}\mathcal{A}(V)$ and $f|_{f^{-1}(V)}\colon f^{-1}(V) \to V$ is induced by the structure morphism $\mathcal{O}_Y(V) \to \mathcal{A}(V)$. Clearly $\mathcal{S}pec$ extends to a functor $\mathcal{S}pec\colon \mathbf{QCohAlg}(Y)^\circ \to \mathbf{Sch}_{/Y}$, and, denoting by $\mathbf{Aff}_Y$ the full subcategory of $\mathbf{Sch}_{/Y}$ whose objects are affine morphisms with target Y (note that $\mathbf{Aff}_Y = \mathbf{AffSch}_{/Y}$ if and only if Y is affine), it is easy to prove the following result.

Proposition 3.38. *For every scheme Y the functor $\mathcal{S}pec$ induces an equivalence $\mathcal{S}pec\colon \mathbf{QCohAlg}(Y)^\circ \to \mathbf{Aff}_Y$, whose quasi-inverse is defined on objects by $(f\colon X \to Y) \mapsto f_*(\mathcal{O}_X)$. Moreover, for every morphism of schemes $f\colon X \to Y$ and every $\mathcal{A} \in \mathbf{QCohAlg}(Y)$ there is a natural bijection*

$$\operatorname{Hom}_{\mathbf{QCohAlg}(Y)}(\mathcal{A}, f_*(\mathcal{O}_X)) \cong \operatorname{Hom}_{\mathbf{Sch}_{/Y}}(X, \mathcal{S}pec\,\mathcal{A}).$$

Given a scheme Y and $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$, there is also a natural way to define a scheme $\mathcal{P}roj\,\mathcal{R}$ together with a structure morphism $p_{\mathcal{R}}\colon \mathcal{P}roj\,\mathcal{R} \to Y$ such that, for every open affine subset $V \subseteq Y$, $p_{\mathcal{R}}^{-1}(V) \cong \operatorname{Proj}\mathcal{R}(V)$ and $p_{\mathcal{R}}|_{p_{\mathcal{R}}^{-1}(V)}\colon p_{\mathcal{R}}^{-1}(V) \to V$ is induced by the structure morphism $\mathcal{O}_Y(V) \to \mathcal{R}_0(V)$. When $\mathcal{R} = \bigoplus_{n\geq 0} \mathrm{S}^n(\mathcal{F})$ is the symmetric algebra over $\mathcal{F} \in \mathbf{QCoh}(Y)$, the scheme $\mathcal{P}roj\,\mathcal{R}$ is denoted by $\mathbb{P}(\mathcal{F})$; note that $\mathbb{P}(\mathcal{O}_Y^{n+1}) \cong \mathbb{P}_Y^n$.

Remark 3.39. The structure morphism $\mathbb{P}(\mathcal{L}) \to Y$ is an isomorphism for every invertible $\mathcal{O}_Y$-module $\mathcal{L}$: this is clear if $\mathcal{L} = \mathcal{O}_Y$, and then it follows from the fact that for every $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$ there is a natural isomorphism $\mathcal{P}roj\,\mathcal{R} \cong \mathcal{P}roj\,\mathcal{R}'$ of $\mathbf{Sch}_{/Y}$, where $\mathcal{R}'$ is defined by $\mathcal{R}'_n := \mathcal{R}_n \otimes \mathcal{L}^{\otimes n}$ for $n \in \mathbb{N}$.

Proposition 3.40. *$p_{\mathcal{R}}\colon \mathcal{P}roj\,\mathcal{R} \to Y$ is separated for every $Y \in \mathbf{Sch}$ and every $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$.*

Proof. Since the property of being separated is local on the codomain, we can assume Y is affine. Then (in view of Corollary 3.12) it follows from the fact that $\operatorname{Proj} R$ is separated for every $R \in \mathbf{GRng}$ ([8, Prop. 2.4.2]). $\square$

If $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$ and $\mathcal{M} \in \mathbf{QCoh}(Y)$ has a structure of graded $\mathcal{R}$-module, we can define $\widetilde{\mathcal{M}} \in \mathbf{QCoh}(\mathcal{P}roj\,\mathcal{R})$ in such a way that, for every open affine subset $V \subseteq Y$, $\widetilde{\mathcal{M}}|_{p_{\mathcal{R}}^{-1}(V)} \cong \widetilde{\mathcal{M}(V)}$ (note that $\mathcal{M}(V) \in \mathcal{R}(V)$-$\mathbf{GMod}$ and $p_{\mathcal{R}}^{-1}(V) \cong \operatorname{Proj}\mathcal{R}(V)$). As usual, we set (for $n \in \mathbb{Z}$) $\mathcal{O}_{\mathcal{P}roj\,\mathcal{R}}(n) := \widetilde{\mathcal{R}(n)}$; they are invertible $\mathcal{O}_{\mathcal{P}roj\,\mathcal{R}}$-modules if $\mathcal{R}$ is generated by $\mathcal{R}_1$ as $\mathcal{R}_0$-algebra (in particular, the $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(n)$ are invertible for every $\mathcal{F} \in \mathbf{QCoh}(Y)$).

A morphism $\varphi\colon \mathcal{R} \to \mathcal{R}'$ of $\mathbf{QCohGAlg}(Y)$ induces in a natural way a morphism $\mathcal{P}roj\,\varphi\colon P(\varphi) \to \mathcal{P}roj\,\mathcal{R}$ of $\mathbf{Sch}_{/Y}$, where the open subset $P(\varphi) \subseteq \mathcal{P}roj\,\mathcal{R}'$ is such that $P(\varphi) \cap p_{\mathcal{R}'}^{-1}(V) = P(\varphi(V)) \subseteq p_{\mathcal{R}'}^{-1}(V) \cong \operatorname{Proj}\mathcal{R}'(V)$ for every open affine subset $V \subseteq Y$.

Proposition 3.41. *For every morphism of schemes $f\colon X \to Y$ and every $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$ there is a natural cartesian diagram*

$$\begin{tikzcd}
\mathcal{P}roj\, f^*(\mathcal{R}) \arrow[r, "f_{\mathcal{R}}"] \arrow[d, "p_{f^*(\mathcal{R})}"'] \arrow[dr, phantom, "\square"] & \mathcal{P}roj\,\mathcal{R} \arrow[d, "p_{\mathcal{R}}"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Moreover, for every morphism $\varphi\colon \mathcal{R} \to \mathcal{R}'$ of $\mathbf{QCohGAlg}(Y)$ the morphism $\mathcal{P}roj\, f^*(\varphi)$ of $\mathbf{Sch}_{/X}$ is identified with the pullback along f of the morphism $\mathcal{P}roj\,\varphi$ of $\mathbf{Sch}_{/Y}$: more precisely, this means that $P(f^*(\varphi)) = f_{\mathcal{R}'}^{-1}(P(\varphi)) \subseteq \mathcal{P}roj\, f^*(\mathcal{R}')$

and there is a cartesian diagram

$$\begin{tikzcd}
\mathcal{P}roj\, f^*(\mathcal{R}') & \supseteq & P(f^*(\varphi)) \arrow[r, "f_{\mathcal{R}'}|_{P(f^*(\varphi))}"] \arrow[d, "\mathcal{P}roj\, f^*(\varphi)"'] \arrow[dr, phantom, "\square"] & P(\varphi) \arrow[d, "\mathcal{P}roj\, \varphi"] & \subseteq & \mathcal{P}roj\, \mathcal{R}' \\
 & & \mathcal{P}roj\, f^*(\mathcal{R}) \arrow[r, "f_{\mathcal{R}}"'] & \mathcal{P}roj\, \mathcal{R}. & &
\end{tikzcd}$$

Proof. The first statement is proved in [8, Prop. 3.5.3], and then the second follows easily from the definitions. □

Assume now that $f\colon X \to Y$ is a morphism of schemes, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, $\mathcal{R} \in \mathbf{QCohGAlg}(Y)$ and $\varphi\colon f^*(\mathcal{R}) \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n} = \bigoplus_{n\geq 0} \mathrm{S}^n(\mathcal{L})$ a morphism of $\mathbf{QCohGAlg}(X)$. Then Proposition 3.41 implies that the morphism $\mathcal{P}roj\, \varphi\colon P(\varphi) \to \mathcal{P}roj\, f^*(\mathcal{R})$ of $\mathbf{Sch}_{/X}$ (where $P(\varphi) \subseteq \mathbb{P}(\mathcal{L}) \cong X$ by Remark 3.39) corresponds to a morphism denoted again by $\mathcal{P}roj\, \varphi\colon P(\varphi) \to \mathcal{P}roj\, \mathcal{R}$ of $\mathbf{Sch}_{/Y}$. In particular, if $\mathcal{R} := \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n}) \in \mathbf{QCohGAlg}(Y)$ (by Proposition 3.17 this is the case if f is quasi-compact and quasi-separated), we can consider the natural morphism

$$\varphi_{f,\mathcal{L}}\colon f^*(\mathcal{R}) \cong \bigoplus_{n\geq 0} f^* \circ f_*(\mathcal{L}^{\otimes n}) \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$$

of $\mathbf{QCohGAlg}(X)$ and the induced morphism $X \supseteq P(\varphi_{f,\mathcal{L}}) \xrightarrow{\mathcal{P}roj\, \varphi_{f,\mathcal{L}}} \mathcal{P}roj(\mathcal{R})$ of $\mathbf{Sch}_{/Y}$.

Definition 3.42. Let X be a scheme. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *ample* if X is quasi-compact and quasi-separated and the following condition is satisfied: for every quasi-coherent $\mathcal{O}_X$-module of finite type $\mathcal{F}$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for every $n \geq n_0$.

Remark 3.43. When X is noetherian the above definition coincides with that of [11], because in this case an $\mathcal{O}_X$-module is quasi-coherent of finite type if and only if it is coherent.

Definition 3.44. Let $f\colon X \to Y$ be a morphism of schemes. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *f-ample* (or *relatively ample*) if there is an open affine cover $\{V_i\}_{i\in I}$ of Y such that $\mathcal{L}|_{f^{-1}(V_i)}$ is ample for every $i \in I$.

Remark 3.45. If $\mathcal{L}$ is f-ample, then f is quasi-compact and quasi-separated (because both properties are local on the codomain, and taking into account Lemma 3.5 and Corollary 3.12). The following result implies (remembering Proposition 3.40) that f is also separated.

Proposition 3.46. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Then an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is f-ample if and only if $\mathcal{R} := \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n}) \in$ $\mathbf{QCohGAlg}(Y)$, $P(\varphi_{f,\mathcal{L}}) = X$ and the natural morphism $\mathcal{P}roj\, \varphi_{f,\mathcal{L}}\colon X \to \mathcal{P}roj\, \mathcal{R}$*

of $\mathbf{Sch}_{/Y}$ *is an open immersion (and then it is also dominant, i.e. its image is dense in* $\mathcal{P}roj\,\mathcal{R}$*). If moreover* Y *is affine, then* $\mathcal{L}$ *is* f*-ample if and only if it is ample.*

Proof. [8, Prop. 4.6.3 and Cor. 4.6.6]. □

Proposition 3.47. *Given a cartesian diagram of schemes*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and an f*-ample invertible* $\mathcal{O}_X$*-module* $\mathcal{L}$*, then* $h^*(\mathcal{L})$ *is* f'*-ample.*

Proof. [8, Prop. 4.6.13]. □

Proposition 3.48. *Given morphisms of schemes* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *with* Z *quasi-compact, an* f*-ample invertible* $\mathcal{O}_X$*-module* $\mathcal{L}$ *and a* g*-ample invertible* $\mathcal{O}_Y$*-module* $\mathcal{M}$*, there exists* $n_0 \in \mathbb{N}$ *such that* $\mathcal{L} \otimes f^*(\mathcal{M})^{\otimes n}$ *is* $(g \circ f)$*-ample for every* $n \geq n_0$*.*

Proof. [8, Prop. 4.6.13]. □

Definition 3.49. A scheme is *quasi-affine* if it is isomorphic to a quasi-compact open subscheme of an affine scheme.

A morphism of schemes $f\colon X \to Y$ is *quasi-affine* if there is an open affine cover $\{V_i\}_{i \in I}$ of Y such that $f^{-1}(V_i) \subseteq X$ is quasi-affine for every $i \in I$.

Proposition 3.50. *For a morphism of schemes* $f\colon X \to Y$ *the following conditions are equivalent:*

1. f *is quasi-affine;*
2. $f = g \circ h$ *with* g *an affine morphism and* h *a quasi-compact open immersion;*
3. f *is quasi-compact,* $f_*(\mathcal{O}_X) \in \mathbf{QCoh}(Y)$ *and the natural morphism* $X \to \mathcal{S}pec\, f_*(\mathcal{O}_X)$ *of* $\mathbf{Sch}_{/Y}$ *(corresponding to* $\mathrm{id}_{f_*(\mathcal{O}_X)}$ *under the bijection of Proposition 3.38) is a (dominant) open immersion;*
4. $\mathcal{O}_X$ *is* f*-ample.*

Proof. (1) ⇒ (4). It is enough to see that, if Z is a quasi-affine scheme (hence it is quasi-compact and there is an open immersion $i\colon Z \hookrightarrow Z'$ with Z' affine), then $\mathcal{O}_Z$ is ample, i.e. that every quasi-coherent $\mathcal{O}_Z$-module of finite type is generated by global sections. More generally, if $\mathcal{F} \in \mathbf{QCoh}(Z)$, then $i_*(\mathcal{F}) \in \mathbf{QCoh}(Z')$ by Proposition 3.17 (i is quasi-compact by Proposition 3.11 and quasi-separated by Example 3.9). Since on an affine scheme every quasi-coherent sheaf is clearly generated by global sections, it follows that $\mathcal{F} \cong i^*(i_*(\mathcal{F}))$ is also generated by global sections.

(3) $\Leftrightarrow$ (4). It follows from Proposition 3.46, since $\mathcal{P}roj\, \varphi_{f,\mathcal{O}_X}$ can be identified with the natural morphism $X \to \mathcal{S}pec\, f_*(\mathcal{O}_X)$.

(3) $\Rightarrow$ (2). By hypothesis f factors as $X \xrightarrow{h} \mathcal{S}pec\, f_*(\mathcal{O}_X) \xrightarrow{g} Y$ with g affine by Proposition 3.38 and h an open immersion, necessarily quasi-compact by Proposition 3.11.

(2) $\Rightarrow$ (1). Clear, since quasi-compact open immersions and affine morphisms are stable under base change. □

Remark 3.51. A quasi-affine morphism is quasi-compact and separated.

Example 3.52. An immersion is quasi-affine if and only if it is quasi-compact.

Corollary 3.53. *For morphisms of schemes the property of being quasi-affine is stable under composition and base change and is local on the codomain.*

Proof. Taking into account that $f\colon X \to Y$ is quasi-affine if and only if $\mathcal{O}_X$ is f-ample, stability under base change follows from Proposition 3.47, and then locality is obvious. As for stability under composition, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are quasi-affine, in order to prove that $g \circ f$ is quasi-affine we can clearly assume that Z is affine, in which case we can apply Proposition 3.48. □

Definition 3.54. A morphism of schemes $f\colon X \to Y$ is *quasi-finite at* $x \in X$ if there exist open neighbourhoods U of x and V of $f(x)$ such that $f(U) \subseteq V$ and $f|_U\colon U \to V$ is of finite type and has discrete fibres. f is *locally quasi-finite* if it is quasi-finite at every point of X (or, equivalently, if it is locally of finite type and has discrete fibres). f is *quasi-finite* if it is locally quasi-finite and quasi-compact.

Example 3.55. A finite morphism of schemes is quasi-finite.

Remark 3.56. It can be proved ([7, Prop. 6.4.4]) that for a morphism of finite type $X \to \operatorname{Spec} \mathbb{K}$ ($\mathbb{K}$ a field) the following conditions are equivalent: $|X|$ is discrete; $|X|$ is finite; every point of X is closed; $X \cong \operatorname{Spec} A$ for some finite $\mathbb{K}$-algebra A. It follows that a morphism of schemes is quasi-finite if and only if it is of finite type and has finite fibres, and that a morphism locally of finite type $f\colon X \to Y$ is locally quasi-finite if and only if $\dim_x f = 0$ for every $x \in X$. It is then also easy to prove the following result.

Proposition 3.57. *For morphisms of schemes the property of being quasi-finite is stable under composition and base change and is local on the codomain; the same is true for the property of being locally quasi-finite, which is also local on the domain.*

Proposition 3.58. *A quasi-finite and separated morphism of schemes is quasi-affine. In particular, a monomorphism of finite type of schemes is quasi-affine.*

Proof. The first statement is proved in [9, Prop. 18.12.12]. As for the second, a monomorphism is separated by Example 3.9 and has discrete fibres because it is injective, so that a monomorphism of finite type is also quasi-finite. □

Definition 3.59. Let $f\colon X \to Y$ be a morphism of schemes. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *f-very ample* (or *relatively very ample*, or simply *very ample*) if there exists $\mathcal{F} \in \mathbf{QCoh}(Y)$ and an immersion $i\colon X \hookrightarrow \mathbb{P}(\mathcal{F})$ of $\mathbf{Sch}_{/Y}$ such that $\mathcal{L} \cong i^*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$.

Remark 3.60. If there exists an f-very ample invertible $\mathcal{O}_X$-module, then f is clearly separated (but in general not quasi-compact).

Proposition 3.61. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is f-very ample, then it is f-ample. If moreover Y is quasi-compact and f is of finite type, then $\mathcal{L}$ is f-ample if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{L}^{\otimes n}$ is f-very ample, if and only if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{L}^{\otimes n}$ is f-very ample for every $n \geq n_0$.*

Proof. [8, Prop. 4.6.2 and Prop. 4.6.11]. □

Definition 3.62. A morphism of schemes $f\colon X \to Y$ is *quasi-projective* if it is of finite type and there exists an invertible $\mathcal{O}_X$-module which is f-ample.

Remark 3.63. By Proposition 3.61 $f\colon X \to Y$ is quasi-projective if it is of finite type and there exists an f-very ample invertible $\mathcal{O}_X$-module, and the viceversa is true if Y is quasi-compact. It can also be proved that f is quasi-affine if and only if it is quasi-compact and $\mathcal{O}_X$ is f-very ample ([8, Prop. 5.1.6]).

Definition 3.64. A morphism of schemes $f\colon X \to Y$ is *projective* if there exists a quasi-coherent $\mathcal{O}_Y$-module of finite type $\mathcal{F}$ and a closed immersion $i\colon X \hookrightarrow \mathbb{P}(\mathcal{F})$ of $\mathbf{Sch}_{/Y}$.

Remark 3.65. In the above notation, if f is projective then $i^*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ is f-very ample.

Proposition 3.66. *If a morphism of schemes $X \to Y$ is projective, then it is quasi-projective and proper, and the viceversa is true if Y is quasi-compact and quasi-separated.*

Proof. [8, Thm. 5.5.3] and [9, Prop. 1.7.19]. □

Remark 3.67. It can be proved that if Y is such that there exists an ample invertible $\mathcal{O}_Y$-module, then a morphism $X \to Y$ is quasi-projective (respectively projective) if and only if there exists a quasi-compact (respectively closed) immersion $X \hookrightarrow \mathbb{P}^n_Y$ of $\mathbf{Sch}_{/Y}$ for some $n \in \mathbb{N}$ ([8, Cor. 5.3.3 and rem. 5.5.4]). Therefore in this case the definition of projective morphism coincides with that of [11], and the same is true for quasi-projective if moreover Y is noetherian (in which case every immersion $X \hookrightarrow \mathbb{P}^n_Y$ is quasi-compact).

Proposition 3.68. *For morphisms of schemes the properties of being quasi-projective and projective are stable under base change.*

Proof. Since the properties of being of finite type and a closed immersion are stable under base change, it follows from Proposition 3.47 and Proposition 3.41. □

Remark 3.69. It is not true in general that the property of being (quasi)projective is stable under composition. However, it is easy to prove (using Proposition 3.48 and Proposition 3.66) that given a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f and g quasi-projective (respectively projective) and Z quasi-compact (respectively quasi-compact and quasi-separated), then $g \circ f$ is also quasi-projective (respectively projective). On the other hand, the property of being (quasi)projective is not local on the codomain, even if one restricts to schemes of finite type over a field.

Example 3.70. Let $\mathbb{K}$ be a field and let $f\colon \tilde{X} \to X$ be a morphism of $\mathbf{Sch}_{/\mathbb{K}}$ as in [11, B, ex. 3.4.1] (the hypothesis $\mathbb{K} = \mathbb{C}$ is not needed). Then $X \to \operatorname{Spec}\mathbb{K}$ is projective and there exist $x_1, x_2 \in X$ (denoted by P and Q in [11]) such that the restriction $f|_{f^{-1}(X \setminus \{x_i\})}\colon f^{-1}(X \setminus \{x_i\}) \to X \setminus \{x_i\}$ is projective for $i = 1, 2$, but f (which is then clearly proper) is not projective (hence not even quasi-projective, by Proposition 3.66), because $\tilde{X} \to \operatorname{Spec}\mathbb{K}$ is not projective.

3.4 (Faithfully) flat morphisms

Definition 3.71. A morphism of schemes $f\colon X \to Y$ is said to be *flat at* $x \in X$ if $f_x^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat morphism of rings. f is *flat* if it is flat at every point of X.

Remark 3.72. A morphism of rings $\phi\colon A \to B$ is flat if and only if the morphism of schemes $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Proposition 3.73. *For morphisms of schemes the property of being flat is stable under composition and base change and is local on the domain and on the codomain.*

Proof. Straightforward. □

Proposition 3.74. *Given a cartesian diagram in* **Sch**

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

such that g is flat and f is quasi-compact and quasi-separated, then for every $\mathcal{F} \in \mathbf{QCoh}(X)$ the natural morphism $g^* \circ f_*(\mathcal{F}) \to f'_* \circ h^*(\mathcal{F})$ is an isomorphism (of $\mathbf{QCoh}(Y')$, by Proposition 3.17).

Proof. [9, Lemme 2.3.1]. □

Definition 3.75. Let A be a ring. An A-module M is *faithfully flat* if the functor $M \otimes_A -\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ is exact (i.e., M is a flat A-module) and faithful. A morphism of rings $A \to B$ is *faithfully flat* if B is a faithfully flat A-module.

Remark 3.76. It is clear that a morphism of rings $\phi\colon A \to B$ is faithfully flat if and only if $\phi_*\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ is exact and faithful.

Lemma 3.77. *For an A-module M the following conditions are equivalent:*

1. *M is faithfully flat;*
2. *a sequence of A-modules $N' \xrightarrow{f} N \xrightarrow{g} N''$ is exact if and only if*
$$M \otimes_A N' \xrightarrow{\mathrm{id}\otimes f} M \otimes_A N \xrightarrow{\mathrm{id}\otimes g} M \otimes_A N''$$
is exact;
3. *M is flat and if $0 \neq N$ is an A-module, then $M \otimes_A N \neq 0$.*

Proof. (1) $\Rightarrow$ (3). Given an A-module $N \neq 0$, obviously $\mathrm{id}_N \neq 0\colon N \to N$; therefore, $\mathrm{id}_{M\otimes_A N} = \mathrm{id}_M \otimes \mathrm{id}_N \neq 0 : M \otimes_A N \to M \otimes_A N$, so that $M \otimes_A N \neq 0$.

(3) $\Rightarrow$ (2). Given a sequence of A-modules $N' \xrightarrow{f} N \xrightarrow{g} N''$ such that
$$M \otimes_A N' \xrightarrow{\mathrm{id}\otimes f} M \otimes_A N \xrightarrow{\mathrm{id}\otimes g} M \otimes_A N''$$
is exact, we have to show that $N' \xrightarrow{f} N \xrightarrow{g} N''$ is exact, too. As $M \otimes_A -$ is exact, we have
$$M \otimes_A \mathrm{im}(g \circ f) \cong \mathrm{im}(\mathrm{id}_M \otimes (g \circ f)) = \mathrm{im}((\mathrm{id}_M \otimes g) \circ (\mathrm{id}_M \otimes f)) = 0.$$
The hypothesis then implies $\mathrm{im}(g \circ f) = 0$, i.e. $g \circ f = 0$. Similarly,
$$M \otimes_A (\ker g/\operatorname{im} f) \cong \ker(\mathrm{id}_M \otimes g)/\operatorname{im}(\mathrm{id}_M \otimes f) = 0,$$
whence $\ker g/\operatorname{im} f = 0$.

(2) $\Rightarrow$ (1). M is clearly flat, and, since $M \otimes_A -$ is an additive functor, it is enough to prove that if $f\colon N' \to N$ is a morphism of A-modules such that $\mathrm{id}_M \otimes f = 0$, then $f = 0$. Now, the sequence $N' \xrightarrow{f} N \xrightarrow{\mathrm{id}} N$ is exact because the sequence
$$M \otimes_A N' \xrightarrow{\mathrm{id}\otimes f=0} M \otimes_A N \xrightarrow{\mathrm{id}} M \otimes_A N$$
is exact, and this implies that $f = 0$. □

Proposition 3.78. *Let $\phi\colon A \to B$ be a flat morphism of rings. Then the following conditions are equivalent:*

1. *ϕ is faithfully flat;*

2. $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective;*

3. *for every maximal ideal* $\mathtt{m} \subset A$, $\phi(\mathtt{m})B \neq B$.

Proof. (1) $\Rightarrow$ (2). Given $\mathtt{p} \in \operatorname{Spec} A$, the fibre of $\operatorname{Spec}\phi$ over $\mathtt{p}$ is not empty because $(\operatorname{Spec}\phi)^{-1}(\mathtt{p}) \cong \operatorname{Spec}(B \otimes_A \kappa(\mathtt{p}))$ and $B \otimes_A \kappa(\mathtt{p}) \neq 0$ by Lemma 3.77.

(2) $\Rightarrow$ (3). Given a maximal ideal $\mathtt{m} \subset A$, there exists $\mathtt{q} \in \operatorname{Spec} B$ such that $\phi^{-1}(\mathtt{q}) = \mathtt{m}$, whence $\phi(\mathtt{m})B \subseteq \mathtt{q} \neq B$.

(3) $\Rightarrow$ (1). Again by Lemma 3.77, we have to prove that, if $0 \neq N$ is an A-module, then $B \otimes_A N \neq 0$. Choosing $0 \neq x \in N$ and setting $N' := (x) \subseteq N$, the natural map $B \otimes_A N' \to B \otimes_A N$ is injective (because ϕ is flat); therefore it is enough to show that $B \otimes_A N' \neq 0$. Now, $N' \cong A/\mathtt{I}$ for some ideal $\mathtt{I} \subsetneq A$, so that $\mathtt{I} \subseteq \mathtt{m}$ for some maximal ideal $\mathtt{m}$ of A. Then $B \otimes_A N' \cong B/\phi(\mathtt{I})B \neq 0$ because $\phi(\mathtt{I})B \subseteq \phi(\mathtt{m})B \neq B$ by hypothesis. □

Corollary 3.79. *Let $\phi\colon A \to B$ be a flat morphism of local rings. Then ϕ is faithfully flat if and only if it is local.*

Remark 3.80. It is easy to see that a faithfully flat morphism of rings $\phi\colon A \to B$ is injective. Indeed, $B \otimes_A \ker\phi = 0$ (for every $b \in B$ and every $a \in \ker\phi$ we have $b \otimes a = \phi(a)b \otimes 1 = 0$), whence $\ker\phi = 0$ by Lemma 3.77. We will see in Proposition 7.1 that a stronger result actually holds.

Proposition 3.81. *If $f\colon X \to Y$ is a flat morphism locally of finite presentation of schemes, then f is open.*

Proof. For every $x \in X$ the local morphism of local rings $f_x^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is faithfully flat by Corollary 3.79, hence $\operatorname{Spec} f_x^\#\colon \operatorname{Spec}\mathcal{O}_{X,x} \to \operatorname{Spec}\mathcal{O}_{Y,f(x)}$ is surjective by Proposition 3.78. Then f is open by Proposition 3.29. □

Example 3.82. The morphism $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ is flat but not open (it is clearly not locally of finite presentation).

Definition 3.83. A morphism of schemes is *faithfully flat* if it is flat and surjective.

Remark 3.84. By Proposition 3.78 a morphism of rings $\phi\colon A \to B$ is faithfully flat if and only if the morphism of schemes $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat.

Remark 3.85. It is not difficult to see that a morphism of schemes $f\colon X \to Y$ is (faithfully) flat if and only if $f^*\colon \mathbf{Mod}(Y) \to \mathbf{Mod}(X)$ (or $f^*\colon \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$) is exact (and faithful).

3.5 Formally unramified, formally smooth and formally étale morphisms

Definition 3.86. A morphism of schemes $X \to Y$ is *formally unramified* (respectively *formally smooth*, respectively *formally étale*) if for every affine scheme W, every closed subscheme $W' \subseteq W$ defined by a nilpotent[6] ideal $\mathcal{I} \subset \mathcal{O}_W$ and every morphism $W \to Y$, the natural map $\mathrm{Hom}_Y(W, X) \to \mathrm{Hom}_Y(W', X)$ is injective (respectively surjective, respectively bijective).

More explicitly, a morphism of schemes $f\colon X \to Y$ is formally unramified (respectively formally smooth, respectively formally étale) if and only if for every $\iota\colon W' \hookrightarrow W$ as in the definition and for every morphisms $k'\colon W' \to X$ and $h\colon W \to Y$ such that $f \circ k' = h \circ \iota$, there exists at most one (respectively there exists, respectively there exists unique) $k\colon W \to X$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ W' \arrow[u, "k'"] \arrow[r, hook, "\iota"'] & W \arrow[u, "h"'] \arrow[ul, dashed, "k"'] \end{tikzcd} \tag{3.1}$$

such that $f \circ k = h$ and $k \circ \iota = k'$.

Definition 3.87. A morphism of rings $\phi\colon A \to B$ is *formally unramified* (respectively *formally smooth*, respectively *formally étale*) if the morphism of schemes $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is formally unramified (respectively formally smooth, respectively formally étale).

Remark 3.88. Obviously a morphism (of schemes or of rings) is formally étale if and only if it is formally unramified and formally smooth.

Remark 3.89. If $W' \subseteq W$ are as in Definition 3.86, then $|W'| = |W|$ (because $\mathcal{I}$ is nilpotent) and, if there exists k such that (3.1) commutes, then $|k| = |k'|$. Since moreover W is affine, say $W = \operatorname{Spec} C$, and $\mathcal{I}$ is quasi-coherent, we have $\mathcal{I} = \widetilde{\mathtt{I}}$ for some (nilpotent) ideal $\mathtt{I}$ of C, so that $W' = \operatorname{Spec} C/\mathtt{I}$ is also affine and the closed immersion $W' \hookrightarrow W$ is induced by $C \twoheadrightarrow C/\mathtt{I}$. In particular, we see that definition Definition 3.87 can be reformulated inside the category of rings.

Remark 3.90. An easy induction argument shows that we would get an equivalent definition if in Definition 3.86 we required $\mathcal{I}^2 = 0$ instead of $\mathcal{I}$ nilpotent. We will freely use this fact in the following.

Example 3.91. A monomorphism (in particular, an immersion) of schemes is formally unramified: indeed, in a diagram like (3.1), by definition of monomorphism there exists at most one $k\colon W \to X$ such that $f \circ k = h$. It is clear that an open immersion is also formally étale. On the other hand, a closed immersion is not formally smooth, in general.

[6] This means that $\mathcal{I}^n = 0$ for some $n \in \mathbb{N}$.

Example 3.92. Obviously the natural morphism $A \to A[t_i]_{i \in I}$ is formally smooth for every ring A and for every set I, whereas it is formally unramified if and only if $I = \emptyset$.

Example 3.93. If A is a ring and $S \subset A$ is a multiplicative system, then the natural morphism $A \to S^{-1}A$ is formally étale: this follows immediately from the universal property of localization, together with the fact that, if $\mathtt{I}$ is a nilpotent ideal of a ring C, then an element of C is invertible if and only if its image in $C/\mathtt{I}$ is invertible.

Lemma 3.94. *If a morphism of schemes $f\colon X \to Y$ is formally unramified (respectively formally étale) then for every scheme Z, for every closed subscheme $Z' \subseteq Z$ defined by a locally nilpotent[7] ideal $\mathcal{I} \subset \mathcal{O}_Z$ and for every morphism $h\colon Z \to Y$, the natural map $\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z', X)$ is injective (respectively bijective).*

Proof. Given a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ Z' \arrow[u, "k'"] \arrow[r, hook] & Z \arrow[u, "h"'] \arrow[ul, dashed, "k"'] \end{tikzcd}$$

we have to show that there exists at most one (respectively there exists unique) $k\colon Z \to X$ such that the whole diagram is commutative. Let $\{W_i\}_{i \in I}$ be an open affine cover of Z such that $\mathcal{I}_i := \mathcal{I}|_{W_i}$ is nilpotent for every $i \in I$. Then $W_i' := Z' \times_Z W_i \subseteq W_i$ is the closed subscheme defined by $\mathcal{I}_i$. Therefore, by definition, for every $i \in I$ there exists at most one (respectively there exists unique) $k_i\colon W_i \to X$ such that $k_i|_{W_i'} = k'|_{W_i'}$ and $f \circ k_i = h|_{W_i}$. It is then clear that there exists at most one k making the diagram commute. If moreover f is formally étale, then for all $i, j \in I$ we have $k_i|_{W_i \cap W_j} = k_j|_{W_i \cap W_j}$ (because, again by definition, $k_i|_W = k_j|_W$ for every open affine subset W of $W_i \cap W_j$), so that k exists because morphisms of schemes can be glued. □

Proposition 3.95. *For morphisms of schemes the properties of being formally unramified, formally smooth and formally étale are stable under composition and base change.*

Proof. Given $f\colon X \to Y$ and $g\colon Y \to Z$ formally unramified (respectively formally smooth), we have to show that given $W' \subseteq W$ as in Definition 3.86 and a

[7] This means that for every $z \in Z$ there exists an open neighbourhood U of z such that $\mathcal{I}|_U$ is nilpotent. If Z is quasi-compact (in particular, if Z is affine) then $\mathcal{I}$ is nilpotent if and only if it is locally nilpotent.

commutative square

$$\begin{tikzcd} X \arrow[r, "g\circ f"] & Z \\ W' \arrow[u, "k'"] \arrow[r, hook] & W \arrow[u, "h"'] \arrow[ul, dashed, "k"'] \end{tikzcd}$$

there exists at most one (respectively there exists) $k\colon W \to X$ such that the whole diagram is commutative. Now, given $k, \tilde{k}\colon W \to X$ such that the above diagram is commutative, we have $f \circ k = f \circ \tilde{k}$ if g is formally unramified, whence $k = \tilde{k}$ if also f is formally unramified. On the other hand, if g is formally smooth, there exists $\bar{k}\colon W \to Y$ such that $g \circ \bar{k} = h$ and $\bar{k}|_{W'} = f \circ k'$. Therefore, if also f is formally smooth, there exists $k\colon W \to X$ such that $f \circ k = \bar{k}$ (whence $g \circ f \circ k = h$) and $k|_{W'} = k'$.

As for base change, if $X' \to Y'$ is obtained by a base change $Y' \to Y$ from $X \to Y$, then, for every morphism $T \to Y'$, there is a natural bijection $\mathrm{Hom}_{Y'}(T, X') \cong \mathrm{Hom}_Y(T, X)$. It follows that, given $W' \subseteq W$ as in Definition 3.86 and a morphism $W \to Y'$, the natural map $\mathrm{Hom}_{Y'}(W, X') \to \mathrm{Hom}_{Y'}(W', X')$ is injective (respectively surjective) if and only if the natural map $\mathrm{Hom}_Y(W, X) \to \mathrm{Hom}_Y(W', X)$ is injective (respectively surjective). □

Proposition 3.96. *For morphisms of schemes, the properties of being formally unramified and formally étale are local on the domain and on the codomain.*

Proof. Taking into account that the property of being formally unramified (respectively formally étale) is stable under composition and base change and that open immersions are formally étale, the only non trivial thing to prove is the following: if $\{U_i\}_{i\in I}$ is an open cover of X and $f\colon X \to Y$ is a morphism of schemes such that $f_i := f|_{U_i}$ is formally unramified (respectively formally étale) for every $i \in I$, then f is formally unramified (respectively formally étale), too. Given a commutative square like (3.1), we have to prove that there exists at most one (respectively there exists unique) k making the diagram commute. For every $i \in I$ let $W'_i := {k'}^{-1}(U_i) \subseteq W'$ and let $W_i \subseteq W$ be the open subscheme defined by $|W_i| = |W'_i|$, so that there is a commutative diagram

$$\begin{tikzcd} U_i \arrow[r, "f_i"] & Y \\ W'_i \arrow[u, "k'_i"] \arrow[r, hook] & W_i \arrow[u, "h_i"'] \end{tikzcd}$$

where k'_i and h_i denote the restrictions of k' and h. By Lemma 3.94 there exists at most one (respectively there exists unique) $k_i\colon W_i \to U_i$ such that $k_i|_{W'_i} = k'_i$ and $f_i \circ k_i = h_i$, and then the uniqueness (respectively the existence and uniqueness) of k follows. □

Remark 3.97. It seems to be an open problem whether Proposition 3.96 is true also for formally smooth (see [17, 4.15], where the term quasi-smooth is used in place of formally smooth): in [9] this is stated, but with an incorrect proof. Notice, however, that if $f\colon X \to Y$ is formally smooth and $U \subseteq X$ and $V \subseteq Y$ are open subschemes such that $f(U) \subseteq V$, then $f|_U\colon U \to V$ is formally smooth, too.

Proposition 3.98. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes.*

1. *If $g \circ f$ is formally unramified, then f is formally unramified, too.*
2. *If $g \circ f$ is formally smooth and g is formally unramified, then f is formally smooth.*

Proof. By Lemma A.5 (and remembering that Δ_g is in any case formally unramified by Example 3.91), it is enough to show that Δ_g is formally smooth (hence formally étale) if g is formally unramified. So we have to prove that, given $W' \subseteq W$ as in Definition 3.86 and a commutative square

$$\begin{tikzcd} Y \arrow[r, "\Delta_g"] & Y \times_Z Y \\ W' \arrow[u, "k'"] \arrow[r, hook] & W \arrow[u, "h"'] \arrow[ul, dashed, "k"'] \end{tikzcd}$$

there exists (unique) k such that the whole diagram is commutative. Indeed, the fact that g is formally unramified implies that $pr_1 \circ h = pr_2 \circ h\colon W \to Y$, and then we can take $k := pr_1 \circ h$. $\square$

Corollary 3.99. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes with g formally étale. Then f is formally unramified (respectively formally smooth, respectively formally étale) if and only if $g \circ f$ is formally unramified (respectively formally smooth, respectively formally étale).*

Lemma 3.100. *A morphism of rings $\phi\colon A \to B$ is formally unramified if and only if $\Omega_{B/A} = 0$.*

Proof. Assume first that $\Omega_{B/A} = 0$. Then in a commutative diagram of rings

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"'] & B \arrow[d, "\psi'"] \arrow[dl, bend right, "\tilde{\psi}"'] \arrow[dl, bend left, "\psi"] \\ C \arrow[r, two heads] & C/\mathtt{I} \end{tikzcd}$$

(where $\mathtt{I} \subset C$ is an ideal such that $\mathtt{I}^2 = 0$) necessarily $\tilde{\psi} = \psi$, since (by Lemma A.25) $\tilde{\psi} - \psi \in \mathrm{Der}_A(B, \mathtt{I}) \cong \mathrm{Hom}_B(\Omega_{B/A}, \mathtt{I}) = 0$. By definition, this proves that ϕ is formally unramified.

Assume conversely that ϕ is formally unramified. Again by Lemma A.25 there is a commutative diagram of rings (where $B \oplus \Omega_{B/A}$ is a ring with multiplication $(b,\omega)(b',\omega') := (bb', b\omega' + b'\omega)$)

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "{(\phi,0)}"'] & B \arrow[d, "\mathrm{id}"] \arrow[dl, bend right, "{(\mathrm{id},d_{B/A})}"'] \arrow[dl, bend left, "{(\mathrm{id},0)}"] \\
B \oplus \Omega_{B/A} \arrow[r] & B.
\end{tikzcd}$$

The hypothesis on ϕ implies that $d_{B/A} = 0$, whence $\Omega_{B/A} = 0$. $\square$

Corollary 3.101. *A morphism of schemes $f\colon X \to Y$ is formally unramified if and only if $\Omega_{X/Y} = 0$.*

Proof. It follows immediately from Proposition 3.96 and the fact that, if $U \subseteq X$ and $V \subseteq Y$ are open subschemes such that $f(U) \subseteq V$, then $\Omega_{X/Y}|_U \cong \Omega_{U/V}$. $\square$

Lemma 3.102. *Let $A \to A'$ be a morphism of rings and $\mathtt{I} \subset A'$ an ideal. If $A \to A'/\mathtt{I}$ is formally smooth, then $d^{\mathtt{I}}_{A'/A}\colon \mathtt{I}/\mathtt{I}^2 \to \Omega_{A'/A} \otimes_{A'} (A'/\mathtt{I})$ is left invertible, and the viceversa is also true if $A \to A'$ is formally smooth.*

Proof. By Lemma A.26 $d^{\mathtt{I}}_{A'/A}$ is left invertible if and only if there is a morphism of A-algebras $\phi\colon A'/\mathtt{I} \to A'/\mathtt{I}^2$ such that $\pi \circ \phi = \mathrm{id}_{A'/\mathtt{I}}$. Now, such a ϕ exists by definition if $A \to A'/\mathtt{I}$ is formally smooth (note that $A'/\mathtt{I} \cong (A'/\mathtt{I}^2)/(\mathtt{I}/\mathtt{I}^2)$ and obviously $\mathtt{I}/\mathtt{I}^2 \subset A'/\mathtt{I}^2$ is a nilpotent ideal). Conversely, assuming that such a ϕ exists and that $A \to A'$ is formally smooth, we have to prove that given a commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A'/\mathtt{I} \arrow[d] \arrow[dl, dashed, "\psi"'] \\
B \arrow[r, two heads] & B/\mathtt{J}
\end{tikzcd}$$

(where $\mathtt{J} \subset B$ is an ideal such that $\mathtt{J}^2 = 0$), there is a morphism ψ such that the diagram remains commutative. Since $A \to A'$ is formally smooth, there is a morphism of A-algebras $\psi'\colon A' \to B$ such that $A' \twoheadrightarrow A'/\mathtt{I} \to B/\mathtt{J}$ factors through ψ'. Then clearly $\psi'(\mathtt{I}) \subseteq \mathtt{J}$, so that ψ' induces a morphism $\tilde{\psi}\colon A'/\mathtt{I}^2 \to B/\mathtt{J}^2 = B$, and it is immediate to see $\psi := \tilde{\psi} \circ \phi$ has the required property. $\square$

Corollary 3.103. *If $A \to B$ is a formally smooth morphism of rings, then $\Omega_{B/A}$ is a projective B-module.*

Proof. Clearly $B \cong A'/\mathtt{I}$ as A-algebras, where $A' = A[t_i]_{i \in I}$ for some set I and $\mathtt{I} \subset A'$ is an ideal. Then $d^{\mathtt{I}}_{A'/A}$ is left invertible, which implies that $\operatorname{coker} d^{\mathtt{I}}_{A'/A} \cong \Omega_{B/A}$ is an direct summand of $\Omega_{A'/A} \otimes_{A'} B \cong B^I$ (hence it is a projective B-module). $\square$

3.6 Unramified, smooth and étale morphisms

Definition 3.104. A morphism of schemes is *unramified* if it is formally unramified and locally of finite type; it is *smooth* (respectively *étale*) if it is formally smooth (respectively formally étale) and locally of finite presentation.

Remark 3.105. In [9] a morphism of schemes is defined to be unramified if it is formally unramified and locally of finite presentation.

Definition 3.106. A morphism of rings $\phi\colon A \to B$ is *unramified* (respectively *smooth*, respectively *étale*) if the morphism of schemes $\operatorname{Spec}\phi\colon \operatorname{Spec}B \to \operatorname{Spec}A$ is unramified (respectively smooth, respectively étale).

Remark 3.107. Obviously a morphism (of schemes or of rings) is étale if and only if it is unramified and smooth.

Remark 3.108. It follows from Example 3.22 and Example 3.91 that an immersion of schemes is unramified and that an open immersion is étale (whereas a closed immersion is not smooth in general).

Proposition 3.109. *For morphisms of schemes, the properties of being unramified, smooth and étale are stable under composition and base change and are local on the domain and on the codomain.*

Proof. By Proposition 3.95, Proposition 3.96 and Corollary 3.25 it remains to show that the property of being smooth is local. So (remembering that open immersions are smooth) we can assume that $\{U_i\}_{i\in I}$ is an open cover of X and $f\colon X \to Y$ is a morphism such that $f_i := f|_{U_i}$ is smooth for every $i \in I$, and we have to prove that f is formally smooth, i.e. that given a commutative square like (3.1), there exists k making the diagram commute. Let $\mathcal{P}$ be the sheaf on $|W'| = |W|$ defined in the following way: for every $U \subseteq W$ open $\mathcal{P}(U)$ is the set of morphisms $k_U\colon U \to X$ such that

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "k'|_{U'}"] \arrow[r, hook] & U \arrow[ul, "k_U"'] \arrow[u, "h|_U"'] \end{tikzcd}$$

commutes (where $U' := U \times_W W' \subseteq W'$). Using Lemma A.25 it is not difficult to prove that $\mathcal{P}$ is in a natural way a pseudo-torsor (see Definition A.28) under the (additive) sheaf of groups $\mathcal{H} := \mathcal{H}om_{W'}(k'^*\Omega_{X/Y}, \mathcal{I})$. f_i being formally smooth means that $\mathcal{P}(U) \neq \emptyset$ if U is affine and $k'(U') \subseteq U_i$, so that $\mathcal{P}$ is actually a $\mathcal{H}$-torsor. As $k'^*\Omega_{X/Y}$ is an $\mathcal{O}_{W'}$-module of finite presentation by Lemma 3.28, it follows from Lemma A.30 that $\mathcal{H}$ is quasi-coherent, whence $H^1(W', \mathcal{H}) = 0$. Therefore $\mathcal{P}$ is trivial by Proposition A.29; in particular, it has global sections, i.e. there exists $k\colon W \to X$ such that (3.1) commutes. $\square$

Definition 3.110. A morphism of schemes $f\colon X \to Y$ is *unramified* (respectively *smooth*, respectively *étale*) *at* $x \in X$ if there exists an open neighbourhood U of x such that $f|_U$ is unramified (respectively smooth, respectively étale).

Remark 3.111. By Proposition 3.109 $f\colon X \to Y$ is unramified (respectively smooth, respectively étale) if and only if it is unramified (respectively smooth, respectively étale) at every point of X. It is also clear from the definition that in any case the set of points where f is unramified (respectively smooth, respectively étale) is open in X.

Remark 3.112. If a morphism of schemes $f\colon X \to Y$ is smooth, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module. Indeed, if f is smooth at $x \in X$, then $\Omega_{X/Y}$ is of finite presentation in a neighbourhood of x by Lemma 3.28, so that $(\Omega_{X/Y})_x$ is a finitely generated projective $\mathcal{O}_{X,x}$-module by Corollary 3.103 (for modules the property of being projective is clearly stable under localization), hence $(\Omega_{X/Y})_x \cong \mathcal{O}_{X,x}^n$ for some $n \in \mathbb{N}$ (it follows easily from Nakayama's lemma that a finitely generated projective module over a local ring is free). Therefore by Corollary A.31 there is an open neighbourhood U of x such that $\Omega_{U/Y} \cong \mathcal{O}_U^n$.

Proposition 3.113. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes.*

1. *If* $g \circ f$ *is unramified, then* f *is unramified, too.*
2. *If* $g \circ f$ *is smooth and* g *is unramified, then* f *is smooth.*

Proof. It follows from Proposition 3.27 and Proposition 3.98. □

Corollary 3.114. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes with* g *étale. Then* f *is unramified (respectively smooth, respectively étale) if and only if* $g \circ f$ *is unramified (respectively smooth, respectively étale).*

We are going to see that unramified, smooth and étale morphisms could be defined in several alternative (equivalent) ways.

Proposition 3.115. *Let* $f\colon X \to Y$ *be a morphism of schemes. Assuming* f *is of finite type at* $x \in X$*, and setting* $y := f(x) \in Y$*, the following conditions are equivalent:*

1. f *is unramified at* x*;*
2. $f_x^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ *is a formally unramified morphism of rings;*
3. *(Jacobian criterion) if* $\operatorname{Spec} B \cong U \subseteq X$ *and* $\operatorname{Spec} A \cong V \subseteq Y$ *are open affine neighbourhoods of* x *and* y *such that* $f(U) \subseteq V$ *and* $B \cong A'/\mathtt{I}$ *as* A*-algebra (where* $A' := A[t_1, \dots, t_n]$ *for some* $n \in \mathbb{N}$ *and* $\mathtt{I} \subset A'$ *is an ideal), then, denoting by* $\mathtt{p}' \in \operatorname{Spec} A'$ *the prime ideal corresponding to* x*, there exist* $P_1, \dots P_n \in \mathtt{I}$ *such that* $\det\left(\frac{\partial P_j}{\partial t_i}\right)_{1 \le i,j \le n} \notin \mathtt{p}'$*;*

4. $(\Omega_{X/Y})_x = 0$*;*

5. $\Delta_f \colon X \to X \times_Y X$ *is a local isomorphism at* x*;*

6. the induced morphism $f^{-1}(y) \to \operatorname{Spec} \kappa(y)$ *is unramified at* x*.*

Proof. (1) $\Leftrightarrow$ (4). It follows from Corollary 3.101, taking into account that, if $(\Omega_{X/Y})_x = 0$, there exists an open neighbourhood U of x such that $\Omega_{X/Y}|_U \cong \Omega_{U/Y} = 0$ (because, by Lemma 3.28, $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite type in a neighbourhood of x).

(2) $\Leftrightarrow$ (4). Since $(\Omega_{X/Y})_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}}$, this follows from Lemma 3.100.

(3) $\Leftrightarrow$ (4). [21, V, Thm. 5].

(4) $\Leftrightarrow$ (5). As Δ_f is an immersion, X can be identified with a closed subscheme of W (defined by some ideal sheaf $\mathcal{I} \subset \mathcal{O}_W$), where W is an open subscheme of $X \times_Y X$. Since $\Omega_{X/Y} \cong \mathcal{I}/\mathcal{I}^2|_X$, we have $(\Omega_{X/Y})_x \cong \mathcal{I}_{(x,x)}/\mathcal{I}^2_{(x,x)}$. Now, if Δ_f is a local isomorphism at x, $\mathcal{I}_{(x,x)} = 0$, so that $(\Omega_{X/Y})_x = 0$. Conversely, if $(\Omega_{X/Y})_x = 0$, then, taking into account that $\mathcal{I}$ is an $\mathcal{O}_W$-module of finite type in a neighbourhood of x (see the proof of Lemma 3.26), $\mathcal{I}_{(x,x)} = 0$ by Nakayama's lemma (obviously $\mathcal{I}_{(x,x)} \subseteq \mathtt{m}_{(x,x)}$), whence $\mathcal{I}|_V = 0$ for some open neighbourhood V of (x,x) in W, which proves that Δ_f is a local isomorphism at x.

(1) $\Rightarrow$ (6). It follows from Proposition 3.109.

(6) $\Rightarrow$ (4). Denoting by $i \colon f^{-1}(y) \hookrightarrow X$ the natural morphism, we have

$$\begin{aligned}(\Omega_{f^{-1}(y)/\operatorname{Spec}\kappa(y)})_x &\cong (i^*\Omega_{X/Y})_x \\ &\cong (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{f^{-1}(y),x} \cong (\Omega_{X/Y})_x/\mathtt{m}_y(\Omega_{X/Y})_x.\end{aligned}$$

On the other hand, we know that $(\Omega_{f^{-1}(y)/\operatorname{Spec}\kappa(y)})_x = 0$ by the already proved equivalence between (1) and (4). Therefore $(\Omega_{X/Y})_x = 0$ by Nakayama's lemma (we have already noted that $(\Omega_{X/Y})_x$ is a finitely generated $\mathcal{O}_{X,x}$-module). $\square$

Corollary 3.116. *A morphism locally of finite type of schemes* $f \colon X \to Y$ *is unramified if and only if* $\Omega_{X/Y} = 0$*, if and only if* $\Delta_f \colon X \to X \times_Y X$ *is an open immersion.*

Remark 3.117. By Proposition 3.115, in order to see if a morphism of schemes is unramified, it is enough to look at the fibres, hence one can reduce to consider morphisms locally of finite type to $\operatorname{Spec} \mathbb{K}$ ($\mathbb{K}$ a field). It can be proved (see [21, III, Prop. 11]) that a morphism of finite type of rings $\mathbb{K} \to A$ is unramified if and only if A is isomorphic (as $\mathbb{K}$-algebra) to a finite product of finite separable field extensions of $\mathbb{K}$. It follows easily that, if a morphism of schemes $f \colon X \to Y$ is of finite type at $x \in X$, then f is unramified at x if and only if $\mathtt{m}_y\mathcal{O}_{X,x} = \mathtt{m}_x$ and the natural map (induced by $f_x^{\#}$) $\kappa(y) = \mathcal{O}_{Y,y}/\mathtt{m}_y \to \mathcal{O}_{f^{-1}(y),x} \cong \mathcal{O}_{X,x}/\mathtt{m}_y\mathcal{O}_{X,x} = \kappa(x)$ is a finite separable extension of fields (which is the definition of [11]). This fact clearly implies that, if f is unramified at x, then it is quasi-finite at x (hence every unramified morphism of schemes is locally quasi-finite).

Definition 3.118. Let $\mathbb{K}$ be a field. A morphism of schemes $f\colon X \to \operatorname{Spec}\mathbb{K}$ (or X, by abuse of notation) is *geometrically regular at* $x \in X$ if $X \times_{\operatorname{Spec}\mathbb{K}} \operatorname{Spec}\overline{\mathbb{K}}$ is regular at every point lying over x. f is *geometrically regular* if it is geometrically regular at every point of X.

Lemma 3.119. *Let $f\colon X \to \operatorname{Spec}\mathbb{K}$ ($\mathbb{K}$ a field) be a morphism of schemes. If f is of finite type at $x \in X$, then f is smooth at x if and only if it is geometrically regular at x. If moreover $\mathbb{K}$ is perfect, then this is the case if and only if X is regular at x.*

Proof. The second statement is proved in [17, Prop. 7.6], and then the first follows from [17, Prop. 4.6]. □

Proposition 3.120. *Let $f\colon X \to Y$ be a morphism of schemes. Assuming f is of finite presentation at $x \in X$, and setting $y := f(x) \in Y$, the following conditions are equivalent:*

1. *f is smooth at x;*
2. *$f^{\#}_x\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a formally smooth morphism of rings;*
3. *(Jacobian criterion) if $\operatorname{Spec}B \cong U \subseteq X$ and $\operatorname{Spec}A \cong V \subseteq Y$ are open affine neighbourhoods of x and y such that $f(U) \subseteq V$ and $B \cong A'/\mathtt{I}$ as A-algebra (where $A' := A[t_1,\dots,t_n]$ for some $n \in \mathbb{N}$ and $\mathtt{I} \subset A'$ is a finitely generated ideal), then, denoting by $\mathtt{p}' \in \operatorname{Spec}A'$ the prime ideal corresponding to x, there exist $m \le n$ (and then $m = n - \dim_x f$), $P_1,\dots,P_m \in \mathtt{I}$ and $1 \le l_1 < \dots < l_m \le n$ such that $(P_1,\dots,P_m)_{\mathtt{p}'} = \mathtt{I}_{\mathtt{p}'}$ and $\det\left(\frac{\partial P_j}{\partial t_{l_i}}\right)_{1\le i,j\le m} \notin \mathtt{p}'$;*
4. *f is flat at x and the induced morphism $f^{-1}(y) \to \operatorname{Spec}\kappa(y)$ is smooth at x;*
5. *f is flat at x and $f^{-1}(y) \to \operatorname{Spec}\kappa(y)$ is geometrically regular at x;*
6. *f is flat at x and $(\Omega_{X/Y})_x$ is a free $\mathcal{O}_{X,x}$-module of rank $\dim_x f$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Let $\mathtt{q} \in \operatorname{Spec}B$ and $\mathtt{p} \in \operatorname{Spec}A$ be the prime ideals corresponding to x and y, so that $\mathtt{p}$ (respectively $\mathtt{p}'$) is the inverse image of $\mathtt{q}$ in A (respectively in A'). Now, $f^{\#}_x$, which can be identified with the natural morphism $A_{\mathtt{p}} \to B_{\mathtt{q}}$, is formally smooth if and only if $A \to B_{\mathtt{q}}$ is formally smooth (as $A \to A_{\mathtt{p}}$ is formally étale by Example 3.93, this follows from Corollary 3.99), and similarly one can show that f is smooth at x if and only if there exists $b \in B \setminus \mathtt{q}$ such that $A \to B_b$ is formally smooth. Since $A \to A'_{\mathtt{p}'}$ is formally smooth (by Example 3.92 and Example 3.93), it follows from Lemma 3.102 that $A \to B_{\mathtt{q}} = A'_{\mathtt{p}'}/\mathtt{I}_{\mathtt{p}'}$ is formally smooth if and only if $d^{\mathtt{I}_{\mathtt{p}'}}_{A'_{\mathtt{p}'}/A}$, which can be identified with the natural morphism of $B_{\mathtt{q}}$-modules $(d^{\mathtt{I}}_{A'/A})_{\mathtt{q}}\colon (\mathtt{I}/\mathtt{I}^2)_{\mathtt{q}} \to (\Omega_{A'/A} \otimes_{A'} B)_{\mathtt{q}}$, is left invertible; in the same way, one sees that $A \to B_b$ is formally smooth if and only if $(d^{\mathtt{I}}_{A'/A})_b\colon (\mathtt{I}/\mathtt{I}^2)_b \to (\Omega_{A'/A} \otimes_{A'} B)_b$ is left invertible. Therefore, by

Lemma A.27 applied to the morphism of B-modules $d^{\mathtt{I}}_{A'/A}\colon \mathtt{I}/\mathtt{I}^2 \to \Omega_{A'/A} \otimes_{A'} B$ (note that $\mathtt{I}/\mathtt{I}^2$ is finitely generated because $\mathtt{I}$ is, and $\Omega_{A'/A} \otimes_{A'} B \cong B^n$ is projective), we obtain that (1) and (2) are both equivalent to the following: there exist $\bar{P}_1, \dots, \bar{P}_m \in \mathtt{I}/\mathtt{I}^2$ and $\bar{\varphi}_1, \dots \bar{\varphi}_m \in (\Omega_{A'/A} \otimes_{A'} B)^\vee$ such that $(\bar{P}_1, \dots, \bar{P}_m)_{\mathtt{q}} = (\mathtt{I}/\mathtt{I}^2)_{\mathtt{q}}$ and $\det\bigl(\bar{\varphi}_i(d^{\mathtt{I}}_{A'/A}(\bar{P}_j))\bigr)_{1\le i,j\le m} \notin \mathtt{q}$. By Nakayama's lemma this last condition is equivalent to the existence of $P_1, \dots, P_m \in \mathtt{I}$ and $\varphi_1, \dots \varphi_m \in \Omega^\vee_{A'/A}$ such that $(P_1, \dots, P_m)_{\mathtt{p}'} = \mathtt{I}_{\mathtt{p}'}$ and $\det\bigl(\varphi_i(d_{A'/A}(P_j))\bigr)_{1\le i,j\le m} \notin \mathtt{p}'$. Then the conclusion follows from the fact that $\Omega^\vee_{A'/A}$ is a free A'-module with base given by the elements defined by $d_{A'/A}(P) \mapsto \frac{\partial P}{\partial t_i}$ (for $i = 1, \dots, n$). It is also clear that $m - n$ coincides with the rank of the free (by Remark 3.112) $\mathcal{O}_{X,x}$-module $(\Omega_{X/Y})_x$, so that the equivalence between (1) and (6) will imply that $m = n - \dim_x f$.

(3) $\Rightarrow$ (4). Since smoothness is stable under base change, it remains to prove that f is flat at x, i.e. (in the above notation) that the morphism of rings $A_{\mathtt{p}} \to B_{\mathtt{q}}$ is flat. For $i = 1, \dots m$, let P'_i (respectively $\bar{P}_i$) be the image of P_i in $\mathtt{I}_{\mathtt{p}'} \subseteq A'_{\mathtt{p}'}$ (respectively in $C := A'_{\mathtt{p}'} \otimes_{A_{\mathtt{p}}} \kappa(\mathtt{p}) \cong A'_{\mathtt{p}'}/\mathtt{p}A'_{\mathtt{p}'}$) and denote by $\mathtt{m} \cong \mathtt{p}'_{\mathtt{p}'}/\mathtt{p}A'_{\mathtt{p}'}$ the maximal ideal of C. Then we claim that the images of $\bar{P}_1, \dots \bar{P}_m$ in $\mathtt{m}/\mathtt{m}^2$ are linearly independent (over $C/\mathtt{m} \cong \kappa(\mathtt{p}')$). Indeed, assume on the contrary they are not: then there exist $R_1, \dots, R_m \in A'_{\mathtt{p}'}$ not all in $\mathtt{p}'_{\mathtt{p}'}$ such that $\sum_{j=1}^m \bar{P}_j \bar{R}_j \in \mathtt{m}^2$ ($\bar{R}_j$ denoting the image of R_j in C), or, equivalently, such that $\sum_{j=1}^m P'_j R_j \in {\mathtt{p}'}^2_{\mathtt{p}'} + \mathtt{p}A'_{\mathtt{p}'}$. Now, we can find $Q_1, \dots, Q_m \in A'$ not all in $\mathtt{p}'$ such that $R_j = Q_j/S$ for some $S \in A' \setminus \mathtt{p}'$ and for every $j = 1, \dots, m$, and then we have $\sum_{j=1}^m P_j Q_j \in {\mathtt{p}'}^2 + \mathtt{p}A'$. Therefore for every $i = 1, \dots, n$

$$\mathtt{p}' \ni \frac{\partial}{\partial t_i} \sum_{j=1}^m P_j Q_j = \sum_{j=1}^m P_j \frac{\partial Q_j}{\partial t_i} + \sum_{j=1}^m Q_j \frac{\partial P_j}{\partial t_i},$$

whence $\sum_{j=1}^m Q_j \frac{\partial P_j}{\partial t_i} \in \mathtt{p}'$ (since each $P_j \in \mathtt{I} \subseteq \mathtt{p}'$). As $\mathtt{p}'$ is prime, it is easy to see that this last fact contradicts the hypotheses that $\det\left(\frac{\partial P_j}{\partial t_{l_i}}\right)_{1\le i,j\le m} \notin \mathtt{p}'$ and that not all the Q_j are in $\mathtt{p}'$; so the claim is proved. Since C is a regular local ring (it is isomorphic to a localization at a prime ideal of $\kappa(\mathtt{p})[t_1, \dots, t_n]$), it follows that $(\bar{P}_1, \dots \bar{P}_m)$ is a regular sequence in C (see e.g. [12, Thm. 169]). Then $A_{\mathtt{p}} \to A'_{\mathtt{p}'}/(P'_1, \dots, P'_m) = A'_{\mathtt{p}'}/\mathtt{I}_{\mathtt{p}'} = B_{\mathtt{q}}$ is flat by [9, Thm. 11.3.8] (equivalence between b) and c), applied to the flat morphism of finite presentation $\operatorname{Spec} A' \to \operatorname{Spec} A$ with $\mathcal{F} = \mathcal{O}_{\operatorname{Spec} A'}$ and $g_i = \widetilde{P_i}$).

(4) $\Rightarrow$ (3). Keeping the above notation, we set $\bar{A}' := A' \otimes_A \kappa(\mathtt{p})$ (which is isomorphic to $\kappa(\mathtt{p})[t_1, \dots, t_n]$), $\bar{B} := B \otimes_A \kappa(\mathtt{p})$, $\bar{\mathtt{I}} := \ker(\bar{A}' \twoheadrightarrow \bar{B})$ (note that there is a natural surjective map $\mathtt{I} \otimes_A \kappa(\mathtt{p}) \twoheadrightarrow \bar{\mathtt{I}}$) and we denote by $\bar{\mathtt{p}}' \in \operatorname{Spec} \bar{A}'$ and $\bar{\mathtt{q}} \in \operatorname{Spec} \bar{B}$ the images of $\mathtt{p}'$ and $\mathtt{q}$. Applying $- \otimes_{A_{\mathtt{p}}} \kappa(\mathtt{p})$ to the exact sequence $0 \to \mathtt{I}_{\mathtt{p}'} \to A'_{\mathtt{p}'} \to B_{\mathtt{q}} \to 0$ we obtain the exact sequence

$$0 \to \mathtt{I}_{\mathtt{p}'}/\mathtt{p}\mathtt{I}_{\mathtt{p}'} \cong \bar{\mathtt{I}}_{\bar{\mathtt{p}}'} \to A'_{\mathtt{p}'}/\mathtt{p}A'_{\mathtt{p}'} \cong \bar{A}'_{\bar{\mathtt{p}}'} \to B_{\mathtt{q}}/\mathtt{p}B_{\mathtt{q}} \cong \bar{B}_{\bar{\mathtt{q}}} \to 0$$

($\mathrm{Tor}_1^{A_{\mathtt{p}}}(B_{\mathtt{q}}, \kappa(\mathtt{p})) = 0$ because $A_{\mathtt{p}} \to B_{\mathtt{q}}$ is flat). As $\kappa(\mathtt{p}) \to \bar{B}_{\bar{\mathtt{q}}}$ is formally smooth, by the already proved equivalence between (2) and (3) there exist $m \le n$, $\bar{P}_1, \dots \bar{P}_m \in \bar{\mathtt{I}}$ and $1 \le l_1 < \cdots < l_m \le n$ such that $(\bar{P}_1, \dots \bar{P}_m)_{\bar{\mathtt{p}}'} = \bar{\mathtt{I}}_{\bar{\mathtt{p}}'}$ and $\det\left(\frac{\partial \bar{P}_j}{\partial t_{l_i}}\right)_{1\le i,j\le m} \notin \bar{\mathtt{p}}'$. We can assume without loss of generality that each $\bar{P}_j$ is the image of some $P_j \in \mathtt{I}$, and then $(P_1, \dots, P_m)_{\mathtt{p}'} = \mathtt{I}_{\mathtt{p}'}$ by Nakayama's lemma and $\det\left(\frac{\partial P_j}{\partial t_{l_i}}\right)_{1\le i,j\le m} \notin \mathtt{p}'$.

(4) ⇔ (5). By Lemma 3.119.

(1) ⇔ (6). See [17, Thm. 8.4]. □

Corollary 3.121. *A morphism locally of finite presentation of schemes $f\colon X \to Y$ is smooth if and only if it is flat and has smooth (or geometrically regular) fibres, if and only if it is flat and $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module whose rank at every point $x \in X$ is $\dim_x f$.*

Remark 3.122. If $f\colon X \to Y$ is a smooth morphism of schemes such that $\dim_x f = n$ for some $n \in \mathbb{N}$ and for every $x \in X$ (this is certainly the case if X is connected, since $\dim_x f$ coincides with the rank of $(\Omega_{X/Y})_x$, which is a locally constant function on X), then f is said to be *smooth of relative dimension* n. For smooth morphisms the relative dimension is stable under base change and is additive under composition.

Remark 3.123. It can be proved (see [17, page 92]) that for morphisms (automatically of finite presentation) between schemes of finite type over a field the definition of smoothness is equivalent to that of [11]: namely, with this hypothesis $f\colon X \to Y$ is smooth of relative dimension n if and only if it is flat, $\dim X' = \dim Y' + n$ for every irreducible components $X' \subseteq X$ and $Y' \subseteq Y$ with $f(X') \subseteq Y'$, and $\dim_{\kappa(x)}(\Omega_{X/Y} \otimes_{\mathcal{O}_X} \kappa(x)) = n$ for every $x \in X$.

Remark 3.124. Every smooth morphism of schemes is open by Proposition 3.81.

Proposition 3.125. *Let $f\colon X \to Y$ be a morphism of schemes. Assuming f is of finite presentation at $x \in X$, and setting $y := f(x) \in Y$, the following conditions are equivalent:*

1. *f is étale at x;*
2. *$f_x^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a formally étale morphism of rings;*
3. *(Jacobian criterion) if $\operatorname{Spec} B \cong U \subseteq X$ and $\operatorname{Spec} A \cong V \subseteq Y$ are open affine neighbourhoods of x and y such that $f(U) \subseteq V$ and $B \cong A'/\mathtt{I}$ as A-algebra (where $A' := A[t_1, \dots, t_n]$ for some $n \in \mathbb{N}$ and $\mathtt{I} \subset A'$ is a finitely generated ideal), then, denoting by $\mathtt{p}' \in \operatorname{Spec} A'$ the prime ideal corresponding to x, there exist $P_1, \dots, P_n \in \mathtt{I}$ such that $(P_1, \dots, P_n)_{\mathtt{p}'} = \mathtt{I}_{\mathtt{p}'}$ and $\det\left(\frac{\partial P_j}{\partial t_i}\right)_{1\le i,j\le n} \notin \mathtt{p}'$;*

4. f is smooth at x and $\dim_x f = 0$*;*

5. f is smooth and quasi-finite at x;

6. f is flat and unramified at x.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (6). It follows from Proposition 3.115 and Proposition 3.120.

(4) $\Leftrightarrow$ (5). It follows from Remark 3.56.

(6) $\Rightarrow$ (1). By Proposition 3.120 it is enough to show that $f^{-1}(y) \to \operatorname{Spec} \kappa(y)$ is geometrically regular at x, hence we have to prove the following: if $\mathbb{K} \to A$ is an unramified morphism of rings, where $\mathbb{K}$ is an algebraically closed field, then $A_{\mathtt{p}}$ is regular for every $\mathtt{p} \in \operatorname{Spec} A$. Now, by Remark 3.117, A is isomorphic (as $\mathbb{K}$-algebra) to the product of a finite number of copies of $\mathbb{K}$, so that $A_{\mathtt{p}} \cong \mathbb{K}$ is obviously regular for every $\mathtt{p} \in \operatorname{Spec} A$. □

Corollary 3.126. *A morphism locally of finite presentation of schemes $f\colon X \to Y$ is étale if and only if it is smooth and locally quasi-finite (or of relative dimension* 0*), if and only if it is flat and unramified.*

Example 3.127. Let A be a ring, $P \in A[t]$ and $B := A[t]/(P)$. If $b \in B$ is such that the image of $\frac{dP}{dt}$ in B_b is invertible, then the natural morphism $A \to B_b$ is étale by the Jacobian criterion. When P is a monic polynomial, a morphism of this form is called *étale standard*. It can be proved (see [21, V, Thm. 1]) that every étale morphism of schemes $f\colon X \to Y$ is locally standard, in the sense that for every $x \in X$ there are open affine neighbourhoods U of x and V of $f(x)$ such that $f(U) \subseteq V$ and the induced morphism of rings $\mathcal{O}_V(V) \to \mathcal{O}_U(U)$ is étale standard.

Proposition 3.128. *A morphism of schemes $f\colon X \to Y$ is unramified if and only if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $f|_{U_i} = g_i \circ h_i$, where g_i is an étale morphism and h_i is a closed immersion for every $i \in I$.*

Proof. [21, V, Thm. 1]. □

Proposition 3.129. *A morphism of schemes $X \to Y$ is smooth if and only if there exist an open cover $\{U_i\}_{i\in I}$ of X and for every $i \in I$ an étale morphism $U_i \to \mathbb{A}^{n_i}_Y$ (for some $n_i \in \mathbb{N}$) in* $\mathbf{Sch}_{/Y}$.

Proof. The other implication being clear, we have to show that every smooth morphism satisfies the stated condition. Since the question is local both on the domain (by definition) and on the codomain (because open immersions are étale), we can assume that $A \to B = A'/\mathtt{I}$ (where $A' := A[t_1, \dots, t_n]$ and $\mathtt{I} \subset A'$ is a finitely generated ideal) is a smooth morphism of rings, and we have to prove that, given $\mathtt{q} \in \operatorname{Spec} B$, there exists $b \in B \setminus \mathtt{q}$ such that $A \to B_b$ is the composition of a polynomial extension $A \to \tilde{A}$ and of an étale morphism $\tilde{A} \to B_b$. Denoting by $\mathtt{p}' \in \operatorname{Spec} A'$ the inverse image of $\mathtt{q}$, by the Jacobian criterion there exist $m \le n$, $P_1, \dots P_m \in \mathtt{I}$ and $1 \le l_1 < \cdots < l_m \le n$ such that $(P_1, \dots, P_m)_{\mathtt{p}'} = \mathtt{I}_{\mathtt{p}'}$ and

$D := \det\left(\frac{\partial P_j}{\partial t_{l_i}}\right)_{1\leq i,j\leq m} \notin \mathfrak{p}'$. As $\mathtt{I}$ is finitely generated, it follows that there exists $P \in A' \setminus \mathfrak{p}'$ such that $(P_1,\dots,P_m)_P = \mathtt{I}_P$ and the image of D in A'_P is invertible. Let $\tilde{A} := A[t_i]_{i\in\{1,\dots,n\}\setminus\{l_1,\dots,l_m\}}$, denote by $b \in B$ the image of P, and set $P_{m+1} := t_{l_{m+1}}P - 1$ (where $t_{l_{m+1}}$ is a new variable): then $B_b \cong \tilde{A}[t_{l_1},\dots,t_{l_{m+1}}]/(P_1,\dots,P_{m+1})$ and $\det\left(\frac{\partial P_j}{\partial t_{l_i}}\right)_{1\leq i,j\leq m+1} = PD$ is such that its image in B_b is invertible, whence $\tilde{A} \to B_b$ is étale again by the Jacobian criterion. □

Corollary 3.130. *Given a smooth and surjective morphism of schemes $f\colon X \to Y$, there exists a morphism $g\colon X' \to X$ such that $f \circ g\colon X' \to Y$ is étale and surjective.*

Proof. First we claim that we can find an open cover $\{U_i\}_{i\in I}$ of X and for every $i \in I$ an étale morphism $h_i\colon U_i \to \mathbb{A}^{n_i}_Y$ (for some $n_i \in \mathbb{N}$) in $\mathbf{Sch}_{/Y}$, with the additional property that there exists a morphism $s_i\colon Y \to \mathbb{A}^{n_i}_Y$ in $\mathbf{Sch}_{/Y}$ such that $s_i(f(U_i)) \subseteq h_i(U_i)$. Indeed, for every $x \in X$ by Proposition 3.129 there is an open neighbourhood U of x and an étale morphism $h\colon U \to \mathbb{A}^n_Y$ (for some $n \in \mathbb{N}$) in $\mathbf{Sch}_{/Y}$. Since $h(U) \subseteq \mathbb{A}^n_Y$ and $f(U) \subseteq Y$ are open (by Remark 3.124), we can find a morphism $s\colon Y \to \mathbb{A}^n_Y$ in $\mathbf{Sch}_{/Y}$ and an open neighbourhood $f(x) \in V \subseteq f(U)$ such that $s(V) \subseteq h(U)$. Then $U' := U \cap f^{-1}(V)$ is such that $s(f(U')) \subseteq h(U')$, and the claim follows.

Now, consider for every $i \in I$ the cartesian diagram

$$\begin{tikzcd} U'_i \arrow[r, "g_i"] \arrow[d, "h'_i"'] \arrow[dr, phantom, "\square"] & U_i \arrow[d, "h_i"] \\ Y \arrow[r, "s_i"'] & \mathbb{A}^{n_i}_Y \end{tikzcd}$$

and let $g\colon X' := \coprod_{i\in I} U'_i \to X$ be the morphism induced by the morphisms $U'_i \xrightarrow{g_i} U_i \subseteq X$. Then (denoting by $p_i\colon \mathbb{A}^{n_i}_Y \to Y$ the structure morphism)

$$(f\circ g)|_{U'_i} = f|_{U_i} \circ g_i = p_i \circ h_i \circ g_i = p_i \circ s_i \circ h'_i = h'_i$$

is étale for every $i \in I$, so that $f \circ g$ is étale. Moreover, for every $y \in Y$ let $i \in I$ and $x \in U_i$ be such that $y = f(x)$: by hypothesis there exists $\tilde{x} \in U_i$ such that $s_i(y) = h_i(\tilde{x})$, whence there exists $x' \in U'_i \subseteq X'$ such that $g_i(x') = \tilde{x}$ and $h'_i(x') = (f \circ g)(x') = y$, which shows that $f \circ g$ is surjective. □

4 Presheaves, sheaves and equivalence relations

4.1 The category of presheaves

Definition 4.1. A *presheaf* (of sets) on a category $\mathbf{C}$ is a functor $\mathbf{C}^\circ \to \mathbf{Set}$. A morphism of presheaves on $\mathbf{C}$ is just a natural transformation of such functors.

The category of presheaves on $\mathbf{C}$ will be denoted by $\widehat{\mathbf{C}} := \mathbf{Fun}(\mathbf{C}^{\circ}, \mathbf{Set})$.

The following example explains why this notion of presheaf is a generalization of the usual notion of presheaf on a topological space.

Example 4.2. If X is a topological space, let $\mathbf{Open}(X)$ be the category having as objects the open subsets of X and as morphisms the inclusions: more precisely $\mathrm{Hom}_{\mathbf{Open}(X)}(U,V)$ is empty if $U \nsubseteq V$, otherwise it contains just one element, the natural inclusion $U \subseteq V$. Then it is immediate to see that $\widehat{\mathbf{Open}(X)}$ is the usual category of presheaves (of sets) on X.

In the following, given a presheaf $F \in \widehat{\mathbf{C}}$, a morphism $f : U \to V$ in $\mathbf{C}$ and $\eta \in F(V)$, if F is clear from the context, $F(f)(\eta) \in F(U)$ will be usually denoted by $f^*(\eta)$, or even by $\eta|_U$ if there can be no doubt about f.

Every object of $\mathbf{C}$ can be naturally viewed as a presheaf: indeed, to each $U \in \mathbf{C}$ we can associate $\mathrm{h}_U := \mathrm{Hom}_{\mathbf{C}}(-,U) \in \widehat{\mathbf{C}}$. Moreover, every morphism $f : U \to V$ of $\mathbf{C}$ induces a morphism $\mathrm{h}_f : \mathrm{h}_U \to \mathrm{h}_V$ of $\widehat{\mathbf{C}}$, i.e. a natural transformation of functors $\mathrm{Hom}_{\mathbf{C}}(-,U) \to \mathrm{Hom}_{\mathbf{C}}(-,V)$, defined by

$$\begin{aligned} \mathrm{h}_f(W) : \mathrm{h}_U(W) = \mathrm{Hom}_{\mathbf{C}}(W,U) &\to \mathrm{Hom}_{\mathbf{C}}(W,V) = \mathrm{h}_V(W) \\ g &\mapsto f \circ g \end{aligned}$$

for every $W \in \mathbf{C}$. It is immediate to see that this defines a functor $\mathrm{h} : \mathbf{C} \to \widehat{\mathbf{C}}$. We are going to see that h identifies $\mathbf{C}$ with a full subcategory of $\widehat{\mathbf{C}}$: this is a consequence of the following fundamental result, whose proof is straightforward.

Proposition 4.3 (Yoneda's lemma)**.** *For every $U \in \mathbf{C}$ and every $F \in \widehat{\mathbf{C}}$ there is a natural bijection (of sets) $F(U) \cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{h}_U, F)$. Explicitly, the map*

$$\begin{aligned} \Phi_{F,U} : \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{h}_U, F) &\to F(U) \\ \alpha &\mapsto \alpha(U)(\mathrm{id}_U) \end{aligned}$$

is bijective and its inverse is the map

$$\begin{aligned} \Psi_{F,U} : F(U) &\to \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{h}_U, F) \\ \xi &\mapsto \tilde{\xi} \end{aligned}$$

where, for every $W \in \mathbf{C}$, $\tilde{\xi}(W) : \mathrm{h}_U(W) = \mathrm{Hom}_{\mathbf{C}}(W,U) \to F(W)$ is defined by $f \mapsto F(f)(\xi)$.

Corollary 4.4. *The functor* $\mathrm{h} : \mathbf{C} \to \widehat{\mathbf{C}}$ *is fully faithful.*

Proof. Just notice that if in Proposition 4.3 we take $F = \mathrm{h}_V$ for some $V \in \mathbf{C}$, then the bijective map $\Psi_{\mathrm{h}_V,U} : \mathrm{h}_V(U) = \mathrm{Hom}_{\mathbf{C}}(U,V) \to \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{h}_U, \mathrm{h}_V)$ coincides with the natural map defined by h. □

Definition 4.5. A presheaf $F \in \widehat{\mathbf{C}}$ is *representable* if it is isomorphic to h_U for some $U \in \mathbf{C}$.

Corollary 4.6. *$F \in \widehat{\mathbf{C}}$ is representable if and only if there exist $U \in \mathbf{C}$ and $\xi \in F(U)$ which is a "universal object" for F in the following sense: for all $V \in \mathbf{C}$ and $\eta \in F(V)$, there exists a unique $f\colon V \to U$ in $\mathbf{C}$ such that $\eta = f^*(\xi)$.*

Proof. If F is representable, there is an isomorphism $\alpha\colon \mathrm{h}_U \xrightarrow{\sim} F$ in $\widehat{\mathbf{C}}$, and it is easy to see that $\Phi_{F,U}(\alpha) = \alpha(U)(\mathrm{id}_U) \in F(U)$ is a universal object for F. Conversely, if $\xi \in F(U)$ is a universal object for F, it is clear by definition that $\Psi_{F,U}(\xi) \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{h}_U, F)$ is an isomorphism. □

Remark 4.7. By Corollary 4.4 the functor h gives an equivalence of categories between $\mathbf{C}$ and the strictly full subcategory of $\widehat{\mathbf{C}}$ whose objects are representable presheaves. In the following we will usually identify these two categories, thus writing, for instance, U instead of h_U. Moreover, we will avoid the explicit mention of the isomorphisms provided by Yoneda's lemma: every $\xi \in F(U)$ will be regarded also as a morphism $\xi\colon U \to F$ in $\widehat{\mathbf{C}}$. In particular, for all $U, V \in \mathbf{C}$ we will often write, accordingly, $V(U)$ instead of $\mathrm{Hom}_{\mathbf{C}}(U, V) = \mathrm{Hom}_{\widehat{\mathbf{C}}}(U, V)$.

Proposition 4.8. *For every category $\mathbf{C}$, the category of presheaves $\widehat{\mathbf{C}}$ has kernels, cokernels, (fibred) products and (fibred) coproducts, and all of them can be computed "componentwise". This means that, for instance in the case of fibred products, given morphisms $\alpha_i\colon F_i \to F$ (for $i = 1, 2$) of $\widehat{\mathbf{C}}$, $G := F_1 \,{}_{\alpha_1}\!\times_{\alpha_2} F_2$ is defined as follows: $G(U) := F_1(U) \,{}_{\alpha_1(U)}\!\times_{\alpha_2(U)} F_2(U)$ (fibred product in $\mathbf{Set}$) for every object U of $\mathbf{C}$ and $G(f) := F_1(f) \times F_2(f)$ for every morphism f of $\mathbf{C}$; the projections $G \to F_i$ are of course given by the projections $G(U) \to F_i(U)$.*

Moreover, if $\mathbf{C}$ has kernels or (fibred) products, then $\mathrm{h}\colon \mathbf{C} \to \widehat{\mathbf{C}}$ preserves them.

Proof. Straightforward from the definitions. □

Definition 4.9. A morphism $\alpha\colon F \to G$ of $\widehat{\mathbf{C}}$ is *representable* if for every $V \in \mathbf{C}$ and every $\eta \in G(V)$ the presheaf $F \,{}_\alpha\!\times_\eta V$ is representable.

Remark 4.10. $\mathbf{C}$ has fibred products if and only if every morphism of $\mathbf{C}$ (regarded as a morphism of $\widehat{\mathbf{C}}$) is representable.

Using Lemma A.1 it is easy to prove the following result.

Proposition 4.11. *For morphisms of $\widehat{\mathbf{C}}$ the property of being representable is stable under composition and base change.*

Definition 4.12. Let P be a property of morphisms of $\mathbf{C}$ which is stable under base change. We will say that a representable morphism $\alpha\colon F \to G$ of $\widehat{\mathbf{C}}$ satisfies P if for every $V \in \mathbf{C}$ and every $\eta \in G(V)$ the induced morphism $F \,{}_\alpha\!\times_\eta V \to V$ (which is a morphism of representable presheaves, hence can be identified with a morphism of $\mathbf{C}$ by Yoneda's lemma) satisfies P.

Remark 4.13. In the hypotheses of the above definition, it is clear that a morphism of $\mathbf{C}$ satisfies P in $\mathbf{C}$ if and only if it satisfies P as representable morphism of presheaves. Moreover, it follows from Proposition 4.11 and Lemma A.1 that, for representable morphisms of $\widehat{\mathbf{C}}$, the property P remains stable under base change (and also under composition, if P is stable under composition for morphisms of $\mathbf{C}$).

Proposition 4.14. *Let $\mathbf{C}$ be a category with fibred products and finite products. Given $F \in \widehat{\mathbf{C}}$, the diagonal morphism $\Delta_F\colon F \to F \times F$ of $\widehat{\mathbf{C}}$ is representable if and only if for every $U \in \mathbf{C}$ and every $\xi \in F(U)$ the morphism $\xi\colon U \to F$ of $\widehat{\mathbf{C}}$ is representable. In this case, moreover, Δ_F satisfies a property P of morphisms of $\mathbf{C}$ which is stable under base change if for all $U \in \mathbf{C}$ and all $\xi_1, \xi_2 \in F(U)$ the natural morphism of representable presheaves $U\,{}_{\xi_1}\!\times_{\xi_2} U \to U \times U$ satisfies P.*

Proof. Assume first that Δ_F is representable. We have to show that for all $U, V \in \mathbf{C}$, all $\xi \in F(U)$ and all $\eta \in F(V)$ the presheaf $U\,{}_{\xi}\!\times_{\eta} V$ is representable. Since $U \times V \in \mathbf{C}$ by hypothesis, this follows from the cartesian diagram

$$\begin{tikzcd}
U\,{}_{\xi}\!\times_{\eta} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d, "\Delta_F"] \\
U \times V \arrow[r, "\xi\times\eta"'] & F \times F.
\end{tikzcd}$$

Conversely, assuming that every morphism from a representable presheaf to F is representable, we have to prove that for all $U \in \mathbf{C}$ and all $\xi_1, \xi_2 \in F(U)$ the presheaf $F\,{}_{\Delta_F}\!\times_{(\xi_1,\xi_2)} U$ is representable. Now, it follows from the hypotheses that in the commutative diagram with cartesian squares

$$\begin{tikzcd}
G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U\,{}_{\xi_1}\!\times_{\xi_2} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d, "\Delta_F"] \\
U \arrow[r, "\Delta_U"'] & U \times U \arrow[r, "\xi_1\times\xi_2"'] & F \times F
\end{tikzcd}$$

all terms of the square on the left are representable. It is then enough to notice that, since $(\xi_1 \times \xi_2) \circ \Delta_U = (\xi_1, \xi_2)$, $F\,{}_{\Delta_F}\!\times_{(\xi_1,\xi_2)} U \cong G$ by Lemma A.1. The last statement is then also clear. $\square$

Proposition 4.15. *Let $\mathbf{C}$ be a category and $\alpha\colon F \to G$ a morphism of $\widehat{\mathbf{C}}$. Then α is a monomorphism (respectively an epimorphism) if and only if the map $\alpha(U)\colon F(U) \to G(U)$ is injective (respectively surjective) for every $U \in \mathbf{C}$. In particular, a morphism of $\mathbf{C}$ is a monomorphism in $\mathbf{C}$ if and only if it is a monomorphism in $\widehat{\mathbf{C}}$.*

Proof. If α is a monomorphism, then for every $U \in \mathbf{C}$ the map $\mathrm{Hom}_{\widehat{\mathbf{C}}}(U, F) \xrightarrow{\alpha\circ} \mathrm{Hom}_{\widehat{\mathbf{C}}}(U, G)$, which can be identified with $\alpha(U)$ by Yoneda's lemma, is injective by definition. If α is an epimorphism, then, since the natural morphisms $j_1, j_2 \colon G \to G \coprod_F G$ are such that $j_1 \circ \alpha = j_2 \circ \alpha$, we have $j_1 = j_2$. This means that $j_1(U) = j_2(U) \colon G(U) \to G(U) \coprod_{F(U)} G(U)$, whence $\alpha(U)$ is surjective for every $U \in \mathbf{C}$. The converse implications are straightforward to check. $\square$

Remark 4.16. Obviously a morphism of $\mathbf{C}$ which is an epimorphism in $\widehat{\mathbf{C}}$ is also an epimorphism in $\mathbf{C}$, but the converse is not true in general (for instance, $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in $\mathbf{Rng}$ but not in $\widehat{\mathbf{Rng}}$).

Corollary 4.17. *A morphism of $\widehat{\mathbf{C}}$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Definition 4.18. If $\alpha \colon F \to G$ is a morphism of $\widehat{\mathbf{C}}$, the *image* of α is the subpresheaf $\operatorname{im} \alpha$ of G defined for every $U \in \mathbf{C}$ by $(\operatorname{im} \alpha)(U) := \operatorname{im} \alpha(U) \subseteq G(U)$.

Remark 4.19. Every morphism $\alpha \colon F \to G$ in $\widehat{\mathbf{C}}$ factors as the composition of an epimorphism and a monomorphism through the natural morphisms $F \twoheadrightarrow \operatorname{im} \alpha \hookrightarrow G$ (such a factorization is clearly unique up to isomorphism). In particular, α is a monomorphism (respectively an epimorphism) if and only if $F \twoheadrightarrow \operatorname{im} \alpha$ is an isomorphism (respectively $\operatorname{im} \alpha = G$).

Given a category $\mathbf{C}$ and a presheaf $H \in \widehat{\mathbf{C}}$, every object $\alpha \colon F \to H$ of $\widehat{\mathbf{C}}_{/H}$ naturally determines a presheaf $G_\alpha \in \widehat{\mathbf{C}_{/H}}$ defined on objects by

$$G_\alpha(\xi \colon U \to H) := \alpha(U)^{-1}(\xi) \subseteq F(U)$$

(and on morphisms, obviously, by $G_\alpha(f) := F(f)$). Conversely, to every presheaf $G \in \widehat{\mathbf{C}_{/H}}$ we can associate an object $\alpha_G \colon F_G \to H$ of $\widehat{\mathbf{C}}_{/H}$ as follows: F_G is defined on objects by

$$F_G(U) := \{(\xi, g) \mid \xi \in H(U),\, g \in G(\xi)\}$$

and on morphisms by

$$\begin{aligned} F_G(f \colon V \to U) \colon F_G(U) &\to F_G(V) \\ (\xi, g) &\mapsto (\xi \circ f, G(f)(g)) \end{aligned}$$

whereas $\alpha_G(U) \colon F_G(U) \to H(U)$ is clearly given by $(\xi, g) \mapsto \xi$. It is then straightforward to prove the following result.

Lemma 4.20. *With the above notation, the assignments $\alpha \mapsto G_\alpha$ and $G \mapsto \alpha_G$ extend to functors $\widehat{\mathbf{C}}_{/H} \to \widehat{\mathbf{C}_{/H}}$ and $\widehat{\mathbf{C}_{/H}} \to \widehat{\mathbf{C}}_{/H}$, which are quasi-inverse equivalences of categories.*

4.2 Grothendieck pretopologies, sites and sheaves on a site

Given a category $\mathbf{C}$ and $U \in \mathbf{C}$, we will denote by $\mathrm{Tar}(U)$ the set of all families $\{f_i\colon U_i \to U\}_{i\in I}$ (where I is an arbitrary set, possibly empty) of morphisms of $\mathbf{C}$ with target U. For later use, we give the following definition.

Definition 4.21. Given $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ and $\mathcal{U}' = \{f'_j\colon U'_j \to U\}_{j\in J}$ in $\mathrm{Tar}(U)$, we will say that $\mathcal{U}'$ is a *refinement* of $\mathcal{U}$ (and we will write $\mathcal{U} \leq \mathcal{U}'$) if for every $j \in J$ there is a morphism $g_j\colon U'_j \to U_{i(j)}$ (for some $i(j) \in I$) such that $f'_j = f_{i(j)} \circ g_j$ (in other words, if every morphism of $\mathcal{U}'$ factors through one of $\mathcal{U}$).

Definition 4.22. A *(Grothendieck) pretopology* τ on a category $\mathbf{C}$ consists of the datum, for each object U of $\mathbf{C}$, of a subset $\mathrm{Cov}(U) = \mathrm{Cov}^\tau(U) \subseteq \mathrm{Tar}(U)$ (whose elements are called *coverings* or *covering families* of U for τ), such that the following axioms are satisfied.

PT0 If $\{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $g\colon V \to U$ is a morphism of $\mathbf{C}$, then the fibred product $V\,{}_g{\times}_{f_i}\,U_i$ exists for every $i \in I$.

PT1 If $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $g\colon V \to U$ is a morphism of $\mathbf{C}$, then $g^*\mathcal{U} := \{pr_i\colon V\,{}_g{\times}_{f_i}\,U_i \to V\}_{i\in I} \in \mathrm{Cov}(V)$ (where pr_i is the natural projection).

PT2 If $\{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $\{f_{i_j}\colon U_{i_j} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(U_i)$ for every $i \in I$, then $\{f_i \circ f_{i_j}\colon U_{i_j} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.

PT3 If $f\colon V \to U$ is an isomorphism of $\mathbf{C}$, then $\{f\} \in \mathrm{Cov}(U)$.

If $f\colon V \to U$ is a morphism of $\mathbf{C}$ such that $\{f\} \in \mathrm{Cov}(U)$, we will say that f is a *covering morphism* (for τ). Note that for morphisms of $\mathbf{C}$ the property of being covering is stable under base change.

Definition 4.23. A *site* is a couple $(\mathbf{C}, \tau)$, where τ is a pretopology on the category $\mathbf{C}$.

Remark 4.24. Of course, as the name pretopology suggests, there is also a notion of (Grothendieck) topology, which we are not going to define here, since its use is not necessary for our purposes. The interested reader can find the basic facts about topologies and relations between topologies and pretopologies in Section B.2. Here it is enough to point out the following facts. Every pretopology generates a topology, but different pretopologies can generate the same topology (in general, not every topology is generated by some pretopology; this happens, however, if the category has fibred products). The correct definition of site is that it is a couple formed by a category and a topology on it. Usually what is really relevant is the topology and not the particular pretopology which generates it: for instance, sheaves (which are the objects one is mainly interested in when dealing with (pre)topologies) can be defined in terms of a pretopology τ, but they actually depend only the topology generated by τ. The classical names which we will give to some pretopologies should be reserved to the topologies generated by them.

Remark 4.25. If $(\mathbf{C},\tau)$ is a site, then for every $U \in \mathbf{C}$ the preordered set $(\mathrm{Cov}(U),\leq)$ is filtered. Indeed, if $\mathcal{U} = \{U_i \to U\}_{i\in I}, \mathcal{U}' = \{U'_j \to U\}_{j\in J} \in \mathrm{Cov}(U)$, then by **PT1** $\{U_i \times_U U'_j \to U_i\}_{j\in J} \in \mathrm{Cov}(U_i)$ for every $i \in I$, whence $\{U_i \times_U U'_j \to U\}_{i\in I, j\in J} \in \mathrm{Cov}(U)$ by **PT2**, and clearly it is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$.

Example 4.26. On every category $\mathbf{C}$ one can consider the pretopology such that the covering families of $U \in \mathbf{C}$ are exactly those formed by a single morphism, which is an isomorphism (with target U); this pretopology is called *chaotic*. On the other hand, if $\mathbf{C}$ has fibre products, one can define a pretopology by setting $\mathrm{Cov}(U) := \mathrm{Tar}(U)$ for every $U \in \mathbf{C}$; this pretopology is called *discrete*.

Example 4.27. If X is a topological space, on the category $\mathbf{Open}(X)$ we can consider the standard pretopology *std*, which is defined as follows: given $U_i \subseteq U$ ($i \in I$) open subsets of X, $\{U_i \subseteq U\}_{i\in I} \in \mathrm{Cov}^{std}(U)$ if and only if $U = \bigcup_{i\in I} U_i$ (i.e., if and only if the U_i are an open cover of U in the usual sense). Indeed, it is immediate to check that *std* satisfies the axioms of pretopology, if one observes that in $\mathbf{Open}(X)$ fibred products exist, and they are given by $U_i \times_U U_j = U_i \cap U_j$.

Example 4.28. There is also a "global" version of the previous example. Namely, on the category **Top** of topological spaces the standard pretopology *std* is defined as follows: $\{f_i \colon U_i \to U\}_{i\in I} \in \mathrm{Cov}^{std}(U)$ if and only if $U = \bigcup_{i\in I} \operatorname{im} f_i$ and f_i is an open immersion for every $i \in I$. The fact that open immersions are stable under composition and base change immediately implies that *std* is indeed a pretopology. With the same definition, the standard pretopology can be put also on other categories, like the category of differentiable manifolds **Diff** (notice that in this case **PT0** is satisfied, even if arbitrary fibred products do not exist in **Diff**) or the category of schemes **Sch** (in this case the standard pretopology is called Zariski, see Example 4.29 below).

Example 4.29. On **Sch** several pretopologies can be defined: for instance, *Zar* (Zariski), *ét* (étale), *sm* (smooth) and *fppf* (faithfully flat and locally of finite presentation). If τ is one of them, then, by definition, $\{f_i \colon U_i \to U\}_{i\in I} \in \mathrm{Cov}^{\tau}(U)$ if and only if $|U| = \bigcup_{i\in I} \operatorname{im}|f_i|$ and moreover for every $i \in I$ the following holds: f_i is an open immersion if $\tau = Zar$; f_i is étale if $\tau = \acute{e}t$; f_i is smooth if $\tau = sm$; f_i is flat and locally of finite presentation if $\tau = fppf$.[8] Again, it is easy to check the axioms, using the fact that each of the above properties (including the "surjectivity" of the family) is stable under composition and base change.

Definition 4.30. Let $(\mathbf{C},\tau)$ be a site. A presheaf $F \in \widehat{\mathbf{C}}$ is *separated* (for τ) if for every covering $\{f_i \colon U_i \to U\}_{i\in I} \in \mathrm{Cov}^{\tau}(U)$ the natural map

$$\tilde{f} := (F(f_i))_{i\in I} \colon F(U) \to \prod_{i\in I} F(U_i)$$

[8]We do not require that each f_i is faithfully flat (i.e., flat and surjective). The name *fppf* comes from the fact that the whole family $\{f_i\}_{i\in I}$ must be "surjective".

is injective. F is a *sheaf* (for τ) if moreover, for every covering as above, the sequence

$$F(U) \xrightarrow{\tilde{f}} \prod_{i\in I} F(U_i) \underset{\widetilde{pr}_2}{\overset{\widetilde{pr}_1}{\rightrightarrows}} \prod_{j,k\in I} F(U_j \times_U U_k)$$

is exact, where $\widetilde{pr}_1$ and $\widetilde{pr}_2$ are the natural maps induced by the projections $pr_1^{j,k}\colon U_j \times_U U_k \to U_j$ and $pr_2^{j,k}\colon U_j \times_U U_k \to U_k$ (more precisely, $\widetilde{pr}_l := (F(pr_l^{j,k}) \circ \pi_l^{j,k})_{j,k\in I}$, where $\pi_1^{j,k}\colon \prod_{i\in I} F(U_i) \to F(U_j)$ and $\pi_2^{j,k}\colon \prod_{i\in I} F(U_i) \to F(U_k)$ are the natural projections).

Remark 4.31. Let $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and let $F \in \widehat{\mathbf{C}}$. Then, in the notation of the above definition, it is clear that in any case $\widetilde{pr}_1 \circ \tilde{f} = \widetilde{pr}_2 \circ \tilde{f}$ (because $f_j \circ pr_1^{j,k} = f_k \circ pr_2^{j,k}$). Therefore, if we define

$$F(\mathcal{U}) := \ker(\prod_{i\in I} F(U_i) \underset{\widetilde{pr}_2}{\overset{\widetilde{pr}_1}{\rightrightarrows}} \prod_{j,k\in I} F(U_j \times_U U_k))$$

(so that $F(\mathcal{U}) = \{\xi \in \prod_{i\in I} F(U_i) \mid \widetilde{pr}_1(\xi) = \tilde{pr}_2(\xi)\}$), we see that $\tilde{f}$ always factors through a map which we will denote by

$$\lambda_{\mathcal{U}}^F\colon F(U) \to F(\mathcal{U})$$

(or simply by $\lambda_{\mathcal{U}}$). Then F is separated (respectively a sheaf) if and only if $\lambda_{\mathcal{U}}^F$ is injective (respectively bijective) for every covering $\mathcal{U}$. We will often use this fact in the following.

For later use, we also fix here some more general notation. Given $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I}$ and $\mathcal{U}' = \{f'_j\colon U'_j \to U\}_{j\in J}$ in $\mathrm{Cov}(U)$ with $\mathcal{U} \le \mathcal{U}'$, for every $F \in \widehat{\mathbf{C}}$ there is a natural induced map $\lambda_{\mathcal{U}',\mathcal{U}} = \lambda_{\mathcal{U}',\mathcal{U}}^F\colon F(\mathcal{U}) \to F(\mathcal{U}')$. Indeed, if $g_j\colon U'_j \to U_{i(j)}$ $(j \in J)$ are such that $f'_j = f_{i(j)} \circ g_j$, it is clear that the map

$$(F(g_j) \circ pr_{i(j)})_{j\in J}\colon \prod_{i\in I} F(U_i) \to \prod_{j\in J} F(U'_j)$$

restricts to a map $\lambda_{\mathcal{U}',\mathcal{U}}(\{g_j\})\colon F(\mathcal{U}) \to F(\mathcal{U}')$. It is also easy to see that $\lambda_{\mathcal{U}',\mathcal{U}} := \lambda_{\mathcal{U}',\mathcal{U}}(\{g_j\})$ is well defined: if $g'_j\colon U'_j \to U_{i'(j)}$ $(j \in J)$ are other morphisms such that $f'_j = f_{i'(j)} \circ g'_j$, then the commutative diagram

$$\begin{tikzcd} U'_j \arrow[r, "g_j"] \arrow[d, "g'_j"'] & U_{i(j)} \arrow[d, "f_{i(j)}"] \\ U_{i'(j)} \arrow[r, "f_{i'(j)}"'] & U \end{tikzcd}$$

implies that there exists a unique $h_j\colon U'_j \to U_{i(j),i'(j)} := U_{i(j)} \times_U U_{i'(j)}$ such that $g_j = pr \circ h_j$ and $g'_j = pr' \circ h_j$ (where $pr\colon U_{i(j),i'(j)} \to U_{i(j)}$ and $pr'\colon U_{i(j),i'(j)} \to U_{i'(j)}$ are the projections). Therefore, given $\xi = (\xi_i)_{i\in I} \in F(\mathcal{U}) \subseteq \prod_{i\in I} F(U_i)$, we have

$$g_j^*(\xi_{i(j)}) = h_j^*(\xi_{i(j)}|_{U_{i(j),i'(j)}}) = h_j^*(\xi_{i'(j)}|_{U_{i(j),i'(j)}}) = {g'_j}^*(\xi_{i'(j)})$$

for every $j \in J$, and this precisely says that $\lambda_{\mathcal{U}',\mathcal{U}}(\{g_j\})(\xi) = \lambda_{\mathcal{U}',\mathcal{U}}(\{g'_j\})(\xi)$. Observe that the sets $F(\mathcal{U})$ together with the maps $\lambda_{\mathcal{U}',\mathcal{U}}$ form a direct system ($\lambda_{\mathcal{U}'',\mathcal{U}} = \lambda_{\mathcal{U}'',\mathcal{U}'} \circ \lambda_{\mathcal{U}',\mathcal{U}}$ if $\mathcal{U} \le \mathcal{U}' \le \mathcal{U}''$ in $\mathrm{Cov}(U)$ and $\lambda_{\mathcal{U},\mathcal{U}} = \mathrm{id}_{F(\mathcal{U})}$) and that, under the natural identification $F(U) \cong F(\{\mathrm{id}_U\})$, $\lambda_{\mathcal{U},\{\mathrm{id}_U\}}$ coincides with $\lambda_{\mathcal{U}}$. Therefore, if F is a sheaf, all the maps $\lambda^F_{\mathcal{U}',\mathcal{U}}$ (and not only the $\lambda^F_{\mathcal{U}}$) are bijective. Similarly, for separated presheaves we have the following generalization.

Lemma 4.32. *If F is a separated presheaf, then for all $U \in \mathbf{C}$ and all $\mathcal{U},\mathcal{U}' \in \mathrm{Cov}(U)$ with $\mathcal{U} \le \mathcal{U}'$, the natural map $\lambda^F_{\mathcal{U}',\mathcal{U}}\colon F(\mathcal{U}) \to F(\mathcal{U}')$ is injective.*

Proof. If $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{U}' = \{U'_j \to U\}_{j\in J}$, then $\mathcal{U} \le \mathcal{U}' \le \mathcal{U}''$, where $\mathcal{U}'' := \{U_i \times_U U'_j \to U\}_{i\in I, j\in J} \in \mathrm{Cov}(U)$. It is then enough to show that the map $\lambda_{\mathcal{U}'',\mathcal{U}} = \lambda_{\mathcal{U}'',\mathcal{U}'} \circ \lambda_{\mathcal{U}',\mathcal{U}}\colon F(\mathcal{U}) \to F(\mathcal{U}'')$ is injective. Given

$$\xi = (\xi_i)_{i\in I}, \eta = (\eta_i)_{i\in I} \in F(\mathcal{U}) \subseteq \prod_{i\in I} F(U_i)$$

such that $\lambda_{\mathcal{U}'',\mathcal{U}}(\xi) = \lambda_{\mathcal{U}'',\mathcal{U}}(\eta)$, for every $i \in I$ we have $\xi_i|_{U_i\times_U U'_j} = \eta_i|_{U_i\times_U U'_j}$ for all $j \in J$, which implies $\xi_i = \eta_i$ (because $\{U_i \times_U U'_j \to U_i\}_{j\in J} \in \mathrm{Cov}(U_i)$ and F is separated), whence $\xi = \eta$. □

Corollary 4.33. *Assume that (for every $U \in \mathbf{C}$) $\mathrm{Cov}'(U) \subseteq \mathrm{Cov}(U)$ is a subset such that for every $\mathcal{U} \in \mathrm{Cov}(U)$ there exists $\mathcal{U}' \in \mathrm{Cov}'(U)$ with $\mathcal{U} \le \mathcal{U}'$. If $F \in \widehat{\mathbf{C}}$ is such that $\lambda^F_{\mathcal{U}'}$ is injective (respectively bijective) for every $U \in \mathbf{C}$ and for every $\mathcal{U}' \in \mathrm{Cov}'(U)$, then F is separated (respectively a sheaf).*

Proof. Given $\mathcal{U} \in \mathrm{Cov}(U)$, let $\mathcal{U}' \in \mathrm{Cov}'(U)$ be such that $\mathcal{U} \le \mathcal{U}'$. Since $\lambda_{\mathcal{U}'} = \lambda_{\mathcal{U}',\mathcal{U}} \circ \lambda_{\mathcal{U}}$ is injective, $\lambda_{\mathcal{U}}$ is injective, too (so that F is separated). Then, by Lemma 4.32, $\lambda_{\mathcal{U}',\mathcal{U}}$ is also injective, and this clearly implies that $\lambda_{\mathcal{U}}$ is bijective if $\lambda_{\mathcal{U}'}$ is bijective. □

Finally, we note that every morphism $f\colon U \to V$ of $\mathbf{C}$ induces, for every $\mathcal{V} \in \mathrm{Cov}(V)$, a natural map $F(f,\mathcal{V})\colon F(\mathcal{V}) \to F(f^*\mathcal{V})$. Such maps are clearly compatible with the direct systems described before (if $\mathcal{V} \le \mathcal{V}' \in \mathrm{Cov}(V)$, then $\lambda_{f^*\mathcal{V}',f^*\mathcal{V}} \circ F(f,\mathcal{V}) = F(f,\mathcal{V}') \circ \lambda_{\mathcal{V}',\mathcal{V}}$).

If $(\mathbf{C},\tau)$ is a site, we will denote by $(\mathbf{C},\tau)^\sim$ (respectively $(\mathbf{C},\tau)^\approx$), or simply by $\mathbf{C}^\sim$ (respectively $\mathbf{C}^\approx$), the full subcategory of $\widehat{\mathbf{C}}$ whose objects are the sheaves (respectively the separated presheaves) for τ; clearly $\mathbf{C}^\sim$ and $\mathbf{C}^\approx$ are strictly full subcategories of $\widehat{\mathbf{C}}$.

Example 4.34. On $\mathbf{C}$ with the chaotic pretopology every presheaf is a sheaf, so that $\mathbf{C}^{\sim} = \widehat{\mathbf{C}}$. Therefore, every statement which is valid for an arbitrary category of sheaves, holds in particular for the categories of presheaves. On the other hand, if $\mathbf{C}$ has fibred products, the only sheaf (up to isomorphism) for the discrete pretopology is the terminal object $* \in \widehat{\mathbf{C}}$ (defined for every $U \in \mathbf{C}$ by $*(U) = \{*\}$): to see this, just consider the empty covering of every $U \in \mathbf{C}$.

Example 4.35. If X is a topological space, $(\mathbf{Open}(X), \mathit{std})^{\sim}$ is the usual category of sheaves (of sets) on X.

Example 4.36. On the category of schemes we have clearly

$$(\mathbf{Sch}, \mathit{fppf})^{\sim} \subseteq (\mathbf{Sch}, \mathit{sm})^{\sim} \subseteq (\mathbf{Sch}, \textit{ét})^{\sim} \subseteq (\mathbf{Sch}, \mathit{Zar})^{\sim}.$$

We claim that actually $(\mathbf{Sch}, \mathit{sm})^{\sim} = (\mathbf{Sch}, \textit{ét})^{\sim}$: given a scheme U and $\mathcal{U} = \{f_i \colon U_i \to U\}_{i\in I} \in \mathrm{Cov}^{sm}(U)$, let $f\colon \coprod_{i\in I} U_i \to U$ be the morphisms induced by the f_i. As f is smooth and surjective, by Corollary 3.130 there exists a morphism $g\colon U' \to \coprod_{i\in I} U_i$ such that $f \circ g\colon U' \to U$ is étale and surjective. Therefore $\{(f \circ g)|_{g^{-1}(U_i)} \colon g^{-1}(U_i) \to U\}_{i\in I} \in \mathrm{Cov}^{\acute{e}t}(U)$ is a refinement of $\mathcal{U}$, and the claim follows from Corollary 4.33. We will see later that the other two inclusions are strict.

Definition 4.37. A pretopology on a category $\mathbf{C}$ is *subcanonical* if $\mathbf{C} \subseteq \mathbf{C}^{\sim}$ (i.e., if every representable presheaf is a sheaf).

Example 4.38. The discrete pretopology on a (non empty) category with fibred products is not subcanonical, unless the category is equivalent to $\{*\}$.

In practice, all "interesting" pretopologies are subcanonical.

Example 4.39. If X is a topological space, the pretopology *std* on $\mathbf{Open}(X)$ is subcanonical: this amounts to the trivial fact that if V and U_i $(i \in I)$ are open subsets of X such that $U_i \subseteq V$ for every $i \in I$, then $\bigcup_{i\in I} U_i \subseteq V$.

Example 4.40. It is a non trivial fact (which we will prove later) that the *fppf* (hence also *sm*, *ét* and *Zar*) pretopology on $\mathbf{Sch}$ is subcanonical. Actually it is easy to prove that *Zar* is subcanonical: it amounts to the fact that morphisms of schemes can be glued.

Proposition 4.41. *Let $F\colon \mathbf{C}' \to \mathbf{C}$ be a functor with the following property: if $V \to U$ and $W \to U$ are morphisms of $\mathbf{C}'$ such that $F(V) \times_{F(U)} F(W)$ exists in $\mathbf{C}$, then $V \times_U W$ exists in $\mathbf{C}'$ and $F(V \times_U W) \cong F(V) \times_{F(U)} F(W)$. If τ is a pretopology on $\mathbf{C}$, then there is a naturally induced pretopology $F^*(\tau)$ on $\mathbf{C}'$, defined in the following way: for every $U \in \mathbf{C}'$, $\{f_i \colon U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}^{F^*(\tau)}(U)$ if and only if $\{F(f_i)\colon F(U_i) \to F(U)\}_{i\in I}$ is in $\mathrm{Cov}^{\tau}(F(U))$. It has the property that the natural functor $\circ F\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}'}$ restricts to functors $\circ F\colon (\mathbf{C}, \tau)^{\approx} \to (\mathbf{C}', F^*(\tau))^{\approx}$ and $\circ F\colon (\mathbf{C}, \tau)^{\sim} \to (\mathbf{C}', F^*(\tau))^{\sim}$.*

Proof. Immediate from the definitions. □

We will apply the above result mainly when F is the inclusion $\mathbf{C}' \subseteq \mathbf{C}$ of a full subcategory (with the property that, if $V \to U$ and $W \to U$ are morphisms of $\mathbf{C}'$ such that $V \times_U W$ exists in $\mathbf{C}$, then $V \times_U W$ is isomorphic to an object of $\mathbf{C}'$) or the forgetful functor $\mathbf{C}'_{/S} \to \mathbf{C}'$ (where S is an object of $\mathbf{C}'$ or, more generally, of $\mathbf{C}$). In these cases the induced pretopology $F^*(\tau)$ will be again denoted by τ. So, for instance, if τ is a pretopology on **Sch**, then τ will denote also the induced pretopology on **QSch**, **AffSch**, $\mathbf{Sch}_{/S}$, $\mathbf{QSch}_{/S}$ and $\mathbf{AffSch}_{/S}$ (S a scheme).

Proposition 4.42. *Let $(\mathbf{C}, \tau)$ be a site and $H \in (\mathbf{C}, \tau)^\sim$. Then the functors of Lemma 4.20 restrict to quasi-inverse equivalences of categories $(\mathbf{C}, \tau)^\sim_{/H} \to (\mathbf{C}_{/H}, \tau)^\sim$ and $(\mathbf{C}_{/H}, \tau)^\sim \to (\mathbf{C}, \tau)^\sim_{/H}$.*

Proof. Given a morphism $\alpha \colon F \to H$ in $\widehat{\mathbf{C}}$, and denoting by $G_\alpha \in \widehat{\mathbf{C}_{/H}}$ the corresponding presheaf, we have to prove that $F \in \mathbf{C}^\sim$ if and only if $G_\alpha \in (\mathbf{C}_{/H})^\sim$.

Assume first that F is a sheaf: given an object $\xi \colon U \to H$ of $\mathbf{C}_{/H}$ and a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$, we have to show that the natural sequence

$$G_\alpha(\xi) \longrightarrow \prod_{i \in I} G_\alpha(\xi_i) \rightrightarrows \prod_{j,k \in I} G_\alpha(\xi_{j,k})$$

(where $\xi_i \colon U_i \to H$ and $\xi_{j,k} \colon U_j \times_U U_k \to H$ are the compositions of ξ with the natural morphisms $U_i \to U$ and $U_j \times_U U_k \to U$) is exact. Now, this sequence is by definition a subsequence of the natural sequence

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{j,k \in I} F(U_j \times_U U_k),$$

which is exact because F is a sheaf. Therefore it is enough to prove the following: if $\eta \in F(U)$ is such that $\eta|_{U_i} \in G_\alpha(\xi_i) \subseteq F(U_i)$ for every $i \in I$, then $\eta \in G_\alpha(\xi)$. Indeed, since $\eta|_{U_i} \in G_\alpha(\xi_i)$, we have $(\alpha \circ \eta)|_{U_i} = \alpha \circ \eta|_{U_i} = \xi_i = \xi|_{U_i}$ (for every $i \in I$), which implies $\alpha \circ \eta = \xi$ (because H is separated), and this precisely says that $\eta \in G_\alpha(\xi)$.

Assume conversely that G_α is a sheaf: we have to prove that given a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ and $\eta_i \in F(U_i)$ (for every $i \in I$) such that $\eta_i|_{U_i \times_U U_j} = \eta_j|_{U_i \times_U U_j}$ for all $i, j \in I$, there exists unique $\eta \in F(U)$ such that $\eta|_{U_i} = \eta_i$ for every $i \in I$. Setting $\xi_i := \alpha \circ \eta_i \in H(U_i)$, we have $\xi_i|_{U_i \times_U U_j} = \xi_j|_{U_i \times_U U_j}$ for all $i, j \in I$, whence there exists a unique $\xi \in H(U)$ such that $\xi|_{U_i} = \xi_i$ for every $i \in I$ (because H is a sheaf). Since, by definition, $\eta_i \in G_\alpha(\xi_i)$ and G_α is a sheaf, there exists a unique $\eta \in G_\alpha(\xi)$ such that $\eta|_{U_i} = \eta_i$ for every $i \in I$. To conclude, if $\eta' \in F(U)$ is another element such that $\eta'|_{U_i} = \eta_i$ for every $i \in I$, we have $(\alpha \circ \eta')|_{U_i} = \xi_i$ for every $i \in I$, which implies that $\eta' \in G_\alpha(\xi)$, and so $\eta' = \eta$. □

Corollary 4.43. *If τ is a subcanonical pretopology on $\mathbf{C}$, then for every $S \in \mathbf{C}$ the induced pretopology τ on $\mathbf{C}_{/S}$ is subcanonical, too.*

Proposition 4.44. *Let $\mathbf{C}' \subseteq \mathbf{C}$ be the inclusion of a full subcategory with the property that, if $V \to U$ and $W \to U$ are morphisms of $\mathbf{C}'$ such that $V \times_U W$ exists in $\mathbf{C}$, then $V \times_U W$ is isomorphic to an object of $\mathbf{C}'$, and let τ be a pretopology on $\mathbf{C}$ which satisfies the following property: for every $U \in \mathbf{C}$ there exists a covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ such that $U_i \in \mathbf{C}'$ for every $i \in I$. Then the natural restriction functor $(\mathbf{C},\tau)^\sim \to (\mathbf{C}',\tau)^\sim$ is an equivalence of categories.*

Proof. For every $U \in \mathbf{C}$ we denote by $\mathrm{Cov}'(U)$ the subset of $\mathrm{Cov}(U)$ given by those coverings $\{U_i \to U\}_{i\in I}$ such that $U_i \in \mathbf{C}'$ for every $i \in I$. Notice that the hypothesis on $\mathbf{C}'$ (together with the axioms of pretopology) implies that for every $U \in \mathbf{C}$ every covering in $\mathrm{Cov}(U)$ admits a refinement in $\mathrm{Cov}'(U)$; it follows that $(\mathrm{Cov}'(U), \leq)$ is a filtered set.

$\mathbf{C}^\sim \to \mathbf{C}'^\sim$ is faithful: given $\alpha, \beta\colon F \to G$ in $\mathbf{C}^\sim$ such that $\alpha|_{\mathbf{C}'} = \beta|_{\mathbf{C}'}$ (i.e., $\alpha(U') = \beta(U')$ for every $U' \in \mathbf{C}'$), we have to prove that $\alpha(U) = \beta(U)$ for every $U \in \mathbf{C}$. Let $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$: then for every $\xi \in F(U)$ and every $i \in I$ we have

$$\alpha(U)(\xi)|_{U_i} = \alpha(U_i)(\xi|_{U_i}) = \beta(U_i)(\xi|_{U_i}) = \beta(U)(\xi)|_{U_i} \in G(U_i).$$

As G is a separated presheaf, this implies $\alpha(U)(\xi) = \beta(U)(\xi)$.

In order to prove that $\mathbf{C}^\sim \to \mathbf{C}'^\sim$ is full, we have to show that for all $F, G \in \mathbf{C}^\sim$ and all $\alpha'\colon F|_{\mathbf{C}'} \to G|_{\mathbf{C}'}$ in $\mathbf{C}'^\sim$, there exists $\alpha\colon F \to G$ in $\mathbf{C}^\sim$ such that $\alpha|_{\mathbf{C}'} = \alpha'$. As before, given $U \in \mathbf{C}$, let $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$. If $\xi \in F(U)$, for all $i, j \in I$ we have $\alpha'(U_i)(\xi|_{U_i})|_{U_i\times_U U_j} = \alpha'(U_j)(\xi|_{U_j})|_{U_i\times_U U_j}$ (this follows from the fact that G is separated and that, if $\{V_k \to U_i \times_U U_j\}_{k\in K} \in \mathrm{Cov}'(U_i \times_U U_j)$, then the restrictions of these two elements to each V_k coincide). Therefore, since G is a sheaf, there exists a unique $\eta \in G(U)$ such that $\eta|_{U_i} = \alpha'(U_i)(\xi|_{U_i})$ for every $i \in I$. It is not difficult to prove (using the fact that $(\mathrm{Cov}'(U), \leq)$ is a filtered set) that η does not depend on the choice of $\mathcal{U} \in \mathrm{Cov}'(U)$. It follows that, if we define $\alpha(U)(\xi) := \eta$, α is a morphism of sheaves, which clearly satisfies $\alpha|_{\mathbf{C}'} = \alpha'$.

It remains to prove that $\mathbf{C}^\sim \to \mathbf{C}'^\sim$ is essentially surjective, i.e. that, given $F' \in \mathbf{C}'^\sim$, there exists $F \in \mathbf{C}^\sim$ such that $F|_{\mathbf{C}'} \cong F'$.

Given $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$, for all $i, j \in I$ we can choose $\mathcal{V}_{i,j} = \{V_{i,j,k} \to U_i \times_U U_j\}_{k\in K_{i,j}} \in \mathrm{Cov}'(U_i \times_U U_j)$, and one can check (using the fact F' is separated and that each $(\mathrm{Cov}'(U_i \times_U U_j), \leq)$ is a filtered set) that

$$F'(\mathcal{U}) := \ker(\textstyle\prod_{i\in I} F'(U_i) \rightrightarrows \prod_{i,j\in I} \prod_{k\in K_{i,j}} F'(V_{i,j,k}))$$

is well defined as a subset of $\prod_{i\in I} F'(U_i)$ (it does not depend on the choices of the $\mathcal{V}_{i,j}$). If $\mathcal{U}' = \{f'_j\colon U'_j \to U\}_{j\in J} \in \mathrm{Cov}'(U)$ is a refinement of $\mathcal{U}$, say $g_j\colon U'_j \to U_{i(j)}$ are such that $f'_j = f_{i(j)} \circ g_j$ for every $j \in J$, the natural map

$$(F'(g_j) \circ pr_{i(j)})_{j\in J}\colon \prod_{i\in I} F'(U_i) \to \prod_{j\in J} F'(U'_j)$$

restricts to a map $\lambda_{\mathcal{U}',\mathcal{U}}\colon F'(\mathcal{U}) \to F'(\mathcal{U}')$, which can be proved to be independent of the choices of the g_j and bijective (because F' is a sheaf). Clearly the sets

$F'(\mathcal{U})$ together with the maps $\lambda_{\mathcal{U}',\mathcal{U}}$ form a direct system, so that we can then define $F(U) := \varinjlim F'(\mathcal{U})$ (the natural map $\mu_{\mathcal{U}}\colon F'(\mathcal{U}) \to F(U)$ is obviously an isomorphism for every $\mathcal{U} \in \mathrm{Cov}'(U)$). If $f\colon U \to V$ is a morphism in $\mathbf{C}$, it is easy to see that, given $\mathcal{V} \in \mathrm{Cov}'(V)$ and $\mathcal{U} \in \mathrm{Cov}'(U)$ such that $f^*\mathcal{V} \leq \mathcal{U}$, there is a naturally induced map $F'(f,\mathcal{V},\mathcal{U})\colon F'(\mathcal{V}) \to F'(\mathcal{U})$ such that $\tilde{F}'(f,\mathcal{V}) := \mu_{\mathcal{U}} \circ F'(f,\mathcal{V},\mathcal{U})\colon F'(\mathcal{V}) \to F(U)$ is well defined (it does not depend on the choice of $\mathcal{U}$). Since $\tilde{F}'(f,\mathcal{V}) = \tilde{F}'(f,\mathcal{V}') \circ \lambda_{\mathcal{V}',\mathcal{V}}$ if $\mathcal{V} \leq \mathcal{V}'$, we can define

$$F(f) := \varinjlim \tilde{F}'(f,\mathcal{V})\colon \varinjlim F'(\mathcal{V}) = F(V) \to F(U),$$

the limit being taken over the filtered set $(\mathrm{Cov}(V), \leq)$. Finally, one can check that with this definition F is really a sheaf, which clearly satisfies $F|_{\mathbf{C}'} \cong F'$. $\square$

Corollary 4.45. *Let S be a scheme and let τ be one of the pretopologies Zar, ét, sm or fppf on $\mathbf{Sch}_{/S}$. Then the natural functors $(\mathbf{Sch}_{/S},\tau)^{\sim} \to (\mathbf{QSch}_{/S},\tau)^{\sim} \to (\mathbf{AffSch}_{/S},\tau)^{\sim}$ are equivalences of categories.*

Definition 4.46. Let $(\mathbf{C},\tau)$ be a site. A property P of objects of $\mathbf{C}$ is *local for τ* if, given $U \in \mathbf{C}$ and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, U satisfies P if and only if U_i satisfies P for every $i \in I$.

Definition 4.47. Let $(\mathbf{C},\tau)$ be a site. A property P of morphisms of $\mathbf{C}$ is *local on the domain* (respectively *local on the codomain*) *for τ* if the following holds: given $U \in \mathbf{C}$ and $\{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, a morphism $g\colon U \to V$ (respectively $V \to U$) satisfies P if and only if $g \circ f_i$ (respectively the projection morphism $V \times_U U_i \to U_i$) satisfies P for every $i \in I$.

Example 4.48. If $(\mathbf{C},\tau) = (\mathbf{Sch}, Zar)$, the above definitions coincide with the usual ones.

Remark 4.49. If $\mathrm{Cov}'(U) \subseteq \mathrm{Cov}(U)$ is a subset (for every $U \in \mathbf{C}$) such that for every $\mathcal{U} \in \mathrm{Cov}(U)$ there exists $\mathcal{U}' \in \mathrm{Cov}'(U)$ with $\mathcal{U} \leq \mathcal{U}'$, then it is easy to see that a property of morphisms of $\mathbf{C}$ which is stable under base change is local on the codomain if it satisfies the condition of the definition for all coverings of $\mathrm{Cov}'(U)$. It follows as in Example 4.36 that a property of morphisms of $\mathbf{Sch}$ which is stable under base change and local on the codomain for *ét* is local on the codomain also for *sm* (but the same is not true for properties local on the domain, or for local properties of objects).

Lemma 4.50. *Assume that $\mathbf{C}$ has fibred products and let P and P' be properties of morphisms of $\mathbf{C}$ such that $f \in \mathrm{Mor}(\mathbf{C})$ satisfies P' if and only if Δ_f satisfies P. If P is local on the codomain for some pretopology τ, then also P' is local on the codomain for τ.*

Proof. Given $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and a morphism $f\colon V \to U$, and denoting by $f_i\colon V_i := V \times_U U_i \to U_i$ the projection morphisms, we have to prove that Δ_f

satisfies P if and only if Δ_{f_i} satisfies P for every $i \in I$. Now, by Lemma A.3, for every $i \in I$ there is a commutative diagram with cartesian squares

$$\begin{tikzcd} V_i \arrow[r, "\Delta_{f_i}"] \arrow[d] \arrow[dr, phantom, "\square"] & V_i \times_{U_i} V_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_i \arrow[d] \\ V \arrow[r, "\Delta_f"'] & V \times_U V \arrow[r] & U \end{tikzcd}$$

and it is then enough to note that $\{V_i \times_{U_i} V_i \to V \times_U V\}_{i\in I} \in \mathrm{Cov}(V \times_U V)$ (this is true by the axioms of pretopology). $\square$

4.3 Sheaf associated to a presheaf

Let $(\mathbf{C}, \tau)$ be a site. To every presheaf $F \in \widehat{\mathbf{C}}$ we can associate a separated presheaf $F^s = F^{s_\tau} \in \mathbf{C}^\approx = (\mathbf{C}, \tau)^\approx$ as follows: for every $U \in \mathbf{C}$

$$F^s(U) := F(U)/\approx,$$

where $\xi \approx \xi'$ if and only if there is $\mathcal{U} \in \mathrm{Cov}(U)$ such that $\lambda_\mathcal{U}(\xi) = \lambda_\mathcal{U}(\xi') \in F(\mathcal{U})$ (the fact that $\mathrm{Cov}(U)$ is a filtered set immediately implies that $\approx$ is an equivalence relation), whereas for every morphism $f : U \to V$ in $\mathbf{C}$, $F^s(f) : F^s(V) \to F^s(U)$ is the map induced by $F(f)$. F^s is obviously a presheaf, and it should be clear by construction that it is separated (it is called the *separated presheaf associated to F*). Notice also that the projections $F(U) \twoheadrightarrow F^s(U)$ define a natural transformation $\sigma_F : F \to F^s$. It is then easy to prove the following result.

Proposition 4.51. *The mapping $F \mapsto F^s$ extends to a functor*

$$-^s = -^{s_\tau} : \widehat{\mathbf{C}} \to \mathbf{C}^\approx,$$

which is left adjoint of the inclusion functor $\mathbf{C}^\approx \subseteq \widehat{\mathbf{C}}$: more explicitly, for every $F \in \widehat{\mathbf{C}}$ and every $G \in \mathbf{C}^\approx$ the natural map

$$\mathrm{Hom}_{\mathbf{C}^\approx}(F^s, G) = \mathrm{Hom}_{\widehat{\mathbf{C}}}(F^s, G) \xrightarrow{\circ\sigma_F} \mathrm{Hom}_{\widehat{\mathbf{C}}}(F, G)$$

is bijective. Moreover, for every $F \in \widehat{\mathbf{C}}$ the natural transformation $\sigma_F : F \to F^s$ is an epimorphism in $\widehat{\mathbf{C}}$, and it is an isomorphism if and only if $F \in \mathbf{C}^\approx$.

In a similar way, to every $F \in \widehat{\mathbf{C}}$ we can associate a sheaf $F^a = F^{a_\tau} \in \mathbf{C}^\sim = (\mathbf{C}, \tau)^\sim$. On each object $U \in \mathbf{C}$ it is defined by $F^a(U) := \varinjlim F^s(\mathcal{U})$, the limit being taken over the filtered set $(\mathrm{Cov}(U), \leq)$; more explicitly,

$$F^a(U) := \{(\mathcal{U}, \xi) \,|\, \mathcal{U} \in \mathrm{Cov}(U), \xi \in F^s(\mathcal{U})\}/\sim,$$

where $(\mathcal{U}, \xi) \sim (\mathcal{U}', \xi')$ if and only if there is $\mathcal{U}'' \in \mathrm{Cov}(U)$ such that $\mathcal{U}, \mathcal{U}' \leq \mathcal{U}''$ and $\lambda^{F^s}_{\mathcal{U}'',\mathcal{U}}(\xi) = \lambda^{F^s}_{\mathcal{U}'',\mathcal{U}'}(\xi') \in F^s(\mathcal{U}'')$. Notice that, by Lemma 4.32, for every

$\mathcal{U} \in \operatorname{Cov}(U)$ the natural map $\mu_{\mathcal{U}} \colon F^s(\mathcal{U}) \to F^a(U)$ is injective (in particular, $F^s(U) \hookrightarrow F^a(U)$), and, if $\mathcal{U} = \{U_i \to U\}_{i\in I}$, $\mathcal{U}' = \{U'_j \to U\}_{j\in J} \in \operatorname{Cov}(U)$ and $\xi = (\xi_i)_{i\in I} \in F(\mathcal{U})$, $\xi' = (\xi'_j)_{j\in J} \in F(\mathcal{U}')$, then $(\mathcal{U},\xi) \sim (\mathcal{U}',\xi')$ if and only if $\xi_i|_{U_i\times_U U'_j} = \xi'_j|_{U_i\times_U U'_j}$ for all $i \in I$ and $j \in J$. If $f\colon U \to V$ is a morphism of $\mathbf{C}$,

$$F^a(f) := \varinjlim(\mu_{f^*\mathcal{V}} \circ F^s(f,\mathcal{V}))\colon \varinjlim F^s(\mathcal{V}) = F^a(V) \to F^a(U),$$

the limit being taken over the filtered set $(\operatorname{Cov}(V),\leq)$. Then it is easy to see that F^a (which is obviously a presheaf) is actually a sheaf (it is called the *sheaf associated to* F); by construction it is also evident that $F^a = (F^s)^a$. As before, the natural maps $F(U) \twoheadrightarrow F^s(U) \hookrightarrow F^a(U)$ define a natural transformation $\rho_F\colon F \to F^a$ (which is therefore the composition of the epimorphism $\sigma_F\colon F \to F^s$ and of the monomorphism $\rho_{F^s}\colon F^s \to (F^s)^a = F^a$). It is not difficult to prove the following result.

Proposition 4.52. *The mapping $F \mapsto F^a$ extends to a functor $-^a = -^{a_\tau}\colon \widehat{\mathbf{C}} \to \mathbf{C}^\sim$, which is left adjoint of the inclusion functor $\mathbf{C}^\sim \subseteq \widehat{\mathbf{C}}$: more explicitly, for all $F \in \widehat{\mathbf{C}}$ and $G \in \mathbf{C}^\sim$ the natural map*

$$\operatorname{Hom}_{\mathbf{C}^\sim}(F^a,G) = \operatorname{Hom}_{\widehat{\mathbf{C}}}(F^a,G) \xrightarrow{\circ\rho_F} \operatorname{Hom}_{\widehat{\mathbf{C}}}(F,G)$$

is bijective. Moreover, for every $F \in \widehat{\mathbf{C}}$ the natural transformation $\rho_F\colon F \to F^a$ is monomorphism in $\widehat{\mathbf{C}}$ if and only if $F \in \mathbf{C}^\approx$, and it is an isomorphism if and only if $F \in \mathbf{C}^\sim$.

Remark 4.53. The associated sheaf can also be defined in a slightly different (but of course equivalent) way. Namely, given $F \in \widehat{\mathbf{C}}$, one can define $L(F) \in \widehat{\mathbf{C}}$ by $L(F)(U) := \varinjlim F(\mathcal{U})$ for every $U \in \mathbf{C}$ (and in the obvious way on morphisms); there is also a natural morphism $\epsilon_F\colon F \to L(F)$ in $\widehat{\mathbf{C}}$. Then it is clear that, if F is separated, $L(F)$ is a sheaf, naturally isomorphic to F^a; moreover, one can prove that in any case $L(F)$ is separated, that there is a natural isomorphism $F^a \cong L(L(F))$ and that, with this identification, $\rho_F\colon F \to F^a$ coincides with $\epsilon_{L(F)} \circ \epsilon_F\colon F \to L(L(F))$.

Proposition 4.54. *For every site $(\mathbf{C},\tau)$ the categories $\mathbf{C}^\approx$ and $\mathbf{C}^\sim$ have kernels, cokernels, (fibred) products and (fibred) coproducts. Moreover, the inclusion functors $\mathbf{C}^\approx \subseteq \widehat{\mathbf{C}}$ and $\mathbf{C}^\sim \subseteq \widehat{\mathbf{C}}$ preserve kernels and (fibred) products, whereas $-^s\colon \widehat{\mathbf{C}} \to \mathbf{C}^\approx$ and $-^a\colon \widehat{\mathbf{C}} \to \mathbf{C}^\sim$ preserve cokernels and (fibred) coproducts.*

Proof. The statements about cokernels and (fibred) coproducts follow easily from the universal property of associated sheaf. As for fibred products (the case of products and kernels is similar) it is enough to prove that, given morphisms $\alpha_i\colon F_i \to F$ (for $i = 1,2$) of $\mathbf{C}^\approx$, $G := F_1\,{}_{\alpha_1}\!\times_{\alpha_2} F_2 \in \widehat{\mathbf{C}}$ is also separated (this is straightforward), and it is a sheaf if $F_1, F_2 \in \mathbf{C}^\sim$. Therefore, given $U \in \mathbf{C}$ and $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \operatorname{Cov}(U)$, we have to prove that the natural map $\lambda_{\mathcal{U}}^G\colon G(U) \to G(\mathcal{U})$ is bijective. So, let $\xi = (\xi_i)_{i\in I} \in G(\mathcal{U}) \subseteq \prod_{i\in I} G(U_i)$, and assume that $\xi_i = (\xi_i^1, \xi_i^2)$ with $\xi_i^j \in F_j(U_i)$. Since $\xi \in G(\mathcal{U})$, we have that $\xi^j := (\xi_i^j)_{i\in I} \in F_j(\mathcal{U})$ for $i = 1, 2$, whence there exists a unique $\eta^j \in F_j(U)$ such that $\lambda_{\mathcal{U}}^{F_j}(\eta^j) = \xi^j$ (because F_j is a sheaf). To conclude, it is enough to note that $\eta := (\eta^1, \eta^2) \in G(U) \subseteq F_1(U) \times F_2(U)$ (so that η is the unique solution of $\lambda_{\mathcal{U}}^G(\eta) = \xi$), i.e. that $\alpha_1(U)(\eta^1) = \alpha_2(U)(\eta^2) \in F(U)$ (this follows from the fact that F is separated). □

Corollary 4.55. *Given $F_k \in \mathbf{C}^\sim$ ($k \in K$), the sheaf $(\coprod_{k\in K} F_k)^a$ (together with the natural morphisms $F_h \to \coprod_{k\in K} F_k \to (\coprod_{k\in K} F_k)^a$ for $h \in K$) gives the coproduct of the F_k in $\mathbf{C}^\sim$. Similar statements hold for fibred coproducts and cokernels.*

Remembering the definition of sheaf associated to a (separated) presheaf, it is immediate to prove the following result.

Lemma 4.56. *Let $\alpha\colon F \hookrightarrow G$ be a monomorphism of $\widehat{\mathbf{C}}$. If G is separated, then F is separated, too. If G is a sheaf, then the induced morphism $\alpha'\colon F^a \to G$ (such that $\alpha' \circ \rho_F = \alpha$) is also a monomorphism.*

In particular, if $\alpha\colon F \to G$ is a morphism of $\mathbf{C}^\sim$, the natural morphism $(\operatorname{im}\alpha)^a \to G$ is a monomorphism of $\widehat{\mathbf{C}}$, so that $(\operatorname{im}\alpha)^a$ can be identified with a subsheaf of G, which we will denote by $\widetilde{\operatorname{im}\alpha}$ and call the *image* of α in $\mathbf{C}^\sim$. Note that, given $U \in \mathbf{C}$ and $\xi \in G(U)$, $\xi \in (\widetilde{\operatorname{im}\alpha})(U)$ if and only if there exists $\{U_i \to U\}_{i\in I} \in \operatorname{Cov}(U)$ such that $\xi|_{U_i} \in (\operatorname{im}\alpha)(U_i)$ for every $i \in I$.

Proposition 4.57. *A morphism $\alpha\colon F \to G$ of $\mathbf{C}^\sim$ is a monomorphism in $\mathbf{C}^\sim$ if and only if it is a monomorphism in $\widehat{\mathbf{C}}$, and it is an epimorphism in $\mathbf{C}^\sim$ if and only if $\widetilde{\operatorname{im}\alpha} = G$.*

Proof. If α is a monomorphism in $\mathbf{C}^\sim$ (the other implication is obvious), we have to show that $\operatorname{Hom}_{\widehat{\mathbf{C}}}(H, F) \xrightarrow{\alpha\circ} \operatorname{Hom}_{\widehat{\mathbf{C}}}(H, G)$ is injective for every $H \in \widehat{\mathbf{C}}$. By Proposition 4.52 this map can be identified with $\operatorname{Hom}_{\mathbf{C}^\sim}(H^a, F) \xrightarrow{\alpha\circ} \operatorname{Hom}_{\mathbf{C}^\sim}(H^a, G)$, which is injective by hypothesis.

Assume now that α is an epimorphism in $\mathbf{C}^\sim$. Let $H := G \coprod_{\widetilde{\operatorname{im}\alpha}} G$ (fibred coproduct in $\widehat{\mathbf{C}}$) and denote by $j_1, j_2\colon G \hookrightarrow H$ the natural morphisms (which satisfy $j_1 \circ \alpha = j_2 \circ \alpha$). Since $\rho_H \circ j_1 \circ \alpha = \rho_H \circ j_2 \circ \alpha \in \operatorname{Hom}_{\mathbf{C}^\sim}(F, H^a)$, it follows that $\rho_H \circ j_1 = \rho_H \circ j_2$. As ρ_H is a monomorphism (because H is separated, as it is immediate to check), this implies that $j_1 = j_2$, whence $\widetilde{\operatorname{im}\alpha} = G$. Conversely, if $\widetilde{\operatorname{im}\alpha} = G$, then, given $\beta_1, \beta_2\colon G \to H$ such that $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, we have in particular $\beta_1|_{\operatorname{im}\alpha} = \beta_2|_{\operatorname{im}\alpha}$, and therefore (by the universal property of associated sheaf) $\beta_1 = \beta_1|_{\widetilde{\operatorname{im}\alpha}} = \beta_2|_{\widetilde{\operatorname{im}\alpha}} = \beta_2$. □

Corollary 4.58. *A morphism of $\mathbf{C}^\sim$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Proof. If $\alpha\colon F \to G$ is a monomorphism and an epimorphism in $\mathbf{C}^{\sim}$, then it is a monomorphism also in $\widehat{\mathbf{C}}$, so that (by Remark 4.19) $F \xrightarrow{\sim} \operatorname{im}\alpha$. Thus $\operatorname{im}\alpha$ is a sheaf and $\operatorname{im}\alpha = \widetilde{\operatorname{im}}\,\alpha = G$, which proves that α is also an epimorphism in $\widehat{\mathbf{C}}$, and the conclusion follows from Corollary 4.17. $\square$

Remark 4.59. Every morphism $\alpha\colon F \to G$ of $\mathbf{C}^{\sim}$ factors as the composition of the epimorphism (of $\mathbf{C}^{\sim}$) $F \twoheadrightarrow \operatorname{im}\alpha \hookrightarrow \widetilde{\operatorname{im}}\,\alpha$ and of the monomorphism $\widetilde{\operatorname{im}}\,\alpha \hookrightarrow G$. It is easy to see that such a factorization is unique up to isomorphism.

Corollary 4.60. *If $\alpha\colon F \to G$ is a representable covering morphism of $\mathbf{C}^{\sim}$, then α is an epimorphism.*

Proof. Given $V \in \mathbf{C}$ and $\eta \in G(V)$, by hypothesis there is a cartesian diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "\eta"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

such that $\{f\} \in \operatorname{Cov}(V)$. Then $\eta|_U = \eta \circ f = \alpha \circ \xi \in (\operatorname{im}\alpha)(U)$, which implies that $\eta \in (\widetilde{\operatorname{im}}\,\alpha)(V)$, so that $\widetilde{\operatorname{im}}\,\alpha = G$. $\square$

4.4 Presheaves and sheaves with values in a category

So far we have dealt only with presheaves and sheaves of sets; of course, one can consider also (pre)sheaves of groups, rings, modules, ... (in general, of objects of an arbitrary category, instead of **Set**). As for presheaves, it is natural to give the following definition.

Definition 4.61. A *presheaf* on a category $\mathbf{C}$ with values in a category $\mathbf{D}$ is a functor $\mathbf{C}^{\circ} \to \mathbf{D}$. A morphism of presheaves on $\mathbf{C}$ with values in $\mathbf{D}$ is just a natural transformation of such functors.

Therefore, the category of presheaves on $\mathbf{C}$ with values in $\mathbf{D}$ is $\mathbf{Fun}(\mathbf{C}^{\circ}, \mathbf{D})$.

Definition 4.62. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A presheaf $F \in \mathbf{Fun}(\mathbf{C}^{\circ}, \mathbf{D})$ is *separated* (respectively a *sheaf*) for a pretopology τ on $\mathbf{C}$ if for every $U \in \mathbf{D}$ the presheaf of sets $\operatorname{Hom}_{\mathbf{D}}(U, F(-)) \in \widehat{\mathbf{C}}$ is separated (respectively a sheaf) for τ.

As usual, we will regard separated presheaves and sheaves as (strictly) full subcategories of $\mathbf{Fun}(\mathbf{C}^{\circ}, \mathbf{D})$.

Remark 4.63. Of course, one has to check that the above definition coincides with the old one in the case $\mathbf{D} = \mathbf{Set}$. This follows from the fact that each functor $\operatorname{Hom}_{\mathbf{Set}}(U, -)$ preserves kernels, products and monomorphisms (as it is easy to see) and that $\operatorname{Hom}_{\mathbf{Set}}(\{*\}, -) \cong \operatorname{id}_{\mathbf{Set}}$.

Remark 4.64. In many common cases, when $\mathbf{D}$ is a category like $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Rng}$, A-$\mathbf{Mod}$ (A a ring), there is a natural forgetful functor $D\colon \mathbf{D} \to \mathbf{Set}$ and a presheaf $F \in \mathbf{Fun}(\mathbf{C}^\circ, \mathbf{D})$ is separated (respectively a sheaf) if and only if $D \circ F \in \widehat{\mathbf{C}}$ is separated (respectively a sheaf). Indeed, if F is separated or a sheaf, the same is true for $D \circ F$ because there exists $X \in \mathbf{D}$ such that $\mathrm{Hom}_{\mathbf{D}}(X, -) \cong D$ ($X = \mathbb{Z}$ for $\mathbf{Grp}$ and $\mathbf{Ab}$, $X = \mathbb{Z}[x]$ for $\mathbf{Rng}$ and $X = A$ for A-$\mathbf{Mod}$). The converse implication follows from the fact that, again, each functor $\mathrm{Hom}_{\mathbf{D}}(U, -)$ preserves kernels, products and monomorphisms and that a morphism $f\colon U \to V$ is a monomorphism (respectively a kernel of $V \underset{g_2}{\overset{g_1}{\rightrightarrows}} W$) in $\mathbf{D}$ if and only if $D(f)$ is a monomorphism (respectively a kernel of $D(V) \underset{D(g_2)}{\overset{D(g_1)}{\rightrightarrows}} D(W)$) in $\mathbf{Set}$.

We are going to see that (pre)sheaves of groups[9] admit alternative (equivalent) descriptions. First, we give a more general definition.

Definition 4.65. Let $\mathbf{C}$ be a category. A *group in* $\mathbf{C}$ is given by an object G of $\mathbf{C}$ together with a group structure on $G(U)$ for every object U of $\mathbf{C}$, such that the map $G(f)\colon G(V) \to G(U)$ is a homomorphism of groups for every morphism $f\colon U \to V$ of $\mathbf{C}$.

In other words, a group in $\mathbf{C}$ is given by an object G of $\mathbf{C}$ together with an isomorphism (in $\widehat{\mathbf{C}}$) $G \cong D \circ F$ for some $F \in \mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Grp})$ (where $D\colon \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor).

Now, it is easy to see that the datum of a sheaf of groups on $\mathbf{C}$ is equivalent to the datum of a group in $\mathbf{C}^\sim$ (and similarly for (separated) presheaves). In fact, if $F\colon \mathbf{C}^\circ \to \mathbf{Grp}$ is a sheaf of groups (so that $G := D \circ F \in \mathbf{C}^\sim$ by Remark 4.64), then G can be given a structure of group in $\mathbf{C}^\sim$ as follows: for every $H \in \mathbf{C}^\sim$ the set $\mathrm{Hom}_{\mathbf{C}^\sim}(H, G)$ is in a natural way a group with multiplication $\alpha \cdot \beta$ defined for every $U \in \mathbf{C}$ by $(\alpha \cdot \beta)(U) := \alpha(U)\beta(U)$ (the latter is multiplication in $\mathrm{Hom}_{\mathbf{Set}}(H(U), G(U))$, which is a group because $G(U) = F(U)$ is a group). Conversely, if G is a group in $\mathbf{C}^\sim$, then the presheaf $F\colon \mathbf{C}^\circ \to \mathbf{Grp}$ defined for every $U \in \mathbf{C}$ by $F(U) := G(U)$ with the natural group structure (notice that $G(U) = \mathrm{Hom}_{\widehat{\mathbf{C}}}(U, G) \cong \mathrm{Hom}_{\mathbf{C}^\sim}(U^a, G)$, which is a group by definition of group in $\mathbf{C}^\sim$) is a sheaf of groups (since $D \circ F = G$).

The following result shows that, if $\mathbf{C}$ has finite products (which is always true for the categories of presheaves and of sheaves), the definition of group in $\mathbf{C}$ can be reformulated in a more intrinsic way.

Proposition 4.66. *Let $\mathbf{C}$ be a category with finite products, and denote by $*$ a terminal object of $\mathbf{C}$. Then to give a group in $\mathbf{C}$ is equivalent to giving morphisms $e\colon * \to G$ ("identity"), $i\colon G \to G$ ("inverse") and $m\colon G \times G \to G$ ("multiplication") of $\mathbf{C}$ such that the following diagrams commute:*

[9] We treat here only the case of groups, but similar considerations can be done for other algebraic structures, like rings, modules, ...

1. $$\begin{tikzcd} & G \times G \arrow[d, "m"] & \\ G \arrow[ur, "{(\mathrm{id}, e\circ *_G)}"] \arrow[r, "\mathrm{id}"'] & G & G \arrow[ul, "{(e\circ *_G, \mathrm{id})}"'] \arrow[l, "\mathrm{id}"] \end{tikzcd};$$

2. $$\begin{tikzcd} G \arrow[r, "{(\mathrm{id}, i)}"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "{(i, \mathrm{id})}"'] \arrow[d] \\ * \arrow[r, "e"'] & G & * \arrow[l, "e"] \end{tikzcd};$$

3. $$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times m"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}.$$

Proof. Given e, i and m such that the diagrams commute, it is clear that for every $U \in \mathbf{C}$ the set $G(U)$ is a group with multiplication given by

$$G(U) \times G(U) \cong (G \times G)(U) \xrightarrow{m(U)} G(U)$$

(the identity is $e(U)(*(U)) \in G(U)$ and the inverse is $i(U)\colon G(U) \to G(U)$) and that in this way $G(f)$ is homomorphism of groups for every $f \in \mathrm{Mor}(\mathbf{C})$.

Conversely, if G is a group in $\mathbf{C}$, then, denoting (for every $U \in \mathbf{C}$) by $m(U)\colon (G \times G)(U) \cong G(U) \times G(U) \to G(U)$ the multiplication, the maps $m(U)$ clearly define a natural transformation (hence a morphism of $\mathbf{C}$, by Yoneda's lemma) $m\colon G \times G \to G$. Moreover, associativity of the multiplication maps (and, again, Yoneda's lemma) implies the commutativity of the last diagram. Similarly, from the existence of identity and inverse one sees that there exists $e\colon * \to G$ and $i\colon G \to G$ such that the other diagrams commute. □

Remark 4.67. The above result shows, in particular, that a group in **Set** (respectively in **Diff**, respectively in **Sch**) is a group in the usual sense (respectively a Lie group, respectively a group scheme).

Definition 4.68. Let G be a group in a category $\mathbf{C}$. An *action (on the right) in* $\mathbf{C}$ of G on an object U of $\mathbf{C}$ is given by actions $\varrho(W)\colon U(W) \times G(W) \to U(W)$ of the group $G(W)$ on the set $U(W)$ (for every $W \in \mathbf{C}$) such that for every morphism $f\colon W \to W'$ of $\mathbf{C}$ the diagram

$$\begin{tikzcd} U(W') \times G(W') \arrow[r, "\varrho(W')"] \arrow[d, "U(f) \times G(f)"'] & U(W') \arrow[d, "U(f)"] \\ U(W) \times G(W) \arrow[r, "\varrho(W)"'] & U(W) \end{tikzcd}$$

commutes. The action ϱ is *free* if $\varrho(W)$ is a free action for every $W \in \mathbf{C}$.

With the same technique of Proposition 4.66 it is easy to prove the following result.

Proposition 4.69. *Let* $\mathbf{C}$ *be a category with finite products and let* G *be a group in* $\mathbf{C}$*. Then to give an action of* G *on an object* U *of* $\mathbf{C}$ *is equivalent to giving a morphism* $\varrho\colon U \times G \to U$ *of* $\mathbf{C}$ *such that the following diagrams commute (where the maps* e *and* m *are as in Proposition 4.66):*

1. $$\begin{tikzcd} U \arrow[rr, "(\mathrm{id}, e\circ *_U)"] \arrow[dr, "\mathrm{id}"'] & & U \times G \arrow[dl, "\varrho"] \\ & U & \end{tikzcd};$$

2. $$\begin{tikzcd} U \times G \times G \arrow[r, "\varrho\times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_U\times m"'] & U \times G \arrow[d, "\varrho"] \\ U \times G \arrow[r, "\varrho"'] & U \end{tikzcd}.$$

In particular, an action in **Set** *is just an action in the usual sense.*

Moreover, an action $\varrho\colon U \times G \to U$ *in* $\mathbf{C}$ *is free if and only if* $(pr_1, \varrho)\colon U \times G \to U \times U$ *is a monomorphism of* $\mathbf{C}$.

4.5 Equivalence relations

Definition 4.70. Let $\mathbf{C}$ be a category. An *equivalence relation in* $\mathbf{C}$ is given by two morphisms $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U$ of $\mathbf{C}$ such that for every $W \in \mathbf{C}$ the induced map of sets

$$(\delta_1(W), \delta_2(W))\colon R(W) \to U(W) \times U(W)$$

is an equivalence relation in the usual sense (in particular, it must be injective).

As in the case of groups, equivalence relations in $\mathbf{C}$ can be described in an alternative way, at least when $\mathbf{C}$ has finite products and fibred products: this is shown in the following result, whose proof is again similar to that of Proposition 4.66.

Proposition 4.71. *Let* $\delta_1, \delta_2\colon R \to U$ *be morphisms in a category* $\mathbf{C}$ *such that* $U \times U$ *and* $R\,{}_{\delta_2}\!\times_{\delta_1} R$ *exist. Then* $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U$ *is an equivalence relation in* $\mathbf{C}$ *if and only if* $\delta := (\delta_1, \delta_2)\colon R \to U \times U$ *is a monomorphism of* $\mathbf{C}$ *and there exist morphisms* $e\colon U \to R$, $i\colon R \to R$ *and* $m\colon R\,{}_{\delta_2}\!\times_{\delta_1} R \to R$ *of* $\mathbf{C}$ *such that the following diagrams commute:*

1. *(reflexivity)*
$$\begin{tikzcd} R \arrow[r, "\delta"] & U \times U \;; \\ & U \arrow[ul, "e"] \arrow[u, "\Delta_U"'] \end{tikzcd}$$

2. *(symmetry)*
$$\begin{tikzcd} U & R \arrow[l, "\delta_1"'] \arrow[r, "\delta_2"] \arrow[d, "i"] & U \;; \\ & R \arrow[ul, "\delta_2"] \arrow[ur, "\delta_1"'] & \end{tikzcd}$$

3. *(transitivity)*
$$\begin{tikzcd} R \,{}_{\delta_2}\!\times_{\delta_1} R \arrow[r, "\delta_1 \times \delta_2"] \arrow[dr, "m"'] & U \times U \;. \\ & R \arrow[u, "\delta"'] \end{tikzcd}$$

In particular, an equivalence relation in **Set** *is just an equivalence relation in the usual sense.*

In case $U \times U$ exists, an equivalence relation $R \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} U$ will be often denoted by $(\delta_1, \delta_2)\colon R \hookrightarrow U \times U$.

Example 4.72. Assume that $\mathbf{C}$ has finite products and let $\varrho\colon U \times G \to U$ be a free action in $\mathbf{C}$. Then it is easy to see that $(pr_1, \varrho)\colon U \times G \to U \times U$ is an equivalence relation in $\mathbf{C}$.

Definition 4.73. A morphism $\pi\colon U \to V$ is a *quotient* of the equivalence relation $R \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} U$ if $\pi \cong \operatorname{coker}(R \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} U)$. π is an *effective quotient* if moreover the induced morphism $R \to U \times_V U$ is an isomorphism; in this case, we will also say that $R \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} U$ is an *effective equivalence relation.*

Remark 4.74. Of course, if a quotient of an equivalence relation exists, then it is unique up to isomorphism, and it is an epimorphism. In general, however, it need not exist, and, if it exists, it need not be effective.

Example 4.75. Every equivalence relation $R \hookrightarrow X \times X$ in **Set** is effective: it is straightforward to check that the natural projection $X \twoheadrightarrow X/R$ is an effective quotient.

Example 4.76. Let X be a set with $\#(X) > 1$. Then it is clear that the equivalence relation $X \times X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X$ (whose quotient in **Set** is $X \to \{*\}$) has no quotient in the full subcategory of **Set** having as objects the sets Y with $\#(Y) > 1$.

Example 4.77. Let $\mathbb{A}^1 := \mathbb{A}^1_{\mathbb{K}} = \operatorname{Spec}\mathbb{K}[t]$ ($\mathbb{K}$ a field of characteristic $\neq 2$) and let $\iota\colon \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec}\mathbb{K}[t, t^{-1}] \to \mathbb{A}^1$ be the inclusion morphism. We consider the equivalence relation $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} \mathbb{A}^1$ in $\mathbf{AffSch}_{/\mathbb{K}}$, where $R := \mathbb{A}^1 \coprod(\mathbb{A}^1 \setminus \{0\})$ and δ_1 (respectively δ_2) is the morphism induced by $\mathrm{id}_{\mathbb{A}^1}$ and ι (respectively $-\iota$); geometrically, R is the disjoint union of the "diagonal" and of the "antidiagonal minus the origin" in $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$.

Algebraically $\delta_i = \operatorname{Spec}\phi_i$, where $\phi_i\colon \mathbb{K}[t] \to \mathbb{K}[t] \times \mathbb{K}[t, t^{-1}]$ are the morphisms of $\mathbb{K}$-**Alg** defined by $\phi_1(t) := (t, t)$ and $\phi_2(t) := (t, -t)$. We claim that $\pi := \operatorname{Spec}\psi\colon \mathbb{A}^1 \to \mathbb{A}^1$, where $\psi\colon \mathbb{K}[t] \to \mathbb{K}[t]$ is defined by $\psi(t) := t^2$, is a non effective quotient of $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} \mathbb{A}^1$ in $\mathbf{AffSch}_{/\mathbb{K}}$. Indeed, by Proposition 4.78 below, this corresponds to the fact that the diagram in $\mathbb{K}$-**Alg**

$$\begin{tikzcd} \mathbb{K}[t] \arrow[r, "\psi"] \arrow[d, "\psi"'] & \mathbb{K}[t] \arrow[d, "\phi_1"] \\ \mathbb{K}[t] \arrow[r, "\phi_2"'] & \mathbb{K}[t] \times \mathbb{K}[t, t^{-1}] \end{tikzcd}$$

is cartesian (which is obvious) but not cocartesian (note that $\mathbb{K}[t]\,{}_{\psi}\!\otimes_{\psi} \mathbb{K}[t] \cong \mathbb{K}[t] \times \mathbb{K}[t]$). It is not difficult to prove that π is a non effective quotient of $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} \mathbb{A}^1$ also in $\mathbf{Sch}_{/\mathbb{K}}$.

Proposition 4.78. *A morphism $\pi\colon U \to V$ is a quotient (respectively an effective quotient) of the equivalence relation $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U$ if and only if the diagram*

$$\begin{tikzcd} R \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & U \arrow[d, "\pi"] \\ U \arrow[r, "\pi"'] & V \end{tikzcd}$$

is cocartesian (respectively cartesian and cocartesian).

Proof. Indeed, if π is a quotient of $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U$, then, given morphisms $f_1, f_2\colon U \to W$ such that $f_1 \circ \delta_1 = f_2 \circ \delta_2$, necessarily $f_1 = f_2 := f$ (as in Proposition 4.71, one can prove that there exists $e\colon U \to R$ such that $\delta_i \circ e = \mathrm{id}_U$ for $1 = 1, 2$, so that $f_1 = f_1 \circ \delta_1 \circ e = f_2 \circ \delta_2 \circ e = f_2$). Therefore, by definition of $\operatorname{coker}(R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U)$, there exists a unique $g\colon V \to W$ such that $g \circ \pi = f$, which proves that the diagram is cocartesian. The rest of the proof is clear from the definitions. $\square$

Example 4.79. Let $f\colon U \to W$ be a morphism in a category $\mathbf{C}$ such that $U \times_W U$ exists. Then it follows immediately from the definition of fibred product that $U \times_W U \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} U$ is an equivalence relation in $\mathbf{C}$. If it has a quotient $\pi\colon U \to V$, then π is actually an effective quotient. Indeed, as π is a quotient, there exists a unique $g\colon V \to W$ such that $f = g \circ \pi\colon U \to W$. Therefore, given morphisms $h_1, h_2\colon Z \to U$ such that $\pi \circ h_1 = \pi \circ h_2$, we have also $f \circ h_1 = f \circ h_2$, whence there exists a unique $h\colon Z \to U \times_W U$ such that $h_i = pr_i \circ h$ for $i = 1, 2$, which proves that the diagram

$$\begin{tikzcd} U \times_W U \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & U \arrow[d, "\pi"] \\ U \arrow[r, "\pi"'] & V \end{tikzcd}$$

is cartesian, (i.e., π is effective).

Taking into account Proposition 4.8 it is then immediate to prove the following result.

Proposition 4.80. *Let $\mathbf{C}$ be a category. Then every equivalence relation $R \hookrightarrow F \times F$ in $\widehat{\mathbf{C}}$ has effective quotient given by the natural projection $F \twoheadrightarrow F/R$ (where $F/R \in \widehat{\mathbf{C}}$ is defined by $(F/R)(W) := F(W)/R(W)$ on objects and in the obvious way on morphisms). Moreover, given a morphism $\alpha\colon F \to G$ in $\widehat{\mathbf{C}}$, the natural morphism $F \times_G F \hookrightarrow F \times F$ is an effective equivalence relation in $\widehat{\mathbf{C}}$, with quotient given by the projection $F \twoheadrightarrow \operatorname{im} \alpha$ (in particular, the quotient is α if and only if α is an epimorphism).*

Remark 4.81. If an equivalence relation $R \hookrightarrow U \times U$ in some category $\mathbf{C}$ has effective quotient $\pi\colon U \to V$, it can happen that π is not the quotient in $\widehat{\mathbf{C}}$. Indeed, if $\pi'\colon U \to F$ is the (effective) quotient in $\widehat{\mathbf{C}}$, then there exists a unique $\alpha\colon F \to V$ such that $\pi = \alpha \circ \pi'$, and effectiveness of the two quotients implies that α is a monomorphism; in general, however, it is not an isomorphism. Consider for instance $R := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \mod n\} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$ (where $n > 1$ is an integer): it is easy to see that this is an equivalence relation in $\mathbf{Grp}$ (and in $\mathbf{Ab}$) with effective quotient given by the natural projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Denoting by $\mathbb{Z} \twoheadrightarrow F$ the quotient in $\widehat{\mathbf{Grp}}$, for every $G \in \mathbf{Grp}$ we have $F(G) \cong \mathbb{Z}(G)/R(G)$. In particular (since $\operatorname{Hom}_{\mathbf{Grp}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \{0\}$) $F(\mathbb{Z}/n\mathbb{Z}) = \{0\}$, whence $F \ncong \mathbb{Z}/n\mathbb{Z}$.

Lemma 4.82. *Let $(\mathbf{C}, \tau)$ be a site and let $R \hookrightarrow F \times F$ be an equivalence relation in $\widehat{\mathbf{C}}$ such that R is a sheaf and F is separated. Then $F/R \in \widehat{\mathbf{C}}$ is separated, too.*

Proof. Given $U \in \mathbf{C}$, $\{U_i \to U\}_{i \in I} \in \operatorname{Cov}(U)$ and $\xi', \eta' \in (F/R)(U)$ such that $\xi'|_{U_i} = \eta'|_{U_i} \in (F/R)(U_i)$ for every $i \in I$, we have to prove that $\xi' = \eta'$. Let $\xi, \eta \in F(U)$ be lifts of ξ' and η'. The fact that $\xi'|_{U_i} = \eta'|_{U_i}$ implies that

$$\zeta_i := (\xi|_{U_i}, \eta|_{U_i}) \in R(U_i) \subseteq F(U_i) \times F(U_i).$$

As clearly $\zeta_i|_{U_i \times_U U_j} = \zeta_j|_{U_i \times_U U_j} \in R(U_i \times_U U_j)$ for all $i, j \in I$, there exists a unique $\zeta \in R(U)$ such that $\zeta_i = \zeta|_{U_i}$ for every $i \in I$ (because R is a sheaf). Since F is separated, it follows that $\zeta = (\xi, \eta)$, whence $\xi' = \eta'$. □

Corollary 4.83. *Let $(\mathbf{C}, \tau)$ be a site. Then every equivalence relation $R \hookrightarrow F \times F$ in $\mathbf{C}^\sim$ has effective quotient given by the natural morphism $F \twoheadrightarrow F/R \xrightarrow{\rho_{F/R}} (F/R)^a$.*

Moreover, given $\alpha\colon F \to G$ in $\mathbf{C}^\sim$, the natural morphism $F \times_G F \hookrightarrow F \times F$ is an effective equivalence relation in $\mathbf{C}^\sim$, with quotient given by the natural morphism $F \to \widetilde{\operatorname{im}}\,\alpha$ (in particular, the quotient is α if and only if α is an epimorphism in $\mathbf{C}^\sim$).

Proof. Since $F \twoheadrightarrow F/R$ is the quotient of $R \hookrightarrow F \times F$ in $\widehat{\mathbf{C}}$, $F \to (F/R)^a$ is the quotient in $\mathbf{C}^\sim$ by Corollary 4.55. As $\rho_{F/R}$ is a monomorphism (because F/R is separated), the natural morphism $F \times_{F/R} F \to F \times_{(F/R)^a} F$ is an isomorphism. It follows that $R \hookrightarrow F \times F$ is effective in $\mathbf{C}^\sim$ because it is effective in $\widehat{\mathbf{C}}$. The second statement can be proved in a similar way. □

Definition 4.84. Let $(\mathbf{C}, \tau)$ be a site. An equivalence relation $R \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} U$ in $\mathbf{C}$ is a τ *equivalence relation* if δ_1 and δ_2 are covering morphisms for τ.

5 Fibred categories

5.1 Fibred categories

We fix a functor $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$: we will regard $\mathbf{F}$ as a category over $\mathbf{C}$ via p. Given a morphism $f\colon U \to V$ of $\mathbf{C}$ and objects ξ, η of $\mathbf{F}$ such that $\mathrm{p}(\xi) = U$ and $\mathrm{p}(\eta) = V$, we define

$$\operatorname{Hom}^f_{\mathbf{F}}(\xi, \eta) := \{\phi \in \operatorname{Hom}_{\mathbf{F}}(\xi, \eta) \mid \mathrm{p}(\phi) = f\}.$$

Moreover, for every object U of $\mathbf{C}$, we will denote by $\mathbf{F}_U$ the subcategory of $\mathbf{F}$ having as objects the objects ξ of $\mathbf{F}$ such that $\mathrm{p}(\xi) = U$ and as morphism the morphisms ϕ of $\mathbf{F}$ such that $\mathrm{p}(\phi) = \mathrm{id}_U$ (i.e., $\operatorname{Hom}_{\mathbf{F}_U}(\xi, \xi') = \operatorname{Hom}^{\mathrm{id}_U}_{\mathbf{F}}(\xi, \xi')$), and call it the *fibre* of $\mathbf{F}$ over U.

Definition 5.1. A morphism $\phi\colon \xi \to \eta$ of $\mathbf{F}$ (say with $\mathrm{p}(\phi) = f\colon U \to V$) is *cartesian* (with respect to p) if for every morphism $\phi'\colon \xi' \to \eta$ of $\mathbf{F}$ (say with $\mathrm{p}(\phi') = f'\colon U' \to V$) and for every morphism $g\colon U' \to U$ of $\mathbf{C}$ such that $f' = f \circ g$, there exists a unique morphism $\psi \in \operatorname{Hom}^g_{\mathbf{F}}(\xi', \xi)$ such that $\phi' = \phi \circ \psi$.

Remark 5.2. This definition of cartesian morphism is not the standard one (see [10, Exp. VI]), which is given by a weaker condition (namely, in the above notation, one restricts to the case $f' = f$ and $g = \mathrm{id}_U$). Let's call *quasi-cartesian* a morphism which is cartesian according to the standard definition. Then the notion of quasi-cartesian morphism can be viewed as a formal generalization of the notion of cartesian diagram. Indeed, if we write $\xi \text{———} U$ to mean that $\mathrm{p}(\xi) = U$ and call "commutative" a diagram like

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, dash] & \eta \arrow[d, dash] \\
U \arrow[r, "f"'] & V
\end{tikzcd}$$

if $\mathrm{p}(\phi) = f$, then the fact that ϕ is quasi-cartesian can be expressed by saying that every "commutative" diagram of continuous arrows

$$\begin{tikzcd}
\xi' \arrow[drr, "\phi'"] \arrow[dr, dotted, "\psi"'] \arrow[ddr, dash] & & \\
& \xi \arrow[r, "\phi"'] \arrow[d, dash] & \eta \arrow[d, dash] \\
& U \arrow[r, "f"'] & V
\end{tikzcd}$$

can be completed with a unique dotted arrow which keeps it "commutative". On the other hand, ϕ is cartesian if and only if the following stronger condition is satisfied: every "commutative" diagram of continuous arrows

$$\begin{tikzcd}
\xi' \arrow[drr, "\phi'"] \arrow[dr, dotted, "\psi"'] \arrow[dd, dash] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, dash] & \eta \arrow[dd, dash] \\
U' \arrow[dr, "g"'] \arrow[drr, "f'"] & & \\
& U \arrow[r, "f"'] & V
\end{tikzcd}$$

can be completed with a unique dotted arrow which keeps it "commutative". Despite this fact, we use the present definition of cartesian morphism because, in our opinion, it simplifies statements and proofs of this section, and we really don't need the notion of quasi-cartesian morphism. Moreover, we are going to consider fibred categories, and in a fibred category every quasi-cartesian morphism is actually cartesian ([10, Exp. VI, Prop. 6.11]).

The following example shows that the relation between cartesian morphisms and cartesian diagrams is not just formal.

Example 5.3. Let $\mathbf{Mor}(\mathbf{C})$ be the category whose objects are morphisms of $\mathbf{C}$ and whose morphisms, say from $h\colon \tilde{U} \to U$ to $k\colon \tilde{V} \to V$, are given by pairs of morphisms of $\mathbf{C}$ $(\tilde{f}\colon \tilde{U} \to \tilde{V}, f\colon U \to V)$ such that the diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tilde{f}"] \arrow[d, "h"'] & \tilde{V} \arrow[d, "k"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

commutes (composition of morphisms is defined in the obvious way). We will regard $\mathbf{Mor}(\mathbf{C})$ as a category over $\mathbf{C}$ via the "target" functor $\mathrm{p}\colon \mathbf{Mor}(\mathbf{C}) \to \mathbf{C}$ defined by sending an object $h\colon \tilde{U} \to U$ to U and a morphism $(\tilde{f}, f)$ as above to f. Then a morphism of $\mathbf{Mor}(\mathbf{C})$ is cartesian if and only if the corresponding commutative diagram of $\mathbf{C}$ is cartesian (it is obvious by definition that the diagram is cartesian if and only if the morphism is quasi-cartesian, and it is easy to see that a quasi-cartesian morphism in $\mathbf{Mor}(\mathbf{C})$ is cartesian). Note that the fibre $\mathbf{Mor}(\mathbf{C})_U$ over $U \in \mathbf{C}$ can be identified with $\mathbf{C}_{/U}$.

Remark 5.4. By definition, a morphism $\phi\colon \xi \to \eta$ of $\mathbf{F}$ (with $\mathrm{p}(\phi) = f\colon U \to V$) is cartesian if and only if for every $g\colon U' \to U$ in $\mathbf{C}$ and every $\xi' \in \mathbf{F}_{U'}$ the map $\mathrm{Hom}^g_{\mathbf{F}}(\xi', \xi) \xrightarrow{\phi\circ} \mathrm{Hom}^{f\circ g}_{\mathbf{F}}(\xi', \eta)$ is bijective. We will often use this fact in the following.

Lemma 5.5. *Let $\xi \xrightarrow{\phi} \eta \xrightarrow{\psi} \zeta$ be morphisms of $\mathbf{F}$ with ψ cartesian. Then ϕ is cartesian if and only if $\psi \circ \phi$ is cartesian.*

Proof. Let $\mathrm{p}(\phi) = f\colon U \to V$ and $\mathrm{p}(\psi) = g\colon V \to W$. Now, ϕ (respectively $\psi \circ \phi$) is cartesian if and only if for every $h\colon U' \to U$ in $\mathbf{C}$ and every $\xi' \in \mathbf{F}_{U'}$ the map $\mathrm{Hom}^h_{\mathbf{F}}(\xi', \xi) \xrightarrow{\phi\circ} \mathrm{Hom}^{f\circ h}_{\mathbf{F}}(\xi', \eta)$ (respectively $\mathrm{Hom}^h_{\mathbf{F}}(\xi', \xi) \xrightarrow{\psi\circ\phi\circ} \mathrm{Hom}^{g\circ f\circ h}_{\mathbf{F}}(\xi', \zeta)$) is bijective. The statement then follows from the fact that the map $\mathrm{Hom}^{f\circ h}_{\mathbf{F}}(\xi', \eta) \xrightarrow{\psi\circ} \mathrm{Hom}^{g\circ f\circ h}_{\mathbf{F}}(\xi', \zeta)$ is bijective because ψ is cartesian. $\square$

Lemma 5.6. *A morphism ϕ of $\mathbf{F}$ is an isomorphism if and only if it is cartesian and* $\mathrm{p}(\phi)$ *is an isomorphism.*

Proof. It is clear from the definition that if ϕ is an isomorphism then it is cartesian (and obviously $\mathrm{p}(\phi)$ is also an isomorphism). So we can assume that $\phi\colon \xi \to \eta$ is cartesian and $\mathrm{p}(\phi) = f\colon U \to V$ is an isomorphism. Then, by definition of cartesian morphism, there exists a unique $\psi \in \mathrm{Hom}^{f^{-1}}_{\mathbf{F}}(\eta, \xi)$ such that $\phi \circ \psi = \mathrm{id}_\eta$. As ϕ and $\phi \circ \psi = \mathrm{id}_\eta$ are cartesian, by Lemma 5.5 ψ is cartesian, too. Since also $\mathrm{p}(\psi) = f^{-1}$ is an isomorphism, by the same argument we get that there exists a unique $\phi' \in \mathrm{Hom}^f_{\mathbf{F}}(\xi, \eta)$ such that $\psi \circ \phi' = \mathrm{id}_\xi$. It follows that $\phi = \phi \circ \psi \circ \phi' = \phi'$ is an isomorphism with inverse ψ. $\square$

Corollary 5.7. *Let $\phi\colon \xi \to \eta$ and $\phi'\colon \xi' \to \eta$ be cartesian morphisms of $\mathbf{F}$ with $\mathrm{p}(\phi) = \mathrm{p}(\phi')\colon U \to V$. Then the morphism $\psi \in \mathrm{Hom}_{\mathbf{F}_U}(\xi', \xi)$ such that $\phi' = \phi \circ \psi$ (ψ exists unique because ϕ is cartesian) is an isomorphism; in particular, $\xi \cong \xi'$ in $\mathbf{F}_U$.*

Proof. ψ is cartesian by Lemma 5.5 and $\mathrm{p}(\psi) = \mathrm{id}_U$ is an isomorphism. □

Definition 5.8. $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ is a *fibred category* (or $\mathbf{F}$ is a fibred category over $\mathbf{C}$) if for every morphism $f\colon U \to V$ of $\mathbf{C}$ and for every object $\eta \in \mathbf{F}_V$, there is a cartesian morphism $\phi\colon \xi \to \eta$ with $\mathrm{p}(\phi) = f$.[10]

Example 5.9. $\mathbf{Mor}(\mathbf{C})$ is a fibred category over $\mathbf{C}$ if and only if $\mathbf{C}$ has fibred products. More generally, if P is a property of morphisms of $\mathbf{C}$ which is stable under base change, then the full subcategory $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ of $\mathbf{Mor}(\mathbf{C})$ (having as objects the morphisms of $\mathbf{C}$ which satisfy P) is a fibred category over $\mathbf{C}$.

Example 5.10. By the previous example $\mathbf{Mor}(\widehat{\mathbf{C}})$ is always a fibred category over $\widehat{\mathbf{C}}$. It follows that its full subcategory $\mathbf{PSh}(\mathbf{C})$ (having as objects the morphisms of $\widehat{\mathbf{C}}$ with target an object of $\mathbf{C}$) is a fibred category over $\mathbf{C}$. Its fibre over $U \in \mathbf{C}$ is $\widehat{\mathbf{C}}_{/U}$, which is equivalent to $\widehat{\mathbf{C}_{/U}}$ by Lemma 4.20: for this reason we call $\mathbf{PSh}(\mathbf{C})$ the *fibred category of presheaves over* $\mathbf{C}$.

Definition 5.11. A functor $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ is a *category fibred in groupoids* (respectively *in equivalence relations*, respectively *in sets*) if it is a fibred category and for every $U \in \mathbf{C}$ the fibre $\mathbf{F}_U$ is a groupoid (respectively an equivalence relation, respectively a set).

Example 5.12. Every presheaf $F \in \widehat{\mathbf{C}}$ naturally determines a category fibred in sets $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ in the following way:

$$\mathrm{Ob}(\mathbf{F}) := \{(U, \xi) \mid U \in \mathbf{C}, \xi \in F(U)\}$$

and, given two objects (U, ξ) and (V, η),

$$\mathrm{Hom}_{\mathbf{F}}((U, \xi), (V, \eta)) := \{f \in \mathrm{Hom}_{\mathbf{C}}(U, V) \mid F(f)(\eta) = \xi\},$$

whereas p is obviously defined by $\mathrm{p}(U, \xi) := U$ and $\mathrm{p}(f) := f$. The fact that F is a functor immediately implies that every morphism in $\mathbf{F}$ is cartesian, and then it is obvious that $\mathbf{F}$ is a fibred category over $\mathbf{C}$ (in sets, as clearly each fibre $\mathbf{F}_U$ is isomorphic to the set $F(U)$). Notice that $\mathbf{F}$ can be naturally identified with the category $\mathbf{C}_{/F}$ and $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ with the forgetful functor $\mathbf{C}_{/F} \to \mathbf{C}$.

Conversely, to every category fibred in sets $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ one can associate a presheaf $F \in \widehat{\mathbf{C}}$, defined on objects by $F(U) := \mathbf{F}_U$ and on morphisms as follows:

[10] In the standard definition of fibred category one requires, in the above notation, that there exists a quasi-cartesian (in the sense of Remark 5.2) ϕ with the same property, and moreover that quasi-cartesian morphisms are stable under composition. The fact that the two definitions are equivalent follows from [10, Exp. VI, Prop. 6.11] and Lemma 5.5.

given $f\colon U \to V$ in $\mathbf{C}$ and $\eta \in \mathbf{F}_V = F(V)$, the fact that $\mathbf{F}$ is fibred in sets implies (by Corollary 5.7) that there exists a unique cartesian morphism $\phi\colon \xi \to \eta$ in $\mathbf{F}$ with $\mathrm{p}(\phi) = f$, and then we define $\mathbf{F}(f)(\eta) := \xi$. It is then easy to check that F is indeed a functor.

Proposition 5.13. $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ *is a category fibred in groupoids if and only if the following two conditions are satisfied:*[11]

1. *for every morphism* $f\colon U \to V$ *of* $\mathbf{C}$ *and for every object* η *of* $\mathbf{F}_V$*, there exists* $\phi\colon \xi \to \eta$ *in* $\mathbf{F}$ *such that* $\mathrm{p}(\phi) = f$*;*

2. *every morphism of* $\mathbf{F}$ *is cartesian.*

Proof. Assuming $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ is a category fibred in groupoids, we have only to prove (2). Given a morphism $\phi\colon \xi \to \eta$ of $\mathbf{F}$ with $\mathrm{p}(\phi) = f\colon U \to V$, we can take a cartesian morphism $\phi'\colon \xi' \to \eta$ with $\mathrm{p}(\phi') = f$. Then, by definition there exists a unique $\psi \in \mathrm{Hom}_{\mathbf{F}_U}(\xi, \xi')$ such that $\phi = \phi' \circ \psi$. As $\mathbf{F}_U$ is a groupoid, ψ is an isomorphism (and in particular it is cartesian), so that ϕ is also cartesian by Lemma 5.5.

Conversely, assuming (1) and (2) hold, we have only to prove that every fibre of $\mathbf{F}$ is a groupoid. Now, if $\phi\colon \xi \to \eta$ is a morphism of $\mathbf{F}_U$, then ϕ is cartesian by hypothesis and $\mathrm{p}(\phi) = \mathrm{id}_U$ is an isomorphism, whence ϕ is an isomorphism by Lemma 5.6. □

Corollary 5.14. *A fibred category* $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ *is fibred in groupoids if and only if it satisfies the following condition: a morphism* ϕ *of* $\mathbf{F}$ *is an isomorphism if and only if* $\mathrm{p}(\phi)$ *is an isomorphism of* $\mathbf{C}$.

Proof. If $\mathbf{F}$ is fibred in groupoids and ϕ is a morphism of $\mathbf{F}$ such that $\mathrm{p}(\phi)$ is an isomorphism, then ϕ is cartesian by Proposition 5.13, whence an isomorphism by Lemma 5.6. Conversely, if the condition is satisfied, then for $U \in \mathbf{C}$ every morphism of $\mathbf{F}_U$ is obviously an isomorphism. □

Corollary 5.15. *Let* $\mathbf{F}$ *be a fibred category over* $\mathbf{C}$ *and let* $\mathbf{F}^{\mathrm{cart}}$ *be the subcategory of* $\mathbf{F}$ *having the same objects as* $\mathbf{F}$ *and as morphisms the cartesian morphisms. Then* $\mathbf{F}^{\mathrm{cart}}$ *is a category fibred in groupoids over* $\mathbf{C}$.

Proof. Notice first that $\mathbf{F}^{\mathrm{cart}}$ is a subcategory of $\mathbf{F}$ by Lemma 5.5. Then it is clear that the natural functor $\mathbf{F}^{\mathrm{cart}} \to \mathbf{C}$ satisfies condition (1) of Proposition 5.13, and as for (2), just observe that every cartesian morphism of $\mathbf{F}$ is cartesian also as a morphism of $\mathbf{F}^{\mathrm{cart}}$ (this follows again from Lemma 5.5). □

[11] This gives an alternative definition of category fibred in groupoids, which is often found in the literature when the general definition of fibred category is not given.

5.2 The 2-category of fibred categories

We fix a base category $\mathbf{C}$; we are going to see that fibred categories over $\mathbf{C}$ are in a natural way the objects of a 2-category (see Section A.2 for the definition of 2-category and notation used).

Definition 5.16. Let $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ and $\mathrm{p}'\colon \mathbf{F}' \to \mathbf{C}$ be two fibred categories. A *morphism* of fibred categories over $\mathbf{C}$ from $\mathbf{F}$ to $\mathbf{F}'$ is a functor $P\colon \mathbf{F} \to \mathbf{F}'$ which sends cartesian morphisms to cartesian morphisms and such that $\mathrm{p}' \circ P = \mathrm{p}$ (strict equality, not just isomorphism of functors). If $P, Q\colon \mathbf{F} \to \mathbf{F}'$ are two morphisms as above, a 2-*morphism* from P to Q is a natural transformation $\alpha\colon P \to Q$ such that $\mathrm{id}_{\mathrm{p}'} \star \alpha = \mathrm{id}_{\mathrm{p}}$ in $\underline{\mathbf{Cat}}$ (more explicitly, this means that for every $\xi \in \mathbf{F}$, say with $\mathrm{p}(\xi) = U$, hence such that $P(\xi), Q(\xi) \in \mathbf{F}'_U$, it is required that $\alpha(\xi) \in \mathrm{Hom}_{\mathbf{F}'_U}(P(\xi), Q(\xi))$, i.e. that $\mathrm{p}'(\alpha(\xi)) = \mathrm{id}_U$).

Proposition 5.17. *Fibred categories over* $\mathbf{C}$ *form the objects of a* 2*-category* $\underline{\mathbf{Fib}}_{\mathbf{C}}$*, with morphisms and* 2*-morphisms as in the above definition and with composition of morphisms and (horizontal and vertical) composition of* 2*-morphisms defined as in* $\underline{\mathbf{Cat}}$*.*

Proof. Straightforward, using the fact that $\underline{\mathbf{Cat}}$ is a 2-category. □

We will denote by $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ (respectively $\underline{\mathbf{Fib}}_{\mathbf{C}}^{equiv}$, respectively $\underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$) the full 2-subcategory of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ having as objects categories fibred in groupoids (respectively in equivalence relations, respectively in sets).

Remark 5.18. Let $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ and $\mathrm{p}'\colon \mathbf{F}' \to \mathbf{C}$ be two fibred categories with $\mathbf{F}' \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$. Then a functor $P\colon \mathbf{F} \to \mathbf{F}'$ is a morphism of fibred categories if and only if $\mathrm{p}' \circ P = \mathrm{p}$ (because every morphism of $\mathbf{F}'$ is cartesian by Proposition 5.13). Moreover, every 2-morphism between two morphisms from $\mathbf{F}$ to $\mathbf{F}'$ is a 2-isomorphism (in other words, $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{F}')$ is a groupoid). It is also clear that if $\mathbf{F}' \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{equiv}$ (respectively $\mathbf{F}' \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$) then $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{F}')$ is an equivalence relation (respectively a set). In particular, we see that $\underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$ is actually an ordinary category, and it is then easy to prove the following result.

Proposition 5.19. *The map* $F \mapsto (\mathbf{C}_{/F} \to \mathbf{C})$ *(see Example 5.12) extends to a functor* $\widehat{\mathbf{C}} \to \underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$*, which is an equivalence of categories (and whose quasi-inverse on objects is also described in Example 5.12).*

From now on we will therefore identify $\widehat{\mathbf{C}}$ with $\underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$; so, for instance, if $F \in \widehat{\mathbf{C}}$, we will often denote again by F the category fibred in sets $\mathbf{C}_{/F} \to \mathbf{C}$.

Example 5.20. If $\{*\}$ is the trivial category (a set with one element), then $\underline{\mathbf{Fib}}_{\{*\}}$ coincides with $\underline{\mathbf{Cat}}$. To see this, just observe that in this case (and more generally when $\mathbf{C}$ is a groupoid) a morphism in a fibred category is cartesian if and only if it is an isomorphism (by Lemma 5.6), and that every functor preserves isomorphisms.

Notice that, if $P \in \mathrm{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{F}')$, then for every $U \in \mathbf{C}$ the restriction of P defines a functor $P_U : \mathbf{F}_U \to \mathbf{F}'_U$.

Lemma 5.21. *Let* $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ *and* $\mathrm{p}'\colon \mathbf{F}' \to \mathbf{C}$ *be fibred categories and let* $P\colon \mathbf{F} \to \mathbf{F}'$ *be a morphism in* $\underline{\mathbf{Fib}}_{\mathbf{C}}$. *Then* P *is faithful (respectively full, respectively essentially surjective) if and only if* $P_U\colon \mathbf{F}_U \to \mathbf{F}'_U$ *is faithful (respectively full, respectively essentially surjective) for every* $U \in \mathbf{C}$.

Proof. It is clear that if P is faithful (respectively full), then each P_U is faithful (respectively full), too. Assume conversely that each P_U is faithful (respectively full): since $\mathrm{p}' \circ P = \mathrm{p}$, it is enough to prove that for every $f\colon U \to V$ in $\mathbf{C}$, every $\xi \in \mathbf{F}_U$ and every $\eta \in \mathbf{F}_V$, the natural map $P^f_{\xi,\eta}\colon \mathrm{Hom}^f_{\mathbf{F}}(\xi, \eta) \to \mathrm{Hom}^f_{\mathbf{F}'}(P(\xi), P(\eta))$ is injective (respectively surjective). Let $\phi\colon \tilde{\xi} \to \eta$ be a cartesian morphism in $\mathbf{F}$ with $\mathrm{p}(\phi) = f$; then in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{F}_U}(\xi, \tilde{\xi}) & \xrightarrow{P^{\mathrm{id}_U}_{\xi,\tilde{\xi}}} & \mathrm{Hom}_{\mathbf{F}'_U}(P(\xi), P(\tilde{\xi})) \\
{\scriptstyle \phi\circ}\big\downarrow & & \big\downarrow{\scriptstyle P(\phi)\circ} \\
\mathrm{Hom}^f_{\mathbf{F}}(\xi, \eta) & \xrightarrow[P^f_{\xi,\eta}]{} & \mathrm{Hom}^f_{\mathbf{F}'}(P(\xi), P(\eta))
\end{array}$$

the vertical maps are isomorphisms because ϕ and $P(\phi)$ are cartesian. As $P^{\mathrm{id}_U}_{\xi,\tilde{\xi}}$ is injective (respectively surjective) by hypothesis, it follows that $P^f_{\xi,\eta}$ is also injective (respectively surjective).

On the other hand, clearly P is essentially surjective if so is each P_U; it thus remains to prove that, if P is essentially surjective, then for every $U \in \mathbf{C}$ the same is true for P_U. Given $\xi' \in \mathbf{F}'_U$, by hypothesis there exists $\eta \in \mathbf{F}_V$ for some $V \in \mathbf{C}$ such that $P(\eta) \in \mathbf{F}'_V$ is isomorphic to ξ' in $\mathbf{F}'$. So, let $\phi'\colon \xi' \to P(\eta)$ be an isomorphism of $\mathbf{F}'$ and let $f := \mathrm{p}'(\phi')\colon U \to V$. Choosing a cartesian morphism $\phi\colon \xi \to \eta$ in $\mathbf{F}$ with $\mathrm{p}(\phi) = f$, we obtain a cartesian morphism $P(\phi)\colon P(\xi) \to P(\eta)$ with $\mathrm{p}'(P(\phi)) = f$. Since also ϕ' is cartesian (because it is an isomorphism), it follows from Corollary 5.7 that $\xi' \cong P(\xi) = P_U(\xi)$ in $\mathbf{F}'_U$. $\square$

Definition 5.22. A morphism $P\colon \mathbf{F} \to \mathbf{F}'$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is a *monomorphism* (respectively an *epimorphism*, respectively an *isomorphism*) if it is fully faithful (respectively essentially surjective, respectively an equivalence of categories) (by Lemma 5.21 this is true if and only if P_U is fully faithful (respectively essentially surjective, respectively an equivalence of categories) for every $U \in \mathbf{C}$).

Remark 5.23. The above definitions require some explanations. First of all, the given definition of isomorphism of fibred categories (which will apply also to stacks) is the most common in the literature, even if it is not the natural one. In fact, in view of Proposition 5.24 below, it would be more appropriate to use the name equivalence instead of isomorphism (see also Remark A.13). In order to avoid possible confusion we will usually specify the meaning in which the term isomorphism is to be intended; in particular, in the (rare) cases in which we have a real isomorphism of categories we will always say isomorphism of categories instead of isomorphism (of fibred categories). As for monomorphism and epimorphism, note first that the notion of fully faithfulness (respectively essential surjectivity) for functors seems to be the natural generalization of the notion of injectivity (respectively surjectivity) for maps between sets (at least, it is true that a functor between two sets, which is always faithful, is full if and only if it is injective, and it is essentially surjective if and only if it is surjective). It would then be natural (in analogy with the usual definition for categories) to define a morphism $f\colon U \to V$ in a 2-category $\underline{\mathbf{C}}$ to be a monomorphism (respectively an epimorphism) if and only if the natural functor $f\circ\colon \mathbf{Hom}_{\underline{\mathbf{C}}}(W,U) \to \mathbf{Hom}_{\underline{\mathbf{C}}}(W,V)$ (respectively $\circ f\colon \mathbf{Hom}_{\underline{\mathbf{C}}}(V,W) \to \mathbf{Hom}_{\underline{\mathbf{C}}}(U,W)$) is fully faithful for every $W \in \underline{\mathbf{C}}$. It is then easy to see that this general definition of monomorphism coincides with the one given above for $\underline{\mathbf{Fib}}_{\mathbf{C}}$, but the same is not true for epimorphism. However, we will use the above definition of epimorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ (which is the obvious generalization of the one used in [15] for stacks) because it allows to extend formally many properties from $\widehat{\mathbf{C}}$ to $\underline{\mathbf{Fib}}_{\mathbf{C}}$. For instance, a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is an isomorphism if and only if it is a monomorphism and an epimorphism. Observe also that a morphism of $\widehat{\mathbf{C}}$ is a monomorphism (respectively an epimorphism, respectively an isomorphism) in $\widehat{\mathbf{C}}$ if and only if it is a monomorphism (respectively an epimorphism, respectively an isomorphism) in $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Proposition 5.24. *A morphism $P\colon \mathbf{F} \to \mathbf{F}'$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is an isomorphism if and only if there exists a morphism $Q\colon \mathbf{F}' \to \mathbf{F}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ together with 2-isomorphisms (of $\underline{\mathbf{Fib}}_{\mathbf{C}}$) $P \circ Q \cong \mathrm{id}_{\mathbf{F}'}$ and $Q \circ P \cong \mathrm{id}_{\mathbf{F}}$.*

Proof. The other implication being trivial, we can assume that P is an isomorphism of fibred categories. For every $U \in \mathbf{C}$ and every $\xi' \in \mathbf{F}'_U$, since $P_U\colon \mathbf{F}_U \to \mathbf{F}'_U$ is essentially surjective, we can choose $Q(\xi') \in \mathbf{F}_U$ and an isomorphism $\alpha(\xi')\colon P \circ Q(\xi') \xrightarrow{\sim} \xi'$ in $\mathbf{F}'_U$. For every morphism $\phi'\colon \xi' \to \eta'$ of $\mathbf{F}'$ (say over $f\colon U \to V$ in $\mathbf{C}$) there exists a unique $Q(\phi') \in \mathrm{Hom}^f_{\mathbf{F}}(Q(\xi'), Q(\eta'))$ which is mapped to ϕ' by the natural map

$$
\begin{tikzcd}
\mathrm{Hom}^f_{\mathbf{F}}(Q(\xi'), Q(\eta')) \arrow[r, "P^f_{Q(\xi'),Q(\eta')}"] & \mathrm{Hom}^f_{\mathbf{F}'}(P \circ Q(\xi'), P \circ Q(\eta')) \arrow[d, "\alpha(\eta')\circ - \circ \alpha(\xi')^{-1}"] \\
& \mathrm{Hom}^f_{\mathbf{F}'}(\xi', \eta')
\end{tikzcd}
$$

(which is bijective because P is a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$). It is clear that Q is a functor and, in order to show that it is a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, it remains to check that it preserves cartesian morphisms. Now, in the above notation, if ϕ' is cartesian, then also $\alpha(\eta')\circ\phi'\circ\alpha(\xi')^{-1}$ is cartesian (by Lemma 5.5), and then it is enough to note that a morphism ϕ of $\mathbf{F}$ with $P(\phi)$ cartesian is cartesian, too (this is also a consequence of fully faithfulness of P). It is also clear that $\alpha\colon P\circ Q\to \mathrm{id}_{\mathbf{F}'}$ is a 2-isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. Moreover, for every $U\in\mathbf{C}$ and every $\xi\in\mathbf{F}_U$, since $P^{\mathrm{id}_U}_{Q\circ P(\xi),\xi}\colon \mathrm{Hom}_{\mathbf{F}_U}(Q\circ P(\xi),\xi)\to\mathrm{Hom}_{\mathbf{F}'_U}(P\circ Q\circ P(\xi),P(\xi))$ is bijective because P_U is fully faithful, there exists a unique $\beta(\xi)\colon Q\circ P(\xi)\to\xi$ in $\mathbf{F}_U$ such that $P(\beta(\xi))=\alpha(P(\xi))$; it is then easy to see that $\beta\colon Q\circ P\to\mathrm{id}_{\mathbf{F}}$ is a 2-isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. □

Remembering that a category equivalent to a groupoid (respectively to an equivalence relation) is a groupoid (respectively an equivalence relation), we have therefore the following result.

Corollary 5.25. $\underline{\mathbf{Fib}}^{gpd}_{\mathbf{C}}$ *and* $\underline{\mathbf{Fib}}^{equiv}_{\mathbf{C}}$ *are strictly full* 2*-subcategories of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

By Lemma A.14 we have also the following characterization of isomorphisms in $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Corollary 5.26. *A morphism* $P\colon\mathbf{F}\to\mathbf{F}'$ *of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$ *is an isomorphism if and only if, for every* $\mathbf{G}\in\underline{\mathbf{Fib}}_{\mathbf{C}}$, *the functor* $P\circ\colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F})\to\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F}')$ *(respectively* $\circ P\colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}',\mathbf{G})\to\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F},\mathbf{G})$*) is an equivalence of categories.*

Proposition 5.27. $\mathbf{F}\in\underline{\mathbf{Fib}}_{\mathbf{C}}$ *is isomorphic to a category fibred in sets if and only if it is fibred in equivalence relations.*

Proof. If $P\colon\mathbf{F}\to\mathbf{F}'$ is an isomorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ and $\mathbf{F}'$ is fibred in sets, then $\mathbf{F}$ is fibred in equivalence relations because $P_U\colon\mathbf{F}_U\to\mathbf{F}'_U$ is an equivalence for every $U\in\mathbf{C}$ (remember that a category is equivalent to a set if and only if it is an equivalence relation). Conversely, assume that $\mathrm{p}\colon\mathbf{F}\to\mathbf{C}$ is fibred in equivalence relations. For every $U\in\mathbf{C}$ let $F(U)$ be the set of equivalence classes of the equivalence relation $\mathbf{F}_U$ and denote by $[\xi]\in F(U)$ the equivalence class of $\xi\in\mathbf{F}_U$. If $f\colon U\to V$ is a morphism of $\mathbf{C}$, for every $\eta\in\mathbf{F}_V$ let $\phi\colon\xi\to\eta$ be a (necessarily cartesian by Proposition 5.13) morphism in $\mathbf{F}$ with $\mathrm{p}(\phi)=f$: it is easy to see that the map $F(f)\colon F(V)\to F(U)$ given by $[\eta]\mapsto[\xi]$ is well defined and that in this way we get a presheaf on $F\in\widehat{\mathbf{C}}$ (which corresponds to the category fibred in sets $\mathbf{C}_{/F}\to\mathbf{C}$). It is then clear that the natural functor $P\colon\mathbf{F}\to\mathbf{C}_{/F}$ (defined on objects by $P(\xi):=[\xi]$ and on morphisms by $P(\phi):=\mathrm{p}(\phi)$) is an isomorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. □

Corollary 5.28. *The inclusion* $\widehat{\mathbf{C}}\subset\underline{\mathbf{Fib}}_{\mathbf{C}}$ *induces a lax* 2*-equivalence between* $\widehat{\mathbf{C}}$ *and* $\underline{\mathbf{Fib}}^{equiv}_{\mathbf{C}}$.

Definition 5.29. If $P\colon\mathbf{F}\to\mathbf{F}'$ is a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, the *image* of P is the (strictly) full subcategory $\operatorname{im}P$ of $\mathbf{F}'$ whose objects are those $\xi'\in\mathbf{F}'$ which are isomorphic to $P(\xi)$ for some $\xi\in\mathbf{F}$.

Remark 5.30. $\operatorname{im} P$, with the restriction of the projection functor $\mathbf{F}' \to \mathbf{C}$, is a fibred category over $\mathbf{C}$ (this follows immediately from the fact that P preserves cartesian morphisms and that every isomorphism of $\mathbf{F}'$ is cartesian). Therefore P factors as the composition of an epimorphism and a monomorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ through the natural functors $\mathbf{F} \to \operatorname{im} P \to \mathbf{F}'$. In particular, P is a monomorphism (respectively an epimorphism) if and only if $\mathbf{F} \to \operatorname{im} P$ is an isomorphism (respectively $\operatorname{im} P = \mathbf{F}'$).

5.3 Fibred categories as lax 2-functors

Let $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ be a fibred category. Given $f\colon U \to V$ in $\mathbf{C}$, let's choose for every $\eta \in \mathbf{F}_V$ a cartesian morphism $\phi\colon \xi \to \eta$ with $\mathrm{p}(\phi) = f$ (such a morphism is unique up to a unique isomorphism by Corollary 5.7), and let's denote it by $\alpha_f(\eta)\colon f^*(\eta) \to \eta$. For every morphism $\psi\colon \eta \to \eta'$ in $\mathbf{F}_V$ there exists unique a morphism $f^*(\psi)\colon f^*(\eta) \to f^*(\eta')$ in $\mathbf{F}_U$ such that

$$\begin{tikzcd}
f^*(\eta) \arrow[r, "\alpha_f(\eta)"] \arrow[d, "f^*(\psi)"'] & \eta \arrow[d, "\psi"] \\
f^*(\eta') \arrow[r, "\alpha_f(\eta')"'] & \eta'
\end{tikzcd}$$

commutes (because $\alpha_f(\eta')$ is cartesian). It is immediate to verify that in this way we obtain a functor $f^*\colon \mathbf{F}_V \to \mathbf{F}_U$. Assume now that cartesian morphisms $\alpha_f(\eta)\colon f^*(\eta) \to \eta$ have been chosen for every morphism f of $\mathbf{C}$ (such a choice is called a *clivage* in [10]). We would like to define a 2-functor $\underline{F}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ by setting $\underline{F}(U) := \mathbf{F}_U$ for every object U of $\mathbf{C}$ and $\underline{F}(f) := f^*$ for every morphism f of $\mathbf{C}$. Unfortunately, this does not yield a strict 2-functor in general. Indeed, while the condition $\underline{F}(\mathrm{id}_U) = \mathrm{id}_{\underline{F}(U)}$ (i.e., $\mathrm{id}_U^* = \mathrm{id}_{\mathbf{F}_U}$) for every $U \in \mathbf{C}$ is satisfied if we choose $\alpha_{\mathrm{id}_U}(\xi) := \mathrm{id}_\xi$ for every $U \in \mathbf{C}$ and every $\xi \in \mathbf{F}_U$ (in this case the clivage is said to be *normalized*), in general there does not exist a clivage such that $\underline{F}(g \circ f) = \underline{F}(f) \circ \underline{F}(g)$ (i.e., $(g \circ f)^* = f^* \circ g^*$) for every couple of composable morphisms f and g of $\mathbf{C}$ (a normalized clivage with this additional property is called a *scindage* in [10]).

Example 5.31. Let $p\colon G \to H$ be a morphism of groups: p (regarded as a functor between two groupoids with one object) is a fibred category if and only if it is surjective (as a morphism of groups). It is easy to see that a clivage for p corresponds to a map (of sets) $s\colon H \to G$ such that $p \circ s = \mathrm{id}_H$, and that such a clivage is a scindage if and only if s is a morphism of groups. So, for instance, a scindage does not exist if p is the natural projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ ($n > 1$).

However, we are going to see that the choice of a clivage c (which we assume to be normalized for simplicity) on $\mathbf{F}$ naturally determines a lax 2-functor $\underline{F} = \underline{F}_{(\mathbf{F},c)}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ (defined as above on objects and on morphisms). Given morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ of $\mathbf{C}$ and $\zeta \in \mathbf{F}_W$, by Corollary 5.7 there exists unique an isomorphism $c_{f,g}(\zeta)\colon f^*(g^*(\zeta)) \to (g\circ f)^*(\zeta)$ in $\mathbf{F}_U$ such that

$$\begin{tikzcd} f^*(g^*(\zeta)) \arrow[r, "\alpha_f(g^*(\zeta))"] \arrow[d, "c_{f,g}(\zeta)"'] & g^*(\zeta) \arrow[d, "\alpha_g(\zeta)"] \\ (g\circ f)^*(\zeta) \arrow[r, "\alpha_{g\circ f}(\zeta)"'] & \zeta \end{tikzcd}$$

commutes ($\alpha_g(\zeta)\circ\alpha_f(g^*(\zeta))$ is cartesian by Lemma 5.5). It is easy to see that the morphisms $c_{f,g}(\zeta)$ define an isomorphism $c_{f,g}\colon f^*\circ g^* \to (g\circ f)^*$ of functors $\mathbf{F}_W \to \mathbf{F}_U$. Since c is normalized, we have $c_{\mathrm{id}_U,f} = c_{f,\mathrm{id}_V} = \mathrm{id}_{f^*}$ for every morphism $f\colon U \to V$. One can check moreover that, given three morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X$ of $\mathbf{C}$, the diagram (in $\mathbf{Fun}(\mathbf{F}_X, \mathbf{F}_U)$)

$$\begin{tikzcd} f^*\circ g^*\circ h^* \arrow[r, "c_{f,g}\star\mathrm{id}_{h^*}"] \arrow[d, "\mathrm{id}_{f^*}\star c_{g,h}"'] & (g\circ f)^*\circ h^* \arrow[d, "c_{g\circ f,h}"] \\ f^*\circ(h\circ g)^* \arrow[r, "c_{f,h\circ g}"'] & (h\circ g\circ f)^* \end{tikzcd}$$

commutes. Therefore $\underline{F}_{(\mathbf{F},c)}$ is a lax 2-functor, according to the following definition.

Definition 5.32. A *lax 2-functor* $\underline{F}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ consists of the data of a category $\underline{F}(U)$ for every object U of $\mathbf{C}$, of a functor $\underline{F}(f)\colon \underline{F}(V) \to \underline{F}(U)$ for every morphism $f\colon U \to V$ of $\mathbf{C}$, and of an isomorphism of functors $c_{f,g} = c^{\underline{F}}_{f,g}\colon \underline{F}(f)\circ\underline{F}(g) \to \underline{F}(g\circ f)$ for every couple of composable morphisms f and g of $\mathbf{C}$, such that $\underline{F}(\mathrm{id}_U) = \mathrm{id}_{\underline{F}(U)}$ for every $U \in \mathbf{C}$, $c_{\mathrm{id}_U,f} = c_{f,\mathrm{id}_V} = \mathrm{id}_{\underline{F}(f)}$ for every morphism $f\colon U \to V$ of $\mathbf{C}$ and $c_{g\circ f,h}\circ(c_{f,g}\star\mathrm{id}_{\underline{F}(h)}) = c_{f,h\circ g}\circ(\mathrm{id}_{\underline{F}(f)}\star c_{g,h})$ if f, g and h are three composable morphisms of $\mathbf{C}$.

Remark 5.33. There exist also more relaxed versions of the notion of lax 2-functor (as we have defined it, it is often called pseudo-functor, for instance in [10]). In particular, instead of the equality $\underline{F}(\mathrm{id}_U) = \mathrm{id}_{\underline{F}(U)}$, one could require the existence of an isomorphism of functors $a_U\colon \underline{F}(\mathrm{id}_U) \to \mathrm{id}_{\underline{F}(U)}$ for every $U \in \mathbf{C}$ such that $c_{\mathrm{id}_U,f} = a_U \star \mathrm{id}_{\underline{F}(f)}$ and $c_{f,\mathrm{id}_V} = \mathrm{id}_{\underline{F}(f)} \star a_V$ for every morphism $f\colon U \to V$ of $\mathbf{C}$ (this is the right definition to use if the clivage is not normalized). An even more general notion is obtained by allowing the natural transformations $c_{f,g}$ (and a_U) not to be isomorphisms.

Conversely, every lax 2-functor $\underline{F}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ determines (this time in a natural way) a fibred category $\mathrm{p}\colon \mathbf{F} = \mathbf{F}(\underline{F}) \to \mathbf{C}$ as follows:

$$\mathrm{Ob}(\mathbf{F}) := \{(U,\xi) \mid U \in \mathbf{C}, \xi \in \underline{F}(U)\},$$
$$\mathrm{Hom}_{\mathbf{F}}((U,\xi),(V,\eta)) := \{(f,\phi) \mid f \in \mathrm{Hom}_{\mathbf{C}}(U,V), \phi \in \mathrm{Hom}_{\underline{F}(U)}(\xi, \underline{F}(f)(\eta))\}.$$

Given $(f,\phi) \in \mathrm{Hom}_{\mathbf{F}}((U,\xi),(V,\eta))$ and $(g,\psi) \in \mathrm{Hom}_{\mathbf{F}}((V,\eta),(W,\zeta))$, we define

$$(g,\psi)\circ(f,\phi) := (g\circ f, c_{f,g}(\zeta)\circ F(f)(\psi)\circ\phi).$$

It is easy to see that $\mathbf{F}$ is a category and that the functor p (obviously defined by $\mathrm{p}(U,\xi) := U$ on objects and $\mathrm{p}(f,\phi) := f$ on morphisms) makes it into a fibred category. Indeed, for every $f\colon U \to V$ in $\mathbf{C}$ and every $\eta \in \underline{F}(V)$, the morphism $(f, \mathrm{id}_{\underline{F}(f)(\eta)})\colon (U, F(f)(\eta)) \to (V,\eta)$ is cartesian; from this we see that $\mathbf{F}$ is also naturally endowed with a (normalized) clivage.

We are going to see that the map $\underline{F} \mapsto \mathbf{F}(\underline{F})$ from lax 2-functors to fibred categories is inverse (up to isomorphism, in a sense to be specified) of the map $\mathbf{F} \mapsto \underline{F}_{(\mathbf{F},c)}$ (for every choice of a normalized clivage c on $\mathbf{F}$). In fact, not only strict 2-functors (see Proposition A.21) but also lax 2-functors form the objects of a (strict) 2-category $\underline{\mathbf{LaxFun}}(\mathbf{C}^\circ, \underline{\mathbf{Cat}})$; its 1-morphisms are given by lax 2-natural transformations and its 2-morphisms by modifications. If $\underline{F}, \underline{G}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ are 2-functors, a *lax 2-natural transformation* $\underline{\alpha}\colon \underline{F} \to \underline{G}$ is given by functors $\underline{\alpha}(U)\colon \underline{F}(U) \to \underline{G}(U)$ for every $U \in \mathbf{C}$ together with isomorphisms

$$\underline{\alpha}_f\colon \underline{G}(f)\circ\underline{\alpha}(V) \to \underline{\alpha}(U)\circ\underline{F}(f)$$

of functors $\underline{F}(V) \to \underline{G}(U)$ for every $f\colon U \to V$ in $\mathbf{C}$, such that $\underline{\alpha}_{\mathrm{id}_U} = \mathrm{id}_{\underline{\alpha}(U)}$ for every $U \in \mathbf{C}$ and, given $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathbf{C}$,

$$c^{G}_{f,g}\circ\underline{\alpha}_{g\circ f} = (\mathrm{id}_{\underline{\alpha}(U)}\star c^{F}_{f,g})\circ(\underline{\alpha}_f\star\mathrm{id}_{\underline{F}(g)})\circ(\mathrm{id}_{\underline{G}(f)}\star\underline{\alpha}_g)$$

as morphisms $\underline{G}(f)\circ\underline{G}(g)\circ\underline{\alpha}(W) \to \underline{\alpha}(U)\circ\underline{F}(f)\circ\underline{F}(g)$. If $\underline{\alpha}, \underline{\beta}\colon \underline{F} \to \underline{G}$ are lax 2-natural transformations, a *modification* $\underline{\epsilon}\colon \underline{\alpha} \to \underline{\beta}$ is given by natural transformations $\underline{\epsilon}(U)\colon \underline{\alpha}(U) \to \underline{\beta}(U)$ for every $U \in \mathbf{C}$ such that

$$\underline{\beta}_f\circ(\mathrm{id}_{\underline{G}(f)}\star\underline{\epsilon}(V)) = (\underline{\epsilon}(U)\star\mathrm{id}_{\underline{F}(f)})\circ\underline{\alpha}_f\colon \underline{G}(f)\circ\underline{\alpha}(V) \to \underline{\beta}(U)\circ\underline{F}(f)$$

for every $f\colon U \to V$ in $\mathbf{C}$. We leave it to the reader to imagine how compositions are defined and to check that $\underline{\mathbf{LaxFun}}(\mathbf{C}^\circ, \underline{\mathbf{Cat}})$ is indeed a 2-category. It is then boring but not difficult to prove the following result.

Proposition 5.34. *The map $\mathbf{F} \mapsto \underline{F}_{(\mathbf{F},c)}$ (for $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ and c an arbitrary normalized clivage on $\mathbf{F}$) extends to a (strict) 2-functor $\underline{\mathbf{Fib}}_{\mathbf{C}} \to \underline{\mathbf{LaxFun}}(\mathbf{C}^\circ, \underline{\mathbf{Cat}})$, which is a (strict) 2-equivalence; a 2-quasi-inverse is given by the natural 2-functor $\underline{\mathbf{LaxFun}}(\mathbf{C}^\circ, \underline{\mathbf{Cat}}) \to \underline{\mathbf{Fib}}_{\mathbf{C}}$ defined on objects by $\underline{F} \mapsto \mathbf{F}(\underline{F})$.*

This shows that it is essentially the same thing to work with fibred categories or with lax 2-functors. Although we will stick to fibred categories (which are much more common in the literature), it will be often useful to endow a fibred category with a clivage (so implicitly using an associated lax 2-functor). For this reason, we will usually tacitly assume that a clivage has been chosen on every fibred category under consideration. So, given $f\colon U \to V$ in $\mathbf{C}$ and $F \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, we will freely use expressions like $f^*(\eta)$ ($\eta \in \mathbf{F}_V$) or $f^*(\psi)$ ($\psi \in \mathrm{Mor}(\mathbf{F}_V)$); in analogy with the notation used for presheaf, we will sometimes write $\eta|_U$ and $\psi|_U$ instead of $f^*(\eta)$ and $f^*(\psi)$, if there can be no doubt about f.

We must also say that in some cases a fibred category is more naturally defined as the fibred category associated to some lax 2-functor. So it is customary to assign a fibred category $\mathbf{F}$ by specifying the fibres $\mathbf{F}_U$ for every $U \in \mathbf{C}$ and the pullback functors f^* for every $f \in \mathrm{Mor}(\mathbf{C})$ (which are often the “obvious” ones).

Example 5.35. If $\mathbf{C} = \mathbf{Sch}_{/S}$ or $\mathbf{C} = \mathbf{AffSch}_{/S}$ (S a scheme), we can define $\underline{Mod}\colon \mathbf{C}^\circ \to \underline{\mathbf{Cat}}$ by $\underline{Mod}(U \to S) := \mathbf{Mod}(U)$ on objects and $\underline{Mod}(f) := f^*$ on morphisms (f^* denotes the usual pullback functor): notice that, if f and g are two composable morphisms of $\mathbf{C}$, the natural isomorphisms $f^* \circ g^* \cong (g \circ f)^*$ satisfy the necessary compatibility relations, so that $\underline{Mod}$ is a lax 2-functor (in the weaker sense of Remark 5.33, since id_U^* is in general only isomorphic to the identity). We will denote by $\overline{\mathbf{Mod}}$ the corresponding fibred category; similarly, $\overline{\mathbf{QCoh}}$ (respectively $\overline{\mathbf{QCohAlg}}$, respectively $\overline{\mathbf{QCohGAlg}}$) will be the fibred category whose fibre over $(U \to S) \in \mathbf{C}$ is $\mathbf{QCoh}(U)$ (respectively $\mathbf{QCohAlg}(U)$, respectively $\mathbf{QCohGAlg}(U)$).

5.4 Yoneda's lemma for fibred categories

Let $\mathbf{C}$ be a category, $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ a fibred category and $U \in \mathbf{C}$ an object (which we will identify with the corresponding category fibred in sets $\mathbf{C}_{/U}$). There is a natural functor $\Phi = \Phi_{\mathbf{F},U}\colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U,\mathbf{F}) \to \mathbf{F}_U$ defined as follows. If $P \in \mathrm{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U,\mathbf{F})$, $\Phi(P) := P(\mathrm{id}_U) = P_U(\mathrm{id}_U)$ (notice that P_U is a functor $U(U) \to \mathbf{F}_U$); similarly, if $\alpha \in \mathrm{Hom}_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U,\mathbf{F})}(P,Q)$ (i.e., if $\alpha\colon P \to Q$ is a 2-morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$), $\Phi(\alpha) := \alpha(\mathrm{id}_U)\colon P(\mathrm{id}_U) \to Q(\mathrm{id}_U)$ ($\alpha(\mathrm{id}_U)$ is a morphism of $\mathbf{F}_U$ by definition of 2-morphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$).

Conversely, it is easy to see that the choice of a clivage c on $\mathbf{F}$ induces a functor $\Psi = \Psi_{(\mathbf{F},c),U}\colon \mathbf{F}_U \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U,\mathbf{F})$. Indeed, for every $\xi \in \mathbf{F}_U$ we define the functor $\Psi(\xi)\colon \mathbf{C}_{/U} \to \mathbf{F}$ as follows: if $f\colon V \to U$ is an object of $\mathbf{C}_{/U}$, $\Psi(\xi)(f) := f^*(\xi) \in \mathbf{F}_V$, whereas if $g\colon V \to V'$ is a morphism of $\mathbf{C}_{/U}$ (say from $f\colon V \to U$ to $f'\colon V' \to U$), $\Psi(\xi)(g) \in \mathrm{Hom}^g_{\mathbf{F}}(f^*(\xi), f'^*(\xi))$ is defined to be the morphism (which exists unique because $\alpha_{f'}(\xi)$ is cartesian) such that

$$\begin{tikzcd}
f^*(\xi) \arrow[rr, "\Psi(\xi)(g)"] \arrow[dr, "\alpha_f(\xi)"'] & & f'^*(\xi) \arrow[dl, "\alpha_{f'}(\xi)"] \\
& \xi &
\end{tikzcd}$$

commutes (note that $\Psi(\xi)(g)$ is cartesian by Lemma 5.5, so that $\Psi(\xi)$ is really a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$). Similarly, if $\phi\colon \xi \to \xi'$ is a morphism of $\mathbf{F}_U$, the natural transformation $\Psi(\phi)\colon \Psi(\xi) \to \Psi(\xi')$ is defined for every object $f\colon V \to U$ of $\mathbf{C}_{/U}$ by $\Psi(\phi)(f) := f^*(\phi)\colon f^*(\xi) \to f^*(\xi')$ (it is clear that $\Psi(\phi)$ is actually a morphism of $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U, \mathbf{F})$). It is then easy to prove the following result, which generalizes Proposition 4.3.

Proposition 5.36. *For every fibred category $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ and every $U \in \mathbf{C}$ the functor $\Phi_{\mathbf{F},U} \colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U, \mathbf{F}) \to \mathbf{F}_U$ is an equivalence of categories. Moreover, for every clivage c on $\mathbf{F}$, the functor $\Psi_{(\mathbf{F},c),U} \colon \mathbf{F}_U \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U, \mathbf{F})$ is a quasi-inverse of $\Phi_{\mathbf{F},U}$.*

Remark 5.37. As we do for presheaves, we will usually identify the categories $\mathbf{F}_U$ and $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U, \mathbf{F})$, without mentioning the functors Φ and Ψ. In particular, every object ξ of $\mathbf{F}_U$ will be identified with the morphism $\Psi(\xi)$ (of course, in this way the morphism ξ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ depends on the chosen clivage, but different choices yield 2-isomorphic morphisms).

5.5 Fibred products of fibred categories

It is not clear a priori how fibred products should be defined in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ (or, more generally, in an arbitrary 2-category). Actually, even in $\underline{\mathbf{Cat}}$ there are (at least) two different possible definitions of fibred product. The more obvious one is the following, which we will call *strict fibred product* and denote by $\times^{\mathrm{s}}$: given functors $F_i \colon \mathbf{C}_i \to \mathbf{C}$ ($i = 1, 2$), one can define the category $\mathbf{C}_1 \,{}_{F_1}{\times^{\mathrm{s}}}_{F_2}\, \mathbf{C}_2$ (or simply $\mathbf{C}_1 \times^{\mathrm{s}}_{\mathbf{C}} \mathbf{C}_2$) by

$$\mathrm{Ob}(\mathbf{C}_1 \times^{\mathrm{s}}_{\mathbf{C}} \mathbf{C}_2) := \{(U_1, U_2) \,|\, U_i \in \mathbf{C}_i, F_1(U_1) = F_2(U_2)\},$$
$$\mathrm{Hom}((U_1, U_2), (V_1, V_2)) := \{(f_1, f_2) \,|\, f_i \in \mathrm{Hom}_{\mathbf{C}_i}(U_i, V_i), F_1(f_1) = F_2(f_2)\}.$$

(composition of morphisms is obviously defined componentwise). It is clear that there are natural projection functors $Pr_i \colon \mathbf{C}_1 \times^{\mathrm{s}}_{\mathbf{C}} \mathbf{C}_2 \to \mathbf{C}_i$ and that $F_1 \circ Pr_1 = F_2 \circ Pr_2$ (true equality, not just isomorphism of functors). It is also easy to see that strict fibred product satisfies the following property: for every $\mathbf{D} \in \underline{\mathbf{Cat}}$ composition with Pr_1 and Pr_2 induces an isomorphism of categories

$$\mathbf{Fun}(\mathbf{D}, \mathbf{C}_1 \times^{\mathrm{s}}_{\mathbf{C}} \mathbf{C}_2) \to \mathbf{Fun}(\mathbf{D}, \mathbf{C}_1) \times^{\mathrm{s}}_{\mathbf{Fun}(\mathbf{D},\mathbf{C})} \mathbf{Fun}(\mathbf{D}, \mathbf{C}_2).$$

However, such a definition of fibred product in $\underline{\mathbf{Cat}}$ (which can be extended to an arbitrary 2-category $\underline{\mathbf{C}}$ defining $U \times^{\mathrm{s}}_W V$ by a similar universal property, namely requiring that the natural functor

$$\mathbf{Hom}_{\underline{\mathbf{C}}}(X, U \times^{\mathrm{s}}_W V) \to \mathbf{Hom}_{\underline{\mathbf{C}}}(X, U) \times^{\mathrm{s}}_{\mathbf{Hom}_{\underline{\mathbf{C}}}(X,W)} \mathbf{Hom}_{\underline{\mathbf{C}}}(X, V)$$

be an isomorphism of categories for every $X \in \underline{\mathbf{C}}$) does not yield a satisfactory notion of fibred product in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. The reasons of this can be understood already in $\underline{\mathbf{Cat}} = \underline{\mathbf{Fib}}_{\{*\}}$: for instance, if F_1 is an equivalence of categories (i.e., an isomorphism of fibred categories), then $Pr_2 \colon \mathbf{C}_1 \times^{\mathrm{s}}_{\mathbf{C}} \mathbf{C}_2 \to \mathbf{C}_2$ is not an equivalence of categories in general. However, although we will use a different notion of fibred product in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, strict fibred products will also play a role in the following, as we are going to explain.

Lemma 5.38. *Let* $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ *be a fibred category and* $F\colon \mathbf{C}' \to \mathbf{C}$ *a functor. Then the projection functor* $\mathrm{p}'\colon \mathbf{F}' := \mathbf{C}' \,{}_F\!\times^{\mathrm{s}}{}_{\mathrm{p}} \mathbf{F} \to \mathbf{C}'$ *is a fibred category, too. Moreover, the projection functor* $\mathbf{F}' \to \mathbf{F}$ *induces an isomorphism of categories* $\mathbf{F}'_{U'} \to \mathbf{F}_{F(U')}$ *for every* $U' \in \mathbf{C}'$ *(in particular, if* $\mathbf{F}$ *is fibred in groupoids or in equivalence relations or in sets, the same is true for* $\mathbf{F}'$*).*

Proof. Given $f'\colon U' \to V'$ in $\mathbf{C}'$ (say with $F(f') = f\colon U \to V$) and $\eta' \in \mathbf{F}'_{V'}$ (by definition of $\mathbf{F}'$, η' must be of the form (V', η) for some $\eta \in \mathbf{F}_V$), we have to show that there exists a cartesian morphism $\phi'\colon \xi' \to \eta'$ in $\mathbf{F}'$ with $\mathrm{p}'(\phi') = f'$. It is straightforward to check that it is enough to take a cartesian morphism $\phi\colon \xi \to \eta$ in $\mathbf{F}$ with $\mathrm{p}(\phi) = f$ and set $\xi' := (U', \xi)$, $\phi' := (f', \phi)$. The second statement is also trivial. $\square$

In the situation of the above lemma, let's denote $\mathbf{F}' \in \underline{\mathbf{Fib}}_{\mathbf{C}'}$ by $F^*(\mathbf{F})$. Using the universal property of strict fibred product and the fact that a morphism (f', ϕ) of $\mathbf{F}'$ is cartesian if and only if ϕ is cartesian in $\mathbf{F}$ (as it is immediate to see), it is then easy to prove the following result.

Proposition 5.39. *For every functor* $F\colon \mathbf{C}' \to \mathbf{C}$ *the map* $\mathbf{F} \mapsto F^*(\mathbf{F})$ *extends to a (strict) 2-functor* $F^*\colon \underline{\mathbf{Fib}}_{\mathbf{C}} \to \underline{\mathbf{Fib}}_{\mathbf{C}'}$*, whose restriction to* $\widehat{\mathbf{C}}$ *coincides with* $\circ F\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}'}$.

Now we fix as usual our base category $\mathbf{C}$ and give the definition of (non strict) fibred product in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ (denoted simply by $\times$). Given morphisms (for $i = 1, 2$) $P_i\colon \mathbf{F}^i \to \mathbf{F}$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, we define $\tilde{\mathbf{F}} := \mathbf{F}^1 \,{}_{P_1}\!\times_{P_2} \mathbf{F}^2$ (or simply $\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$) as follows: for every $U \in \mathbf{C}$

$$\mathrm{Ob}(\tilde{\mathbf{F}}_U) := \{(\xi_1, \xi_2, \lambda) \mid \xi_i \in \mathbf{F}^i_U, \lambda \in \mathrm{Isom}_{\mathbf{F}_U}(P_1(\xi_1), P_2(\xi_2))\}$$

and for every morphism $f\colon U \to V$ of $\mathbf{C}$

$$\begin{aligned}\mathrm{Hom}^f_{\tilde{\mathbf{F}}}&((\xi_1, \xi_2, \lambda), (\eta_1, \eta_2, \mu)) \\ &:= \{(\phi_1, \phi_2) \mid \phi_i \in \mathrm{Hom}^f_{\mathbf{F}^i}(\xi_i, \eta_i), \mu \circ P_1(\phi_1) = P_2(\phi_2) \circ \lambda\}.\end{aligned}$$

It is clear that in this way we obtain a category $\tilde{\mathbf{F}}$ together with a functor $\tilde{\mathbf{F}} \to \mathbf{C}$, and we have to prove that it is a fibred category. Given $f\colon U \to V$ in $\mathbf{C}$ and $(\eta_1, \eta_2, \mu) \in \tilde{\mathbf{F}}_V$, for $i = 1, 2$ we can choose a cartesian morphism $\phi_i\colon \xi_i \to \eta_i$ in $\mathbf{F}^i$ over f. Since $P_2(\phi_2)$ is a cartesian morphism, there exists a unique $\lambda \in \mathrm{Hom}_{\mathbf{F}_U}(P_1(\xi_1), P_2(\xi_2))$ such that the diagram

$$\begin{array}{ccc} P_1(\xi_1) & \xrightarrow{P_1(\phi_1)} & P_1(\eta_1) \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} \\ P_2(\xi_2) & \xrightarrow[P_2(\phi_2)]{} & P_2(\eta_2) \end{array}$$

commutes. Now, λ is cartesian by Lemma 5.5 ($P_2(\phi_2)\circ\lambda = \mu\circ P_1(\phi_1)$ is cartesian again by Lemma 5.5), hence it is an isomorphism by Lemma 5.6. Therefore, by definition, $(\xi_1,\xi_2,\lambda)\in\tilde{\mathbf{F}}_U$ and $(\phi_1,\phi_2)\in\mathrm{Hom}_{\tilde{\mathbf{F}}}((\xi_1,\xi_2,\lambda),(\eta_1,\eta_2,\mu))$, and the following lemma implies that (ϕ_1,ϕ_2) is cartesian, so that $\tilde{\mathbf{F}}$ is actually a fibred category over $\mathbf{C}$.

Lemma 5.40. *A morphism (ϕ_1,ϕ_2) of $\tilde{\mathbf{F}}=\mathbf{F}^1\times_{\mathbf{F}}\mathbf{F}^2$ is cartesian if and only if ϕ_1 is cartesian in $\mathbf{F}^1$ and ϕ_2 is cartesian in $\mathbf{F}^2$.*

Proof. Let's say $(\phi_1,\phi_2)\in\mathrm{Hom}^f_{\tilde{\mathbf{F}}}((\xi_1,\xi_2,\lambda),(\eta_1,\eta_2,\mu))$, and assume first that ϕ_1 and ϕ_2 are cartesian. Then, given $g\colon U'\to U$ in $\mathbf{C}$ and

$$(\phi'_1,\phi'_2)\in\mathrm{Hom}^{f\circ g}_{\tilde{\mathbf{F}}}((\xi'_1,\xi'_2,\lambda'),(\eta_1,\eta_2,\mu)),$$

for $i=1,2$ there exist unique $\psi_i\in\mathrm{Hom}^g_{\mathbf{F}^i}(\xi'_i,\xi_i)$ such that $\phi'_i=\phi_i\circ\psi_i$; it follows that, in order to prove that (ϕ_1,ϕ_2) is cartesian, it is enough to show that $(\psi_1,\psi_2)\in\mathrm{Hom}^g_{\tilde{\mathbf{F}}}((\xi'_1,\xi'_2,\lambda'),(\xi_1,\xi_2,\lambda))$, i.e. that $\lambda\circ P_1(\psi_1)=P_2(\psi_2)\circ\lambda'$. Since (ϕ_1,ϕ_2) and (ϕ'_1,ϕ'_2) are morphisms of $\tilde{\mathbf{F}}$,

$$\begin{tikzcd}
P_1(\xi_1) \arrow[r, "P_1(\phi_1)"] \arrow[d, "\lambda"'] & P_1(\eta_1) \arrow[d, "\mu"] & P_1(\xi'_1) \arrow[l, "P_1(\phi'_1)"'] \arrow[d, "\lambda'"] \\
P_2(\xi_2) \arrow[r, "P_2(\phi_2)"'] & P_2(\eta_2) & P_2(\xi'_2) \arrow[l, "P_2(\phi'_2)"]
\end{tikzcd}$$

is a commutative diagram (in $\mathbf{F}$). Therefore we have

$$\begin{aligned} P_2(\phi_2)\circ\lambda\circ P_1(\psi_1) &= \mu\circ P_1(\phi_1)\circ P_1(\psi_1) \\ &= \mu\circ P_1(\phi'_1) = P_2(\phi'_2)\circ\lambda' = P_2(\phi_2)\circ P_2(\psi_2)\circ\lambda', \end{aligned}$$

which implies $\lambda\circ P_1(\psi_1)=P_2(\psi_2)\circ\lambda'$ because $P_2(\phi_2)$ is cartesian.

Assume conversely that (ϕ_1,ϕ_2) is cartesian. Let $\bar\phi_i\in\mathrm{Hom}^f_{\mathbf{F}^i}(\bar\xi_i,\eta_i)$ be cartesian morphisms for $i=1,2$: we already know that there exists unique $\bar\lambda\in\mathrm{Isom}_{\mathbf{F}_U}(P_1(\bar\xi_1),P_2(\bar\xi_2))$ such that $(\bar\phi_1,\bar\phi_2)\in\mathrm{Hom}^f_{\tilde{\mathbf{F}}}((\bar\xi_1,\bar\xi_2,\bar\lambda),(\eta_1,\eta_2,\mu))$, and $(\bar\phi_1,\bar\phi_2)$ is cartesian by the already proved implication. Then by Corollary 5.7 there exists unique $(\psi_1,\psi_2)\in\mathrm{Isom}_{\tilde{\mathbf{F}}_U}((\xi_1,\xi_2,\lambda),(\bar\xi_1,\bar\xi_2,\bar\lambda))$ such that $(\phi_1,\phi_2)=(\bar\phi_1,\bar\phi_2)\circ(\psi_1,\psi_2)$; this means that (for $i=1,2$) ψ_i is an isomorphism such that $\phi_i=\bar\phi_i\circ\psi_i$, so that ϕ_i is cartesian by Lemma 5.5. □

It follows that for $i=1,2$ the natural functor $Pr_i\colon\tilde{\mathbf{F}}\to\mathbf{F}^i$ (defined on objects by $(\xi_1,\xi_2,\lambda)\mapsto\xi_i$ and on morphisms by $(\phi_1,\phi_2)\mapsto\phi_i$) is a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. It is also clear that setting for every $(\xi_1,\xi_2,\lambda)\in\tilde{\mathbf{F}}$

$$\gamma(\xi_1,\xi_2,\lambda):=\lambda\colon P_1\circ Pr_1(\xi_1,\xi_2,\lambda)=P_1(\xi_1)\to P_2(\xi_2)=P_2\circ Pr_2(\xi_1,\xi_2,\lambda)$$

defines a (tautological) natural transformation $\gamma\colon P_1 \circ Pr_1 \to P_2 \circ Pr_2$, which is obviously a 2-isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. In other words, there is a 2-commutative diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}$

$$\begin{tikzcd} \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2 \arrow[r, "Pr_1"] \arrow[d, "Pr_2"'] \arrow[dr, phantom, "\swarrow\gamma"] & \mathbf{F}^1 \arrow[d, "P_1"] \\ \mathbf{F}^2 \arrow[r, "P_2"'] & \mathbf{F}. \end{tikzcd} \tag{5.1}$$

Remark 5.41. It is clear by construction that each fibre $(\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2)_U$ is isomorphic (as a category) to $\mathbf{F}^1_U \times_{\mathbf{F}_U} \mathbf{F}^2_U$ (fibred product in $\underline{\mathbf{Cat}} = \underline{\mathbf{Fib}}_{\{*\}}$). It follows that if $\mathbf{F}^1$ and $\mathbf{F}^2$ are fibred in groupoids or in equivalence relations or in sets, the same is true for $\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$; notice also that in this case $\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$ is isomorphic (as a category) to $\mathbf{F}^1 \times_{\mathbf{F}^{\mathrm{cart}}} \mathbf{F}^2$. It is easy to see that if $\mathbf{F}^1$, $\mathbf{F}^2$ and $\mathbf{F}$ are fibred in sets (so that they can be identified with presheaves), then this new definition of fibred product coincides (up to isomorphism) with the usual one in $\widehat{\mathbf{C}}$.

Remark 5.42. Writing $\mathbf{C}$ for the trivial fibred category $\mathrm{id}\colon \mathbf{C} \to \mathbf{C}$ (it corresponds to the terminal object of $\widehat{\mathbf{C}}$ and it is terminal also in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, in the sense that for every fibred category $\mathrm{p}\colon \mathbf{F} \to \mathbf{C}$ the category $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{C})$ is a set with only one element, namely p), then $\mathbf{F}^1 \times_{\mathbf{C}} \mathbf{F}^2$ (which can be naturally identified with $\mathbf{F}^1 \times^{\mathrm{s}}{}_{\mathbf{C}} \mathbf{F}^2$) will be denoted simply by $\mathbf{F}^1 \times \mathbf{F}^2$ (from what we are going to say in general about fibred products, it will be clear that this is the right notion of product in $\underline{\mathbf{Fib}}_{\mathbf{C}}$).

Now we want to study the properties of the fibred product $\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$. We start by observing that for every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the fibred product (in $\underline{\mathbf{Cat}}$)

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^2)$$

is a category whose objects can be naturally identified with 2-commutative diagrams in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ of the form

$$\begin{tikzcd} \mathbf{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathbf{F}^1 \arrow[d, "P_1"] \\ \mathbf{F}^2 \arrow[r, "P_2"'] & \mathbf{F}. \end{tikzcd}$$

Notice moreover that given a 2-commutative diagrams in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ of the form

$$\begin{tikzcd} \mathbf{F}' \arrow[r, "P'_1"] \arrow[d, "P'_2"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathbf{F}^1 \arrow[d, "P_1"] \\ \mathbf{F}^2 \arrow[r, "P_2"'] & \mathbf{F}, \end{tikzcd} \tag{5.2}$$

for every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ there is a naturally induced functor

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}') \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^2)$$

defined on objects by $Q \mapsto (P_1' \circ Q, P_2' \circ Q, \alpha \star \mathrm{id}_Q)$ and on morphisms by $\beta \mapsto (\mathrm{id}_{P_1'} \star \beta, \mathrm{id}_{P_2'} \star \beta)$. It is then straightforward to prove the following result.

Lemma 5.43. *For every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the natural functor (induced by (5.1))*

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^2)$$

is an isomorphism of categories. Its inverse is the natural functor defined on objects by sending

$$\begin{tikzcd}
\mathbf{G} \arrow[r, "Q_1"] \arrow[d, "Q_2"'] & \mathbf{F}^1 \arrow[d, "P_1"] \arrow[dl, Rightarrow, "\beta"'] \\
\mathbf{F}^2 \arrow[r, "P_2"'] & \mathbf{F}
\end{tikzcd}$$

to $\tilde{Q} \colon \mathbf{G} \to \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$, where $\tilde{Q}(\xi) := (Q_1(\xi), Q_2(\xi), \beta(\xi))$ for every object ξ of $\mathbf{G}$ and $\tilde{Q}(\phi) := (Q_1(\phi), Q_2(\phi))$ for every morphism ϕ of $\mathbf{G}$.

Definition 5.44. Assume that (5.2) is a 2-commutative diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. Then the diagram is 2-*cartesian* if the naturally induced functor

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}') \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G},\mathbf{F})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^2) \quad (5.3)$$

is an equivalence of categories for every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$.

For later use, we give here also the generalization to fibred categories of the notion of cocartesian diagram, which can be expressed in terms of fibred products. Observe that given the 2-commutative diagram (5.2) for every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ there is a naturally induced functor

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{G}) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}^1, \mathbf{G}) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}',\mathbf{G})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}^2, \mathbf{G}) \quad (5.4)$$

defined on objects by $Q \mapsto (Q \circ P_1, Q \circ P_2, \mathrm{id}_Q \star \alpha)$ and on morphisms by $\beta \mapsto (\beta \star \mathrm{id}_{P_1}, \beta \star \mathrm{id}_{P_2})$. Again, it is useful to observe that the objects of the category on the right-hand side of (5.4) can be naturally identified with 2-commutative diagrams in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ of the form

$$\begin{tikzcd}
\mathbf{F}' \arrow[r, "P_1'"] \arrow[d, "P_2'"'] & \mathbf{F}^1 \arrow[d] \arrow[dl, Rightarrow] \\
\mathbf{F}^2 \arrow[r] & \mathbf{G}.
\end{tikzcd}$$

Definition 5.45. Assume that (5.2) is a 2-commutative diagram in a full 2-subcategory $\underline{\mathbf{F}}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. Then the diagram is 2-*cocartesian* (in $\underline{\mathbf{F}}$) if the naturally induced functor (5.4) is an equivalence of categories for every $\mathbf{G} \in \underline{\mathbf{F}}$.

Remark 5.46. Of course, if a 2-commutative diagram in $\underline{\mathbf{F}}$ is 2-cocartesian in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, then it is 2-cocartesian in $\underline{\mathbf{F}}$, too, but the converse is false in general (even in the cases of interest to us, like $\underline{\mathbf{F}} = \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}, \underline{\mathbf{Fib}}_{\mathbf{C}}^{equiv}, \underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$ or the 2-category of (pre)stacks, which we will introduce later). We didn't need to give a similar definition for 2-cartesian diagrams because all the relevant 2-subcategories of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ are closed under fibred products.

Remark 5.47. Since $\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{G})$ is a set if $\mathbf{G}$ is fibred in sets, it is clear that the notion of 2-cocartesianity in $\underline{\mathbf{Fib}}_{\mathbf{C}}^{set}$ coincides with the usual notion of cocartesianity in $\widehat{\mathbf{C}}$.

Returning to fibred products and 2-cartesian diagrams, we observe that a diagram like (5.2) induces (by the isomorphism of Lemma 5.43 for $\mathbf{G} = \mathbf{F}'$) a morphism which we denote by $(P_1', P_2'; \alpha) \colon \mathbf{F}' \to \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$. Moreover, in analogy with the notation used in ordinary categories, for every morphism $P \colon \mathbf{F}' \to \mathbf{F}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ we will denote by $\Delta_P := (\mathrm{id}_{\mathbf{F}'}, \mathrm{id}_{\mathbf{F}'}; \mathrm{id}_P) \colon \mathbf{F}' \to \mathbf{F}' \times_{\mathbf{F}} \mathbf{F}'$ the diagonal morphism; similarly, for every fibred category $\mathrm{p} \colon \mathbf{F} \to \mathbf{C}$, we set $\Delta_{\mathbf{F}} := \Delta_{\mathrm{p}} \colon \mathbf{F} \to \mathbf{F} \times \mathbf{F}$.

Proposition 5.48. *The* 2*-commutative diagram* (5.2) *is* 2*-cartesian if and only if the corresponding morphism* $(P_1', P_2'; \alpha) \colon \mathbf{F}' \to \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$ *is an isomorphism of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Proof. It is easy to check that for every $\mathbf{G} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the functor (5.3) coincides (under the isomorphism of Lemma 5.43) with the functor

$$(P_1', P_2', \alpha)\circ \colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}') \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{G}, \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2),$$

and then the statement follows from Corollary 5.26. $\square$

Keeping the same notation used for cartesian diagrams, we will often use the symbol $\square$ to denote 2-cartesian diagrams.

Definition 5.49. A property P of morphisms of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is *stable under base change* if for every 2-cartesian diagram

$$\begin{tikzcd} \mathbf{F}' \arrow[r] \arrow[d, "P'"'] \arrow[dr, phantom, "\square"] & \mathbf{F} \arrow[d, "P"] \\ \mathbf{G}' \arrow[r] & \mathbf{G} \end{tikzcd}$$

such that P satisfies P, then P' satisfies P, too.

Remark 5.50. If a property P of morphisms of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is stable under base change and $P, P' \colon \mathbf{F} \to \mathbf{G}$ are morphisms which are 2-isomorphic, then P satisfies P if and only if P' satisfies P. Indeed, if $\alpha \colon P \to P'$ is a 2-isomorphism, there is clearly a 2-cartesian diagram

$$\begin{tikzcd} \mathbf{F} \arrow[r, "\mathrm{id}"] \arrow[d, "P'"'] \arrow[dr, phantom, "\Box_{\alpha}\ \swarrow"] & \mathbf{F} \arrow[d, "P"] \\ \mathbf{G} \arrow[r, "\mathrm{id}"'] & \mathbf{G}. \end{tikzcd}$$

Lemma 5.51. *Let $\alpha_i \colon P_i \to Q_i$ (for $i = 1, 2$) be 2-isomorphisms and let*

$$\begin{tikzcd} \mathbf{F}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box\ \swarrow"] & \mathbf{F}^1 \arrow[d, "P_1"] \\ \mathbf{F}^2 \arrow[r, "P_2"'] & \mathbf{F}, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{F}'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box\ \swarrow"] & \mathbf{F}^1 \arrow[d, "Q_1"] \\ \mathbf{F}^2 \arrow[r, "Q_2"'] & \mathbf{F} \end{tikzcd}$$

be 2-cartesian diagrams in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. Then $\mathbf{F}' \cong \mathbf{F}''$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Proof. It is immediate to see that the functor $\mathbf{F}^1 {}_{P_1}\!\times_{P_2} \mathbf{F}^2 \to \mathbf{F}^1 {}_{Q_1}\!\times_{Q_2} \mathbf{F}^2$ defined on objects by $(\xi_1, \xi_2, \lambda) \mapsto (\xi_1, \xi_2, \alpha_2(\xi_2) \circ \lambda \circ \alpha_1(\xi_1)^{-1})$ and which is the identity on morphisms is an isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ (and also an isomorphism of categories). The statement then follows from Proposition 5.48. $\Box$

Lemma 5.52. *If the 2-commutative diagram (5.2) is 2-cartesian and $P_1 \colon \mathbf{F}^1 \to \mathbf{F}$ is an isomorphism (of $\underline{\mathbf{Fib}}_{\mathbf{C}}$), then $P_2' \colon \mathbf{F}' \to \mathbf{F}^2$ is an isomorphism, too.*

Proof. By Proposition 5.48 it is enough to prove that the projection functor $Pr_2 \colon \tilde{\mathbf{F}} = \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2 \to \mathbf{F}^2$ is an isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, i.e. that $(Pr_2)_U \colon \tilde{\mathbf{F}}_U \to \mathbf{F}^2_U$ is an equivalence of categories for every $U \in \mathbf{C}$. $(Pr_2)_U$ is fully faithful: given $(\xi_1, \xi_2, \lambda), (\xi_1', \xi_2', \lambda') \in \tilde{\mathbf{F}}_U$ and $\phi_2 \colon \xi_2 \to \xi_2'$ in $\mathbf{F}^2_U$, there exists unique $\phi_1 \colon \xi_1 \to \xi_1'$ in $\mathbf{F}^1_U$ such that $(\phi_1, \phi_2) \in \mathrm{Hom}_{\tilde{\mathbf{F}}_U}((\xi_1, \xi_2, \lambda), (\xi_1', \xi_2', \lambda'))$, i.e. such that $P_1(\phi_1) = \lambda'^{-1} \circ P_2(\phi_2) \circ \lambda$ (because $(P_1)_U$ is fully faithful). $(Pr_2)_U$ is essentially surjective (actually even surjective on objects): given $\xi_2 \in \mathbf{F}^2_U$, since $(P_1)_U$ is essentially surjective, there exists $\xi_1 \in \mathbf{F}^1_U$ and $\lambda \in \mathrm{Isom}_{\mathbf{F}_U}(P_1(\xi_1), P_2(\xi_2))$, whence $(\xi_1, \xi_2, \lambda) \in \tilde{\mathbf{F}}_U$ and $Pr_2(\xi_1, \xi_2, \lambda) = \xi_2$. $\Box$

Lemma 5.53. *Given a 2-commutative diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}$*

$$\begin{tikzcd} \mathbf{G}^3 \arrow[r, "Q_2"] \arrow[d, "R_3"'] \arrow[dr, phantom, "\alpha'\ \swarrow"] & \mathbf{G}^2 \arrow[r, "Q_1"] \arrow[d] \arrow[dr, phantom, "\Box_{\alpha}\ \swarrow"] & \mathbf{G}^1 \arrow[d, "R_1"] \\ \mathbf{F}^3 \arrow[r, "P_2"'] & \mathbf{F}^2 \arrow[r, "P_1"'] & \mathbf{F}^1 \end{tikzcd}$$

such that the square on the right is 2-cartesian, then the square on the left is 2-cartesian if and only if the composition

$$\begin{tikzcd} \mathbf{G}^3 \arrow[r, "Q_1 \circ Q_2"] \arrow[d, "R_3"'] & \mathbf{G}^1 \arrow[d, "R_1"] \arrow[dl, Rightarrow, "\alpha''"'] \\ \mathbf{F}^3 \arrow[r, "P_1 \circ P_2"'] & \mathbf{F}^1 \end{tikzcd}$$

(where $\alpha'' := (\mathrm{id}_{P_1} \star \alpha') \circ (\alpha \star \mathrm{id}_{Q_2})$*) is* 2*-cartesian.*

Proof. The square on the left (respectively the composition square) is 2-cartesian if and only if the natural functor

$$F_2 \colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{G}^3) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{G}^2) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{F}^2)} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{F}^3)$$

(respectively

$$F_1 \colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{G}^3) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{G}^1) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{F}^1)} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{F}^3))$$

is an equivalence of categories for every $\mathbf{H} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$. It is easy to see that, setting $\mathbf{C}_i := \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{F}^i)$ and $\mathbf{D}_i := \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{H}, \mathbf{G}^i)$, there is a natural commutative diagram in $\underline{\mathbf{Cat}}$

$$\begin{tikzcd} \mathbf{D}_3 \arrow[r, "F_2"] \arrow[dr, "F_1"'] & \mathbf{D}_2 \times_{\mathbf{C}_2} \mathbf{C}_3 \arrow[r, "G"] & (\mathbf{D}_1 \times_{\mathbf{C}_1} \mathbf{C}_2) \times_{\mathbf{C}_2} \mathbf{C}_3 \arrow[dl, "H"] \\ & \mathbf{D}_1 \times_{\mathbf{C}_1} \mathbf{C}_3. & \end{tikzcd}$$

G, being induced by the natural functor $\mathbf{D}_2 \to \mathbf{D}_1 \times_{\mathbf{C}_1} \mathbf{C}_2$ (which is an equivalence because the square on the right is cartesian), is easily seen to be an equivalence. Therefore, in order to prove that F_2 is an equivalence if and only if F_1 is, it is enough to check that the natural functor H is an equivalence, which is straightforward. □

5.6 Representability for fibred categories

We fix as usual the base category $\mathbf{C}$. We are going to see that the notion of representability (both of objects and of morphisms) can be naturally extended from $\widehat{\mathbf{C}}$ to $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Definition 5.54. $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is *representable* if it is isomorphic (in $\underline{\mathbf{Fib}}_{\mathbf{C}}$) to some $U \in \mathbf{C}$ (i.e., to the fibred category $\mathbf{C}_{/U} \to \mathbf{C}$).

Remark 5.55. Since $\widehat{\mathbf{C}}$ is a full 2-subcategory of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, $F \in \widehat{\mathbf{C}}$ is representable as a fibred category if and only if it is representable as a presheaf. Notice that if

$\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is representable, then (by Proposition 5.27) $\mathbf{F}$ is fibred in equivalence relations, but not necessarily in sets. On the other hand, the lax 2-equivalence of Corollary 5.28 restricts to a lax 2-equivalence between $\mathbf{C}$ and the strictly full 2-subcategory of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ whose objects are representable fibred categories.

Definition 5.56. A morphism $P\colon \mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is *representable* if for every morphism $Q\colon \mathbf{H} \to \mathbf{G}$, where $\mathbf{H}$ is a representable fibred category, the fibred product $\mathbf{F}\,{}_P\!\times_Q \mathbf{H}$ is representable, too.

Remark 5.57. Using Lemma 5.52 and Lemma 5.53 it is easy to see that P as above is representable if and only if the condition of the definition is satisfied for every morphism $Q\colon V \to \mathbf{G}$, where $V \in \mathbf{C}$. Notice that in this case there is always (by Proposition 5.48) a 2-cartesian diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathbf{F} \arrow[d, "P"] \\ V \arrow[r, "Q"'] & \mathbf{G} \end{tikzcd}$$

with $U \in \mathbf{C}$ (the morphism f of $\mathbf{C}$ is obviously unique up to composition with an isomorphism $U' \to U$ of $\mathbf{C}$). It is also clear that for morphisms of $\widehat{\mathbf{C}}$ this new definition of representability coincides with the old one.

The following result is again an immediate consequence of Lemma 5.53.

Proposition 5.58. *For morphisms of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, the property of being representable is stable under composition and base change.*

Definition 5.59. Let P be a property of morphisms of $\mathbf{C}$ which is stable under base change. We will say that a representable morphism $P\colon \mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ satisfies P if for every $V \in \mathbf{C}$ and every morphism $Q\colon V \to \mathbf{G}$ the induced morphism $U \cong \mathbf{F}\,{}_P\!\times_Q V \to V$ of $\mathbf{C}$ satisfies P.

Remark 5.60. In the hypotheses of the above definition, it is clear that a representable morphism of $\widehat{\mathbf{C}}$ satisfies P in $\widehat{\mathbf{C}}$ if and only if it satisfies P in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. Moreover, it follows from Proposition 5.58 and Lemma 5.53 that, for representable morphisms of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, the property P remains stable under base change (and also under composition, if P is stable under composition for morphisms of $\mathbf{C}$).

We are going to see that also the criterion for representability of the diagonal extends to fibred categories. Before, however, we need to prove a result, which was obvious in the case of presheaves. Given morphisms $P_i\colon U_i \to \mathbf{F}$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ with $U_i \in \mathbf{C}$ (for $i = 1, 2$), recall that by definition there is a natural 2-cartesian diagram

$$\begin{tikzcd} U_1\,{}_{P_1}\!\times_{P_2} U_2 \arrow[r, "Pr_1"] \arrow[d, "Pr_2"'] \arrow[dr, phantom, "\square\ \gamma"] & U_1 \arrow[d, "P_1"] \\ U_2 \arrow[r, "P_2"'] & \mathbf{F} \end{tikzcd}$$

and observe that $U_1 \,{}_{P_1}\!\times_{P_2} U_2 \in \widehat{\mathbf{C}}$ by Remark 5.41.

Lemma 5.61. *In the above notation, the 2-commutative diagram of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$

$$\begin{tikzcd} U_1 \,{}_{P_1}\!\times_{P_2} U_2 \arrow[r, "P_1 \circ Pr_1"] \arrow[d, "(Pr_1, Pr_2)"'] & \mathbf{F} \arrow[d, "\Delta_{\mathbf{F}}"] \arrow[dl, phantom, "\swarrow (\mathrm{id}, \gamma)"] \\ U_1 \times U_2 \arrow[r, "P_1 \times P_2"'] & \mathbf{F} \times \mathbf{F} \end{tikzcd}$$

is 2*-cartesian.*

Proof. Setting $\mathbf{G} := \mathbf{F} \,{}_{\Delta_{\mathbf{F}}}\!\times_{(P_1 \times P_2)} (U_1 \times U_2)$, by Proposition 5.48 we have to show that the morphism

$$R := (P_1 \circ Pr_1, (Pr_1, Pr_2); (\mathrm{id}, \gamma)) \colon U_1 \,{}_{P_1}\!\times_{P_2} U_2 \to \mathbf{G}$$

is an isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, i.e., that $R_W \colon (U_1 \,{}_{P_1}\!\times_{P_2} U_2)_W \to \mathbf{G}_W$ is an equivalence of categories for every $W \in \mathbf{C}$. Now, by definition of fibred product,

$$(U_1 \,{}_{P_1}\!\times_{P_2} U_2)_W = \{(f_1, f_2, \lambda) \,|\, f_i \in U_i(W), \lambda \in \mathrm{Isom}_{\mathbf{F}_W}(P_1(f_1), P_2(f_2))\}$$

(it is just a set, since $U_1 \,{}_{P_1}\!\times_{P_2} U_2 \in \widehat{\mathbf{C}}$), whereas

$$\mathrm{Ob}(\mathbf{G}_W) = \{(\zeta, g_1, g_2, \mu_1, \mu_2) \,|\, \zeta \in \mathbf{F}_W, g_i \in U_i(W), \mu_i \in \mathrm{Isom}_{\mathbf{F}_W}(\zeta, P_i(g_i))\}$$

and

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{G}_W}((\zeta, g_1, g_2, \mu_1, \mu_2), (\zeta', g_1', g_2', \mu_1', \mu_2')) \\ &\qquad = \begin{cases} \{\phi \in \mathrm{Hom}_{\mathbf{F}_W}(\zeta, \zeta') \,|\, \mu_i = \mu_i' \circ \phi\} & \text{if } g_i = g_i' \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

(notice that the last set contains only the isomorphism ${\mu_1'}^{-1} \circ \mu_1$ if $g_i = g_i'$ and ${\mu_1'}^{-1} \circ \mu_1 = {\mu_2'}^{-1} \circ \mu_2$, and is empty otherwise, so that $\mathbf{G}_W$ is an equivalence relation, as expected). It is then very easy to see that R_W, which is defined by $(f_1, f_2, \lambda) \mapsto (P_1(f_1), f_1, f_2, \mathrm{id}_{P_1(f_1)}, \lambda)$, is an equivalence of categories. □

Using Lemma 5.61 the proof of the following proposition is the same as that of Proposition 4.14, except that, of course, cartesian diagrams are replaced by 2-cartesian diagrams and Lemma 5.53 must be used instead of Lemma A.1.

Proposition 5.62. *Assume that* $\mathbf{C}$ *has fibred products and finite products. Given* $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$*, the diagonal morphism* $\Delta_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F} \times \mathbf{F}$ *of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$ *is representable if and only if every morphism* $U \to \mathbf{F}$ *of* $\underline{\mathbf{Fib}}_{\mathbf{C}}$ *(with* $U \in \mathbf{C}$*) is representable. In this case, moreover,* $\Delta_{\mathbf{F}}$ *satisfies a property* P *of morphisms of* $\mathbf{C}$ *which is stable under base change if for all* $U \in \mathbf{C}$ *and all* $\xi_1, \xi_2 \colon U \to \mathbf{F}$ *the natural morphism of representable presheaves* $U \,{}_{\xi_1}\!\times_{\xi_2} U \to U \times U$ *satisfies* P.

Remark 5.63. As (by Proposition 5.36) every morphism $U \to \mathbf{F}$ is 2-isomorphic to one defined by an object of $\mathbf{F}_U$, by Lemma 5.51 we can reformulate the above result by saying that $\Delta_{\mathbf{F}}$ is representable if and only if for all $U, V \in \mathbf{C}$, all $\xi \in \mathbf{F}_U$ and all $\eta \in \mathbf{F}_V$ the presheaf $U\,{}_\xi\!\times_\eta V$ is representable. We are going to see how presheaves of this form can be described.

Definition 5.64. Given $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, $U \in \mathbf{C}$ and $\xi_1, \xi_2 \in \mathbf{F}_U$, we define the presheaf $\mathcal{H}om_U(\xi_1, \xi_2) \in \widehat{\mathbf{C}_{/U}}$ as follows: for every object $f \colon V \to U$ of $\mathbf{C}_{/U}$

$$\mathcal{H}om_U(\xi_1, \xi_2)(f) := \mathrm{Hom}_{\mathbf{F}_V}(f^*(\xi_1), f^*(\xi_2))$$

and for every morphism $g \in \mathrm{Hom}_{\mathbf{C}_{/U}}(f \colon V \to U, f' \colon V' \to U)$

$$\begin{aligned} \mathcal{H}om_U(\xi_1, \xi_2)(g) \colon \mathrm{Hom}_{\mathbf{F}_{V'}}({f'}^*(\xi_1), {f'}^*(\xi_2)) &\to \mathrm{Hom}_{\mathbf{F}_V}(f^*(\xi_1), f^*(\xi_2)) \\ \phi &\mapsto c_{g,f'}(\xi_2) \circ g^*(\phi) \circ c_{g,f'}(\xi_1)^{-1} \end{aligned}$$

(we are using the functorial isomorphism $c_{g,f'} \colon g^* \circ {f'}^* \xrightarrow{\sim} (f' \circ g)^* = f^*$).

We also denote by $\mathcal{I}som_U(\xi_1, \xi_2) \in \widehat{\mathbf{C}_{/U}}$ the subpresheaf of $\mathcal{H}om_U(\xi_1, \xi_2)$ defined by

$$\mathcal{I}som_U(\xi_1, \xi_2)(f \colon V \to U) := \mathrm{Isom}_{\mathbf{F}_V}(f^*(\xi_1), f^*(\xi_2)).$$

Remark 5.65. The above definition depends on the clivage, but it is easy to see that different clivages yield isomorphic presheaves. It is also clear that if $\xi_i \cong \xi_i'$ in $\mathbf{F}_U$, then $\mathcal{H}om_U(\xi_1, \xi_2) \cong \mathcal{H}om_U(\xi_1', \xi_2')$ and $\mathcal{I}som_U(\xi_1, \xi_2) \cong \mathcal{I}som_U(\xi_1', \xi_2')$.

Lemma 5.66. *Given $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, $U \in \mathbf{C}$ and $\xi_1, \xi_2 \in \mathbf{F}_U$, let $\pi \colon I \to U$ be the morphism of $\widehat{\mathbf{C}}$ corresponding to $\mathcal{I}som_U(\xi_1, \xi_2) \in \widehat{\mathbf{C}_{/U}}$ (under the equivalence of Lemma 4.20). Then there is a natural 2-cartesian diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}$*

$$\begin{tikzcd} I \arrow[r, "P"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \mathbf{F} \arrow[d, "\Delta_{\mathbf{F}}"] \arrow[dl, phantom, "\swarrow\alpha"] \\ U \arrow[r, "(\xi_1,\xi_2)"'] & \mathbf{F} \times \mathbf{F}, \end{tikzcd}$$

where, for every $(f, \phi) \in I(V)$ (with $f \in U(V)$ and $\phi \in \mathrm{Isom}_{\mathbf{F}_V}(f^*(\xi_1), f^*(\xi_2))$), $P(f, \phi) := f^*(\xi_1) \in \mathbf{F}_V$ and $\alpha(f, \phi) := (\mathrm{id}_{f^*(\xi_1)}, \phi)$.

Moreover, given $V \in \mathbf{C}$ such that $U \times V \in \mathbf{C}$, $\xi \in \mathbf{F}_U$ and $\eta \in \mathbf{F}_V$, the natural morphism $U\,{}_\xi\!\times_\eta V \to U \times V$ of $\widehat{\mathbf{C}}$ corresponds to $\mathcal{I}som_{U \times V}(pr_1^*(\xi), pr_2^*(\eta)) \in \widehat{\mathbf{C}_{/U \times V}}$.

Proof. Similar to that of Lemma 5.61. $\square$

Corollary 5.67. *Given $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, $\Delta_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F} \times \mathbf{F}$ is representable (and satisfies a property P of morphisms of $\mathbf{C}$ which is stable under base change) if and only if the presheaf $\mathcal{I}som_U(\xi_1, \xi_2)$ is representable (and the structure morphism $\mathcal{I}som_U(\xi_1, \xi_2) \to U$ satisfies P) for all $U \in \mathbf{C}$ and all $\xi_1, \xi_2 \in \mathbf{F}_U$.*

5.7 Categories fibred in groupoids and groupoids in a category

The notions of group and equivalence relation in a category can be naturally generalized to a notion of groupoid in a category (recall that groups are just groupoids with one object and equivalence relations are groupoids with only trivial automorphisms). Assuming for simplicity that the category has fibred products, we will define it in the style of the characterizations given in Proposition 4.66 and Proposition 4.71 for groups and equivalence relations. The following definition should then be clear, if one observes that a groupoid $\mathbf{D}$ is completely described by the two sets $\mathrm{Ob}(\mathbf{D})$, $\mathrm{Mor}(\mathbf{D})$ and by the five maps

$$\begin{array}{ccc} s\colon \mathrm{Mor}(\mathbf{D}) \to \mathrm{Ob}(\mathbf{D}), & t\colon \mathrm{Mor}(\mathbf{D}) \to \mathrm{Ob}(\mathbf{D}), & \mathrm{Ob}(\mathbf{D}) \to \mathrm{Mor}(\mathbf{D}), \\ (f\colon U \to V) \mapsto U & (f\colon U \to V) \mapsto V & U \mapsto \mathrm{id}_U \end{array}$$

$$\begin{array}{cc} \mathrm{Mor}(\mathbf{D}) \to \mathrm{Mor}(\mathbf{D}), & \mathrm{Mor}(\mathbf{D})\, {}_s{\times}_t \,\mathrm{Mor}(\mathbf{D}) \to \mathrm{Mor}(\mathbf{D}) \\ f \mapsto f^{-1} & (g\colon V \to W, f\colon U \to V) \mapsto g \circ f \end{array}$$

(subject to some suitable compatibilities, expressing the fact that $\mathbf{D}$ is really a groupoid).

Definition 5.68. Let $\mathbf{C}$ be a category with fibred products. A *groupoid in* $\mathbf{C}$ is given by two objects U ("objects") and V ("morphisms") and five morphisms $s\colon V \to U$ ("source"), $t\colon V \to U$ ("target"), $e\colon U \to V$ ("identity"), $i\colon V \to V$ ("inverse") and $m\colon V\, {}_s{\times}_t\, V \to V$ ("composition") of $\mathbf{C}$ such that $s \circ e = t \circ e = \mathrm{id}_U$, $s = t \circ i$, $t = s \circ i$, $s \circ m = s \circ pr_2$, $t \circ m = t \circ pr_1$ (where $pr_i\colon V\, {}_s{\times}_t\, V \to V$ are the natural projections) and the following diagrams commute:

1.
$$\begin{tikzcd} & V\, {}_s{\times}_t\, V \arrow[d, "m"] & \\ V \arrow[ur, "(\mathrm{id}, e\circ s)"] \arrow[r, "\mathrm{id}"'] & V & V \arrow[ul, "(e\circ t, \mathrm{id})"'] \arrow[l, "\mathrm{id}"] \end{tikzcd} ;$$

2.
$$\begin{tikzcd} V \arrow[r, "(\mathrm{id}, i)"] \arrow[d, "t"'] & V\, {}_s{\times}_t\, V \arrow[d, "m"] & V \arrow[l, "(i, \mathrm{id})"'] \arrow[d, "s"] \\ U \arrow[r, "e"'] & V & U \arrow[l, "e"] \end{tikzcd} ;$$

3.
$$\begin{tikzcd} V\, {}_s{\times}_t\, V\, {}_s{\times}_t\, V \arrow[r, "\mathrm{id}_V \times m"] \arrow[d, "m \times \mathrm{id}_V"'] & V\, {}_s{\times}_t\, V \arrow[d, "m"] \\ V\, {}_s{\times}_t\, V \arrow[r, "m"'] & V \end{tikzcd} .$$

A groupoid in $\mathbf{C}$ as in the above definition will be usually denoted simply by $V \overset{t}{\underset{s}{\rightrightarrows}} U$.

Remark 5.69. If $\mathbf{C}$ has also a terminal object $*$, a group G in $\mathbf{C}$ defines a groupoid $G \rightrightarrows *$ in $\mathbf{C}$ (with e, i and m as in Proposition 4.66). Similarly, it is easy to see that an equivalence relation $R \overset{\delta_1}{\underset{\delta_2}{\rightrightarrows}} U$ in $\mathbf{C}$ is a groupoid in $\mathbf{C}$ (again, with e, i and m as in Proposition 4.71).

Example 5.70. Assuming $\mathbf{C}$ has also a terminal object $*$, let $\varrho \colon U \times G \to U$ be an action in $\mathbf{C}$, and denote by $e_G \colon * \to G$, $i_G \colon G \to G$ and $m_G \colon G \times G \to G$ the identity, inverse and multiplication morphisms. Then it is straightforward to check that ϱ naturally determines a groupoid $U \times G \overset{pr_1}{\underset{\varrho}{\rightrightarrows}} U$ in $\mathbf{C}$ with

$$\begin{aligned} e &= (\mathrm{id}, e_G \circ *_U) \colon U \to U \times G, \\ i &= (\varrho, i_G \circ pr_2) \colon U \times G \to U \times G, \\ m &= \mathrm{id}_U \times m_G \colon U \times G \times G \cong (U \times G)\, {}_{\varrho}{\times}_{pr_1} (U \times G) \to U \times G. \end{aligned}$$

Assume now that $F_\bullet = (F_1 \overset{t}{\underset{s}{\rightrightarrows}} F_0)$ is a groupoid in $\widehat{\mathbf{C}}$. Then it is clear that (for $U \in \mathbf{C}$) $[F_\bullet]'_U := F_\bullet(U) = (F_1(U) \overset{t(U)}{\underset{s(U)}{\rightrightarrows}} F_0(U))$ is a groupoid in **Set** (i.e., an ordinary groupoid) and that for every morphism $f \colon U \to V$ of $\mathbf{C}$ the maps $F_0(f) \colon \mathrm{Ob}([F_\bullet]'_V) \to \mathrm{Ob}([F_\bullet]'_U)$ and $F_1(f) \colon \mathrm{Mor}([F_\bullet]'_V) \to \mathrm{Mor}([F_\bullet]'_U)$ define a functor $f^* \colon [F_\bullet]'_V \to [F_\bullet]'_U$. If $U \xrightarrow{f} V \xrightarrow{g} W$ are two morphisms of $\mathbf{C}$, the fact that $F_i(g \circ f) = F_i(f) \circ F_i(g)$ implies that $(g \circ f)^* = f^* \circ g^* \colon [F_\bullet]'_W \to [F_\bullet]'_U$ (true equality, not just isomorphism of functors); similarly, for every $U \in \mathbf{C}$ we have $\mathrm{id}_U^* = \mathrm{id}_{[F_\bullet]'_U}$. This means that the groupoids $[F_\bullet]'_U$ together with the functors f^* define a (strict) 2-functor $\mathbf{C}^\circ \to \underline{\mathbf{Gpd}}$, whose corresponding category fibred in groupoids (naturally endowed with a scindage) is denoted by $[F_\bullet]' = [F_1 \overset{t}{\underset{s}{\rightrightarrows}} F_2]' \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$.

Remark 5.71. Conversely, it is easy to see that if $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ admits a scindage, then it is of the form $[F_\bullet]'$ for some groupoid in $\widehat{\mathbf{C}}$ (with $F_0(U) = \mathrm{Ob}(\mathbf{F}_U)$ and $F_1(U) = \mathrm{Mor}(\mathbf{F}_U)$ for every $U \in \mathbf{C}$). However, we will not need this fact.

The natural functors $\pi'_U \colon F_0(U) \to [F_\bullet]'_U$ (defined to be the identity on objects and in the unique possible way on morphisms) clearly determine a morphism $\pi' \colon F_0 \to [F_\bullet]'$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$, which is obviously an epimorphism. Notice also that there is a natural 2-commutative diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$

$$\begin{tikzcd} F_1 \arrow[r, "s"] \arrow[d, "t"'] & F_0 \arrow[d, "\pi'"] \arrow[dl, phantom, "\swarrow\gamma"] \\ F_0 \arrow[r, "\pi'"'] & {[F_\bullet]'} \end{tikzcd} \tag{5.5}$$

where the tautological 2-isomorphism γ is defined for all $U \in \mathbf{C}$ and all $\xi \in F_1(U)$ by

$$\gamma(\xi) := \xi \colon \pi' \circ s(\xi) = s(U)(\xi) \to t(U)(\xi) = \pi' \circ t(\xi).$$

Proposition 5.72. *If $F_\bullet = (F_1 \overset{t}{\underset{s}{\rightrightarrows}} F_2)$ is a groupoid in $\widehat{\mathbf{C}}$, then the 2-commutative diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ (5.5) is 2-cartesian and 2-cocartesian in $\underline{\mathbf{Fib}}_{\mathbf{C}}$.*

Proof. It follows immediately from the definitions that not only the diagram is 2-cartesian, but the morphism $(s,t;\gamma)\colon F_1 \to F_0 \times_{[F_\bullet]'} F_0$ is an isomorphism of categories and not just of fibred categories (they are both categories fibred in sets). Similarly, it turns out that for every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, setting

$$\mathbf{H} := \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(F_0, \mathbf{F}) \times_{\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(F_1,\mathbf{F})} \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(F_0, \mathbf{F}),$$

the naturally induced functor $H\colon \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}([F_\bullet]', \mathbf{F}) \to \mathbf{H}$ (defined on objects by $H(P) := (P \circ \pi', P \circ \pi', \mathrm{id}_P \star \gamma))$ is an isomorphism of categories (and not just an equivalence, as it is required in the definition of 2-cocartesian diagram). Indeed, given an object $(P_1, P_2, \alpha) \in \mathbf{H}$ ($\alpha\colon P_1 \circ s \xrightarrow{\sim} P_2 \circ t$ is a 2-isomorphism, so that, for every $U \in \mathbf{C}$ and every $\phi \in F_1(U)$, $\alpha(\phi)\colon P_1(s(\phi)) \xrightarrow{\sim} P_2(t(\phi))$ is an isomorphism of $\mathbf{F}_U$), we can naturally define an object $P \in \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}([F_\bullet]', \mathbf{F})$ as follows. For every $U \in \mathbf{C}$ the functor $P_U\colon [F_\bullet]'_U = F_\bullet(U) \to \mathbf{F}_U$ is defined on objects by $P(\xi) := P_1(\xi)$ for every $\xi \in F_0(U)$ and on morphisms by $P(\phi) := \alpha(e \circ t(\phi))^{-1} \circ \alpha(\phi)$ for every $\phi \in F_1(U)$. It is then easy to see that the functors P_U yield a morphism $P\colon [F_\bullet]' \to \mathbf{F}$ and that the mapping $(P_1, P_2, \alpha) \mapsto P$ extends to a functor $\mathbf{H} \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}([F_\bullet]', \mathbf{F})$, which is the inverse of H. □

The above result can be regarded as a generalization of the first part of Proposition 4.80 (if the groupoid $F_\bullet$ is an equivalence relation, then $[F_\bullet]'$, which is fibred in equivalence relations, is isomorphic to the quotient presheaf of $F_\bullet$ and (5.5) can be identified with the corresponding cartesian and cocartesian diagram in $\widehat{\mathbf{C}}$), and we are going to see that also the second part can be generalized in a similar way.

Given a morphism $P\colon F_0 \to \mathbf{F}$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$ with $F_0 \in \widehat{\mathbf{C}}$, let $F_1 := F_0 \times_{\mathbf{F}} F_0$ and consider the natural 2-cartesian diagram in $\underline{\mathbf{Fib}}_{\mathbf{C}}$

$$\begin{tikzcd} F_1 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & F_0 \arrow[d, "P"] \arrow[dl, phantom, "\square_\gamma\swarrow"] \\ F_0 \arrow[r, "P"'] & \mathbf{F}. \end{tikzcd} \tag{5.6}$$

We claim that there is a natural groupoid $F_\bullet = (F_1 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} F_0)$ in $\widehat{\mathbf{C}}$. In fact, $F_1 \in \widehat{\mathbf{C}}$ by Remark 5.41 and it is easy to check that the axioms of groupoids are satisfied by the morphisms

$$e := \Delta_P \colon F_0 \to F_1,$$
$$i := (pr_1, pr_2; \gamma^{-1}) \colon F_1 \to F_1,$$
$$m := pr_{1,3} \colon F_0 \times_{\mathbf{F}} F_0 \times_{\mathbf{F}} F_0 \cong F_1 \,{}_{pr_2}\!\times_{pr_1} F_1 \to F_1 = F_0 \times_{\mathbf{F}} F_0$$

(where $pr_{1,3}$ is the projection onto the first and third factors). By Proposition 5.72 the 2-commutative diagram (5.6) induces a morphism (well defined up to 2-isomorphism) $P' \colon [F_\bullet]' \to \mathbf{F}$ such that (denoting by $\pi' \colon F_0 \to [F_\bullet]'$ the natural morphism) $P' \circ \pi' \cong P \colon F_0 \to \mathbf{F}$ (actually the proof of Proposition 5.72 implies that there exists a unique P' such that $P' \circ \pi' = P$, but we will not need this fact).

Lemma 5.73. *With the above notation, for every $U \in \mathbf{C}$ and every $\xi, \eta \in F_0(U) = \mathrm{Ob}([F_\bullet]'_U)$ the functor $P'_U \colon [F_\bullet]'_U \to \mathbf{F}_U$ induces a bijection*

$$\mathrm{Hom}_{[F_\bullet]'_U}(\xi, \eta) \xrightarrow{\sim} \mathrm{Isom}_{\mathbf{F}_U}(P'(\xi), P'(\eta)).$$

In particular, P' is a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ if $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$.

Proof. Immediate from the definitions. □

Corollary 5.74. *If $P \colon F_0 \to \mathbf{F}$ is a morphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ with $F_0 \in \widehat{\mathbf{C}}$, then the induced morphism $P' \colon [F_\bullet]' := [F_0 \times_{\mathbf{F}} F_0 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} F_0]' \to \mathbf{F}$ yields an isomorphism $[F_\bullet]' \cong \mathrm{im}\, P$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$; in particular, $[F_\bullet]' \cong \mathbf{F}$ if and only if P is an epimorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$.*

Proof. Since $\mathrm{im}\, P = \mathrm{im}\, P'$ (because $\pi' \colon F_0 \to [F_\bullet]'$ is an epimorphism), it follows from Remark 5.30. □

6 Stacks

6.1 Prestacks and stacks

We fix a site $(\mathbf{C}, \tau)$. In the following, given a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U) = \mathrm{Cov}^\tau(U)$, for all $i, j, k \in I$ we will write $U_{i,j}$ instead of $U_i \times_U U_j$ and $U_{i,j,k}$ instead of $U_i \times_U U_j \times_U U_k$.

Definition 6.1. Given $U \in \mathbf{C}$, $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ and $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ (on which we assume, as usual, that a clivage has been chosen), a *descent datum* (in $\mathbf{F}$) relative to $\mathcal{U}$ consists of a collection of objects $\xi_i \in \mathbf{F}_{U_i}$ (for $i \in I$) together with isomorphisms $\phi_{i,j} \colon \xi_j|_{U_{i,j}} \xrightarrow{\sim} \xi_i|_{U_{i,j}}$ of $\mathbf{F}_{U_{i,j}}$ (for $i, j \in I$), such that the diagram in $\mathbf{F}_{U_{i,j,k}}$

$$\begin{tikzcd}
(\xi_k|_{U_{i,k}})|_{U_{i,j,k}} \arrow[rr, "\phi_{i,k}|_{U_{i,j,k}}"] \arrow[d, "\wr"'] & & (\xi_i|_{U_{i,k}})|_{U_{i,j,k}} \arrow[d, "\wr"] \\
\xi_k|_{U_{i,j,k}} & & \xi_i|_{U_{i,j,k}} \\
(\xi_k|_{U_{j,k}})|_{U_{i,j,k}} \arrow[u, "\wr"] \arrow[d, "\phi_{j,k}|_{U_{i,j,k}}"'] & & (\xi_i|_{U_{i,j}})|_{U_{i,j,k}} \arrow[u, "\wr"'] \\
(\xi_j|_{U_{j,k}})|_{U_{i,j,k}} \arrow[r, "\sim"'] & \xi_j|_{U_{i,j,k}} & (\xi_j|_{U_{i,j}})|_{U_{i,j,k}} \arrow[l, "\sim"] \arrow[u, "\phi_{i,j}|_{U_{i,j,k}}"']
\end{tikzcd}$$

commutes for all $i, j, k \in I$ (the unnamed arrows are the natural isomorphisms).

A descent datum $(\{\xi_i\}, \{\phi_{i,j}\})_{i,j \in I}$ relative to $\mathcal{U}$ as above is *effective* if there exists an object $\xi \in \mathbf{F}_U$ together with isomorphisms $\psi_i \colon \xi|_{U_i} \xrightarrow{\sim} \xi_i$ of $\mathbf{F}_{U_i}$ (for $i \in I$), such that the diagram

$$\begin{tikzcd}
\xi_j|_{U_{i,j}} \arrow[rr, "\phi_{i,j}"] & & \xi_i|_{U_{i,j}} \\
(\xi|_{U_j})|_{U_{i,j}} \arrow[u, "\psi_j|_{U_{i,j}}"] \arrow[r, "\sim"'] & \xi|_{U_{i,j}} & (\xi|_{U_i})|_{U_{i,j}} \arrow[l, "\sim"] \arrow[u, "\psi_i|_{U_{i,j}}"']
\end{tikzcd}$$

(where again the unnamed arrows are the natural isomorphisms) commutes for all $i, j \in I$.

Remark 6.2. It is customary to formulate the above definition in a less precise but simpler way: if one pretends that all the natural isomorphisms are just identities, we see that $(\{\xi_i\}, \{\phi_{i,j}\})_{i,j \in I}$ is a descent datum if and only if it satisfies the cocycle condition

$$\phi_{i,k}|_{U_{i,j,k}} = \phi_{i,j}|_{U_{i,j,k}} \circ \phi_{j,k}|_{U_{i,j,k}} \colon \xi_k|_{U_{i,j,k}} \to \xi_i|_{U_{i,j,k}}$$

for all $i, j, k \in I$, and it is effective if and only if there exist $\xi \in \mathbf{F}_U$ and $\psi_i \colon \xi|_{U_i} \xrightarrow{\sim} \xi_i$ in $\mathbf{F}_{U_i}$ such that $\phi_{i,j} = \psi_i|_{U_{i,j}} \circ (\psi_j|_{U_{i,j}})^{-1}$ for all $i, j \in I$. We will often make this abuse of notation in the following.

Definition 6.3. A fibred category $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is a *prestack* (for τ) if for all $U \in \mathbf{C}$ and all $\xi_1, \xi_2 \in \mathbf{F}_U$ the presheaf $\mathcal{H}om_U(\xi_1, \xi_2) \in \widehat{\mathbf{C}_{/U}}$ is a sheaf. $\mathbf{F}$ is a *stack* (for τ) if it is a prestack and moreover, for every $U \in \mathbf{C}$ and every $\mathcal{U} \in \mathrm{Cov}(U)$, every descent datum relative to $\mathcal{U}$ is effective.

Remark 6.4. By Remark 5.65 the definition of prestack is independent of the choice of the clivage, and it is easy to see that the same is true for the definition of stack.

Remark 6.5. A (pre)stack $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ is called a (pre)stack of groupoids, whereas an arbitrary (pre)stack is also called a (pre)stack of categories. Sometimes in the literature the word (pre)stack is reserved to (pre)stacks of groupoids.

Example 6.6. The fibred category $\mathbf{Mor}(\mathbf{Sch}_{/S}) \in \underline{\mathbf{Fib}}_{\mathbf{Sch}_{/S}}$ (S a scheme) defined in Example 5.9 is a stack for *Zar*: the fact that morphisms of schemes can be glued clearly implies that it is a prestack, and using the fact that also schemes can be glued, it is easy to see that it is actually a stack. Indeed, let $\mathcal{U} = \{U_i \hookrightarrow U\}_{i\in I} \in \mathrm{Cov}^{Zar}(U)$ (where we can always assume that each U_i is an open subscheme of U) and let $(\{p_i \colon X_i \to U_i\}, \{f_{i,j}\})_{i,j\in I}$ be a descent datum relative to $\mathcal{U}$ (the p_i are morphisms of $\mathbf{Sch}_{/S}$ and $f_{i,j} \colon V_{j,i} := p_j^{-1}(U_{i,j}) \xrightarrow{\sim} V_{i,j} := p_i^{-1}(U_{i,j})$ are isomorphisms of $\mathbf{Sch}_{/U_{i,j}}$); note that the cocycle condition applied when $i = j = k$ implies that $f_{i,i} = \mathrm{id}$ and then the same condition applied when $i = k$ implies that $f_{j,i} = f_{i,j}^{-1}$ for all $i, j \in I$. Therefore the schemes X_i can be glued along the open subschemes $V_{i,j}$ (for $i \neq j \in I$) using the isomorphisms $f_{i,j}$, and we get a scheme X together with an open cover $\{V_i\}_{i\in I}$ of X and isomorphisms $g_i \colon V_i \xrightarrow{\sim} X_i$ such that $g_i(V_i \cap V_j) = V_{i,j}$ and $f_{i,j} = g_i|_{V_i\cap V_j} \circ (g_j|_{V_i\cap V_j})^{-1}$ for all $i, j \in I$. The fact that the $f_{i,j}$ are morphisms of $\mathbf{Sch}_{/U_{i,j}}$ implies that the morphisms $p'_i := p_i \circ g_i \colon V_i \to U$ satisfy $p'_i|_{V_i\cap V_j} = p'_j|_{V_i\cap V_j}$ for all $i, j \in I$, hence there exists a unique $p \in \mathrm{Hom}_{\mathbf{Sch}_{/S}}(X, U)$ such that $p|_{V_i} = p'_i$ for every $i \in I$. Then the object $p \colon X \to U$ together with the isomorphisms g_i shows that the descent datum is effective.

We will see later that $\mathbf{Mor}(\mathbf{Sch}_{/S})$ is a prestack (but not a stack) also for *fppf*.

Example 6.7. Similarly, using the fact that sheaves (and morphisms of sheaves) can be glued (see [11, II, exer. 1.22]), it is easy to prove that the fibred categories $\overline{\mathbf{Mod}}, \overline{\mathbf{QCoh}} \in \underline{\mathbf{Fib}}_{\mathbf{Sch}_{/S}}$ defined in Example 5.35 are stacks for *Zar*. We will see later that $\overline{\mathbf{QCoh}}$ is a stack also for *fppf*.

We will denote by $\underline{\mathbf{PSt}}_{(\mathbf{C},\tau)}$ (respectively $\underline{\mathbf{St}}_{(\mathbf{C},\tau)}$), or simply by $\underline{\mathbf{PSt}}_{\mathbf{C}}$ (respectively $\underline{\mathbf{St}}_{\mathbf{C}}$) the full 2-subcategory of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ whose objects are prestacks (respectively stacks) for τ. Similarly, $\underline{\mathbf{Pst}}_{\mathbf{C}}^{gpd}$ (respectively $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$) will be the full 2-subcategory of $\underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ whose objects are prestacks (respectively stacks).

We are going to see that prestacks and stacks are the natural generalizations to fibred categories of the notions of separated presheaf and of sheaf on a site.[12] In order to do that, it is useful to give an alternative characterization of (pre)stacks, which generalizes that of Remark 4.31 for separated presheaves and sheaves. Given

[12]We are using the terminology which is more common in the literature. However, it must be said that (more coherently) some authors use the term prestack for fibred category and call separated prestack what we have defined to be a prestack.

$\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, we define a category $\mathbf{F}_{\mathcal{U}}$ (called the *category of descent data* in $\mathbf{F}$ relative to $\mathcal{U}$), whose objects are descent data relative to $\mathcal{U}$ and whose morphisms are defined as follows. A morphism in $\mathbf{F}_{\mathcal{U}}$ from $(\{\xi_i\},\{\phi_{i,j}\})_{i,j\in I}$ to $(\{\xi'_i\},\{\phi'_{i,j}\})_{i,j\in I}$ is given by a collection of morphisms $\psi_i \in \mathrm{Hom}_{\mathbf{F}_{U_i}}(\xi_i,\xi'_i)$ (for $i\in I$) such that the diagram

$$\begin{tikzcd} \xi_j|_{U_{i,j}} \arrow[r, "\phi_{i,j}"] \arrow[d, "\psi_j|_{U_{i,j}}"'] & \xi_i|_{U_{i,j}} \arrow[d, "\psi_i|_{U_{i,j}}"] \\ \xi'_j|_{U_{i,j}} \arrow[r, "\phi'_{i,j}"'] & \xi'_i|_{U_{i,j}} \end{tikzcd}$$

commutes for all $i,j\in I$. There is a natural functor $\lambda_{\mathcal{U}} = \lambda^{\mathbf{F}}_{\mathcal{U}} \colon \mathbf{F}_U \to \mathbf{F}_{\mathcal{U}}$, defined on objects by $\xi \mapsto (\{\xi|_{U_i}\},\{\theta_{i,j}\})_{i,j\in I}$ (where $\theta_{i,j}$ denotes the natural isomorphism $(\xi|_{U_j})|_{U_{i,j}} \cong \xi|_{U_{i,j}} \cong (\xi|_{U_i})|_{U_{i,j}}$) and on morphisms by $\psi \mapsto \{\psi|_{U_i}\}_{i\in I}$.

Remark 6.8. If $\mathbf{F}$ is fibred in groupoids (respectively in equivalence relations, respectively in sets), then each category $\mathbf{F}_{\mathcal{U}}$ is a groupoid (respectively an equivalence relation, respectively a set). In particular, if $F \in \widehat{\mathbf{C}} \subset \underline{\mathbf{Fib}}_{\mathbf{C}}$, it is clear that $F_{\mathcal{U}}$ can be identified with the set $F(\mathcal{U})$ defined in Remark 4.31, and that in this way the new definition of $\lambda^F_{\mathcal{U}}$ coincides with the old one.

Proposition 6.9. $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ *is a prestack (respectively a stack) if and only if the functor* $\lambda^{\mathbf{F}}_{\mathcal{U}} \colon \mathbf{F}_U \to \mathbf{F}_{\mathcal{U}}$ *is fully faithful (respectively an equivalence of categories) for every* $U \in \mathbf{C}$ *and every* $\mathcal{U} \in \mathrm{Cov}(U)$. *In particular, a presheaf* $F \in \widehat{\mathbf{C}}$ *is a prestack (respectively a stack) if and only if it is separated (respectively a sheaf).*

Proof. As it is obvious by definition that $\lambda_{\mathcal{U}}$ is essentially surjective if and only if every descent datum relative to $\mathcal{U}$ is effective, it is clear that it is enough to prove that $\lambda_{\mathcal{U}}$ is fully faithful if and only if the natural sequence of sets

$$\mathrm{Hom}_{\mathbf{F}_U}(\xi,\eta) \longrightarrow H \underset{\widetilde{pr}_2}{\overset{\widetilde{pr}_1}{\rightrightarrows}} K$$

(where $H := \prod_{i\in I} \mathrm{Hom}_{\mathbf{F}_{U_i}}(\xi|_{U_i},\eta|_{U_i})$ and $K := \prod_{j,k\in I} \mathrm{Hom}_{\mathbf{F}_{U_{j,k}}}(\xi|_{U_{j,k}},\eta|_{U_{j,k}})$) is exact for every $\xi,\eta \in \mathbf{F}_U$. Indeed, it is easy to see that $\mathrm{Hom}_{\mathbf{F}_{\mathcal{U}}}(\lambda_{\mathcal{U}}(\xi),\lambda_{\mathcal{U}}(\eta))$, as a subset of H, coincides with $\ker(H \underset{\widetilde{pr}_2}{\overset{\widetilde{pr}_1}{\rightrightarrows}} K)$.

Taking into account Remark 6.8, the last statement follows from the fact that a map between two sets is a fully faithful functor (respectively an equivalence) if and only if it is injective (respectively bijective). □

As in the case of presheaves, we are going to see that the functors $\lambda_{\mathcal{U}} = \lambda^{\mathbf{F}}_{\mathcal{U}} \colon \mathbf{F}_U \to \mathbf{F}_{\mathcal{U}}$ admit generalizations to functors $\lambda_{\mathcal{U}',\mathcal{U}} = \lambda^{\mathbf{F}}_{\mathcal{U}',\mathcal{U}} \colon \mathbf{F}_{\mathcal{U}} \to \mathbf{F}_{\mathcal{U}'}$ for

$\mathcal{U} \le \mathcal{U}' \in \operatorname{Cov}(U)$. Indeed, if $\mathcal{U} = \{f_i \colon U_i \to U\}_{i \in I}$, $\mathcal{U}' = \{f'_{i'} \colon U'_{i'} \to U\}_{i' \in I'}$ and $g_{i'} \colon U'_{i'} \to U_{i(i')}$ ($i' \in I'$) are such that $f'_{i'} = f_{i(i')} \circ g_{i'}$, we can define a functor $\lambda_{\mathcal{U}',\mathcal{U}}(\{g_{i'}\}) \colon \mathbf{F}_{\mathcal{U}} \to \mathbf{F}_{\mathcal{U}'}$ by

$$(\{\xi_i\}, \{\phi_{i,j}\})_{i,j \in I} \mapsto (\{g_{i'}^*(\xi_{i(i')})\}, \{\phi_{i(i'),i(j')}|_{U_{i',j'}}\})_{i',j' \in I'}$$

on objects (and in the obvious way on morphisms). It is easy to see that if $\tilde{g}_{i'} \colon U'_{i'} \to U_{\tilde{i}(i')}$ ($i' \in I'$) are other morphisms such that $f'_{i'} = f_{\tilde{i}(i')} \circ \tilde{g}_{i'}$, then the functors $\lambda_{\mathcal{U}',\mathcal{U}}(\{g_{i'}\})$ and $\lambda_{\mathcal{U}',\mathcal{U}}(\{\tilde{g}_{i'}\})$ are naturally isomorphic, so that a functor $\lambda_{\mathcal{U}',\mathcal{U}} \colon \mathbf{F}_{\mathcal{U}} \to \mathbf{F}_{\mathcal{U}'}$ is well defined up to isomorphism. It is then also clear that there are natural isomorphisms $\lambda_{\mathcal{U}'',\mathcal{U}} \cong \lambda_{\mathcal{U}'',\mathcal{U}'} \circ \lambda_{\mathcal{U}',\mathcal{U}}$ if $\mathcal{U} \le \mathcal{U}' \le \mathcal{U}''$ in $\operatorname{Cov}(U)$ and $\lambda_{\mathcal{U},\mathcal{U}} \cong \mathrm{id}_{\mathbf{F}_{\mathcal{U}}}$ and that, under the natural equivalence between $\mathbf{F}_U$ and $\mathbf{F}_{\{\mathrm{id}_U\}}$, $\lambda_{\mathcal{U},\{\mathrm{id}_U\}}$ coincides with $\lambda_{\mathcal{U}}$. It is not difficult to prove the following results.

Lemma 6.10. *If $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is a prestack (respectively a stack), then for all $U \in \mathbf{C}$ and all $\mathcal{U},\mathcal{U}' \in \operatorname{Cov}(U)$ with $\mathcal{U} \le \mathcal{U}'$, the functor $\lambda_{\mathcal{U}',\mathcal{U}} \colon \mathbf{F}_{\mathcal{U}} \to \mathbf{F}_{\mathcal{U}'}$ is fully faithful (respectively an equivalence).*

Corollary 6.11. *Assume that (for every $U \in \mathbf{C}$) $\operatorname{Cov}'(U) \subseteq \operatorname{Cov}(U)$ is a subset such that for every $\mathcal{U} \in \operatorname{Cov}(U)$ there exists $\mathcal{U}' \in \operatorname{Cov}'(U)$ with $\mathcal{U} \le \mathcal{U}'$. If $\mathbf{F} \in \widehat{\mathbf{C}}$ is such that $\lambda_{\mathcal{U}'}^{\mathbf{F}}$ is fully faithful (respectively an equivalence) for every $U \in \mathbf{C}$ and every $\mathcal{U}' \in \operatorname{Cov}'(U)$, then $\mathbf{F}$ is a prestack (respectively a stack).*

Proof. Using Corollary 4.33 it is easy to see that the presheaf $\mathcal{H}om_U(\xi_1, \xi_2)$ is a sheaf for all $U \in \mathbf{C}$ and all $\xi_1, \xi_2 \in \mathbf{F}_U$ (i.e., $\mathbf{F}$ is a prestack). Moreover, given $\mathcal{U} \in \operatorname{Cov}(U)$, let $\mathcal{U}' \in \operatorname{Cov}'(U)$ be such that $\mathcal{U} \le \mathcal{U}'$; since $\lambda_{\mathcal{U}',\mathcal{U}}$ is fully faithful by Lemma 6.10, it is clear that $\lambda_{\mathcal{U}}$ is an equivalence if $\lambda_{\mathcal{U}'} \cong \lambda_{\mathcal{U}',\mathcal{U}} \circ \lambda_{\mathcal{U}}$ is an equivalence. □

Example 6.12. The same argument of Example 4.36 shows that $\underline{\mathbf{St}}_{(\mathbf{Sch},sm)} = \underline{\mathbf{St}}_{(\mathbf{Sch},\acute{e}t)}$.

Lemma 6.13. *A morphism $f \colon U \to V$ of $\mathbf{C}$ induces functors $f_{\mathcal{V}}^* \colon \mathbf{F}_{\mathcal{V}} \to \mathbf{F}_{f^*\mathcal{V}}$ for every $\mathcal{V} \in \operatorname{Cov}(V)$ such that there are natural isomorphisms $\lambda_{f^*\mathcal{V}',f^*\mathcal{V}} \circ f_{\mathcal{V}}^* \cong f_{\mathcal{V}'}^* \circ \lambda_{\mathcal{V}',\mathcal{V}}$ if $\mathcal{V} \le \mathcal{V}' \in \operatorname{Cov}(V)$.*

Lemma 6.14. *Every morphism $P \colon \mathbf{F} \to \mathbf{F}'$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ induces functors $P_{\mathcal{U}} \colon \mathbf{F}_{\mathcal{U}} \to \mathbf{F}'_{\mathcal{U}}$ for every $U \in \mathbf{C}$ and every $\mathcal{U} \in \operatorname{Cov}(U)$ such that there are natural isomorphisms $\lambda_{\mathcal{U}',\mathcal{U}}^{\mathbf{F}'} \circ P_{\mathcal{U}} \cong P_{\mathcal{U}'} \circ \lambda_{\mathcal{U}',\mathcal{U}}^{\mathbf{F}}$ if $\mathcal{U} \le \mathcal{U}' \in \operatorname{Cov}(U)$. Moreover, if P is a monomorphism (respectively an epimorphism, respectively an isomorphism), then each $P_{\mathcal{U}}$ is fully faithful (respectively essentially surjective, respectively an equivalence).*

Corollary 6.15. *$\underline{\mathbf{PSt}}_{\mathbf{C}}$ and $\underline{\mathbf{St}}_{\mathbf{C}}$ are strictly full 2-subcategories of $\underline{\mathbf{Fib}}_{\mathbf{C}}$.*

Proof. If $P\colon \mathbf{F} \to \mathbf{F}'$ is an isomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, then for every $U \in \mathbf{C}$ and every $\mathcal{U} \in \mathrm{Cov}(U)$ the 2-commutative diagram of $\underline{\mathbf{Cat}}$

$$\begin{tikzcd}
\mathbf{F}_U \arrow[r, "\lambda_{\mathcal{U}}^{\mathbf{F}}"] \arrow[d, "P_U"'] & \mathbf{F}_{\mathcal{U}} \arrow[d, "P_{\mathcal{U}}"] \arrow[dl, phantom, "\swarrow"] \\
\mathbf{F}'_U \arrow[r, "\lambda_{\mathcal{U}}^{\mathbf{F}'}"'] & \mathbf{F}'_{\mathcal{U}}
\end{tikzcd}$$

(where P_U and $P_{\mathcal{U}}$ are equivalences) implies that $\lambda_{\mathcal{U}}^{\mathbf{F}}$ is fully faithful (respectively an equivalence) if and only if $\lambda_{\mathcal{U}}^{\mathbf{F}'}$ is fully faithful (respectively an equivalence). $\square$

Corollary 6.16. *If $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is a (pre)stack, then $\mathbf{F}^{\mathrm{cart}}$ is a (pre)stack (of groupoids).*

Proof. Just observe that the inclusion functor $\mathbf{F}^{\mathrm{cart}} \subseteq \mathbf{F}$ is a morphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ and that it induces an identification of $\mathbf{F}_U^{\mathrm{cart}}$ (respectively $\mathbf{F}_{\mathcal{U}}^{\mathrm{cart}}$) with the subcategory of $\mathbf{F}_U$ (respectively $\mathbf{F}_{\mathcal{U}}$) having the same objects and whose morphisms are the isomorphisms. $\square$

Remark 6.17. Of course, if $\mathbf{F}^{\mathrm{cart}}$ is a (pre)stack, then $\mathbf{F}$ need not be a (pre)stack. However, it is clear that if $\mathbf{F}$ is a prestack and $\mathbf{F}^{\mathrm{cart}}$ is a stack, then $\mathbf{F}$ is a stack, too.

Proposition 6.18. *If $\mathbf{F}^i \to \mathbf{F}$ (for $1 = 1, 2$) are morphisms in $\underline{\mathbf{PSt}}_{\mathbf{C}}$ (respectively in $\underline{\mathbf{St}}_{\mathbf{C}}$), then $\mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2 \in \underline{\mathbf{PSt}}_{\mathbf{C}}$ (respectively $\underline{\mathbf{St}}_{\mathbf{C}}$), too.*

Proof. Setting $\tilde{\mathbf{F}} := \mathbf{F}^1 \times_{\mathbf{F}} \mathbf{F}^2$, we have to prove that the functor $\lambda_{\mathcal{U}}^{\tilde{\mathbf{F}}}\colon \tilde{\mathbf{F}}_U \to \tilde{\mathbf{F}}_{\mathcal{U}}$ is fully faithful (respectively an equivalence) for every $U \in \mathbf{C}$ and every $\mathcal{U} \in \mathrm{Cov}(U)$. Now, by definition $\tilde{\mathbf{F}}_U$ is isomorphic to $\mathbf{F}_U^1 \times_{\mathbf{F}_U} \mathbf{F}_U^2$, and it is easy to see that $\tilde{\mathbf{F}}_{\mathcal{U}}$ is naturally equivalent to $\mathbf{F}_{\mathcal{U}}^1 \times_{\mathbf{F}_{\mathcal{U}}} \mathbf{F}_{\mathcal{U}}^2$. With these identification we can regard $\lambda_{\mathcal{U}}^{\tilde{\mathbf{F}}}$ as a functor $\mathbf{F}_U^1 \times_{\mathbf{F}_U} \mathbf{F}_U^2 \to \mathbf{F}_{\mathcal{U}}^1 \times_{\mathbf{F}_{\mathcal{U}}} \mathbf{F}_{\mathcal{U}}^2$ and in this way (pretending that the natural isomorphisms of Lemma 6.14 are all identities) it is defined on objects by $(\xi_1, \xi_2, \alpha) \mapsto (\lambda_{\mathcal{U}}^{\mathbf{F}^1}(\xi_1), \lambda_{\mathcal{U}}^{\mathbf{F}^2}(\xi_2), \lambda_{\mathcal{U}}^{\mathbf{F}}(\alpha))$ (and similarly on morphisms). Then it is clear from the definition of fibred product (in $\underline{\mathbf{Cat}}$) that $\lambda_{\mathcal{U}}^{\tilde{\mathbf{F}}}$ is fully faithful if so are $\lambda_{\mathcal{U}}^{\mathbf{F}^1}$, $\lambda_{\mathcal{U}}^{\mathbf{F}^2}$ and $\lambda_{\mathcal{U}}^{\mathbf{F}}$. If moreover $\lambda_{\mathcal{U}}^{\mathbf{F}^1}$ and $\lambda_{\mathcal{U}}^{\mathbf{F}^2}$ are essentially surjective, then the same is true for $\lambda_{\mathcal{U}}^{\tilde{\mathbf{F}}}$ (even if $\lambda_{\mathcal{U}}^{\mathbf{F}}$ is not): given $(\xi'_1, \xi'_2, \alpha') \in \mathbf{F}_{\mathcal{U}}^1 \times_{\mathbf{F}_{\mathcal{U}}} \mathbf{F}_{\mathcal{U}}^2$ there exist (for $i = 1, 2$) $\xi_i \in \mathbf{F}_U^i$ such that $\xi'_i \cong \lambda_{\mathcal{U}}^{\mathbf{F}^i}(\xi_i)$ and the fact that $\lambda_{\mathcal{U}}^{\mathbf{F}}$ is fully faithful implies that there exists a unique α such that $(\xi_1, \xi_2, \alpha) \in \mathbf{F}_U^1 \times_{\mathbf{F}_U} \mathbf{F}_U^2$ and $(\xi'_1, \xi'_2, \alpha') \cong \lambda_{\mathcal{U}}^{\tilde{\mathbf{F}}}(\xi_1, \xi_2, \alpha)$. $\square$

We are going to extend from sheaves to stacks also the results of Proposition 4.41 and Proposition 4.44.

Proposition 6.19. *Let $F\colon \mathbf{C}' \to \mathbf{C}$ be a functor as in Proposition 4.41. Then the 2-functor $F^*\colon \underline{\mathbf{Fib}}_{\mathbf{C}} \to \underline{\mathbf{Fib}}_{\mathbf{C}'}$ (defined in Proposition 5.39) restricts to 2-functors $F^*\colon \underline{\mathbf{PSt}}_{(\mathbf{C},\tau)} \to \underline{\mathbf{PSt}}_{(\mathbf{C}',F^*(\tau))}$ and $F^*\colon \underline{\mathbf{St}}_{(\mathbf{C},\tau)} \to \underline{\mathbf{St}}_{(\mathbf{C}',F^*(\tau))}$*

Proof. Just observe that, given $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, $U \in \mathbf{C}'$ and $\mathcal{U} = \{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}^{F^*(\tau)}(U)$ (hence, by definition, $F(\mathcal{U}) := \{F(f_i)\}_{i\in I} \in \mathrm{Cov}^{\tau}(F(U))$), $F^*(\mathbf{F})_U$ is naturally isomorphic to $\mathbf{F}_{F(U)}$ (by Lemma 5.38), and similarly it is easy to see that $F^*(\mathbf{F})_{\mathcal{U}}$ is naturally isomorphic to $\mathbf{F}_{F(\mathcal{U})}$ and that in this way $\lambda_{\mathcal{U}}^{F^*(\mathbf{F})}\colon F^*(\mathbf{F})_U \to F^*(\mathbf{F})_{\mathcal{U}}$ is identified with $\lambda^{\mathbf{F}}_{F(\mathcal{U})}\colon \mathbf{F}_{F(U)} \to \mathbf{F}_{F(\mathcal{U})}$. □

Proposition 6.20. *Let $\mathbf{C}' \subseteq \mathbf{C}$ be the inclusion of a full subcategory with the property that, if $V \to U$ and $W \to U$ are morphisms of $\mathbf{C}'$ such that $V \times_U W$ exists in $\mathbf{C}$, then $V \times_U W$ is isomorphic to an object of $\mathbf{C}'$, and assume that τ satisfies the following property: for every $U \in \mathbf{C}$ there exists a covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ such that $U_i \in \mathbf{C}'$ for every $i \in I$. Then the natural restriction (strict) 2-functor $\underline{\mathbf{St}}_{(\mathbf{C},\tau)} \to \underline{\mathbf{St}}_{(\mathbf{C}',\tau)}$ is a lax 2-equivalence.*

Proof. We give just a sketch of the proof, which is similar (but with more technical complications) to that of Proposition 4.44. With the same notation used there, we will write $-|_{\mathbf{C}'}$ for the restriction 2-functor and we will denote (for $U \in \mathbf{C}$) by $\mathrm{Cov}'(U)$ the subset of $\mathrm{Cov}(U)$ given by those coverings $\{U_i \to U\}_{i\in I}$ such that $U_i \in \mathbf{C}'$ for every $i \in I$.

$\underline{\mathbf{St}}_{\mathbf{C}} \to \underline{\mathbf{St}}_{\mathbf{C}'}$ is 2-fully faithful: given morphisms $P, Q\colon \mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{St}}_{\mathbf{C}}$ and a 2-morphism $\alpha'\colon P|_{\mathbf{C}'} \to Q|_{\mathbf{C}'}$ of $\underline{\mathbf{St}}_{\mathbf{C}'}$, we have to show that there exists a unique 2-morphism $\alpha\colon P \to Q$ such that $\alpha' = \alpha|_{\mathbf{C}'}$. Now, for every $U \in \mathbf{C}$ let's choose $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$. Setting (for every $\xi \in \mathbf{F}_U$) $\xi_i := \xi|_{U_i}$, it is easy to see that the morphisms (of $\mathbf{G}_{U_i}$) $\alpha'(\xi_i)\colon P(\xi_i) \cong P(\xi)|_{U_i} \to Q(\xi)|_{U_i} \cong Q(\xi_i)$ are such that $\alpha'(\xi_i)|_{U_{i,j}} = \alpha'(\xi_j)|_{U_{i,j}}$ for all $i, j \in I$. Therefore (since $\mathcal{H}om_U(P(\xi), Q(\xi))$ is a sheaf because $\mathbf{G}$ is a prestack) there exists a unique $\alpha(\xi) \in \mathrm{Hom}_{\mathbf{G}_U}(P(\xi), Q(\xi))$ such that $\alpha(\xi)|_{U_i} = \alpha'(\xi_i)$ for every $i \in I$. Then one can check without difficulties that $\alpha(\xi)$ is well defined and that $\alpha\colon P \to Q$ is a 2-morphism (and obviously it is the unique such that $\alpha' = \alpha|_{\mathbf{C}'}$).

$\underline{\mathbf{St}}_{\mathbf{C}} \to \underline{\mathbf{St}}_{\mathbf{C}'}$ is essentially full: given $\mathbf{F}, \mathbf{G} \in \underline{\mathbf{St}}_{\mathbf{C}}$ and $P'\colon \mathbf{F}|_{\mathbf{C}'} \to \mathbf{G}|_{\mathbf{C}'}$ in $\underline{\mathbf{St}}_{\mathbf{C}'}$, we have to prove that there exists $P\colon \mathbf{F} \to \mathbf{G}$ such that $P|_{\mathbf{C}'} \cong P'$ in $\mathbf{Hom}_{\underline{\mathbf{St}}_{\mathbf{C}'}}(\mathbf{F}|_{\mathbf{C}'}, \mathbf{G}|_{\mathbf{C}'})$. Choosing again (for $U \in \mathbf{C}$) a covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$, we define (for $\xi \in \mathbf{F}_U$) $P(\xi) \in \mathbf{G}_U$ as follows. Setting $\eta_i := P'(\xi|_{U_i}) \in \mathbf{G}_{U_i}$, we claim that there exists a natural isomorphism $\psi_{i,j}\colon \eta_j|_{U_{i,j}} \to \eta_i|_{U_{i,j}}$ for all $i, j \in I$: indeed, if $\{V_k \to U_{i,j}\}_{i,j\in I} \in \mathrm{Cov}'(U_{i,j})$, the fact that $\mathbf{G}^{\mathrm{cart}}$ is a prestack implies that there exists a unique $\psi_{i,j} \in \mathrm{Isom}_{\mathbf{G}_{U_{i,j}}}(\eta_j|_{U_{i,j}}, \eta_i|_{U_{i,j}})$ such that $\psi_{i,j}|_{V_k}$ is the natural isomorphism $\eta_j|_{V_k} \cong P'(\xi|_{V_k}) \cong \eta_i|_{V_k}$. Then it should be clear that $(\{\eta_i\}, \{\psi_{i,j}\})_{i,j\in I}$ is a descent datum in $\mathbf{G}$ relative to $\mathcal{U}$, so that (since $\mathbf{G}$ is a stack) there exists (unique up to isomorphism) $\eta \in \mathbf{G}_U$ such that $(\{\eta_i\}, \{\psi_{i,j}\})_{i,j\in I} \cong \lambda^{\mathbf{G}}_{\mathcal{U}}(\eta)$. Then one can check that there is a natural way to define a functor $P\colon \mathbf{F} \to \mathbf{G}$ which is given on objects by $P(\xi) := \eta$, and that P is a morphism of $\underline{\mathbf{St}}_{\mathbf{C}}$ such that $P|_{\mathbf{C}'} \cong P'$.

$\underline{\mathbf{St}}_{\mathbf{C}} \to \underline{\mathbf{St}}_{\mathbf{C}'}$ is essentially surjective: given $\mathbf{F}' \in \underline{\mathbf{St}}_{\mathbf{C}'}$, we have to find $\mathbf{F} \in \underline{\mathbf{St}}_{\mathbf{C}}$ such that $\mathbf{F}|_{\mathbf{C}'} \cong \mathbf{F}'$. First, for every $U \in \mathbf{C}$ and every $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}'(U)$ we can define a category $\mathbf{F}'_{\mathcal{U}}$ as follows. Given objects $\xi_i \in \mathbf{F}'_{U_i}$, for all $i, j \in I$ the presheaf $G'_{i,j} \in \widehat{\mathbf{C}'_{/U_{i,j}}}$, defined on objects by $G'_{i,j}(V \to U_{i,j}) := \mathrm{Isom}_{\mathbf{F}'_V}(\xi_j|_V, \xi_i|_V)$ is a sheaf (this follows easily from the fact that $\mathbf{F}'$ is a prestack), whence (by Proposition 4.44) there exists (unique up to isomorphism) $G_{i,j} \in (\mathbf{C}'_{/U_{i,j}})^{\sim}$ such that $G'_{i,j} \cong G_{i,j}|_{\mathbf{C}'}$, and we define $J_{\xi_j,\xi_i} := G_{i,j}(\mathrm{id}_{U_{i,j}})$ (notice that, in case $U_{i,j} \in \mathbf{C}'$, this set is bijective to $\mathrm{Isom}_{\mathbf{F}'_{U_{i,j}}}(\xi_j|_{U_{i,j}}, \xi_i|_{U_{i,j}})$). Then the objects of $\mathbf{F}'_{\mathcal{U}}$ are collections of the form $(\{\xi_i\}, \{\phi_{i,j}\})_{i,j\in I}$ where $\xi_i \in \mathbf{F}'_{U_i}$, $\phi_{i,j} \in J_{\xi_j,\xi_i}$ and the $\phi_{i,j}$ satisfy the cocycle condition "$\phi_{i,k}|_{U_{i,j,k}} = \phi_{i,j}|_{U_{i,j,k}} \circ \phi_{j,k}|_{U_{i,j,k}}$" (it should be clear how to give a precise meaning to this expression); morphisms in $\mathbf{F}'_{\mathcal{U}}$ from $(\{\xi_i\}, \{\phi_{i,j}\})_{i,j\in I}$ to $(\{\xi'_i\}, \{\phi'_{i,j}\})_{i,j\in I}$ are of course given by collections of morphisms $\psi_i \in \mathrm{Hom}_{\mathbf{F}'_{U_i}}(\xi_i, \xi'_i)$ which are compatible (in the obvious sense) with the $\phi_{i,j}$ and $\phi'_{i,j}$. One can prove that, up to equivalence, the category $\mathbf{F}'_{\mathcal{U}}$ is independent of the choice of $\mathcal{U} \in \mathrm{Cov}'(U)$ and that the map $U \mapsto \mathbf{F}'_{\mathcal{U}}$ can be extended to a lax 2-functor $\mathbf{C}^\circ \to \underline{\mathbf{Cat}}$, whose corresponding fibred category $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is really a stack such that $\mathbf{F}|_{\mathbf{C}'} \cong \mathbf{F}'$. □

Corollary 6.21. *Let S be a scheme and let τ be one of the pretopologies Zar, ét, sm or fppf on $\mathbf{Sch}_{/S}$. Then the natural 2-functors*

$$\underline{\mathbf{St}}_{(\mathbf{Sch}_{/S},\tau)} \to \underline{\mathbf{St}}_{(\mathbf{QSch}_{/S},\tau)} \to \underline{\mathbf{St}}_{(\mathbf{AffSch}_{/S},\tau)}$$

are lax 2-equivalences.

6.2 Stack associated to a fibred category

We are going to see that the construction of separated presheaf (respectively sheaf) associated to a presheaf described in Section 4.3 can be generalized to a construction of prestack (respectively stack) associated to a fibred category, with similar universal properties.

We fix a site $(\mathbf{C}, \tau)$. For every fibred category $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$, the *prestack associated to* $\mathbf{F}$, denoted by $\mathbf{F}^s = \mathbf{F}^{s_\tau} \in \underline{\mathbf{PSt}}_{\mathbf{C}} = \underline{\mathbf{PSt}}_{(\mathbf{C},\tau)}$, is defined as follows. For every $U \in \mathbf{C}$ the fibre $\mathbf{F}^s_U$ has $\mathrm{Ob}(\mathbf{F}^s_U) := \mathrm{Ob}(\mathbf{F}_U)$ (so that $\mathrm{Ob}(\mathbf{F}^s) = \mathrm{Ob}(\mathbf{F})$) and for all $\xi_1, \xi_2 \in \mathbf{F}_U$

$$\mathrm{Hom}_{\mathbf{F}^s_U}(\xi_1, \xi_2) := \mathcal{H}om_U(\xi_1, \xi_2)^a(\mathrm{id}_U).$$

For every morphism $f\colon U \to V$ of $\mathbf{C}$ one can define a functor $f^*\colon \mathbf{F}^s_V \to \mathbf{F}^s_U$, which is the same as $f^*\colon \mathbf{F}_V \to \mathbf{F}_U$ on objects and which is given on morphism by the restriction map

$$\mathcal{H}om_V(\eta_1, \eta_2)^a(\mathrm{id}_V) \to \mathcal{H}om_V(\eta_1, \eta_2)^a(f) \cong \mathcal{H}om_U(f^*(\eta_1), f^*(\eta_2))^a(\mathrm{id}_U)$$

(the latter natural identification is clear from the definition of associated sheaf). It is then easy to see that the categories $\mathbf{F}^s_U$ together with the functors f^* naturally extend to a lax 2-functor $\mathbf{C}^\circ \to \underline{\mathbf{Cat}}$, and that the corresponding fibred category $\mathbf{F}^s$ is really a prestack. It is also clear that the natural functors $\mathbf{F}_U \to \mathbf{F}^s_U$ (given by the identity on objects and by the maps $\rho_{\mathcal{H}om_U(\xi_1,\xi_2)}(\mathrm{id}_U)$ on morphisms) determine a morphism $\sigma_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^s$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, and it is not difficult to prove the following result.

Proposition 6.22. *For every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ and every $\mathbf{G} \in \underline{\mathbf{PSt}}_{\mathbf{C}}$ the natural functor*

$$\circ\sigma_{\mathbf{F}} \colon \mathbf{Hom}_{\underline{\mathbf{PSt}}_{\mathbf{C}}}(\mathbf{F}^s, \mathbf{G}) = \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}^s, \mathbf{G}) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{G})$$

is an equivalence of categories. Moreover, for every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the morphism $\sigma_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^s$ is an epimorphism in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, and it is an isomorphism if and only if $\mathbf{F} \in \underline{\mathbf{PSt}}_{\mathbf{C}}$.

Similarly, for every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the *stack associated to* $\mathbf{F}$, denoted by $\mathbf{F}^a = \mathbf{F}^{a_\tau} \in \underline{\mathbf{St}}_{\mathbf{C}} = \underline{\mathbf{St}}_{(\mathbf{C},\tau)}$, is defined as follows. For every $U \in \mathbf{C}$ the fibre $\mathbf{F}^a_U$ has

$$\mathrm{Ob}(\mathbf{F}^a_U) := \{(\mathcal{U}, \xi) \,|\, \mathcal{U} \in \mathrm{Cov}(U), \xi \in \mathbf{F}^s_{\mathcal{U}}\}$$

and for all $(\mathcal{U}_1, \xi_1), (\mathcal{U}_2, \xi_2) \in \mathrm{Ob}(\mathbf{F}^a_U)$, choosing $\mathcal{U} \in \mathrm{Cov}(U)$ such that $\mathcal{U}_1, \mathcal{U}_2 \le \mathcal{U}$,

$$\mathrm{Hom}_{\mathbf{F}^a_U}((\mathcal{U}_1, \xi_1), (\mathcal{U}_2, \xi_2)) := \mathrm{Hom}_{\mathbf{F}^s_{\mathcal{U}}}(\lambda^{\mathbf{F}^s}_{\mathcal{U},\mathcal{U}_1}(\xi_1), \lambda^{\mathbf{F}^s}_{\mathcal{U},\mathcal{U}_2}(\xi_2))$$

(note that, by Lemma 6.10, this definition is independent, up to natural bijection, of the choice of the common refinement $\mathcal{U}$ of $\mathcal{U}_1$ and $\mathcal{U}_2$, so that, defining composition of morphisms in $\mathbf{F}^a_U$ in the obvious way, $\mathbf{F}^a_U$ is really a category). Again, it is easy to check (using Lemma 6.13) that every morphism $f \colon U \to V$ of $\mathbf{C}$ induces a functor $f^* \colon \mathbf{F}^a_V \to \mathbf{F}^a_U$ (defined on objects by $(\mathcal{V}, \eta) \mapsto (f^*\mathcal{V}, f^*_{\mathcal{V}}(\eta))$), and that the data of the categories $\mathbf{F}^a_U$ and of the functors f^* extend to a lax 2-functor $\mathbf{C}^\circ \to \underline{\mathbf{Cat}}$, such that the corresponding fibred category $\mathbf{F}^a$ is really a stack. It is evident by definition that $(\mathbf{F}^s)^a = \mathbf{F}^a$. It is also clear that the natural (fully faithful) functors $\mathbf{F}^s_U \cong \mathbf{F}^s_{\{\mathrm{id}_U\}} \to \mathbf{F}^a_U$ determine a monomorphism $\mathbf{F}^s \to \mathbf{F}^a$ in $\underline{\mathbf{Fib}}_{\mathbf{C}}$, which, composed with the epimorphism $\sigma_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^s$, yields a morphism $\rho_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^a$, and one can prove the following result.

Proposition 6.23. *For every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ and every $\mathbf{G} \in \underline{\mathbf{St}}_{\mathbf{C}}$ the natural functor*

$$\circ\rho_{\mathbf{F}} \colon \mathbf{Hom}_{\underline{\mathbf{St}}_{\mathbf{C}}}(\mathbf{F}^a, \mathbf{G}) = \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}^a, \mathbf{G}) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(\mathbf{F}, \mathbf{G})$$

is an equivalence of categories. Moreover, for every $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ the morphism $\rho_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^a$ of $\underline{\mathbf{Fib}}_{\mathbf{C}}$ is a monomorphism if and only if $\mathbf{F} \in \underline{\mathbf{PSt}}_{\mathbf{C}}$, and it is an isomorphism if and only if $\mathbf{F} \in \underline{\mathbf{St}}_{\mathbf{C}}$.

Remark 6.24. It follows easily from the definitions that if $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is fibred in groupoids (respectively in equivalence relations), then $\mathbf{F}^s$ and $\mathbf{F}^a$ are fibred in groupoids (respectively in equivalence relations), too. On the other hand, if $\mathbf{F}$ is fibred in sets (say $\mathbf{F}$ corresponds to $F \in \widehat{\mathbf{C}}$), then $\mathbf{F}^s$ and $\mathbf{F}^a$ need not be fibred in sets; however, by Proposition 6.22 and Proposition 6.23 (taking into account Proposition 6.9), there are induced morphisms $\mathbf{F}^s \to F^s$ and $\mathbf{F}^a \to F^a$, which are immediately seen to be isomorphisms of $\underline{\mathbf{Fib}}_{\mathbf{C}}$.

Lemma 6.25. *Let $P \colon \mathbf{F} \to \mathbf{G}$ be a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$. If $\mathbf{G}$ is a prestack, then $\mathbf{F}$ is a prestack, too. If $\mathbf{G}$ is a stack, then an induced morphism $P' \colon \mathbf{F}^a \to \mathbf{G}$ such that $P' \circ \rho_{\mathbf{F}} \cong P$ (by Proposition 6.23 P' exists unique up to 2-isomorphism) is also a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$.*

Proof. Straightforward from the definitions. □

In particular, if $P \colon \mathbf{F} \to \mathbf{G}$ is a morphism of $\underline{\mathbf{St}}_{\mathbf{C}}$, a morphism $Q \colon (\operatorname{im} P)^a \to \mathbf{G}$ (induced by the natural monomorphism $\operatorname{im} P \subseteq \mathbf{G}$) is a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$, so that $\widetilde{\operatorname{im}} P := \operatorname{im} Q \subseteq \mathbf{G}$ is a stack (it is isomorphic to $(\operatorname{im} P)^a$ by Remark 5.30) which will be called the *image* of P in $\underline{\mathbf{St}}_{\mathbf{C}}$. It is clear that for every $U \in \mathbf{C}$ an object $\xi \in \mathbf{G}_U$ is in $(\widetilde{\operatorname{im}} P)_U$ if and only if there exists a covering $\{U_i \to U\}_{i \in I} \in \operatorname{Cov}(U)$ such that $\xi|_{U_i} \in (\operatorname{im} P)_{U_i}$ for every $i \in I$.

Definition 6.26. A morphism $P \colon \mathbf{F} \to \mathbf{F}'$ of $\underline{\mathbf{St}}_{\mathbf{C}}$ is a *monomorphism* (respectively an *isomorphism*) in $\underline{\mathbf{St}}_{\mathbf{C}}$ if and only if it is a monomorphism (respectively an isomorphism) in $\underline{\mathbf{Fib}}_{\mathbf{C}}$. P is an *epimorphism* in $\underline{\mathbf{St}}_{\mathbf{C}}$ if and only if $\widetilde{\operatorname{im}} P = \mathbf{F}'$.

Lemma 6.27. *A morphism of $\underline{\mathbf{St}}_{\mathbf{C}}$ is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Proof. Completely similar to that of Corollary 4.58. □

Remark 6.28. Every morphism $P \colon \mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{St}}_{\mathbf{C}}$ factors as the composition of the epimorphism (of $\underline{\mathbf{St}}_{\mathbf{C}}$) $\mathbf{F} \to \widetilde{\operatorname{im}} P$ and of the monomorphism $\widetilde{\operatorname{im}} P \subseteq \mathbf{G}$.

Lemma 6.29. *Every representable covering morphism of $\underline{\mathbf{St}}_{\mathbf{C}}$ is an epimorphism.*

Proof. Similar to that of Corollary 4.58, replacing the cartesian diagram by a 2-cartesian diagram and equalities by isomorphisms. □

6.3 The stack of sheaves

Let $(\mathbf{C}, \tau)$ be a site. We will denote by $\mathbf{Sh}(\mathbf{C}, \tau)$ (or simply by $\mathbf{Sh}(\mathbf{C})$) the full subcategory of $\mathbf{PSh}(\mathbf{C})$ (the category of presheaves over $\mathbf{C}$ defined in Example 5.10) whose objects are those morphisms $\alpha \colon F \to U$ of $\widehat{\mathbf{C}}$ (with $U \in \mathbf{C}$) such that the presheaf $G_\alpha \in \widehat{\mathbf{C}_{/U}}$ corresponding to $\alpha \in \widehat{\mathbf{C}}_{/U}$ (under the equivalence of Lemma 4.20) is a sheaf (for τ). We claim that $\mathbf{Sh}(\mathbf{C})$ is a fibred category over $\mathbf{C}$

(which we will call the *fibred category of sheaves over* $\mathbf{C}$, since each fibre $\mathbf{Sh}(\mathbf{C})_U$ is naturally equivalent to $(\mathbf{C}_{/U})^{\sim}$). Indeed, given a cartesian diagram in $\widehat{\mathbf{C}}$

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "\alpha'"'] & F \arrow[d, "\alpha"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

(with f in $\mathbf{C}$), and denoting by $G_\alpha \in \widehat{\mathbf{C}_{/U}}$ and $G_{\alpha'} \in \widehat{\mathbf{C}_{/U'}}$ the presheaves corresponding to α and α', it is easy to see that $G_{\alpha'}$ is defined on objects by $G_{\alpha'}(g) = G_\alpha(f \circ g)$ for every $g \in \mathbf{C}_{/U'}$ (and in the obvious way on morphisms). Then it is clear that $G_{\alpha'}$ is a sheaf if G_α is a sheaf, and this means that $\mathbf{Sh}(\mathbf{C})$ is a fibred category.

Remark 6.30. If the pretopology τ is subcanonical (i.e., if $\mathbf{C} \subseteq \mathbf{C}^{\sim}$), then (by Proposition 4.42) a morphism $F \to U$ of $\widehat{\mathbf{C}}$ (with $U \in \mathbf{C}$) is an object of $\mathbf{Sh}(\mathbf{C})$ if and only if $F \in \mathbf{C}^{\sim}$; in this case $\mathbf{Sh}(\mathbf{C})_U$ is just $\mathbf{C}^{\sim}_{/U}$.

Proposition 6.31. *For every site* $(\mathbf{C}, \tau)$ *the fibred category* $\mathbf{Sh}(\mathbf{C}, \tau)$ *is a stack for* τ *(the* stack of sheaves over $\mathbf{C}$*).*

Proof. We have to prove that for every covering $\mathcal{U} = \{p_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ the natural functor $\lambda_{\mathcal{U}} : \mathbf{Sh}(\mathbf{C})_U \to \mathbf{Sh}(\mathbf{C})_{\mathcal{U}}$ is an equivalence. We first show that $\lambda_{\mathcal{U}}$ is fully faithful: given two objects $\alpha : F \to U$ and $\alpha' : F' \to U$ of $\mathbf{Sh}(\mathbf{C})_U$ and denoting (for $i, j \in I$) by $\alpha_i : F_i := F \times_U U_i \to U_i$ and by $\alpha_{i,j} : F_{i,j} := F \times_U U_{i,j} \to U_{i,j}$ the morphisms induced by α (and defining similarly α'_i and $\alpha'_{i,j}$), we have to prove that given morphisms $\phi_i \in \mathrm{Hom}_{\mathbf{Sh}(\mathbf{C})_{U_i}}(\alpha_i, \alpha'_i) = \mathrm{Hom}_{\widehat{\mathbf{C}}_{/U_i}}(\alpha_i, \alpha'_i)$ such that $\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}} \in \mathrm{Hom}_{\widehat{\mathbf{C}}_{/U_{i,j}}}(\alpha_{i,j}, \alpha'_{i,j})$ for all $i, j \in I$, there exists unique $\phi \in \mathrm{Hom}_{\widehat{\mathbf{C}}_{/U}}(\alpha, \alpha')$ such that $\phi|_{U_i} = \phi_i$ for every $i \in I$. Let $G \in (\mathbf{C}_{/U})^{\sim}$, $G_i \in (\mathbf{C}_{/U_i})^{\sim}$ and $G_{i,j} \in (\mathbf{C}_{/U_{i,j}})^{\sim}$ be the sheaves corresponding to α, α_i and $\alpha_{i,j}$ (and define similarly G', G'_i and $G'_{i,j}$), and let $\varphi_i \in \mathrm{Hom}_{(\mathbf{C}_{/U_i})^{\sim}}(G_i, G'_i)$ be the morphisms corresponding to ϕ_i: then $\varphi_i|_{U_{i,j}} = \varphi_j|_{U_{i,j}} \in \mathrm{Hom}_{(\mathbf{C}_{/U_{i,j}})^{\sim}}(G_{i,j}, G'_{i,j})$ for all $i, j \in I$ and we have to prove that there exists a unique $\varphi \in \mathrm{Hom}_{(\mathbf{C}_{/U})^{\sim}}(G, G')$ such that $\varphi|_{U_i} = \varphi_i$ for every $i \in I$. Given an object $f : V \to U$ of $\mathbf{C}_{/U}$, consider the cartesian diagrams (for $i, j \in I$)

$$\begin{tikzcd} V_{i,j} \arrow[r] \arrow[d, "f_{i,j}"'] \arrow[dr, phantom, "\square"] & V_i \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] \\ U_{i,j} \arrow[r, "pr_1^{i,j}"'] & U_i \arrow[r, "p_i"'] & U \end{tikzcd} \tag{6.1}$$

and recall that $G_i(f_i) = G(p_i \circ f_i)$ and $G_{i,j}(f_{i,j}) = G(p_{i,j} \circ f_{i,j})$ (where $p_{i,j} := p_i \circ pr_1^{i,j}$), and similarly for G'. Given $\xi \in G(f)$, let $\xi_i := \xi|_{p_i \circ f_i} \in G(p_i \circ f_i) = G_i(f_i)$

and $\xi'_i := \varphi_i(f_i)(\xi_i) \in G'_i(f_i) = G'(p_i \circ f_i)$. As clearly $\xi'_i|_{p_{i,j}\circ f_{i,j}} = \xi'_j|_{p_{i,j}\circ f_{i,j}}$ for all $i, j \in I$, there exists a unique $\xi' \in G'(f)$ such that $\xi'_i = \xi'|_{p_i \circ f_i}$ for every $i \in I$ (because G' is a sheaf). It is then straightforward to see that, defining $\varphi(f)\colon G(f) \to G'(f)$ by $\xi \mapsto \xi'$, we obtain a morphism $\varphi \in \mathrm{Hom}_{(\mathbf{C}_{/U})^\sim}(G, G')$ which satisfies $\varphi|_{U_i} = \varphi_i$ for every $i \in I$, and that φ is unique with this property.

It remains to prove that $\lambda_{\mathcal{U}}$ is essentially surjective, i.e. that every descent datum relative to $\mathcal{U}$ is effective. Now, a descent datum relative to $\mathcal{U}$ is given by objects $\alpha_i\colon F_i \to U_i$ of $\mathbf{Sh}(\mathbf{C})_{U_i}$ together with isomorphisms $\phi_{i,j}\colon \alpha_j|_{U_{i,j}} \xrightarrow{\sim} \alpha_i|_{U_{i,j}}$ of $\mathbf{Sh}(\mathbf{C})_{U_{i,j}}$ satisfying the cocycle condition $\phi_{i,k}|_{U_{i,j,k}} = \phi_{i,j}|_{U_{i,j,k}} \circ \phi_{j,k}|_{U_{i,j,k}}$ (for $i, j, k \in I$). Denoting by $G_i \in (\mathbf{C}_{/U_i})^\sim$ the sheaves corresponding to α_i and by $\varphi_{i,j}\colon G_j|_{U_{i,j}} \xrightarrow{\sim} G_i|_{U_{i,j}}$ the isomorphisms corresponding to $\phi_{i,j}$, it is clear that also the $\varphi_{i,j}$ satisfy the cocyle condition and that, in order to prove that the descent datum is effective, we have to show that there exists a sheaf $G \in (\mathbf{C}_{/U})^\sim$ together with isomorphisms $\psi_i\colon G|_{U_i} \xrightarrow{\sim} G_i$ such that $\varphi_{i,j} = \psi_i|_{U_{i,j}} \circ (\psi_j|_{U_{i,j}})^{-1}$ for all $i, j \in I$. For every object object $f\colon V \to U$ of $\mathbf{C}_{/U}$, using the notation of the above diagram (6.1), we define

$$G(f) := \ker\Big(\prod_{i\in I} G_i(f_i) \underset{b_f}{\overset{a_f}{\rightrightarrows}} \prod_{j,k\in I} G_j(pr_1^{j,k} \circ f_{j,k})\Big),$$

where, for every $\xi = (\xi_i)_{i\in I} \in \prod_{i\in I} G_i(f_i)$,

$$a_f(\xi) := (\xi_j|_{pr_1^{j,k}\circ f_{j,k}})_{j,k\in I},$$
$$b_f(\xi) := (\varphi_{j,k}(f_{j,k})(\xi_k|_{pr_2^{j,k}\circ f_{j,k}}))_{j,k\in I}$$

(regarding $\varphi_{j,k}(f_{j,k})$ as a map $G_k(pr_2^{j,k} \circ f_{j,k}) \to G_j(pr_1^{j,k} \circ f_{j,k})$). If $f'\colon V' \to U$ is another object of $\mathbf{C}_{/U}$ and $g \in \mathrm{Hom}_{\mathbf{C}_{/U}}(f', f)$ (i.e., $g\colon V' \to V$ is such that $f \circ g = f'$), then, defining f'_i similarly to f_i and denoting by $g_i\colon V'_i \to V_i$ the morphisms induced by g, it is easy to see that the map

$$\prod_{i\in I} G_i(g_i)\colon \prod_{i\in I} G_i(f_i) \to \prod_{i\in I} G_i(f'_i)$$

restricts to a map $G(g)\colon G(f) \to G(f')$, and that in this way we obtain a presheaf $G \in \widehat{\mathbf{C}_{/U}}$. We claim that actually $G \in (\mathbf{C}_{/U})^\sim$. Indeed, given $\{V'_{i'} \to V\}_{i'\in I'} \in \mathrm{Cov}(V)$, let $f'_{i'}\colon V'_{i'} \to U$ and $f'_{i',j'}\colon V'_{i',j'} \to U$ be the compositions of f with the natural morphisms $V'_{i'} \to V$ and $V'_{i',j'} \to V$, and define (for $(i, j \in I$ and $i', j' \in I')$ $f'_{i',i}$ and $f'_{i',i,j}$ (respectively $f'_{i',j',i}$ and $f'_{i',j',i,j}$) from $f'_{i'}$ (respectively from $f'_{i',j'}$)

as f_i and $f_{i,j}$ are defined from f. Then in the natural commutative diagram

$$
\begin{tikzcd}
G(f) \arrow[r] \arrow[d] & \prod_{i\in I} G_i(f_i) \arrow[r, shift left, "a_f"] \arrow[r, shift right, "b_f"'] \arrow[d] & \prod_{j,k\in I} G_j(\bar{f}_{j,k}) \arrow[d] \\
\prod_{i'\in I'} G(f'_{i'}) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i'\in I',\, i\in I} G_i(f'_{i',i}) \arrow[r, shift left, "\prod_{i'\in I'} a_{f'_{i'}}"] \arrow[r, shift right, "\prod_{i'\in I'} b_{f'_{i'}}"'] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i'\in I',\, j,k\in I} G_j(\bar{f}'_{i',j,k}) \\
\prod_{j',k'\in I'} G(f'_{j',k'}) \arrow[r] & \prod_{j',k'\in I',\, i\in I} G_i(f'_{j',k',i}) &
\end{tikzcd}
$$

(where $\bar{f}_{j,k} := pr_1^{j,k} \circ f_{j,k}$ and $\bar{f}'_{i',j,k} := pr_1^{j,k} \circ f'_{i',j,k}$) the rows are exact by definition of G, the column in the middle is exact and the map on the right is injective (because each G_i is a sheaf), which implies that the column on the left is exact, too, so that G is a sheaf. To conclude, for every $l \in I$ we can define a natural morphism $\psi'_l \colon G_l \to G|_{U_l}$ in $(\mathbf{C}_{/U_l})^\sim$ as follows. Given an object $h \colon W \to U_l$ of $\mathbf{C}_{/U_l}$, let $h_i \colon W \times_{U_l} U_{l,i} \to U_{l,i}$ be the induced morphisms for $i \in I$. Then it is easy to see that the map

$$
\begin{aligned}
\psi'_l(h) \colon G_l(h) \to G|_{U_l}(h) = G(p_l \circ h) \subseteq \prod_{i\in I} G_i(pr_2^{l,i} \circ h_i) \\
\eta \mapsto (\varphi_{i,l}(h_i)(\eta|_{pr_1^{l,i} \circ h_i}))_{i\in I}
\end{aligned}
$$

is well defined (this follows from the fact that the $\varphi_{i,j}$ satisfy the cocycle condition) and bijective (because G_l is a sheaf) and that, setting $\psi_l := (\psi'_l)^{-1}$, we have $\varphi_{i,j} = \psi_i|_{U_{i,j}} \circ (\psi_j|_{U_{i,j}})^{-1}$ for all $i, j \in I$. $\square$

Definition 6.32. Let P be a property of morphisms of $\mathbf{C}$ which is stable under base change. Then P satisfies *descent* (respectively *effective descent*) for τ if the fibred category $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ (defined in Example 5.9) is a prestack (respectively a stack) for τ.

Corollary 6.33. *Assume that τ is a subcanonical pretopology on a category $\mathbf{C}$. Then a property P of morphisms of $\mathbf{C}$ which is stable under base change always satisfies descent for τ (in particular, $\mathbf{Mor}(\mathbf{C})$ is a prestack), and it satisfies effective descent for τ if and only if the following holds: if $\alpha \colon F \to U$ is a morphism of $\mathbf{C}^\sim$ with $U \in \mathbf{C}$ and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ is a covering such that for every $i \in I$ the sheaf $F_i := F \times_U U_i$ is representable and the induced morphism $\alpha_i \colon F_i \to U_i$ satisfies P, then F is representable and α satisfies P, too. Moreover, if P satisfies effective descent, then it is local on the codomain, and the viceversa holds if $\mathbf{Mor}(\mathbf{C})$ is a stack.*

Proof. The fibred category $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ is a prestack by Lemma 6.25 (observe that it is a full subcategory of the prestack $\mathbf{Sh}(\mathbf{C})$ by Remark 6.30). If the condition is satisfied then $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ is a stack: a descent datum $\tilde{\alpha} = (\{\alpha_i\}, \{\phi_{i,j}\})_{i,j\in I}$ in $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ relative to a covering $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ is effective as a descent datum in $\mathbf{Sh}(\mathbf{C})$ (because $\mathbf{Sh}(\mathbf{C})$ is a stack), so that there exists a morphism $\alpha\colon F \to U$ in $\mathbf{C}^{\sim}$ such that $\tilde{\alpha} \cong \lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}(\alpha)$; now, the hypothesis implies that α is isomorphic to an object of $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$, whence $\tilde{\alpha}$ is effective also in $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$. Assume conversely that $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ is a stack, let $\mathcal{U}$ be as above and let $\alpha\colon F \to U$ in $\mathbf{C}^{\sim}$ be such that F_i is representable and α_i satisfies P for every $i \in I$. Then $\lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}(\alpha) \in \mathbf{Sh}(\mathbf{C})_{\mathcal{U}}$ is isomorphic to an object $\tilde{\alpha} \in \mathbf{Mor}^{\mathsf{P}}(\mathbf{C})_{\mathcal{U}}$. Since $\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})$ is a stack, there exists $f \in \mathbf{Mor}^{\mathsf{P}}(\mathbf{C})_U$ (f is a morphism of $\mathbf{C}$ with target U and which satisfies P) such that $\tilde{\alpha} \cong \lambda_{\mathcal{U}}^{\mathbf{Mor}^{\mathsf{P}}(\mathbf{C})}(f) \cong \lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}(f)$. Therefore $\lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}(\alpha) \cong \lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}(f)$, which implies (as $\lambda_{\mathcal{U}}^{\mathbf{Sh}(\mathbf{C})}$ is an equivalence) that $\alpha \cong f$ in $\mathbf{Sh}(\mathbf{C})_U = \mathbf{C}^{\sim}_{/U}$, and this precisely says that F is representable and α satisfies P. The last statement is then clear. □

6.4 Stacks of groupoids and quotient stacks

We fix as usual a site $(\mathbf{C}, \tau)$.

Lemma 6.34. *Let $F_\bullet = (F_1 \overset{t}{\underset{s}{\rightrightarrows}} F_0)$ be a groupoid in $\widehat{\mathbf{C}}$ such that F_1 is a sheaf and F_0 is separated. Then $[F_\bullet]' \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ is a prestack.*

Proof. Given $U \in \mathbf{C}$, $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $\xi, \eta \in F_0(U) = \mathrm{Ob}([F_\bullet]'_U)$, we have to prove that the natural sequence of sets

$$\mathrm{Hom}_{[F_\bullet]'_U}(\xi,\eta) \longrightarrow H := \prod_{i\in I} \mathrm{Hom}_{[F_\bullet]'_{U_i}}(\xi|_{U_i}, \eta|_{U_i}) \overset{\widetilde{pr}_1}{\underset{\widetilde{pr}_2}{\rightrightarrows}} K$$

(where $K := \prod_{j,k\in I} \mathrm{Hom}_{[F_\bullet]'_{U_{j,k}}}(\xi|_{U_{j,k}}, \eta|_{U_{j,k}})$) is exact. If $\tilde{\phi} := (\phi_i)_{i\in I} \in H$ (by definition, $\phi_i \in F_1(U_i)$ is such that $s(U_i)(\phi_i) = \xi|_{U_i}$ and $t(U_i)(\phi_i) = \eta|_{U_i}$) is such that $\widetilde{pr}_1(\tilde{\phi}) = \widetilde{pr}_2(\tilde{\phi})$, then (since F_1 is a sheaf) there exists a unique $\phi \in F_1(U)$ such that $\phi|_{U_i} = \phi_i$ for every $i \in I$, and it remains to show that $\phi \in \mathrm{Hom}_{[F_\bullet]'_U}(\xi, \eta) \subseteq F_1(U)$ (i.e., that $s(U)(\phi) = \xi$ and $t(U)(\phi) = \eta$). As $s(U)(\phi)|_{U_i} = s(U_i)(\phi_i) = \xi|_{U_i} \in F_0(U_i)$ and $t(U)(\phi)|_{U_i} = t(U_i)(\phi_i) = \eta|_{U_i} \in F_0(U_i)$ for every $i \in I$, this follows from the fact that F_0 is separated. □

In particular, if $F_\bullet$ is a groupoid in $\mathbf{C}^{\sim}$, we will denote by $[F_\bullet]$ the associated stack $[F_\bullet]'^{a}$ (recall that, by Proposition 6.23, the natural morphism $\rho_{[F_\bullet]'}\colon [F_\bullet]' \to [F_\bullet]$ is a monomorphism of $\underline{\mathbf{Fib}}_{\mathbf{C}}$). We have then the following result, which generalizes Corollary 4.83.

Corollary 6.35. *Let $F_\bullet = (F_1 \overset{t}{\underset{s}{\rightrightarrows}} F_0)$ be a groupoid in $\mathbf{C}^{\sim}$. Then the natural*

2-commutative diagram in $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$

$$\begin{tikzcd} F_1 \arrow[r, "s"] \arrow[d, "t"'] & F_0 \arrow[d, "\pi"] \arrow[dl, phantom, "\tilde{\gamma}\ \swarrow"] \\ F_0 \arrow[r, "\pi"'] & {[F_\bullet]} \end{tikzcd}$$

(where, in the notation of (5.5)*,* $\pi := \rho_{[F_\bullet]'} \circ \pi'$ *and* $\tilde{\gamma} := \mathrm{id}_{\rho_{[F_\bullet]'}} \star \gamma$*) is* 2*-cartesian and* 2*-cocartesian in* $\underline{\mathbf{St}}_{\mathbf{C}}$*.*

Moreover, if $P\colon F_0 \to \mathbf{F}$ *is a morphism in* $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$ *with* $F_0 \in \mathbf{C}^\sim$*, then the induced morphism* $\tilde{P}\colon [F_\bullet] := [F_0 \times_{\mathbf{F}} F_0 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} F_0] \to \mathbf{F}$ *yields an isomorphism* $[F_\bullet] \cong \widetilde{\mathrm{im}}\, P$ *in* $\underline{\mathbf{St}}_{\mathbf{C}}$*; in particular,* $[F_\bullet] \cong \mathbf{F}$ *if and only if* P *is an epimorphism in* $\underline{\mathbf{St}}_{\mathbf{C}}$*.*

Proof. Taking into account Proposition 5.72, the diagram is 2-cocartesian by the property of associated stack, and it is 2-cartesian because $\rho_{[F_\bullet]'}$ is a monomorphism. The second statement can be proved similarly from Corollary 5.74. □

Now we want to give an alternative description of the stacks of the form $[F \times G \overset{pr_1}{\underset{\varrho}{\rightrightarrows}} F]$, where $\varrho\colon F \times G \to F$ is an action in $\mathbf{C}^\sim$. Before we do that, however, we need to recall the notion of torsor on a site (see Definition A.28 for the usual definition of torsor on a topological space).

In the following, given $F \in \mathbf{C}^\sim$, we will denote by $F|_U$ the object of $\widehat{\mathbf{C}_{/U}}$ corresponding to $pr_2\colon F \times U \to U$ under the natural equivalence between $\widehat{\mathbf{C}_{/U}}$ and $\widehat{\mathbf{C}}_{/U}$; recall that it is defined on object by $F|_U(V \to U) = F(V)$ and that $F|_U \in (\mathbf{C}_{/U})^\sim$ (see the discussion before Remark 6.30).

Definition 6.36. Let G be a group in $\mathbf{C}^\sim$. A *G-pseudo-torsor* is a sheaf $H \in \mathbf{C}^\sim$ together with an action (in $\mathbf{C}^\sim$) $\varrho_H\colon H \times G \to H$ such that for every $U \in \mathbf{C}$ either $H(U)$ is empty or $\varrho_H(U)$ is free and transitive.

A morphism between two G-pseudo-torsors H and H' is a G-equivariant morphism of sheaves $\alpha\colon H \to H'$ (i.e., $\alpha \in \mathrm{Hom}_{\mathbf{C}^\sim}(H, H')$ is such that $\alpha \circ \varrho_H = \varrho_{H'} \circ (\alpha \times \mathrm{id}_G)$); this clearly defines a category G-**PTor** of G-pseudo-torsors.

A G-pseudo-torsor is *trivial* if it is isomorphic (in G-**PTor**) to G with action given by multiplication $G \times G \to G$.

Note that, if H is a G-pseudo-torsor as above, then $H|_U$ is a $G|_U$-pseudo-torsor for every $U \in \mathbf{C}$ (with action $\varrho_H|_U$).

Definition 6.37. If $U \in \mathbf{C}$ and G is a group in $(\mathbf{C}_{/U})^\sim$, a G-pseudo-torsor H is a *G-torsor* if it is locally trivial, i.e. if there exists a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that $H|_{U_i}$ is a trivial $G|_{U_i}$-pseudo-torsor for every $i \in I$ (in particular, trivial pseudo-torsors are torsors). A morphism of G-torsors is just a morphism of G-pseudo-torsors; G-**Tor** will be the full subcategory of G-**PTor** whose objects are G-torsors.

Remark 6.38. It is not difficult to prove that in the definition of G-torsor the hypothesis that H is a G-pseudo-torsor could be replaced by the weaker assumption that H is just a sheaf with an action of G: namely, if (H, ϱ_H) is locally trivial (again, in the sense that there exists $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that $H|_{U_i}$ is $G|_{U_i}$-equivariantly isomorphic to $G|_{U_i}$ for every $i \in I$), then it is necessarily a G-pseudo-torsor (hence a G-torsor).

It is also easy to see that, if $(\mathbf{C}, \tau) = (\mathbf{Open}(X), \mathit{std})$ (X a topological space), then the above definition is equivalent to Definition A.28.

Lemma 6.39. *G-**Tor** is a groupoid for every $U \in \mathbf{C}$ and every group G in $(\mathbf{C}_{/U})^\sim$.*

Proof. It is clear by definition that, if $\alpha \in \mathrm{Hom}_{G\text{-}\mathbf{PTor}}(H, H')$, then $\alpha(f)$ is bijective for every $f \in \mathbf{C}_{/U}$ such that $H(f) \neq \emptyset$. It is thus enough to prove that if $H \in G$-**Tor** and $f\colon V \to U$ is such that $H'(f) \neq \emptyset$, then $H(f) \neq \emptyset$, too. Since H is a G-torsor, there exists $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that $H|_{U_i}$ is a trivial $G|_{U_i}$-torsor for every $i \in I$, and this implies that $f^*\mathcal{U} = \{V_i \xrightarrow{pr_i} V\}_{i \in I} \in \mathrm{Cov}(V)$ is such that $H(f_i) \neq \emptyset$ for every $i \in I$ (where $f_i := f \circ pr_i\colon V_i \to U$). Choose $\xi' \in H'(f)$ and set $\xi'_i := \xi'|_{f_i} \in H'(f_i)$: by what we said before it follows that there exists a unique $\xi_i \in H(f_i)$ such that $\xi'_i = \alpha(f_i)(\xi_i)$ for every $i \in I$. Since $\xi_i|_{f_{i,j}} = \xi_j|_{f_{i,j}}$ (where $f_{i,j} := f_i \times_f f_j\colon V_{i,j} \to U$) for all $i, j \in I$ (because $\alpha(f_{i,j})(\xi_i|_{f_{i,j}}) = \xi'|_{f_{i,j}} = \alpha(f_{i,j})(\xi_j|_{f_{i,j}})$) and H is a sheaf, there exists a unique $\xi \in H(f)$ such that $\xi_i = \xi|_{f_i}$ for every $i \in I$, and this shows that $H(f) \neq \emptyset$. □

Given an action $F \times G \to F$ in $\mathbf{C}^\sim$ we can now define a category fibred in groupoids $[F/G] \in \underline{\mathbf{Fib}}_{\mathbf{C}}^{gpd}$ as follows. For every $U \in \mathbf{C}$ the objects of the fibre $[F/G]_U$ are the pairs (H, φ) where H is a $G|_U$-torsor and $\varphi\colon H \to F|_U$ is a $G|_U$-equivariant morphism in $(\mathbf{C}_{/U})^\sim$, whereas

$$\mathrm{Hom}_{[F/G]_U}((H, \varphi), (H', \varphi')) := \{\alpha \in \mathrm{Hom}_{G|_U\text{-}\mathbf{Tor}}(H, H') \mid \varphi' \circ \alpha = \varphi\}.$$

Defining (for every morphism $f\colon U \to V$ of $\mathbf{C}$) the functor $f^*\colon [F/G]_V \to [F/G]_U$ in the obvious way, it is clear that we really obtain a fibred category $[F/G]$, and Lemma 6.39 implies that each fibre $[F/G]_U$ is a groupoid.

Proposition 6.40. *$[F/G] \in \underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$ (it is called the* quotient stack *of F by G) for every action $F \times G \to F$ in $\mathbf{C}^\sim$.*

Proof. Fix a covering $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$.

$[F/G]$ is a prestack: given two objects $(H, \varphi), (H', \varphi') \in [F/G]_U$ and morphisms $\alpha_i \in \mathrm{Hom}_{[F/G]_{U_i}}((H, \varphi)|_{U_i}, (H', \varphi')|_{U_i})$ such that $\alpha_i|_{U_{i,j}} = \alpha_j|_{U_{i,j}}$ for all $i, j \in I$, by Proposition 6.31 there exists a unique $\alpha\colon H \to H'$ in $(\mathbf{C}_{/U})^\sim$ such that $\alpha|_{U_i} = \alpha_i$ for all $i \in I$, and it is easy to see that α is $G|_U$-equivariant and $\varphi' \circ \alpha = \varphi$, so that actually $\alpha \in \mathrm{Hom}_{[F/G]_U}((H, \varphi), (H', \varphi'))$.

$[F/G]$ is a stack: given objects $(H_i, \varphi_i) \in [F/G]_{U_i}$ and isomorphisms

$$\alpha_{i,j} \colon (H_j, \varphi_j)|_{U_{i,j}} \xrightarrow{\sim} (H_i, \varphi_i)|_{U_{i,j}}$$

of $[F/G]_{U_{i,j}}$ such that $\alpha_{i,k}|_{U_{i,j,k}} = \alpha_{i,j}|_{U_{i,j,k}} \circ \alpha_{j,k}|_{U_{i,j,k}}$ for all $i, j, k \in I$, again by Proposition 6.31 there exist $H \in (\mathbf{C}_{/U})^\sim$ and isomorphisms $\beta_i \colon H|_{U_i} \xrightarrow{\sim} H_i$ of $(\mathbf{C}_{/U_i})^\sim$ such that $\alpha_{i,j} = \beta_i|_{U_{i,j}} \circ (\beta_j|_{U_{i,j}})^{-1}$ for all $i, j \in I$. Setting $\varrho_i := \beta_i^{-1} \circ \varrho_{H_i} \circ (\beta_i \times \mathrm{id}) \colon H|_{U_i} \times G|_{U_i} \to H|_{U_i}$, it is clear that $\varrho_i|_{U_{i,j}} = \varrho_j|_{U_{i,j}}$ for all $i, j \in I$, whence there exists a unique $\varrho_H \colon H \times G|_U \to H$ in $(\mathbf{C}_{/U})^\sim$ such that $\varrho_H|_{U_i} = \varrho_i$ for all $i \in I$, and it is easy to see that ϱ_H is an action which makes H into a $G|_U$-torsor (and obviously each β_i is $G|_{U_i}$-equivariant). Similarly, the morphisms $\varphi'_i := \varphi_i \circ \beta_i \colon H|_{U_i} \to F|_{U_i}$ satisfy $\varphi_i|_{U_{i,j}} = \varphi_j|_{U_{i,j}}$ for all $i, j \in I$, so that there exists a unique $\varphi \in \mathrm{Hom}_{(\mathbf{C}_{/U})^\sim}(H, F|_U)$ such that $\varphi|_{U_i} = \varphi'_i$ for all $i \in I$, and one can easily check that φ is $G|_U$-equivariant (hence $(H, \varphi) \in [F/G]_U$) and that $\beta_i \in \mathrm{Hom}_{[F/G]_{U_i}}((H, \varphi)|_{U_i}, (H_i, \varphi_i))$ for every $i \in I$. □

Example 6.41. When $F = *$ (the terminal object of $\widehat{\mathbf{C}}$, which is always a sheaf), the quotient stack $[*/G]$ is denoted by BG and is called the *classifying stack* of G (each fibre BG_U can be identified with $G|_U$-**Tor**).

We can now prove that for every action $\varrho \colon F \times G \to F$ in $\mathbf{C}^\sim$ the quotient stack $[F/G]$ is naturally isomorphic to $[F \times G \overset{pr_1}{\underset{\varrho}{\rightrightarrows}} F]$. Observe first that there is a natural morphism $Q \colon F \to [F/G]$ in $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$ such that $Q_U \colon F(U) \to [F/G]_U$ is given by $\xi \mapsto (G|_U, \varphi_\xi)$, where the $G|_U$-equivariant morphism $\varphi_\xi \colon G|_U \to F|_U$ is defined for every object $f \colon V \to U$ of $\mathbf{C}_{/U}$ by

$$\begin{aligned} \varphi_\xi(f) \colon G|_U(f) = G(V) &\to F(V) = F|_U(f). \\ g &\mapsto \varrho(V)(f^*(\xi), g) \end{aligned}$$

It is also clear that there is a natural 2-commutative diagram in $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$

$$\begin{tikzcd} F \times G \arrow[r, "\varrho"] \arrow[d, "pr_1"'] & F \arrow[d, "Q"] \arrow[dl, Rightarrow, "\beta"'] \\ F \arrow[r, "Q"'] & {[F/G]} \end{tikzcd} \tag{6.2}$$

where (for every $U \in \mathbf{C}$ and every $(\xi, g) \in (F \times G)(U)$) the morphism

$$\beta(\xi, g) \colon Q \circ \varrho(\xi, g) = (G|_U, \varphi_{\varrho(\xi,g)}) \to Q \circ pr_1(\xi, g) = (G|_U, \varphi_\xi)$$

of $[F/G]_U$ is given by the $G|_U$-equivariant morphism $G|_U \to G|_U$ induced by left multiplication by g (more precisely, for every object $f \colon V \to U$ of $\mathbf{C}_{/U}$ the map $G|_U(f) = G(V) \to G(V)$ is given by $h \mapsto f^*(g)h$).

Proposition 6.42. *The diagram* (6.2) *induces a morphism*

$$P\colon [F_\bullet] := [F \times G \overset{pr_1}{\underset{\varrho}{\rightrightarrows}} F] \to [F/G]$$

(such that $P \circ \pi \cong Q$, where $\pi\colon F \to [F_\bullet]$ is the natural morphism), which is an isomorphism in $\underline{\mathbf{St}}_{\mathbf{C}}^{gpd}$; therefore (6.2) *is* 2*-cartesian and* 2*-cocartesian in $\underline{\mathbf{St}}_{\mathbf{C}}$.*

Proof. The morphism P such that $P \circ \pi \cong Q$ exists by the first part of Corollary 6.35; similarly, there is an induced morphism $P'\colon [F_\bullet]' \to [F/G]$ such that $P' \circ \pi' \cong Q$ (and clearly $P' \cong P \circ \rho_{[F_\bullet]'}$), by Proposition 5.72. Taking into account that $P'_U\colon [F_\bullet]'_U \to [F/G]_U$ is defined as Q on objects and as β on morphisms, it is not difficult to prove that P' is a monomorphism, whence also P is a monomorphism by Lemma 6.25. Moreover, $\widetilde{\operatorname{im}} Q = [F/G]$ (this follows easily from the fact that torsors are locally trivial), which clearly implies that P is an epimorphism in $\underline{\mathbf{St}}_{\mathbf{C}}$, hence an isomorphism by Lemma 6.27. $\square$

7 Faithfully flat descent

7.1 Descent for modules

Given a morphism of rings $\phi\colon A \to B$, there is a natural complex of A-modules

$$C_\phi^\bullet = \cdots \to 0 \to A \xrightarrow{d^0=\phi} B \xrightarrow{d^1} B \otimes_A B \xrightarrow{d^2} B \otimes_A B \otimes_A B \to \cdots$$

with $C_\phi^i = B^{\otimes_A i}$ and $d^i := \sum_{j=0}^{i} (-1)^j e_j^i\colon B^{\otimes_A i} \to B^{\otimes_A (i+1)}$, where the morphisms of A-algebras $e_j^i\colon B^{\otimes_A i} \to B^{\otimes_A (i+1)}$ are defined by

$$e_j^i(b_1 \otimes \cdots \otimes b_i) := b_1 \otimes \cdots \otimes b_j \otimes 1 \otimes b_{j+1} \otimes \cdots \otimes b_i$$

(it is straightforward to check that $C_\phi^\bullet$ is really a complex).

Proposition 7.1. *If $\phi\colon A \to B$ is a faithfully flat morphism of rings, then the complex $C_\phi^\bullet$ defined above is exact. More generally, for every A-module M, the complex of A-modules $C_\phi^\bullet \otimes_A M$ is also exact.*

Proof. Given a morphism of rings $\psi\colon A \to A'$, let $B' := A' \otimes_A B$ and let $\phi'\colon A' \to B'$ be the induced morphism of rings. If M is an A-module, then, denoting by M' the A'-module $A' \otimes_A M$, there is a natural isomorphism of complexes of A'-modules $A' \otimes_A C_\phi^\bullet \otimes_A M \cong C_{\phi'}^\bullet \otimes_{A'} M'$ (this follows easily from the associativity of tensor product). Thus, by Lemma 3.77, in order to prove that $C_\phi^\bullet \otimes_A M$ is exact, it is enough to show that $C_{\phi'}^\bullet \otimes_{A'} M'$ is exact for some faithfully flat morphism $\psi\colon A \to A'$. In particular, we can take $\psi = \phi\colon A \to B$, and in this case $\phi'\colon B \to B \otimes_A B$ is the map given by $b \mapsto b \otimes 1$; notice that there is a morphism of rings

$\mu\colon B\otimes_A B\to B$ (defined by $\mu(b\otimes b'):=bb'$) such that $\mu\circ\phi'=\mathrm{id}_B$. Therefore, we can assume without loss of generality that $\phi\colon A\to B$ satisfies the following property: there exists a morphism of rings $\sigma\colon B\to A$ such that $\sigma\circ\phi=\mathrm{id}_A$. We are going to show that with this additional hypothesis the complex $C_\phi^\bullet$ is actually homotopic to 0 (hence, for every A-module M, $C_\phi^\bullet\otimes_A M$ is also homotopic to 0, and in particular exact). Indeed, the morphisms of A-modules $k^i\colon B^{\otimes_A i}\to B^{\otimes_A (i-1)}$ defined by $k^i(b_1\otimes\cdots\otimes b_i):=\sigma(b_1)b_2\otimes\cdots\otimes b_i$ are such that $e_j^{i-1}\circ k^i=k^{i+1}\circ e_{j+1}^i$ for $0\le j<i$ and $k^{i+1}\circ e_0^i=\mathrm{id}_{B^{\otimes_A i}}$. It follows that

$$k^{i+1}\circ d^i+d^{i-1}\circ k^i=\sum_{j=0}^{i}(-1)^j k^{i+1}\circ e_j^i+\sum_{j=0}^{i-1}(-1)^j e_j^{i-1}\circ k^i=$$
$$=k^{i+1}\circ e_0^i+\sum_{j=0}^{i-1}(-1)^j(e_j^{i-1}\circ k^i-k^{i+1}\circ e_{j+1}^i)=\mathrm{id}_{B^{\otimes_A i}},$$

which shows that $C_\phi^\bullet$ is homotopic to 0. $\square$

Corollary 7.2. *If $\phi\colon A\to B$ is a faithfully flat morphism of rings and M,M' are two A-modules, then there is an exact sequence of A-modules*[13]

$$0\to\mathrm{Hom}_A(M',M)\xrightarrow{\phi_*}\mathrm{Hom}_B(B\otimes_A M',B\otimes_A M)\to$$
$$\xrightarrow{e_{0*}^1-e_{1*}^1}\mathrm{Hom}_{B\otimes_A B}(B\otimes_A B\otimes_A M',B\otimes_A B\otimes_A M).$$

Proof. The given sequence can be naturally identified with the sequence

$$0\to\mathrm{Hom}_A(M',M)\xrightarrow{(\phi\otimes\mathrm{id}_M)\circ}\mathrm{Hom}_A(M',B\otimes_A M)$$
$$\xrightarrow{(d^1\otimes\mathrm{id}_M)\circ}\mathrm{Hom}_A(M',B\otimes_A B\otimes_A M),$$

which is exact because $0\to M\cong A\otimes_A M\xrightarrow{\phi\otimes\mathrm{id}_M}B\otimes_A M\xrightarrow{d^1\otimes\mathrm{id}_M}B\otimes_A B\otimes_A M$ is exact by Proposition 7.1 and $\mathrm{Hom}_A(M',-)$ is left exact. $\square$

Given a morphism of rings $\phi\colon A\to B$, besides the morphisms e_j^i introduced before, we define morphisms $\epsilon_i\colon B\to B\otimes_A B\otimes_A B$ for $i=0,1,2$ by $\epsilon_0(b):=b\otimes1\otimes1$, $\epsilon_1(b):=1\otimes b\otimes1$, $\epsilon_2(b):=1\otimes1\otimes b$; for $j=0,1$ and $j'=0,1,2$, we have

$$e_{j'}^2\circ e_j^1=\begin{cases}\epsilon_{\{0,1,2\}\setminus\{j,j'\}} & \text{if } j<j'\\ \epsilon_{\{0,1,2\}\setminus\{j',j+1\}} & \text{if } j\ge j'.\end{cases}$$

Observe that for every $N\in B$-**Mod** there are canonical isomorphisms of $(B\otimes_A B)$-modules $e_{0*}^1(N):=(B\otimes_A B)_{e_0^1}\otimes_B N\cong B\otimes_A N$ (given by $(b_1\otimes b_2)\otimes n\mapsto$

[13] We are using the canonical identifications $e_{0*}^1\circ\phi_*\cong s_*\cong e_{1*}^1\circ\phi_*$, where $s\colon A\to B\otimes_A B$ is the structure morphism.

$b_1 \otimes (b_2 n))$ and $e^1_{1*}(N) \cong N \otimes_A B$, and similarly there are canonical isomorphisms of $(B \otimes_A B \otimes_A B)$-modules $\epsilon_{0*}(N) \cong N \otimes_A B \otimes_A B$, $\epsilon_{1*}(N) \cong B \otimes_A N \otimes_A B$ and $\epsilon_{2*}(N) \cong B \otimes_A B \otimes_A N$. With these identifications (and taking into account that, if $e^2_{j'} \circ e^1_j = \epsilon_i$, then there is a canonical isomorphism $e^2_{j'\,*}(e^1_{j\,*}(N)) \cong \epsilon_{i*}(N))$, a morphism $\alpha\colon N \otimes_A B \to B \otimes_A N$ of $(B \otimes_A B)$-**Mod** induces morphisms of $(B \otimes_A B \otimes_A B)$-**Mod**

$$\begin{aligned} e^2_{0*}(\alpha)&\colon B \otimes_A N \otimes_A B \to B \otimes_A B \otimes_A N, \\ e^2_{1*}(\alpha)&\colon N \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A N, \\ e^2_{2*}(\alpha)&\colon N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B \end{aligned}$$

($e^2_{j\,*}(\alpha)$ is given by id_B on the $(j+1)^{\mathrm{th}}$ term and by α on the other two terms).

Remark 7.3. It is immediate to see that for every $M \in A$-**Mod** the natural isomorphism of $(B \otimes_A B)$-modules $\alpha_M\colon \phi_*(M) \otimes_A B \xrightarrow{\sim} B \otimes_A \phi_*(M)$ (defined by $(b \otimes m) \otimes b' \mapsto b \otimes (b' \otimes m)$, where $b \otimes m \in \phi_*(M) = B \otimes_A M$) satisfies the cocycle condition $e^2_{1*}(\alpha_M) = e^2_{0*}(\alpha_M) \circ e^2_{2*}(\alpha_M)$.

Proposition 7.4. *In the above notation, if $\phi\colon A \to B$ is a faithfully flat morphism of rings, N is a B-module and $\alpha\colon N \otimes_A B \xrightarrow{\sim} B \otimes_A N$ is an isomorphism of $B \otimes_A B$-modules such that*

$$e^2_{1*}(\alpha) = e^2_{0*}(\alpha) \circ e^2_{2*}(\alpha)\colon N \otimes_A B \otimes_A B \xrightarrow{\sim} B \otimes_A B \otimes_A N,$$

then $M := \{m \in N \mid \alpha(m \otimes 1) = 1 \otimes m\} \subseteq N$ is an A-submodule and the natural map $\mu\colon \phi_*(M) = B \otimes_A M \to N$ (defined by $b \otimes m \mapsto bm$) is an isomorphism of B-modules such that

$$\begin{tikzcd} \phi_*(M) \otimes_A B \arrow[r, "\alpha_M"] \arrow[d, "\mu\otimes\mathrm{id}"'] & B \otimes_A \phi_*(M) \arrow[d, "\mathrm{id}\otimes\mu"] \\ N \otimes_A B \arrow[r, "\alpha"'] & B \otimes_A N \end{tikzcd}$$

is a commutative diagram of $B \otimes_A B$-modules.

Proof. It is obvious that M is an A-submodule of N and μ is a morphism of B-modules. Moreover, since α is a morphism of $(B \otimes_A B)$-modules, for all $m \in M$ and all $b, b' \in B$ we have

$$\begin{aligned} &(\mathrm{id} \otimes \mu) \circ \alpha_M\big((b \otimes m) \otimes b'\big) = (\mathrm{id} \otimes \mu)\big(b \otimes (b' \otimes m)\big) = b \otimes (b'm) = \\ &= (b \otimes b')(1 \otimes m) = (b \otimes b')\alpha(m \otimes 1) = \alpha\big((bm) \otimes b'\big) = \alpha \circ (\mu \otimes \mathrm{id})\big((b \otimes m) \otimes b'\big) \end{aligned}$$

(hence the diagram commutes). Note that so far we have not used the hypotheses that ϕ is faithfully flat and that α is an isomorphism which satisfies the cocycle condition: they are needed only to prove that μ is an isomorphism. Denoting by

$e_0^1(N)\colon N \to B \otimes_A N$ and $e_1^1(N)\colon N \to N \otimes_A B$ the maps defined, respectively, by $n \mapsto 1 \otimes n$ and $n \mapsto n \otimes 1$, and setting $\gamma := e_0^1(N) - \alpha \circ e_1^1(N)\colon N \to B \otimes_A N$, by definition there is an exact sequence of A-modules $0 \to M \xrightarrow{\iota} N \xrightarrow{\gamma} B \otimes_A N$ (where ι is the natural inclusion). Then in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \phi_*(M) \cong M \otimes_A B \arrow[r, "\iota\otimes\mathrm{id}_B"] \arrow[d, "\mu"'] & N \otimes_A B \arrow[r, "\gamma\otimes\mathrm{id}_B"] \arrow[d, "\alpha", "\wr"'] & B \otimes_A N \otimes_A B \arrow[d, "e^2_{0*}(\alpha)", "\wr"'] \\ 0 \arrow[r] & N \cong A \otimes_A N \arrow[r, "\phi\otimes\mathrm{id}_N"'] & B \otimes_A N \arrow[r, "d^1\otimes\mathrm{id}_N"'] & B \otimes_A B \otimes_A N \end{tikzcd}$$

the rows are exact (the first because ϕ is flat and the second by Proposition 7.1). Therefore (since α and $e^2_{0*}(\alpha)$ are isomorphisms) μ will be an isomorphism by the five lemma, provided the diagram is commutative. Now, it is immediate to see that the square on the left is commutative, and as for the square on the right, it is enough to observe (remembering that $d^1 = e_0^1 - e_1^1$ and $\gamma = e_0^1(N) - \alpha \circ e_1^1(N)$) that $(e_0^1 \otimes \mathrm{id}_N) \circ \alpha = e^2_{0*}(\alpha) \circ (e_0^1(N) \otimes \mathrm{id}_B)$ and

$$\begin{aligned}(e_1^1 \otimes \mathrm{id}_N) \circ \alpha &= e^2_{1*}(\alpha) \circ (e_1^1(N) \otimes \mathrm{id}_B) \\ &= e^2_{0*}(\alpha) \circ e^2_{2*}(\alpha) \circ (e_1^1(N) \otimes \mathrm{id}_B) = e^2_{0*}(\alpha) \circ \big((\alpha \circ e_1^1(N)) \otimes \mathrm{id}_B\big).\end{aligned}$$

□

7.2 Descent for schemes and for quasi-coherent sheaves

We fix a base scheme $S \neq \emptyset$.

Lemma 7.5. *Given $U \in \mathbf{Sch}_{/S}$ and $\mathcal{U} = \{g_j\colon W_j \to U\}_{j\in J} \in \mathrm{Cov}^{fppf}(U)$, there exists $\mathcal{V} = \{f_i\colon V_i \to U\}_{i\in I} \in \mathrm{Cov}^{fppf}(U)$ such that for every $i \in I$ there exists a local isomorphism $h_i\colon V_i \to W_{j(i)}$ (for some $j(i) \in J$) with $f_i = g_{j(i)} \circ h_i$ (in particular, $\mathcal{U} \leq \mathcal{V}$) and f_i is the composition of a faithfully flat morphism of finite presentation of affine schemes $f_i'\colon V_i \to U_i$ and of an open immersion $U_i \hookrightarrow U$.*

Proof. Since each g_j is open by Proposition 3.81 and $U = \bigcup_{j\in J} \operatorname{im} g_j$, it is easy to see that there exists $\{U_i \subseteq U\}_{i\in I} \in \mathrm{Cov}^{Zar}(U)$ such that, for every $i \in I$, U_i is affine and $U_i \subseteq \operatorname{im} g_{j(i)}$ for some $j(i) \in J$. Now, for every $x \in U_i$ let $W_x \subseteq W_{j(i)}$ be an open affine subscheme such that $x \in g_{j(i)}(W_x) \subseteq U_i$: as U_i is quasi-compact, there exist $x_1, \ldots, x_n \in U_i$ such that $U_i = \bigcup_{k=1}^n g_{j(i)}(W_{x_k})$. Setting $V_i := \coprod_{k=1}^n W_{x_k}$, it is then clear that the morphism $f_i'\colon V_i \to U_i$ induced by the restrictions of $g_{j(i)}$ has the required properties. □

Proposition 7.6. *If $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{Sch}_{/S}}$ is a prestack (respectively a stack) for Zar such that for every faithfully flat morphism of finite presentation of affine schemes $f\colon V \to U$ the functor $\lambda^{\mathbf{F}}_{\{f\}}\colon \mathbf{F}_U \to \mathbf{F}_{\{f\}}$ is fully faithful (respectively an equivalence), then $\mathbf{F}$ is a prestack (respectively a stack) for fppf, too. In particular, if*

$F \in \widehat{\mathbf{Sch}_{/S}}$ is a separated presheaf (respectively a sheaf) for Zar such that for every f as above the map $\lambda^F_{\{f\}}\colon F(U) \to F(\{f\})$ is injective (respectively bijective), then F is separated (respectively a sheaf) for fppf, too.

Proof. Taking into account Corollary 6.11 and Lemma 7.5, it is enough to prove that $\lambda^{\mathbf{F}}_{\mathcal{V}}$ is fully faithful (respectively an equivalence) for every $U \in \mathbf{Sch}_{/S}$ and every $\mathcal{V} = \{f_i\colon V_i \to U\}_{i\in I} \in \mathrm{Cov}^{fppf}(U)$ such that each f_i is the composition of a faithfully flat morphism of finite presentation of affine schemes $f'_i\colon V_i \to U_i$ and of an open immersion $U_i \hookrightarrow U$.

$\lambda_{\mathcal{V}}$ is fully faithful: given $\xi,\xi' \in \mathbf{F}_U$ and $\psi \in \mathrm{Hom}_{\mathbf{F}_{\mathcal{V}}}(\lambda_{\mathcal{V}}(\xi),\lambda_{\mathcal{V}}(\xi'))$, we have to prove that there exists a unique $\phi \in \mathrm{Hom}_{\mathbf{F}_U}(\xi,\xi')$ such that $\psi = \lambda_{\mathcal{V}}(\phi)$. Now, by definition, ψ is given by morphisms $\psi_i \in \mathrm{Hom}_{\mathbf{F}_{V_i}}(\xi|_{V_i},\xi'|_{V_i})$ such that $pr_1^*(\psi_i) = pr_2^*(\psi_j) \in \mathrm{Hom}_{\mathbf{F}_{V_i\times_U V_j}}(\xi|_{V_i\times_U V_j},\xi'|_{V_i\times_U V_j})$ for all $i,j \in I$. In particular, for every $i \in I$ we have $pr_1^*(\psi_i) = pr_2^*(\psi_i) \in \mathrm{Hom}_{\mathbf{F}_{V_i\times_U V_i}}(\xi|_{V_i\times_U V_i},\xi'|_{V_i\times_U V_i})$; as $\lambda_{\{f_i\}}$ is fully faithful (and taking into account that $V_i\times_U V_i = V_i\times_{U_i} V_i$), this implies that there exists a unique $\phi_i \in \mathrm{Hom}_{\mathbf{F}_{U_i}}(\xi|_{U_i},\xi'|_{U_i})$ such that $\psi_i = \phi_i|_{V_i}$. For every open affine subset $U' \subseteq U_{i,j}$ the natural morphism $W' := (V_i \times_U V_j) \times_{U_{i,j}} U' \to U'$ is faithfully flat and of finite presentation, so that (by the same argument used in the proof of Lemma 7.5) there exists a morphism $V' \to W'$ with V' affine such that the induced morphism $V' \to U'$ is also faithfully flat and of finite presentation. As $\phi_i|_{V'} = \phi_j|_{V'}$ (because $\phi_i|_{W'} = \psi_i|_{W'} = \psi_j|_{W'} = \phi_j|_{W'}$), this implies that $\phi_i|_{U'} = \phi_j|_{U'}$. Therefore, since $\mathbf{F}$ is a prestack for *Zar*, we have $\phi_i|_{U_{i,j}} = \phi_j|_{U_{i,j}}$ for all $i,j \in I$, whence there exists a unique $\phi \in \mathrm{Hom}_{\mathbf{F}_U}(\xi,\xi')$ such that $\phi_i = \phi|_{U_i}$ for every $i \in I$. Then $\psi = \lambda_{\mathcal{V}}(\phi)$ and ϕ is clearly unique with this property.

It remains to show that $\lambda_{\mathcal{V}}$ is essentially surjective if $\mathbf{F}$ is a stack for *Zar* and $\lambda_{\{f'_i\}}$ is an equivalence for every $i \in I$. Given $\eta = (\{\eta_i\},\{\psi_{i,j}\})_{i,j\in I} \in \mathbf{F}_{\mathcal{V}}$ ($\eta_i \in \mathbf{F}_{V_i}$ and $\psi_{i,j}\colon \eta_j|_{V_i\times_U V_j} \xrightarrow{\sim} \eta_i|_{V_i\times_U V_j}$ are isomorphisms of $\mathbf{F}_{V_i\times_U V_j}$ satisfying the cocycle condition), for every $i \in I$ (since $\lambda_{\{f'_i\}}$ is an equivalence) there exists (unique up to isomorphism) $\xi_i \in \mathbf{F}_{U_i}$ together with an isomorphism $\beta_i\colon \xi_i|_{V_i} \xrightarrow{\sim} \eta_i$ in $\mathbf{F}_{V_i}$ such that $\psi_{i,i} = pr_1^*(\beta_i)\circ pr_2^*(\beta_i)^{-1}$. For all $i,j \in I$ we define $\psi'_{i,j}$ to be the isomorphism of $\mathbf{F}_{V_i\times_U V_j}$ which makes the diagram

$$\begin{tikzcd}
\xi_j|_{V_i\times_U V_j} \arrow[r, "\psi'_{i,j}"] \arrow[d, "\beta_j|_{V_i\times_U V_j}"'] & \xi_i|_{V_i\times_U V_j} \arrow[d, "\beta_i|_{V_i\times_U V_j}"] \\
\eta_j|_{V_i\times_U V_j} \arrow[r, "\psi_{i,j}"'] & \eta_i|_{V_i\times_U V_j}
\end{tikzcd}$$

commute (in particular, $\psi'_{i,i} = \mathrm{id}$). As $\{V_i \times_U V_j \to U_{i,j}\} \in \mathrm{Cov}^{fppf}(U_{i,j})$ and as we already know that $\mathbf{F}$ is a prestack for *fppf*, there exists a unique $\phi_{i,j}\colon \xi_j|_{U_{i,j}} \xrightarrow{\sim} \xi_i|_{U_{i,j}}$ in $\mathbf{F}_{U_{i,j}}$ such that $\psi'_{i,j} = \phi_{i,j}|_{V_i\times_U V_j}$ (in particular, $\phi_{i,i} = \mathrm{id}$). It is immediate to see that the $\phi_{i,j}$ satisfy the cocycle condition, so that (since $\mathbf{F}$ is a stack for *Zar*) there exists (unique up to isomorphism) $\xi \in \mathbf{F}_U$ together with isomorphisms $\alpha_i \colon \xi|_{U_i} \xrightarrow{\sim} \xi_i$ in $\mathbf{F}_{U_i}$ such that $\phi_{i,j} = \alpha_i|_{U_{i,j}} \circ (\alpha_j|_{U_{i,j}})^{-1}$. Setting $\gamma_i := \beta_i \circ \alpha_i|_{V_i} \colon \xi|_{V_i} \xrightarrow{\sim} \eta_i$ in $\mathbf{F}_{V_i}$, we have for all $i, j \in I$

$$\begin{aligned}\gamma_i|_{V_i \times_U V_j} \circ (\gamma_j|_{V_i \times_U V_j})^{-1} &= \beta_i|_{V_i \times_U V_j} \circ \alpha_i|_{V_i \times_U V_j} \circ (\alpha_j|_{V_i \times_U V_j})^{-1} \circ (\beta_j|_{V_i \times_U V_j})^{-1} \\ &= \beta_i|_{V_i \times_U V_j} \circ \phi_{i,j}|_{V_i \times_U V_j} \circ (\beta_j|_{V_i \times_U V_j})^{-1} \\ &= \beta_i|_{V_i \times_U V_j} \circ \psi'_{i,j} \circ (\beta_j|_{V_i \times_U V_j})^{-1} = \psi_{i,j}\end{aligned}$$

and this precisely says that $\lambda_{\mathcal{V}}(\xi) \cong \eta$. $\square$

Remark 7.7. Using Lemma 7.5 it is also easy to see that a property P of objects of $\mathbf{Sch}_{/S}$ is local for *fppf* if it is local for *Zar* and the following holds: if $U' \to U$ is a faithfully flat morphism of finite presentation of affine schemes, then U satisfies P if and only if U' satisfies P. Similarly, one can show that a property P of morphisms of $\mathbf{Sch}_{/S}$ is local on the codomain (respectively domain) for *fppf* if it is local on the codomain (respectively domain) for *Zar* and the following holds: given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "g"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

(respectively given morphisms $U' \xrightarrow{f} U \xrightarrow{g} X$) of $\mathbf{Sch}_{/S}$ such that f is a faithfully flat morphism of finite presentation of affine schemes, then g satisfies P if and only if g' (respectively $g \circ f$) satisfies P. The same is true if *fppf* is replaced by *sm* (respectively *ét*) and f is assumed smooth (respectively étale) and surjective.

Theorem 7.8. *On* $\mathbf{Sch}_{/S}$ *the fppf pretopology is subcanonical.*

Proof. By Corollary 4.43 we can assume $S = \operatorname{Spec} \mathbb{Z}$ and, since we know that *Zar* is subcanonical on $\mathbf{Sch}$ (see Example 4.40), by Proposition 7.6 it is enough to prove the following: if $\phi \colon A \to B$ is a faithfully flat morphism of finite presentation of rings, then, setting $f := \operatorname{Spec} \phi \colon V := \operatorname{Spec} B \to U := \operatorname{Spec} A$, the sequence of sets

$$\operatorname{Hom}_{\mathbf{Sch}}(U, Z) \xrightarrow{\circ f} \operatorname{Hom}_{\mathbf{Sch}}(V, Z) \underset{\circ pr_2}{\overset{\circ pr_1}{\rightrightarrows}} \operatorname{Hom}_{\mathbf{Sch}}(V \times_U V, Z)$$

is exact for every $Z \in \mathbf{Sch}$. Note that this follows from Proposition 7.1 if $Z = \operatorname{Spec} C$ is affine, as in this case the above sequence can be identified with the sequence

$$\operatorname{Hom}_{\mathbf{Rng}}(C, A) \xrightarrow{\phi \circ} \operatorname{Hom}_{\mathbf{Rng}}(C, B) \underset{e^1_1 \circ}{\overset{e^1_0 \circ}{\rightrightarrows}} \operatorname{Hom}_{\mathbf{Rng}}(C, B \otimes_A B),$$

which is exact because the sequence of rings $A \xrightarrow{\phi} B \overset{e_0^1}{\underset{e_1^1}{\rightrightarrows}} B \otimes_A B$ is exact (as a sequence of sets). Passing to the general case, we first show that $\circ f$ is injective. Indeed, given $g_1, g_2 \colon U \to Z$ such that $g_1 \circ f = g_2 \circ f$, we clearly have $|g_1| = |g_2|$, since f is surjective. For every $x \in U$ let $W \subseteq Z$ be an open affine neighbourhood of $g_1(x) = g_2(x)$; as $|g_1|$ is continuous, there exists $a \in A$ such that $x \in D(a) \cong \operatorname{Spec} A_a$ and $|g_1|(D(a)) = |g_2|(D(a)) \subseteq W$ (hence $g_i|_{D(a)}$ can be regarded as morphisms $D(a) \to W$). Moreover, $f^{-1}(D(a)) = D(\phi(a)) \cong \operatorname{Spec} B_{\phi(a)}$ and $f|_{D(\phi(a))} \colon D(\phi(a)) \to D(a)$ (which is also a faithfully flat morphism of finite presentation of affine schemes) is such that $g_1|_{D(a)} \circ f|_{D(\phi(a))} = g_2|_{D(a)} \circ f|_{D(\phi(a))}$, so that $g_1|_{D(a)} = g_2|_{D(a)}$ by the already proved case, and clearly it follows that $g_1 = g_2$. It remains to show that, given $h \colon V \to Z$ such that $h \circ pr_1 = h \circ pr_2$, there exists $g \colon U \to Z$ such that $h = g \circ f$, and (as we have already proved uniqueness) it is enough to prove that for every $x \in U$ there is an open neighbourhood $U' \subseteq U$ of x and a morphism $g' \colon U' \to Z$ such that $g' \circ f|_{f^{-1}(U')} = h|_{f^{-1}(U')}$. So, given $x \in U$, we choose $y \in f^{-1}(x)$ and an open affine neighbourhood $W \subseteq Z$ of $h(y)$. Since $f(h^{-1}(W))$ is open in U by Proposition 3.81, there exists $a \in A$ such that $x \in D(a) \subseteq f(h^{-1}(W))$. We claim that $f^{-1}(D(a)) = D(\phi(a)) \subseteq h^{-1}(W)$: indeed, given $y' \in D(\phi(a))$, there exists $y'' \in h^{-1}(W)$ such that $f(y'') = f(y')$ (because $f(y') \in D(a) \subseteq f(h^{-1}(W))$), whence there exists $\tilde{y} \in V \times_U V$ such that $pr_1(\tilde{y}) = y'$ and $pr_2(\tilde{y}) = y''$; it follows that $h(y') = h(pr_1(\tilde{y})) = h(pr_2(\tilde{y})) = h(y'') \in W$, i.e. $y' \in h^{-1}(W)$. Therefore $h|_{D(\phi(a))}$ can be regarded as a morphism $D(\phi(a)) \to W$, so that (by the already proved case, applied to the faithfully flat morphism of finite presentation of affine schemes $f|_{D(\phi(a))} \colon D(\phi(a)) \to D(a)$) there exists a unique $g' \colon D(a) \to Z$ such that $g' \circ f|_{D(\phi(a))} = h|_{D(\phi(a))}$. $\square$

Thus, taking into account Example 4.36,

$$\mathbf{Sch}_{/S} \subseteq (\mathbf{Sch}_{/S}, \mathit{fppf})^{\sim} \subseteq (\mathbf{Sch}_{/S}, \mathit{sm})^{\sim} = (\mathbf{Sch}_{/S}, \mathit{\acute{e}t})^{\sim} \subseteq (\mathbf{Sch}_{/S}, \mathit{Zar})^{\sim}$$

and we are going to give examples which show that all the inclusions are strict.

Example 7.9. Let $P = P_S \in \widehat{\mathbf{Sch}_{/S}}$ be the presheaf defined on objects by $P(U) := \{X \subseteq |U|\}$ (all subsets of $|U|$) and on morphisms by $P(f)(X) := |f|^{-1}(X)$: then P is a non representable sheaf for *fppf*.

We first prove that $P \in (\mathbf{Sch}_{/S}, \mathit{fppf})^{\sim}$. Given a covering $\{f_i \colon U_i \to U\}_{i \in I} \in \operatorname{Cov}^{\mathit{fppf}}(U)$ and $X_i \in P(U_i)$ (i.e., $X_i \subseteq |U_i|$) such that $|pr_1|^{-1}(X_i) = |pr_2|^{-1}(X_j) \subseteq |U_{i,j}|$ for all $i, j \in I$, let $X := \bigcup_{i \in I} |f_i|(X_i) \subseteq |U|$. We claim that $|f_i|^{-1}(X) = X_i$ for every $i \in I$: as it is obvious that $X_i \subseteq |f_i|^{-1}(X)$, we have only to prove that $|f_i|^{-1}(X) \subseteq X_i$. So fix $i \in I$ and $x_i \in |f_i|^{-1}(X)$; then $f_i(x_i) \in X$, so that, by definition of X, there exist $j \in I$ and $x_j \in X_j$ such that $f_i(x_i) = f_j(x_j)$. Therefore there exists $y \in U_{i,j}$ such that $pr_1(y) = x_i$ and $pr_2(y) = x_j$, which implies that $y \in |pr_2|^{-1}(X_j) = |pr_1|^{-1}(X_i)$, whence $x_i = pr_1(y) \in X_i$. In order to conclude that P is a sheaf it is then enough to observe that $X \in P(U)$ is unique with the property that $|f_i|^{-1}(X) = X_i$ for every $i \in I$ (i.e., that P is separated): this follows immediately from the fact that $|U| = \bigcup_{i\in I} \mathrm{im}|f_i|$ (notice that we did not use any hypothesis on the morphisms f_i, but only the "surjectivity" of the covering).

It remains to prove that $P = P_S$ is not representable. So we assume on the contrary that P_S is representable: since then clearly $P_{S'}$ is also representable for every scheme S' such that $\mathrm{Hom}_{\mathbf{Sch}}(S', S) \neq \emptyset$, we can reduce to the case $S = \operatorname{Spec}\mathbb{K}$ ($\mathbb{K}$ a field). By Corollary 4.6 there exist $U \in \mathbf{Sch}_{/\mathbb{K}}$ and $A \subseteq |U|$ such that for every $V \in \mathbf{Sch}_{/\mathbb{K}}$ and every $X \subseteq |V|$, there exists a unique $f \in \mathrm{Hom}_{\mathbf{Sch}_{/\mathbb{K}}}(V, U)$ such that $X = |f|^{-1}(A)$. As for any extension of fields $\mathbb{K} \subseteq \mathbb{K}'$ the set $P(\operatorname{Spec}\mathbb{K}')$ has 2 elements, it is clear that $|U|$ consists of exactly 2 points (both with residue field $\mathbb{K}$), say $|U| = \{u_1, u_2\}$. Then A is either open or closed (if $\{u_i\}$ is not open, then U is affine, hence $\{u_i\}$ is closed because a ring has at least one maximal ideal), and we get a contradiction taking V such that not all subsets of $|V|$ are either open or closed (e.g., $V = \operatorname{Spec}\mathbb{K}[t]$).

Example 7.10. Given a subcanonical pretopology τ on $\mathbf{Sch}_{/S}$ and a morphism $f\colon X \to Y$ of $\mathbf{Sch}_{/S}$, we define $F_{f,\tau} \in (\mathbf{Sch}_{/S}, \tau)^{\sim}$ to be the subpresheaf of the sheaf Y given by the morphisms which locally (for τ) factor through f: explicitly, $g\colon U \to Y$ is in $F_{f,\tau}(U)$ if and only if there exist $\{p_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and morphisms $h_i\colon U_i \to X$ such that $g \circ p_i = f \circ h_i$ for every $i \in I$ (using the axioms of pretopology, it is very easy to see that $F_{f,\tau}$ is really a sheaf).

Now, let $f\colon X := \operatorname{Spec}\mathbb{K}' \to Y := \operatorname{Spec}\mathbb{K}$ be a morphism of $\mathbf{Sch}_{/S}$ corresponding to a finite separable non trivial extension of fields $\mathbb{K} \subset \mathbb{K}'$ (there is always such a morphism: you can take $s \in S$, $\mathbb{K} := \kappa(s)(x^n)$ and $\mathbb{K}' := \kappa(s)(x)$, where $n > 1$ is not a multiple of the characteristic of $\kappa(s)$). Then $F_{f,Zar}$ is a sheaf for *Zar* but not for *ét*. Indeed, $\mathrm{id}_Y \in Y(Y)$ is clearly not in $F_{f,Zar}(Y)$, but $Y(f)(\mathrm{id}_Y) = f \in F_{f,Zar}(X) \subseteq Y(X)$ and $\{f\} \in \mathrm{Cov}^{\acute{e}t}(Y)$ (by Remark 3.117).

On the other hand, let $\mathbb{K} := \overline{\kappa(s)}$ for some $s \in S$ and let $A := \mathbb{K}[t]/(t^2)$. Then the morphism $f\colon X \to Y$ of $\mathbf{Sch}_{/S}$ induced by the natural morphism of rings $\phi\colon A \to B := A[x]/(x^2 - t)$ is such that $F_{f,\acute{e}t}$ is a sheaf for *ét* but not for *fppf*. As before, since $\{f\} \in \mathrm{Cov}^{fppf}(Y)$, it is enough to show that $\mathrm{id}_Y \notin F_{f,\acute{e}t}(Y)$. So, assume on the contrary that $\mathrm{id}_Y \in F_{f,\acute{e}t}(Y)$: then there is a morphism $g\colon X' \to X$ such that $f' := f \circ g\colon X' \to Y$ is étale and surjective. Hence the induced morphism $\tilde{f}'\colon \tilde{X}' := X' \times_Y \operatorname{Spec}\mathbb{K} \to \operatorname{Spec}\mathbb{K}$ is also étale and surjective, so that (by Remark 3.117) there is an open subset $\tilde{U} \cong \operatorname{Spec}\mathbb{K} \subseteq \tilde{X}'$ such that $f|_{\tilde{U}}$ is induced by $\mathrm{id}_{\mathbb{K}}$. As $|X'| \cong |\tilde{X}'|$, the open subscheme $U \subseteq X'$ inducing $\tilde{U}$ on $\tilde{X}'$ has exactly one point (hence is affine) and $f'|_U$ is induced by an étale morphism of rings $\phi'\colon A \to B'$, where $\phi' = \psi \circ \phi$ for some $\psi\colon B \to B'$. Now, B' is a free A-module because it is flat and, since $B' \otimes_A \mathbb{K} \cong \mathbb{K}$, we conclude that ϕ' is an isomorphism, which is impossible because t is not a square in A.

By Corollary 6.33 we have also the following result.

Corollary 7.11. $\mathbf{Mor}(\mathbf{Sch}_{/S}) \in \underline{\mathbf{PSt}}_{(\mathbf{Sch}_{/S}, fppf)}$ *and every property of morphisms of* $\mathbf{Sch}_{/S}$ *which is stable under base change satisfies descent for fppf.*

Theorem 7.12. *In the notation of Example 5.35,* $\overline{\mathbf{QCoh}} \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, fppf)}$.

Proof. As we already know that $\overline{\mathbf{QCoh}} \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, Zar)}$ (see Example 6.7), by Proposition 7.6 it is enough to prove that for every faithfully flat morphism of finite presentation of rings $\phi\colon A \to B$ the functor $\lambda^{\overline{\mathbf{QCoh}}}_{\{\operatorname{Spec}\phi\}}$ is an equivalence. Now, taking into account the natural equivalence (for every ring C) between C-**Mod** and **QCoh**$(\operatorname{Spec} C)$, Corollary 7.2 implies that $\lambda_{\{\operatorname{Spec}\phi\}}$ is fully faithful and Proposition 7.4 that it is essentially surjective. $\square$

Remark 7.13. In the above proof we did not use the hypothesis ϕ of finite presentation, so that $\lambda^{\overline{\mathbf{QCoh}}}_{\{f\}}$ is an equivalence if f is a faithfully flat morphism of affine schemes (and then it would be easy to prove that, more generally, it is an equivalence if f is a faithfully flat and quasi-compact morphism of schemes). On the other hand, in the proof of Theorem 7.8, in order to show that $\lambda^{Z}_{\{f\}}$ is bijective for every $Z \in \mathbf{Sch}$, we used the hypothesis that the faithfully flat morphism of affine schemes $f\colon V \to U$ was of finite presentation (namely, we used the fact that such a morphism is open). However, it can be proved (see [10, Exp. VIII, Cor. 5.3]) that actually $\lambda^{Z}_{\{f\}}$ is bijective for every $Z \in \mathbf{Sch}$ and every faithfully flat morphism of affine schemes $f\colon V \to U$ (and more generally when f is a faithfully flat and quasi-compact morphism of schemes), using the fact that in this case U has the quotient topology induced by the surjective map f (see [10, Exp. VIII, Cor. 4.3]). Nevertheless, it is not true that the maps $\lambda^{Z}_{\{f\}}$ are bijective and the functor $\lambda^{\overline{\mathbf{QCoh}}}_{\{f\}}$ is an equivalence for every faithfully flat morphism of schemes f (for this reason we use the *fppf* pretopology instead of the faithfully flat pretopology *fp*, defined by $\{f_i\colon U_i \to U\}_{i\in I} \in \operatorname{Cov}^{fp}(U)$ if and only if $|U| = \bigcup_{i\in I} \operatorname{im}|f_i|$ and f_i is flat for every $i \in I$).[14]

Example 7.14. Let $X := \coprod_{\mathfrak{p}\in\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_{\mathfrak{p}} \in \mathbf{Sch}$ and denote by $f\colon X \to \operatorname{Spec}\mathbb{Z}$ the (unique) morphism, which is clearly faithfully flat (but neither locally of finite presentation nor quasi-compact). Then the scheme Z obtained by gluing the schemes $\operatorname{Spec}\mathbb{Z}_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec}\mathbb{Z}$) along the open subschemes $\{(0)\} \subseteq \operatorname{Spec}\mathbb{Z}_{\mathfrak{p}}$ (each of them can be naturally identified with $\operatorname{Spec}\mathbb{Q}$) is such that $\lambda^{Z}_{\{f\}}$ is not bijective (i.e., the sequence

$$\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec}\mathbb{Z}, Z) \xrightarrow{\circ f} \operatorname{Hom}_{\mathbf{Sch}}(X, Z) \overset{\circ pr_1}{\underset{\circ pr_2}{\rightrightarrows}} \operatorname{Hom}_{\mathbf{Sch}}(X \times X, Z)$$

is not exact). Indeed, the morphism $g\colon X \to Z$ induced by the natural morphisms $i_{\mathfrak{p}}\colon \operatorname{Spec}\mathbb{Z}_{\mathfrak{p}} \hookrightarrow Z$ satisfies $g \circ pr_1 = g \circ pr_2$ (this follows from the fact that if $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}\mathbb{Z}$ and $\mathfrak{p} \neq \mathfrak{q}$, then $\mathbb{Z}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{q}} \cong \mathbb{Q}$), but there is no morphism

[14] What we have just said suggests that it should be possible to replace the hypothesis "locally of finite presentation" with "quasi-compact". Actually this is true, but, in order to do that in a reasonably simple way, one has to use topologies instead of pretopologies (see Example B.27).

$h\colon \operatorname{Spec}\mathbb{Z} \to Z$ such that $h \circ f = g$. Suppose on the contrary there is such a morphism h: necessarily $h(\mathtt{p}) = i_{\mathtt{p}}(\mathtt{p}\mathbb{Z}_{\mathtt{p}})$ for every $\mathtt{p} \in \operatorname{Spec}\mathbb{Z}$, but then h would not be continuous ($\{(0)\} = i_{\mathtt{p}}(\{(0)\}) \subseteq Z$ is open, but $h^{-1}(\{(0)\}) = \{(0)\} \subseteq \operatorname{Spec}\mathbb{Z}$ is not open). This implies that the *fp* pretopology on **Sch** (defined in Remark 7.13) is not subcanonical, and also that $Z \notin (\mathbf{AffSch}, \mathit{fp})^{\sim}$ (in the example, just replace the covering $\{f\}$ with $\{\operatorname{Spec}\mathbb{Z}_{\mathtt{p}} \to \operatorname{Spec}\mathbb{Z}\}_{\mathtt{p}\in\operatorname{Spec}\mathbb{Z}} \in \operatorname{Cov}^{\mathit{fp}}(\operatorname{Spec}\mathbb{Z})$). However, *fp* is subcanonical on **AffSch**: this was shown in the first part of the proof of Theorem 7.8 (the case Z affine, where the hypothesis ϕ of finite presentation was not used).

Corollary 7.15. *In the notation of Example 5.35,* $\overline{\mathbf{QCohAlg}}, \overline{\mathbf{QCohGAlg}} \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, fppf)}$.

Proof. We prove only that $\mathbf{F} := \overline{\mathbf{QCohAlg}} \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, fppf)}$, since the other case $\overline{\mathbf{QCohGAlg}}$ can be dealt with in a completely similar way. Notice first that objects of $\mathbf{F}_U = \mathbf{QCohAlg}(U)$ ($U \in \mathbf{Sch}_{/S}$) can be identified with triples $(\mathcal{F}, \mu, \epsilon)$, where $\mathcal{F} \in \mathbf{QCoh}(U)$ and $\mu\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ ("multiplication") and $\epsilon\colon \mathcal{O}_U \to \mathcal{F}$ ("identity") are morphisms of $\mathbf{QCoh}(U)$ satisfying a list of axioms (like associativity and commutativity of multiplication, ...), which can all be expressed by requiring the commutativity of some diagrams in $\mathbf{QCoh}(U)$ (in the style of Proposition 4.66 for group structure). In this way morphisms of $\mathbf{F}_U$ are given by

$$\begin{aligned}&\operatorname{Hom}_{\mathbf{F}_U}((\mathcal{F}, \mu, \epsilon), (\mathcal{F}', \mu', \epsilon'))\\ &\qquad = \{\alpha \in \operatorname{Hom}_{\mathbf{QCoh}(U)}(\mathcal{F}, \mathcal{F}') \mid \alpha \circ \mu = \mu' \circ (\alpha \otimes \alpha), \alpha \circ \epsilon = \epsilon'\}\end{aligned}$$

and, if $f\colon V \to U$ is a morphism of $\mathbf{Sch}_{/S}$, the pullback functor $f^*\colon \mathbf{F}_U \to \mathbf{F}_V$ can be defined in the obvious way (since there are natural isomorphisms $f^*(\mathcal{O}_U) \cong \mathcal{O}_V$ and $f^*(\mathcal{F} \otimes \mathcal{F}) \cong f^*(\mathcal{F}) \otimes f^*(\mathcal{F})$). Now, for every covering $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \operatorname{Cov}^{fppf}(U)$, we have to prove that $\lambda^{\mathbf{F}}_{\mathcal{U}}\colon \mathbf{F}_U \to \mathbf{F}_{\mathcal{U}}$ is an equivalence.

$\lambda^{\mathbf{F}}_{\mathcal{U}}$ is fully faithful: as $\lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}$ is fully faithful, given $\tilde{\mathcal{F}} = (\mathcal{F}, \mu, \epsilon), \tilde{\mathcal{F}}' = (\mathcal{F}', \mu', \epsilon') \in \mathbf{F}_U$ and

$$\beta \in \operatorname{Hom}_{\mathbf{F}_{\mathcal{U}}}(\lambda^{\mathbf{F}}_{\mathcal{U}}(\tilde{\mathcal{F}}), \lambda^{\mathbf{F}}_{\mathcal{U}}(\tilde{\mathcal{F}}')) \subseteq \operatorname{Hom}_{\overline{\mathbf{QCoh}}_{\mathcal{U}}}(\lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}(\mathcal{F}), \lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}(\mathcal{F}')),$$

there exists a unique $\alpha \in \operatorname{Hom}_{\mathbf{QCoh}(U)}(\mathcal{F}, \mathcal{F}')$ such that $\beta = \lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}(\alpha)$, and it is straightforward to check that $\alpha \circ \mu = \mu' \circ (\alpha \otimes \alpha)$ and $\alpha \circ \epsilon = \epsilon'$ (i.e., $\alpha \in \operatorname{Hom}_{\mathbf{F}_U}(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}')$).

$\lambda^{\mathbf{F}}_{\mathcal{U}}$ is essentially surjective: given $(\tilde{\mathcal{F}}_i, \beta_{i,j})_{i,j\in I} \in \mathbf{F}_{\mathcal{U}}$ (with $\tilde{\mathcal{F}}_i = (\mathcal{F}_i, \mu_i, \epsilon_i)$), there exist $\mathcal{F} \in \mathbf{QCoh}(U)$ and $\alpha_i\colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ in $\mathbf{QCoh}(U_i)$ such that $\beta_{i,j} = \alpha_i|_{U_{i,j}} \circ (\alpha_j|_{U_{i,j}})^{-1}$ for all $i, j \in I$ (because $(\mathcal{F}_i, \beta_{i,j})_{i,j\in I} \in \overline{\mathbf{QCoh}}_{\mathcal{U}}$ and $\lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}$ is essentially surjective). Again using the fact that $\lambda^{\overline{\mathbf{QCoh}}}_{\mathcal{U}}$ is fully faithful, and setting $\mu'_i := \alpha_i^{-1} \circ \mu_i \circ (\alpha_i \otimes \alpha_i)\colon \mathcal{F}|_{U_i} \otimes \mathcal{F}|_{U_i} \to \mathcal{F}|_{U_i}$ and $\epsilon'_i := \alpha_i^{-1} \circ \epsilon_i\colon \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$, it is clear that there exist unique $\mu\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ and $\epsilon\colon \mathcal{O}_U \to \mathcal{F}$ in $\mathbf{QCoh}(U)$

such that $\mu'_i = \mu|_{U_i}$ and $\epsilon'_i = \epsilon|_{U_i}$ for every $i \in I$, and then it is easy to prove that $\tilde{\mathcal{F}} := (\mathcal{F}, \mu, \epsilon) \in \mathbf{F}_U$ and that the α_i give an isomorphism $\lambda^{\mathbf{F}}_{\mathcal{U}}(\tilde{\mathcal{F}}) \xrightarrow{\sim} (\tilde{\mathcal{F}}_i, \beta_{i,j})_{i,j \in I}$ in $\mathbf{F}_{\mathcal{U}}$. □

7.3 Properties of effective descent

Proposition 7.16. *For morphisms of $\mathbf{Sch}_{/S}$ the property of being affine satisfies effective descent for fppf.*

Proof. We have to prove that $\mathbf{Aff} := \mathbf{Mor}^{\mathsf{P}}(\mathbf{Sch}_{/S}) \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, \mathit{fppf})}$, where P is the property of being affine. This follows immediately from Corollary 7.15, as $\mathbf{Aff}_U$ is naturally equivalent to $\mathbf{QCohAlg}(U)^{\circ}$ (by Proposition 3.38) for every $U \in \mathbf{Sch}_{/S}$. □

Lemma 7.17. *Let $O \in \widehat{\mathbf{Sch}_{/S}}$ be the the subpresheaf of P (see Example 7.9) defined on objects by $O(U) := \{X \subseteq |U| \text{ open}\}$ (all open subsets of $|U|$). Then O is a sheaf for fppf.*

Proof. Since O is a subpresheaf of the sheaf P, it is enough to show that if $\{f_i \colon U_i \to U\}_{i \in I} \in \mathrm{Cov}^{\mathit{fppf}}(U)$ and $X \subseteq |U|$ is such that $X_i := |f_i|^{-1}(X) \subseteq |U_i|$ is open for every $i \in I$, then X is open in $|U|$. This is true because $X = \bigcup_{i \in I} |f_i|(X_i)$ and each $|f_i|(X_i)$ is open by Proposition 3.81. □

Proposition 7.18. *For morphisms of $\mathbf{Sch}_{/S}$ the properties of being an open immersion and a closed immersion satisfy effective descent for fppf.*

Proof. By Corollary 6.33 we have to prove the following: if $F \to U$ is a morphism of $(\mathbf{Sch}_{/S}, \mathit{fppf})^{\sim}$ with $U \in \mathbf{Sch}_{/S}$ and $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}^{\mathit{fppf}}(U)$ is a covering such that $F \times_U U_i \cong V_i \in \mathbf{Sch}_{/S}$ and the induced morphism $f_i \colon V_i \to U_i$ is an open (respectively closed) immersion for every $i \in I$, then $F \cong V \in \mathbf{Sch}_{/S}$ and $V \to U$ is an open (respectively closed) immersion. If the f_i are open immersions, then $X_i := \mathrm{im}|f_i| \subseteq |U_i|$ are open subsets such that $X_i|_{U_{i,j}} = X_j|_{U_{i,j}}$ for all $i, j \in I$, whence (by Lemma 7.17) there exists a unique $X \subseteq |U|$ such that $X|_{U_i} = X_i$ for every $i \in I$. On the other hand, if the f_i are closed immersions, then the corresponding quasi-coherent ideals $\mathcal{I}_i$ of $\mathcal{O}_{U_i}$ are such that $\mathcal{I}_i|_{U_{i,j}} = \mathcal{I}_j|_{U_{i,j}}$ for all $i, j \in I$, whence (by Theorem 7.12) there exists a unique $\mathcal{I} \in \mathbf{QCoh}(U)$ such that $\mathcal{I}|_{U_i} \cong \mathcal{I}_i$ for every $i \in I$, and it is easy to see that $\mathcal{I}$ is in a natural way an ideal of $\mathcal{O}_U$. It is then clear that the open (respectively closed) immersion $V \hookrightarrow U$ corresponding to X (respectively $\mathcal{I}$) has the required property. □

Lemma 7.19. *For morphisms of $\mathbf{Sch}_{/S}$ the property of being quasi-compact is local on the codomain for fppf.*

Proof. Since (by Proposition 3.4) it is a property stable under base change and local on the codomain for *Zar*, by Remark 7.7 it is enough to prove that given a cartesian diagram in $\mathbf{Sch}_{/S}$

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "g"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

such that f is a faithfully flat morphism of finite presentation of affine schemes and g' is quasi-compact, then g is quasi-compact, too. Now, $X' = {g'}^{-1}(U')$ is quasi-compact, so that $X = h(X')$ is also quasi-compact, and the conclusion follows from Lemma 3.5. $\square$

Although the property of being (quasi)projective is certainly not of effective descent for *fppf* (it is not even local on the codomain for *Zar*, see Example 3.70), we are going to see that a stack for *fppf* is obtained if one considers morphisms together with a relatively ample invertible sheaf.

Given $U \in \mathbf{Sch}_{/S}$, we define $\mathbf{A}_U$ to be the category whose object are pairs $(q\colon X \to U, \mathcal{L})$ where q is a morphism of $\mathbf{Sch}_{/S}$ and $\mathcal{L}$ is a q-ample $\mathcal{O}_X$-module (recall that then q is quasi-compact and quasi-separated by Remark 3.45) and whose morphisms are given by

$$\begin{aligned} &\operatorname{Hom}_{\mathbf{A}_U}((q\colon X \to U, \mathcal{L}), (q'\colon X' \to U, \mathcal{L}')) \\ &\qquad\qquad := \{(s\colon X \to X', \alpha\colon s^*(\mathcal{L}') \to \mathcal{L}) \mid q' \circ s = q\} \end{aligned}$$

(composition of morphisms is of course defined by $(t,\beta)\circ(s,\alpha) := (t\circ s, \alpha \circ s^*(\beta))$, using as usual the canonical identification $s^* \circ t^* \cong (t \circ s)^*$). A morphism $f\colon V \to U$ of $\mathbf{Sch}_{/S}$ induces a natural functor $f^*\colon \mathbf{A}_U \to \mathbf{A}_V$, defined on objects by

$$(X \to U, \mathcal{L}) \mapsto (pr_2\colon X \times_U V \to V, pr_1^*(\mathcal{L}))$$

($pr_1^*(\mathcal{L})$ is pr_2-ample by Proposition 3.47), and as usual it is easy to see that in this way we obtain a fibred category $\mathbf{A} \in \underline{\mathbf{Fib}}_{\mathbf{Sch}_{/S}}$.

Proposition 7.20. *In the above notation,* $\mathbf{A} \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, fppf)}$.

Proof. Using the fact that $\mathbf{Mor}(\mathbf{Sch}_{/S})$ and $\overline{\mathbf{QCoh}}$ are prestacks (by Corollary 7.11 and Theorem 7.12), it is very easy to see that $\mathbf{A} \in \underline{\mathbf{PSt}}_{(\mathbf{Sch}_{/S}, fppf)}$. Then, given $U \in \mathbf{Sch}_{/S}$ and $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \operatorname{Cov}^{fppf}(U)$, we have to show that $\lambda^{\mathbf{A}}_{\mathcal{U}}\colon \mathbf{A}_U \to \mathbf{A}_{\mathcal{U}}$ is essentially surjective. For every descent datum $(\{(q_i\colon X_i \to U, \mathcal{L}_i)\}, \{(s_{i,j}, \alpha_{i,j})\})_{i,j\in I} \in \mathbf{A}_{\mathcal{U}}$, for all $i,j \in I$ we have a commutative diagram with cartesian squares

$$\begin{tikzcd} X_j \arrow[d, "q_j"'] \arrow[dr, phantom, "\square"] & X_j \times_{U_j} U_{i,j} \arrow[l, "pr_2'"'] \arrow[r, "s_{i,j}", "\sim"'] \arrow[dr, "q_j'"] & X_i \times_{U_i} U_{i,j} \arrow[r, "pr_1'"] \arrow[d, "q_i'"'] \arrow[dr, phantom, "\square"] & X_i \arrow[d, "q_j"] \\ U_j & U_{i,j} \arrow[l, "pr_2"] \arrow[rr, "pr_1"'] & & U_i \end{tikzcd}$$

and with this notation $\alpha_{i,j}\colon (pr'_1 \circ s_{i,j})^*(\mathcal{L}_i) \xrightarrow{\sim} {pr'_2}^*(\mathcal{L}_j)$. By Proposition 3.46 $\mathcal{R}_i := \bigoplus_{n\geq 0} q_{i*}(\mathcal{L}_i^{\otimes n}) \in \mathbf{QCohGAlg}(U_i)$ and q_i factors as the composition of an open immersion $f_i := \mathcal{P}roj\,\varphi_{q_i,\mathcal{L}_i}\colon X_i \hookrightarrow \mathcal{P}roj\,\mathcal{R}_i$ and of the structure morphism $p_{\mathcal{R}_i}\colon \mathcal{P}roj\,\mathcal{R}_i \to U_i$. By Proposition 3.74 for all $i,j \in I$ there are natural isomorphisms

$$pr_2^*(\mathcal{R}_j) \cong \bigoplus_{n\geq 0} pr_2^*\big(q_{j*}(\mathcal{L}_j^{\otimes n})\big) \cong \bigoplus_{n\geq 0} {q'_j}_*\big({pr'_2}^*(\mathcal{L}_j)^{\otimes n}\big) \cong$$
$$\cong \bigoplus_{n\geq 0} {q'_j}_*\big((pr'_1 \circ s_{i,j})^*(\mathcal{L}_i)^{\otimes n}\big) \cong \bigoplus_{n\geq 0} pr_1^*\big(q_{i*}(\mathcal{L}_i^{\otimes n})\big) \cong pr_1^*(\mathcal{R}_i)$$

(the middle one being induced by $\alpha_{i,j}^{-1}$) and it is straightforward to check that the resulting isomorphisms $\beta_{i,j}\colon pr_2^*(\mathcal{R}_j) \xrightarrow{\sim} pr_1^*(\mathcal{R}_i)$ of $\mathbf{QCohGAlg}(U_{i,j})$ satisfy the cocycle condition. Thus (by Corollary 7.15) there exist $\mathcal{R} \in \mathbf{QCohGAlg}(U)$ and isomorphisms $\gamma_i\colon \mathcal{R}|_{U_i} \xrightarrow{\sim} \mathcal{R}_i$ of $\mathbf{QCohGAlg}(U_i)$ such that $\beta_{i,j} = \gamma_i|_{U_{i,j}} \circ (\gamma_j|_{U_{i,j}})^{-1}$ for all $i,j \in I$. Setting $P := \mathcal{P}roj\,\mathcal{R}$ and $P_i := P \times_U U_i$ (note that $\{P_i \to P\}_{i\in I} \in \mathrm{Cov}^{fppf}(P)$), each γ_i induces an isomorphism $\tilde{\gamma}_i\colon P_i \xrightarrow{\sim} \mathcal{P}roj\,\mathcal{R}_i$ of $\mathbf{Sch}_{/U_i}$ (by Proposition 3.41), hence an open immersion $\tilde{f}_i := \tilde{\gamma}_i^{-1} \circ f_i\colon X_i \hookrightarrow P_i$. The isomorphisms

$$\tilde{s}_{i,j}\colon X_j \times_{P_j} P_{i,j} \cong X_j \times_{U_j} U_{i,j} \xrightarrow{s_{i,j}} X_i \times_{U_i} U_{i,j} \cong X_i \times_{P_i} P_{i,j}$$

of $\mathbf{Sch}_{/P_{i,j}}$ are easily seen to satisfy the cocycle condition, and this implies (by Proposition 7.18) that there exist an open immersion $f\colon X \hookrightarrow P$ and isomorphisms $\tilde{t}_i\colon X \times_P P_i \xrightarrow{\sim} X_i$ of $\mathbf{Sch}_{/P_i}$ such that $\tilde{s}_{i,j} = \tilde{t}_i|_{P_{i,j}} \circ (\tilde{t}_j|_{P_{i,j}})^{-1}$ for all $i,j \in I$. Then the induced isomorphisms $t_i\colon X \times_U U_i \cong X \times_P P_i \xrightarrow{\tilde{t}_i} X_i$ of $\mathbf{Sch}_{/U_i}$ satisfy $s_{i,j} = t_i|_{U_{i,j}} \circ (t_j|_{U_{i,j}})^{-1}$ for all $i,j \in I$. Note that $q := p_{\mathcal{R}} \circ f\colon X \to U$ is quasi-compact by Lemma 7.19 (the projection $pr_2 = q_i \circ t_i\colon X \times_U U_i \to U_i$ is quasi-compact for every $i \in I$) and quasi-separated (because $p_{\mathcal{R}}$ and f are quasi-separated), and that $\mathcal{U}' := \{g_i := pr_1 \circ t_i^{-1}\colon X_i \to X\}_{i\in I} \in \mathrm{Cov}^{fppf}(X)$. Moreover, the isomorphisms $\tilde{\alpha}_{i,j}\colon \mathcal{L}_j|_{X_{i,j}} \xrightarrow{\sim} \mathcal{L}_i|_{X_{i,j}}$ induced by $\alpha_{i,j}$ (under the natural isomorphisms $X_{i,j} \cong X_j \times_{U_j} U_{i,j}$) satisfy the cocycle condition, so that (by Theorem 7.12) there exists $\mathcal{L} \in \mathbf{QCoh}(X)$ such that $\lambda_{\mathcal{U}'}^{\overline{\mathbf{QCoh}}}(\mathcal{L}) \cong (\{\mathcal{L}_i\},\{\tilde{\alpha}_{i,j}\})_{i,j\in I}$. So it remains to prove that $\mathcal{L}$ is a q-ample invertible $\mathcal{O}_X$-module (because then clearly $\lambda_{\mathcal{U}}^{\mathbf{A}}(q,\mathcal{L}) \cong (\{(q_i,\mathcal{L}_i)\},\{(s_{i,j},\alpha_{i,j})\})_{i,j\in I})$.

Now, for every $x \in X$ there exist $i \in I$ and $x_i \in X_i$ such that $g_i(x_i) = x$, hence, for every open beighbourhood V_i of x_i, $g'_i := g_i|_{V_i}\colon V_i \to V := g_i(V_i)$ is faithfully flat locally of finite presentation and V is an open neighbourhood of x (by Proposition 3.81). Choosing V_i such that $\mathcal{L}_i|_{V_i} \cong \mathcal{O}_{V_i}$, we have then ${g'_i}^*(\mathcal{L}|_V) \cong \mathcal{O}_{V_i}$, which implies that $\mathcal{L}|_V \cong \mathcal{O}_V$ (again by Theorem 7.12), i.e. $\mathcal{L}$ is invertible. Now, note that $\mathcal{R} \cong \bigoplus_{n\geq 0} q_*(\mathcal{L}^{\otimes n}) \in \mathbf{QCohGAlg}(U)$ by Corollary 7.15, since their pullbacks to each U_i are isomorphic to $\mathcal{R}_i$ (this can be seen again using Proposition 3.74). In order to show that $\mathcal{L}$ is q-ample, by Proposition 3.46 it is enough to prove that $P(\varphi_{q,\mathcal{L}}) = X$ and that the natural morphism $\mathcal{P}roj\, \varphi_{q,\mathcal{L}} \colon X \to \mathcal{P}roj\, \mathcal{R}$ is an open immersion. Indeed, using Proposition 3.41 it is easy to see that $g_i^{-1}(P(\varphi_{q,\mathcal{L}})) = P(\varphi_{q_i,\mathcal{L}_i}) = X_i$ and that there is a cartesian diagram

$$\begin{tikzcd} X_i \arrow[r, "g_i"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\square"] & P(\varphi_{q,\mathcal{L}}) \arrow[d, "\mathcal{P}roj\, \varphi_{q,\mathcal{L}}"] \arrow[r, phantom, "\subseteq"] & X \\ \mathcal{P}roj\, \mathcal{R}_i \arrow[r] & \mathcal{P}roj\, \mathcal{R} & \end{tikzcd}$$

for every $i \in I$, which clearly implies that $P(\varphi_{q,\mathcal{L}}) = X$ and $\mathcal{P}roj\, \varphi_{q,\mathcal{L}} = f$. □

Corollary 7.21. *For morphisms of* $\mathbf{Sch}_{/S}$ *the property of being quasi-affine satisfies effective descent for fppf.*

Proof. We have to prove that $\mathbf{F} := \mathbf{Mor}^{\mathsf{P}}(\mathbf{Sch}_{/S}) \in \underline{\mathbf{St}}_{(\mathbf{Sch}_{/S}, fppf)}$, where P is the property of being quasi-affine; by Corollary 7.11 we already know that $\mathbf{F}$ is a prestack. Using Proposition 3.50 it is easy to see that there is a natural morphism of fibred categories $P \colon \mathbf{F} \to \mathbf{A}$, defined on objects by $(p \colon X \to U) \mapsto (p, \mathcal{O}_X)$. Now, we have to show that for every $U \in \mathbf{Sch}_{/S}$ and every $\mathcal{U} \in \mathrm{Cov}^{fppf}(U)$ the functor $\lambda^{\mathbf{F}}_{\mathcal{U}} \colon \mathbf{F}_U \to \mathbf{F}_{\mathcal{U}}$ is essentially surjective. Given $\xi \in \mathbf{F}_{\mathcal{U}}$, let $(p \colon X \to U, \mathcal{L}) \in \mathbf{A}_U$ be such that $\lambda^{\mathbf{A}}_{\mathcal{U}}(p, \mathcal{L}) \cong P_{\mathcal{U}}(\xi)$: it is then clear that (by Theorem 7.12) $\mathcal{L} \cong \mathcal{O}_X$ (so that p is quasi-affine, again by Proposition 3.50) and that $\lambda^{\mathbf{F}}_{\mathcal{U}}(p) \cong \xi$. □

Remark 7.22. The above result could be proved also directly, without using Proposition 7.20 (but with a similar technique of proof). Indeed, since (by Proposition 3.50) a quasi-compact morphism of $\mathbf{Sch}_{/S}$ is quasi-affine if and only if the natural morphism $X \to \mathcal{S}pec\, f_*(\mathcal{O}_X)$ is an open immersion, one can use the fact that $\overline{\mathbf{QCohAlg}}$ is a stack (and, again, that the property of being an open immersion satisfies effective descent and the property of being quasi-compact is local on the codomain). We leave the details to the reader.

7.4 Local properties

Proposition 7.23. *The following properties of morphisms of* $\mathbf{Sch}_{/S}$ *are stable under base change and local on the codomain for fppf: open immersion, closed immersion, affine, quasi-affine, quasi-compact, separated, quasi-separated, (locally) of finite type (and with fibres of dimension* n*), (locally) of finite presentation, finite, (locally) quasi-finite, surjective, universally injective, universally open, universally closed, proper, (faithfully) flat, unramified, smooth (of relative dimension* n*), étale, with geometrically connected or geometrically irreducible or geometrically reduced fibres.*

Proof. The case of geometrically connected or geometrically irreducible or geometrically reduced fibres follows from the fact that, if $\mathbb{K}$ is a field, then $X \in \mathbf{Sch}_{/\mathbb{K}}$

is geometrically connected (respectively geometrically irreducible, respectively geometrically reduced) if and only if $X \times_{\operatorname{Spec} \mathbb{K}} \operatorname{Spec} \mathbb{K}'$ is connected (respectively irreducible, respectively reduced) for some algebraically closed extension $\mathbb{K}'$ of $\mathbb{K}$ (see [9, Prop. 4.5.1. and Prop. 4.6.1]). Similarly, once we have proved the statement for locally of finite type, the case of n-dimensional fibres will follow from the fact that, if $X \to \operatorname{Spec} \mathbb{K}$ is a morphism locally of finite type, then $\dim X = \dim(X \times_{\operatorname{Spec} \mathbb{K}} \operatorname{Spec} \mathbb{K}')$ for every field extension $\mathbb{K}'$ of $\mathbb{K}$ (by [9, Cor. 4.1.4. and Cor. 5.2.2]). We already know that all the other properties are stable under base change and local on the codomain for *Zar*. Recall also that we have already treated open or closed immersions (in Proposition 7.18), affine (in Proposition 7.16), quasi-affine (in Corollary 7.21) and quasi-compact (in Lemma 7.19). Then separated and quasi-separated follow from Lemma 4.50, since a morphism f of $\mathbf{Sch}_{/S}$ is separated (respectively quasi-separated) if and only if Δ_f is a closed immersion (respectively quasi-compact). Let P be one of the remaining properties: by Remark 7.7 it is enough to show that, if

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "g"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

is a cartesian diagram in $\mathbf{Sch}_{/S}$ such that f is a faithfully flat morphism locally of finite presentation and g' satisfies P, then g satisfies P, too.

As for surjective, universally injective, universally open and universally closed, it is clearly enough to prove that, if g' is surjective, or injective, or open or closed, then the same is true for g. As f is surjective, it is very easy to see that if g' is surjective (respectively injective), then g is surjective (respectively injective). Since moreover f is open (by Proposition 3.81), it is clear that $W \subseteq |U|$ is open (respectively closed) if and only if $|f|^{-1}(W) \subseteq |U'|$ is open (respectively closed). Thus, if g' is open (respectively closed) and $Z \subseteq |X|$ is open (respectively closed), then the same is true for $|g|(Z)$, as $|f|^{-1}(|g|(Z)) = |g'|(|h|^{-1}(Z))$ is clearly open (respectively closed).

For the rest of the proof we assume (always by Remark 7.7) that U and U' are affine. Notice moreover that (except for finite) we are only left with properties local on the domain for *Zar* (those which are not, namely of finite type, of finite presentation, quasi-finite, proper and faithfully flat, are combinations of properties which either are local on the domain for *Zar* or have already been proved), so that in any case we can assume that X (hence also X') is affine, too. Let $\phi\colon A \to A'$, $\psi\colon A \to B$ and $\psi'\colon A' \to B' := A' \otimes_A B$ be morphisms of rings inducing f, g and g' (then ϕ is faithfully flat and of finite presentation). As for locally of finite type or presentation, one has to show that ψ is of finite type (respectively presentation) if ψ' is of finite type (respectively presentation), which is done in [9, Lemme 2.7.1.1]. Similarly, in the finite case, we need to show that ψ is finite if ψ' is finite. More generally, $M \in A$-$\mathbf{Mod}$ is finitely generated if $M' := A' \otimes_A M \in A'$-$\mathbf{Mod}$ is finitely generated: indeed, it is easy to see that there exist $x_1, \ldots, x_n \in M$ such that $M' = (1 \otimes x_1, \ldots, 1 \otimes x_n)$, and then the morphism $\alpha\colon A^n \to M$ defined by the x_i is surjective by Lemma 3.77 ($A' \otimes_A \alpha$ is surjective by hypothesis), whence $M = (x_1, \ldots, x_n)$. If g' is locally quasi-finite, then g is locally quasi-finite by Remark 3.56: we have to prove that, given an extension of fields $\mathbb{K} \subseteq \mathbb{K}'$ and a finitely generated $\mathbb{K}$-algebra C, then $\dim_{\mathbb{K}} C < \infty$ if $\dim_{\mathbb{K}'}(\mathbb{K}' \otimes_{\mathbb{K}} C) < \infty$, which is obvious. As for flat, we have to prove that ψ is flat if ψ' is flat; given an exact sequence of A-modules $M^\bullet$, $B' \otimes_A M^\bullet \cong B' \otimes_B (B \otimes_A M^\bullet)$ is exact because $\psi' \circ \phi\colon A \to B'$ is flat, whence $B \otimes_A M^\bullet$ is exact by Lemma 3.77 (note that $B \to B'$ is faithfully flat). If g' is smooth, then g is smooth by [17, Prop. 4.6], and it is clear that relative dimension is preserved (being the rank of the sheaf of relative differentials); in particular, if g' is étale (i.e., smooth of relative dimension 0), then g is étale. Finally, in the unramified case, by Proposition 3.115 we can look at the fibres and then it is enough to observe that a morphism with target $\operatorname{Spec} \mathbb{K}$ (being always flat) by Proposition 3.125 is unramified if and only if it is étale. □

Remark 7.24. The above result implies that the property of being finite also satisfies effective descent for *fppf*, since the same is true for the property of being affine (by Proposition 7.16) and a finite morphism is affine.

Remark 7.25. We do not know if the property of being an immersion is local on the codomain for *fppf* (or if it even satisfies effective descent, as stated in [14]): this is claimed in [15], but referring to [9, Prop. 2.7.1], where it is proved only for quasi-compact immersion (which is then also a property of effective descent for *fppf*, as a quasi-compact immersion is quasi-affine).

Example 7.26. Let $\mathbf{F} := \mathbf{Mor}^{\mathsf{P}}(\mathbf{Sch}_{/S}) \in \underline{\mathbf{Fib}}_{\mathbf{Sch}_{/S}}$, where P is the property of being proper, smooth and with fibres which are geometrically irreducible curves of genus g (for some $g \geq 2$). Then $\mathbf{F}$ is a stack for *fppf*, and (by Corollary 6.16) the same is true for $\mathbf{M}_g := \mathbf{F}^{\mathrm{cart}}$, which is called the *moduli stack of (smooth) curves of genus* g. Using the fact that, if $p\colon X \to U$ is an object of $\mathbf{F}$, then $\Omega_{X/U}$ is a p-ample invertible $\mathcal{O}_X$-module (since $\Omega_{X/U}^{\otimes n}$ is p-very ample for $n \geq 3$ by [4, Cor. to Thm. 1.2], this follows from Proposition 3.61), and remembering that the sheaf of relative differentials is preserved under base change, the proof is similar to that of Corollary 7.21. Indeed, $\mathbf{F}$ is a prestack, there is a morphism of fibred categories $P\colon \mathbf{F} \to \mathbf{A}$ defined on objects by $(p\colon X \to U) \mapsto (p, \Omega_{X/U})$, and in the same way it follows that, given $\mathcal{U} \in \mathrm{Cov}^{fppf}(U)$ and $\xi \in \mathbf{F}_{\mathcal{U}}$, there exists a morphism $p\colon X \to U$ such that $\lambda_{\mathcal{U}}^{\mathbf{Mor}(\mathbf{Sch}_{/S})}(p) \cong \xi$. To conclude, we have only to show that p satisfies P: by Proposition 7.23 p is proper, smooth and its fibres are geometrically irreducible curves, and then it is obvious that they are of genus g.

Proposition 7.27. *The following properties of objects of* $\mathbf{Sch}_{/S}$ *are local for sm: locally noetherian (it is local also for fppf), normal, reduced, regular.*

Proof. We prove here only the case of locally notherian; for the other properties, see [9, Prop. 17.5.7 and Prop. 17.5.8]. By Remark 7.7 we have to prove that if

$\phi\colon A \to B$ is a faithfully flat morphism of finite presentation of rings, then A is noetherian if and only if B is noetherian. The other implication being clear, we can assume that B is noetherian. If $\mathtt{I}_1 \subseteq \mathtt{I}_2 \subseteq \cdots$ is an ascending chain of ideals in A, then, since ϕ is flat, each $\mathtt{J}_i := B \otimes_A \mathtt{I}_i$ is an ideal of B and $\mathtt{J}_1 \subseteq \mathtt{J}_2 \subseteq \cdots$ is an ascending chain in B. As B is noetherian, this implies that there exists $n \in \mathbb{N}$ such that $\mathtt{J}_n = \mathtt{J}_m$ for every $m \geq n$. Therefore, for $m \geq n$ we have $0 = \mathtt{J}_m/\mathtt{J}_n \cong B \otimes_A (\mathtt{I}_m/\mathtt{I}_n)$, whence $\mathtt{I}_m = \mathtt{I}_n$ (by Lemma 3.77), which proves that A is noetherian. □

Proposition 7.28. *The following properties of morphisms of* $\mathbf{Sch}_{/S}$ *are local on the domain for ét: universally open, locally of finite type, locally of finite presentation, flat (these are local also for fppf), smooth (it is local also for sm), locally quasi-finite, unramified, smooth of relative dimension* n*, étale.*

Proof. Since each property P in the above list is local on the domain for *Zar*, by Remark 7.7 we have to prove the following: given morphisms $U' \xrightarrow{f} U \xrightarrow{g} X$ of $\mathbf{Sch}_{/S}$ with f flat of finite presentation (or smooth, or étale, according to the cases) and surjective of affine schemes, g satisfies P if and only if $g \circ f$ satisfies P. As P is local on the codomain for *Zar*, we can assume that also X is affine, and as (except for smooth of relative dimension n, which will be clear, however) f satisfies P (f is open, hence universally open, by Proposition 3.81, and an étale morphism is locally quasi-finite by Proposition 3.125) and P is stable under composition, it is enough to show that g satisfies P if $g \circ f$ does. If $g \circ f$ is open then g is also open (if $V \subseteq U$ is open, then $g(V) = (g \circ f)(f^{-1}(V))$ is open), and this implies that g is universally open if so is $g \circ f$. In the case of locally of finite type or presentation, we have to prove that a morphism of rings $A \to B$ is of finite type (respectively presentation) if there exists a faithfully flat morphism of finite presentation $B \to B'$ such that $A \to B'$ is of finite type (respectively presentation), which is done in [9, Cor. 11.3.17]. As for flat, in the same situation, we have to show that $A \to B$ is flat if $A \to B'$ is flat, which we have already done in the proof of Proposition 7.23. The smooth case follows from [9, Cor. 17.7.7] and smooth of relative dimension n (in particular, étale) is then clear; again as in the proof of Proposition 7.23, unramified can be reduced to étale by looking at the fibres. So it remains to consider locally quasi-finite: by Remark 3.56 it is enough to prove that if C is a finitely generated $\mathbb{K}$-algebra ($\mathbb{K}$ a field) and $C \to C'$ is a faithfully flat morphism such that $\dim_{\mathbb{K}} C' < \infty$, then $\dim_{\mathbb{K}} C' < \infty$, which follows from the fact that $C \to C'$ is injective by Proposition 7.1. □

8 Algebraic spaces

8.1 Algebraic spaces and étale equivalence relations

We fix a base quasi-separated scheme S. Observe that (by Corollary 3.12) an object $U \to S$ of $\mathbf{Sch}_{/S}$ is in $\mathbf{QSch}_{/S}$ (i.e., U is a quasi-separated scheme) if and only if it is quasi-separated (as a morphism of schemes). Following [15] we will denote by $\mathbf{Sp}_{/S}$ the category $(\mathbf{QSch}_{/S}, \acute{e}t)^{\sim}$, and call its objects *spaces over* S (or S*-spaces*, or simply *spaces*).

Remark 8.1. Since $\acute{e}t$ is a subcanonical pretopology (by Theorem 7.8), $\mathbf{Sp}_{/S}$ is naturally equivalent to $(\mathbf{QSch}, \acute{e}t)^{\sim}_{/S}$ by Proposition 4.42, so that $\mathbf{Sp}_{/S}$ can be really identified with the quotient category of $\mathbf{Sp} := \mathbf{Sp}_{/\mathbb{Z}}$ over S. Notice also that (by Corollary 4.45) $\mathbf{Sp}_{/S}$ is naturally equivalent to $(\mathbf{AffSch}_{/S}, \acute{e}t)^{\sim}$ and to $(\mathbf{Sch}_{/S}, \acute{e}t)^{\sim}$; hence we could as well use $\mathbf{AffSch}_{/S}$ (as in [15]) or $\mathbf{Sch}_{/S}$ as base category. Thus in $\mathbf{Sp}_{/S}$ there are several possible notions of representability, and we give the following definition (corresponding to representability in $\widehat{\mathbf{QSch}_{/S}}$), reserving another meaning to the term representable (see Definition 8.23).

Definition 8.2. An S-space F is *schematic* if there exists $U \in \mathbf{QSch}_{/S}$ such that $F \cong U$ in $\mathbf{Sp}_{/S}$. A morphism $\alpha\colon F \to G$ of $\mathbf{Sp}_{/S}$ is *schematic* if for every $V \in \mathbf{QSch}_{/S}$ and every $\eta \in G(V)$ the S-space $F\,{}_{\alpha}{\times}_{\eta}\,V$ is schematic.

Recall from Proposition 4.11 and Remark 4.13 that for morphisms of $\mathbf{Sp}_{/S}$ the property of being schematic is stable under composition and base change, and that properties of morphisms of $\mathbf{QSch}_{/S}$ which are stable under base change can be naturally extended to schematic morphisms of $\mathbf{Sp}_{/S}$.

Remark 8.3. A morphism of $\mathbf{Sp}_{/S}$ is schematic if the condition of the definition is satisfied for every $V \in \mathbf{AffSch}_{/S}$. Indeed, taking into account that $\mathbf{Mor}(\mathbf{Sch}_{/S})$ is a stack for *Zar* (see Example 6.6) and that for morphisms of $\mathbf{Sch}_{/S}$ the property of being quasi-separated is local on the codomain for *Zar*, this follows from Corollary 6.33.

Definition 8.4. $X \in \mathbf{Sp}_{/S}$ is an *algebraic space over* S (or an S*-algebraic space*, or simply an *algebraic space*) if the following conditions are satisfied:

1. $\Delta_X\colon X \to X \times X$ is schematic and quasi-compact;
2. there exist $U \in \mathbf{QSch}_{/S}$ and a morphism $\pi\colon U \to X$ of $\mathbf{Sp}_{/S}$ (necessarily schematic, by Proposition 4.14) which is étale and surjective.

We will denote by $\mathbf{AlgSp}_{/S}$ the full subcategory of $\mathbf{Sp}_{/S}$ whose objects are algebraic spaces; it is clearly a strictly full subcategory.

Remark 8.5. Obviously a scheme is an algebraic space if and only if it is quasi-separated. In particular, $\mathbf{QSch}_{/S}$ can be identified with a full subcategory of $\mathbf{AlgSp}_{/S}$.

Remark 8.6. Let's call a morphism $\alpha\colon F \to G$ of $\mathbf{Sp}_{/S}$ *weakly schematic* if for every $V \in \mathbf{Sch}_{/S}$ and every $\eta \in G(V)$ the S-space $F\,{}_{\alpha}{\times}_{\eta}\,V$ is a scheme. As in Remark 8.3, it is enough to check the above condition for $V \in \mathbf{AffSch}_{/S}$, whence every schematic morphism is weakly schematic (of course, the converse is false).

It is easy to see that one obtains an equivalent definition of algebraic space if schematic is replaced by weakly schematic and U is assumed to be an object of $\mathbf{Sch}_{/S}$ instead of $\mathbf{QSch}_{/S}$. Indeed, Δ_X is schematic if it is weakly schematic: since Δ_X is a monomorphism, for every morphism $V \to X \times X$ with $V \in \mathbf{QSch}_{/S}$ the induced morphism $W := V \times_{X \times X} X \to V$ is also a monomorphism, hence quasi-separated by Example 3.9, and then $W \in \mathbf{QSch}_{/S}$. Moreover, for every $U \in \mathbf{Sch}_{/S}$ there exists an étale and surjective morphism $f\colon U' \to U$ with U' quasi-separated (we can clearly take U' to be a disjoint union of affine schemes), so that $\pi \circ f\colon U' \to X$ is étale and surjective, if the same is true for π.

Lemma 8.7. *Let* P *be a property of morphisms of schemes which is stable under base change and local on the codomain for ét. If* $f\colon X \to Y$ *is a schematic morphism of* $\mathbf{AlgSp}_{/S}$ *and there exists a schematic, étale and surjective morphism* $V \to Y$ *(with* V *schematic) such that the projection* $g\colon U := X \times_Y V \to V$ *satisfies* P*, then* f *satisfies* P*.*

Proof. Given a morphism $V' \to Y$ with V' schematic, we have to prove that the projection $g'\colon U' := X \times_Y V' \to V'$ satisfies P. Setting $V'' := V \times_Y V'$ and $U'' := X \times_Y V''$, there is a natural commutative diagram with cartesian squares

$$\begin{tikzcd} U \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & U'' \arrow[l] \arrow[r] \arrow[d, "g''"'] \arrow[dr, phantom, "\square"] & U' \arrow[d, "g'"] \\ V & V'' \arrow[l] \arrow[r, "pr_2"'] & V' \end{tikzcd}$$

(all terms of which are schematic). As g satisfies P by hypothesis, the same is true for g'' (because P is stable under base change), and then also for g' (because P is local on the codomain for *ét* and pr_2 is étale and surjective). □

Lemma 8.8. *Let* $\alpha\colon F \to G$ *and* $\beta\colon Y \to G$ *be morphisms of* $\mathbf{Sp}_{/S}$ *such that* α *is an epimorphism and* Y *is an algebraic space. Then there is a commutative diagram in* $\mathbf{Sp}_{/S}$

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d] & Y \arrow[d, "\beta"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

such that V *is schematic and* π *is (schematic) étale and surjective.*

Proof. Up to composing β with a schematic, étale and surjective morphism $U \to Y$ with U schematic, we can assume $Y \in \mathbf{QSch}_{/S}$. As $\beta \in G(Y) = (\widetilde{\operatorname{im} \alpha})(Y)$ (by Proposition 4.57), there exists $\{V_i \to Y\}_{i \in I} \in \operatorname{Cov}^{\acute{e}t}(Y)$ such that each $\beta|_{V_i}$ factors through α. Then the induced morphism $\pi\colon V := \coprod_{i \in I} V_i \to Y$ is étale and surjective and $\beta \circ \pi$ factors through α. □

Proposition 8.9. *The category* $\mathbf{AlgSp}_{/S}$ *has coproducts and fibred products (and then also finite products, since it has a terminal object, namely* S*) and the inclusion functors* $\mathbf{QSch}_{/S} \subseteq \mathbf{AlgSp}_{/S} \subseteq \mathbf{Sp}_{/S}$ *preserve them.*

Proof. Given $U_i \in \mathbf{QSch}_{/S}$ $(i \in I)$, $U := \coprod_{i\in I} U_i$ represents the coproduct also in $\mathbf{Sp}_{/S}$ (actually even in $(\mathbf{QSch}_{/S}, Zar)^{\sim}$): this follows from the fact that, setting $\mathcal{U} := \{U_i \to U\}_{i\in I} \in \mathrm{Cov}^{Zar}(U)$, $\lambda^F_{\mathcal{U}} \colon F(U) \xrightarrow{\sim} F(\mathcal{U}) \cong \prod_{i\in I} F(U_i)$ for every S-space F, since $U_i \times_U U_j = \emptyset$ if $i \neq j$ and $F(\emptyset)$ has exactly one element (to see this, consider the empty covering of $\emptyset$), whereas $U_i \times_U U_i \cong U_i$. To conclude the proof about coproducts, we have to show that, given $X_i \in \mathbf{AlgSp}_{/S}$ $(i \in I)$, $X := \coprod_{i\in I} X_i \in \mathbf{Sp}_{/S}$ is an algebraic space: it is very easy to see that Δ_X is schematic and quasi-compact and that, if $U_i \to X_i$ (with $U_i \in \mathbf{QSch}_{/S}$) are étale and surjective, then the induced morphism $\coprod_{i\in I} U_i \to X$ is étale and surjective.

As for fibred products, since we already know (by Proposition 4.8 and Proposition 4.54) that the inclusion functor $\mathbf{QSch}_{/S} \subseteq \mathbf{Sp}_{/S}$ preserves them, it is enough to prove that, given morphisms $f_i \colon X_i \to X$ of $\mathbf{AlgSp}_{/S}$ (for $i = 1, 2$), $X_1 \times_X X_2 \in \mathbf{Sp}_{/S}$ is an algebraic space. Now, the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\Delta_{X_1}"'] & X \arrow[d, "\Delta_X"'] & X_2 \arrow[l, "f_2"'] \arrow[d, "\Delta_{X_2}"] \\
X_1 \times X_1 \arrow[r, "f_1\times f_1"'] & X \times X & X_2 \times X_2 \arrow[l, "f_2\times f_2"]
\end{tikzcd}$$

obviously commutes, and so $\Delta_{X_1\times_X X_2}$, which can be identified with

$$\Delta_{X_1} \times \Delta_{X_2} \colon X_1 \times_X X_2 \to (X_1 \times X_1) \times_{(X\times X)} (X_2 \times X_2),$$

is schematic and quasi-compact by Lemma A.6. Moreover, given a schematic, étale and surjective morphism $\pi \colon U \to X$ with U schematic (π is then an epimorphism of $\mathbf{Sp}_{/S}$ by Corollary 4.60), by Lemma 8.8 there exists a commutative diagram

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d, "\pi_1"'] & U \arrow[d, "\pi"'] & U_2 \arrow[l] \arrow[d, "\pi_2"] \\
X_1 \arrow[r, "f_1"'] & X & X_2 \arrow[l, "f_2"]
\end{tikzcd}$$

such that U_i is schematic and π_i is étale and surjective for $i = 1, 2$. Then $U_1 \times_U U_2 \in \mathbf{QSch}_{/S}$ and $\pi_1 \times \pi_2 \colon U_1 \times_U U_2 \to X_1 \times_X X_2$ is étale and surjective, again by Lemma A.6. □

Remark 8.10. If X is an algebraic space and $\pi \colon U \to X$ (with U schematic) is étale and surjective, then, in particular, $X \times X$ is also an algebraic space and

$\pi \times \pi \colon U \times U \to X \times X$ is étale and surjective. Since there is a cartesian diagram

$$\begin{tikzcd} U \times_X U \arrow[r] \arrow[d, "\delta"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_X"] \\ U \times U \arrow[r, "\pi \times \pi"'] & X \times X, \end{tikzcd}$$

by Lemma 8.7 Δ_X satisfies a property P of morphisms of schemes which is stable under base change and local on the codomain for *ét* if (and only if) δ satisfies P. So, taking into account Proposition 7.23, we see that Δ_X is locally of finite type (δ is locally of finite type by Proposition 3.27, since $pr_1 \circ \delta \colon U \times_X U \to U$ is étale), hence of finite type (it is quasi-compact by hypothesis); then (δ being clearly a monomorphism) we obtain that Δ_X is also quasi-finite, separated and quasi-affine by Proposition 3.58 and unramified by Example 3.91. More generally, if $f \colon X \to Y$ is a morphism of $\mathbf{AlgSp}_{/S}$, then $\Delta_f \colon X \to X \times_Y X$ satisfies all the above properties (in particular, it is schematic and quasi-compact): indeed, if P is one of them, as for morphisms of $\mathbf{AlgSp}_{/S}$ the property of having diagonal which satisfies P is stable under composition and base change by Lemma A.4, Δ_f satisfies P by Lemma A.5, since Δ_X satisfies P and the same is true for Δ_{Δ_Y} (which is an isomorphism by Lemma A.2). With this argument it is also easy to see that (as expected) $\mathbf{AlgSp}_{/S}$ can be really identified with the quotient category of $\mathbf{AlgSp} := \mathbf{AlgSp}_{/\mathbb{Z}}$ over S.

Theorem 8.11. *If $X \in \mathbf{AlgSp}_{/S}$ and $\pi \colon U \to X$ (with $U \in \mathbf{QSch}_{/S}$) is (schematic) étale and surjective, then the natural morphism*

$$\delta = (\delta_1, \delta_2) \colon U \times_X U \to U \times U$$

is quasi-compact and it defines an étale equivalence relation in $\mathbf{QSch}_{/S}$ with quotient π in $\mathbf{Sp}_{/S}$.

Conversely, if $\delta = (\delta_1, \delta_2) \colon R \to U \times U$ is an étale equivalence relation in $\mathbf{QSch}_{/S}$ such that δ is quasi-compact and with quotient $\pi \colon U \to X$ in $\mathbf{Sp}_{/S}$, then $X \in \mathbf{AlgSp}_{/S}$, π is (schematic) étale and surjective, and $R \cong U \times_X U$.

Proof. If X is an algebraic space and π is étale and surjective, then δ_1 and δ_2 are étale and surjective in $\mathbf{QSch}_{/S}$ (hence the equivalence relation is étale) and π is an epimorphism of $\mathbf{Sp}_{/S}$ by Corollary 4.60, so that π is a quotient of δ by Corollary 4.83. Moreover, δ is quasi-compact by Remark 8.10.

Conversely, we observe that δ is quasi-affine (by the same argument of Remark 8.10), and we are going to see that also Δ_X is schematic and quasi-affine. To this purpose it suffices to prove (by Proposition 4.14) that, given morphisms $f \colon V \to X$ and $g \colon W \to X$ with $V, W \in \mathbf{QSch}_{/S}$, $V \times_X W$ is schematic and the natural morphism $\delta^{f,g} = (\delta_1^{f,g}, \delta_2^{f,g}) \colon V \times_X W \to V \times W$ is quasi-affine; notice moreover that π will be étale and surjective, provided we show that $\delta_1^{f,\pi} \colon V \times_X U \to V$ is étale and surjective for every f as above. First we assume that $g = \pi \colon U \to X$

and that f factors through π, say $f = \pi \circ f'$ for some $f' \colon V \to U$; then there is a natural commutative diagram with cartesian squares

$$\begin{tikzcd} V \times_X U \arrow[r, "\delta^{f,\pi}"] \arrow[d] \arrow[dr, phantom, "\square"] & V \times U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "f'"] \\ U \times_X U \arrow[r, "\delta^{\pi,\pi}"'] & U \times U \arrow[r, "pr_1"'] & U. \end{tikzcd}$$

Note that $U \times_X U \cong R$ (by Corollary 4.83) and $\delta^{\pi,\pi}$ can be identified with δ, so that $V \times_X U$ is schematic, $\delta^{f,\pi}$ is quasi-affine and $\delta_1^{f,\pi}$ is étale and surjective by base change. For arbitrary f (and always $g = \pi$) by Lemma 8.8 we can find a commutative diagram

$$\begin{tikzcd} V' \arrow[r, "p"] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r, "\pi"'] & X \end{tikzcd}$$

such that V' is schematic and p is étale and surjective. Again, there is a natural commutative diagram with cartesian squares

$$\begin{tikzcd} V' \times_X U \arrow[r, "\delta^{f\circ p,\pi}"] \arrow[d] \arrow[dr, phantom, "\square"] & V' \times U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V' \arrow[d, "p"] \\ V \times_X U \arrow[r, "\delta^{f,\pi}"'] & V \times U \arrow[r] & V. \end{tikzcd}$$

By the already proved case ($f \circ p$ factors through π) $V' \times_X U$ is schematic, $\delta^{f\circ p,\pi}$ is quasi-affine and $\delta_1^{f\circ p,\pi}$ is étale and surjective. Since the property of being quasi-affine satisfies effective descent for *ét* (by Corollary 7.21), it follows from Corollary 6.33 that $V \times_X U$ is schematic and $\delta^{f,\pi}$ is quasi-affine; then $\delta_1^{f,\pi}$ is étale and surjective because the property of being étale and surjective is local on the codomain for *ét* (by Proposition 7.23). The general case can be dealt with in a similar way (first when g factors through π and then for arbitrary g). □

Remark 8.12. With the same proof one can show that, more generally, a quotient in $\mathbf{Sp}_{/S}$ of an étale equivalence relation $\delta \colon R \to U \times U$ in $\mathbf{Sch}_{/S}$ with δ quasi-compact is also an algebraic space (according to the equivalent definition of Remark 8.6).

Corollary 8.13. *$X \in \mathbf{Sp}_{/S}$ is an algebraic space if and only if there exists a schematic, étale and surjective morphism $\pi \colon U \to X$ with U schematic and such that the natural morphism (of $\mathbf{QSch}_{/S}$) $\delta \colon U \times_X U \to U \times U$ is quasi-compact.*

Proof. If π satisfies the hypothesis, then it is an epimorphism of $\mathbf{Sp}_{/S}$ by Corollary 4.60, hence it is a quotient of δ (which is an étale equivalence relation in $\mathbf{QSch}_{/S}$) by Corollary 4.83; thus X is an algebraic space. The other implication is clear, since δ is quasi-compact by Remark 8.10. $\square$

Example 8.14. Let $\delta\colon R \to \mathbb{A}^1 \times \mathbb{A}^1$ be as in Example 4.77 (we are assuming that there exists $s \in S$ such that the characteristic of $\kappa(s)$ is not 2): it is clearly an étale equivalence relation in $\mathbf{QSch}_{/S}$ such that δ is quasi-compact. It follows that, if $\pi\colon \mathbb{A}^1 \to X$ is a quotient in $\mathbf{Sp}_{/S}$, then X is an algebraic space (by Theorem 8.11) which is not a scheme (Δ_X is not an immersion because δ, which is obtained from Δ_X by the base change $\pi \times \pi$, is not an immersion).

8.2 Extension to algebraic spaces of some properties of schemes

Definition 8.15. A morphism $f\colon X \to Y$ of $\mathbf{AlgSp}_{/S}$ is *separated* (respectively *locally separated*) if $\Delta_f\colon X \to X \times_Y X$ (which is a schematic morphism by Remark 8.10) is a closed immersion (respectively an immersion). $X \in \mathbf{AlgSp}_{/S}$ is *(locally) separated* if the structure morphism $X \to S$ is (locally) separated.

Remark 8.16. Of course, every morphism f of algebraic spaces is quasi-separated, in the sense that (again by Remark 8.10) Δ_f is always quasi-compact (although not necessarily an immersion). Every scheme is locally separated and X as in Example 8.14 is an example of an algebraic space which is not locally separated. There exist also separated algebraic spaces which are not schematic (for examples, see the introduction of [14]).

Definition 8.17. Let P be a property of schemes which is local for *ét* (e.g., one of those of Proposition 7.27). Then $X \in \mathbf{AlgSp}_{/S}$ satisfies P if there exists a schematic, étale and surjective morphism $U \to X$ such that $U \in \mathbf{QSch}_{/S}$ satisfies P.

Remark 8.18. If X satisfies P and $U' \to X$ (with U' schematic) is étale and surjective, then U' satisfies P: this follows from the fact that $U'' := U \times_X U'$ is schematic and the projections $U'' \to U$ and $U'' \to U'$ are étale and surjective. In particular, the new definition coincides with the usual one if $X \in \mathbf{QSch}_{/S}$.

Definition 8.19. Let P be a property of morphisms of schemes which is local on the domain and on the codomain for *ét* (e.g., one of those of Proposition 7.28) or the property of being surjective.[15] Then a morphism $f\colon X \to Y$ of $\mathbf{AlgSp}_{/S}$

[15] This property (which is not even local on the domain for *Zar*) is usually (e.g., in [14] and [15]) considered to be local on the domain for *ét* (and also for *fppf*), since it clearly satisfies the condition of the definition for coverings consisting of a single morphism (which is what is needed here).

satisfies P if there exists a commutative diagram with cartesian square in $\mathbf{AlgSp}_{/S}$

$$\begin{tikzcd} U \arrow[r, "\tilde{\pi}"] \arrow[dr, "g"'] & \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & V \arrow[r, "\pi"'] & Y \end{tikzcd}$$

such that g is a morphism of $\mathbf{QSch}_{/S}$ which satisfies P and π and $\tilde{\pi}$ are étale and surjective.

Remark 8.20. If f satisfies P and

$$\begin{tikzcd} U' \arrow[r, "\tilde{\pi}'"] \arrow[dr, "g'"'] & \tilde{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & V' \arrow[r, "\pi'"'] & Y \end{tikzcd}$$

is another commutative diagram with cartesian square in $\mathbf{AlgSp}_{/S}$ such that g' is a morphism of $\mathbf{QSch}_{/S}$ and π' and $\tilde{\pi}'$ are étale and surjective, then g' satisfies P: this can be easily proved using the fact that, if f is a morphism of $\mathbf{QSch}_{/S}$, then f satisfies P if and only if g (or g') does (which implies, taking into account Lemma 8.7, that the new definition coincides with the usual one for schematic morphisms). It is also easy to see that, if P is stable under base change (and composition) for morphisms of $\mathbf{QSch}_{/S}$, then P remains stable under base change (and composition) for morphisms of $\mathbf{AlgSp}_{/S}$.

Then we can extend in a natural way to $\mathbf{AlgSp}_{/S}$ the pretopologies *ét*, *sm* and *fppf*. Namely, if τ is one of them, by definition $\{f_i \colon X_i \to X\}_{i\in I} \in \mathrm{Cov}^{\tau}(X)$ if and only if the induced morphism $\coprod_{i\in I} X_i \to X$ is surjective and moreover for every $i \in I$ the following holds: f_i is étale if $\tau = \textit{ét}$; f_i is smooth if $\tau = \textit{sm}$; f_i is flat and locally of finite presentation if $\tau = \textit{fppf}$.

Remark 8.21. If P is a property of schemes which is local for τ, then its extension to objects of $\mathbf{AlgSp}_{/S}$ is again local for τ. Similarly, if P is a property of morphisms of schemes which is local on the domain and on the codomain for τ, then its extension to morphisms of $\mathbf{AlgSp}_{/S}$ is again local on the domain and on the codomain for τ.

Remark 8.22. The natural functor $(\mathbf{AlgSp}_{/S}, \tau)^{\sim} \to (\mathbf{QSch}_{/S}, \tau)^{\sim}$ is an equivalence of categories by Proposition 4.44 (and, similarly, the natural 2-functor $\underline{\mathbf{St}}_{(\mathbf{AlgSp}_{/S},\tau)} \to \underline{\mathbf{St}}_{(\mathbf{QSch}_{/S},\tau)}$ is a lax 2-equivalence by Proposition 6.20). In particular, $\mathbf{Sp}_{/S}$ can be identified with $(\mathbf{AlgSp}_{/S}, \textit{ét})^{\sim}$, which justifies the following definition (corresponding to representability in $\widehat{\mathbf{AlgSp}_{/S}}$).

Definition 8.23. An S-space F is *representable* if it is an algebraic space. A morphism $\alpha\colon F \to G$ of $\mathbf{Sp}_{/S}$ is *representable* if for every morphism $\eta\colon Y \to G$ with $Y \in \mathbf{AlgSp}_{/S}$ the S-space $F\,{}_{\alpha}\!\times_{\eta} Y$ is representable.

Remark 8.24. By Proposition 8.9 every morphism of $\mathbf{AlgSp}_{/S}$ is representable.

Lemma 8.25. *Given a cartesian diagram in* $\mathbf{Sp}_{/S}$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ U \arrow[r, "\pi"'] & X, \end{tikzcd}$$

such that $U, X, Y \in \mathbf{AlgSp}_{/S}$ *and* π *is étale and surjective, then* $F \in \mathbf{AlgSp}_{/S}$*, too.*

Proof. Up to composing π with an étale and surjective morphism $U' \to U$ such that $U' \in \mathbf{QSch}_{/S}$, we can assume that $U \in \mathbf{QSch}_{/S}$ (hence π is schematic). Let $V \to Y$ be an étale and surjective morphism with $V \in \mathbf{QSch}_{/S}$: since the induced morphism $V \to F$ is schematic, étale and surjective, by Corollary 8.13 it is enough to show that the natural morphism (of $\mathbf{QSch}_{/S}$) $V \times_F V \to V \times V$ is quasi-compact. The natural commutative diagram with cartesian squares

$$\begin{tikzcd} V \times_F V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \times V \arrow[d] \\ Y \times_F Y \arrow[r, "\alpha"'] & Y \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times Y \arrow[d] \\ & X \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd}$$

implies that it suffices to prove that α is schematic and quasi-compact (remember that, by definition, Δ_X has the same property, which is stable under composition and base change). Now, denoting by $\beta\colon Y \times_F Y \to U \times_X U$ the natural morphism, by Lemma A.3 α can be identified with Δ_β, which is schematic and quasi-compact by Remark 8.10 (note that $Y \times_F Y \cong Y \times_U X$ is an algebraic space by Proposition 8.9). $\square$

Corollary 8.26. *A morphism* $F \to G$ *of* $\mathbf{Sp}_{/S}$ *is representable if* $F \times_G U \in \mathbf{AlgSp}_{/S}$ *for every morphism* $U \to G$ *with* $U \in \mathbf{AffSch}_{/S}$*. In particular, every schematic morphism is representable.*

Proof. Given a morphism $X \to G$ with $X \in \mathbf{AlgSp}_{/S}$, we have to prove that $F' := F \times_G X$ is an algebraic space. Clearly there exists an étale and surjective morphism $U \to X$ such that $U = \coprod_{i\in I} U_i$ with $U_i \in \mathbf{AffSch}_{/S}$ for every $i \in I$.

Now, it is easy to see (using Proposition 4.54) that $F' \times_X U \cong \coprod_{i\in I}(F \times_G U_i)$, which is an algebraic space by Proposition 8.9; then F' is an algebraic space by Lemma 8.25. $\square$

Corollary 8.27. $\mathbf{Mor}(\mathbf{AlgSp}_{/S}) \in \underline{\mathbf{St}}_{(\mathbf{AlgSp}_{/S},\acute{e}t)}$ *and every property of morphisms of* $\mathbf{AlgSp}_{/S}$ *which is stable under base change and local on the codomain for ét is of effective descent for ét (hence also for sm, by Example 6.12).*

Proof. By Corollary 6.33 it is enough to prove the following: if $F \to X$ is a morphism of $\mathbf{Sp}_{/S}$ with $X \in \mathbf{AlgSp}_{/S}$ and there exists $\{U_i \to X\}_{i\in I} \in \mathrm{Cov}^{et}(X)$ such that $F \times_X U_i$ is an algebraic space for every $i \in I$, then F is an algebraic space, too. Now, the natural morphism $U := \coprod_{i\in I} U_i \to X$ is étale and surjective and (as before) $F \times_X U \cong \coprod_{i\in I}(F \times_X U_i) \in \mathbf{AlgSp}_{/S}$, so that the conclusion follows again from Lemma 8.25. $\square$

Definition 8.28. Let P be a property of morphisms of schemes which satisfies effective descent for *ét* (e.g., affine, quasi-affine, finite, open or closed or quasi-compact immersion). Then a morphism $f\colon X \to Y$ of $\mathbf{AlgSp}_{/S}$ satisfies P if it is schematic and satisfies P.

Remark 8.29. If τ is one of *fppf*, *sm* or *ét* and P is a property of morphisms of schemes which satisfies effective descent for τ, it is easy to see that the extension of P to morphisms of $\mathbf{AlgSp}_{/S}$ satisfies effective descent for τ, too.

Definition 8.30. $X \in \mathbf{AlgSp}_{/S}$ is *quasi-compact* if there exists an étale and surjective morphism $U \to X$ with $U \in \mathbf{QSch}_{/S}$ quasi-compact. X is *noetherian* if it is locally noetherian and quasi-compact.

Remark 8.31. Since a continuous map of topological spaces sends quasi-compact subsets to quasi-compact subsets, it is clear that for objects of $\mathbf{QSch}_{/S}$ the new definitions coincide with the usual ones.

Lemma 8.32. *If* $f\colon X \to X'$ *is a surjective morphism of* $\mathbf{AlgSp}_{/S}$ *with* X *quasi-compact, then* X' *is quasi-compact, too.*

Proof. Let $\pi\colon U \to X$ and $\pi'\colon U' \to X'$ be étale and surjective morphisms with $U, U' \in \mathbf{QSch}_{/S}$ and U quasi-compact. Consider the cartesian diagram

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "f\circ\pi"] \\ U' \arrow[r, "\pi'"'] & X' \end{tikzcd}$$

and observe that V is schematic and p is étale and surjective. As p is open and U is quasi-compact, it is easy to see that there exists $V' \subseteq V$ open and quasi-compact such that $p|_{V'}$ is again surjective. Since $|g|(|V'|) \subseteq |U'|$ is quasi-compact, there exists $U'' \subseteq U'$ open and quasi-compact such that $|g|(|V'|) \subseteq |U''|$ (we can take U'' to be a finite union of open affine subsets), and then it is clear that $\pi'|_{U''}$ is étale and surjective. $\square$

Definition 8.33. A morphism $X \to Y$ of $\mathbf{AlgSp}_{/S}$ is *quasi-compact* if for every morphism $V \to Y$ with $Y \in \mathbf{QSch}_{/S}$ quasi-compact, the algebraic space $X \times_Y V$ is quasi-compact.

Remark 8.34. If $X \to Y$ is quasi-compact, then $X \times_Y Y'$ is quasi-compact for every morphism $Y' \to Y$ of $\mathbf{AlgSp}_{/S}$ with Y' quasi-compact: this follows easily from Lemma 8.32. Then it is immediate to see that for morphisms of $\mathbf{AlgSp}_{/S}$ the property of being quasi-compact is stable under composition and base change. It is also clear that the new definition coincides with the usual one for schematic morphisms.

Proposition 8.35. *For morphisms of* $\mathbf{AlgSp}_{/S}$ *the property of being quasi-compact is local on the codomain for fppf.*

Proof. It is clearly enough to prove the following: if

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is a cartesian diagram in $\mathbf{AlgSp}_{/S}$ with f' quasi-compact and g faithfully flat (i.e., flat and surjective) and locally of finite presentation, then f is quasi-compact. Given a morphism $h\colon V \to Y$ with $V \in \mathbf{QSch}_{/S}$ quasi-compact, we can find a commutative diagram

$$\begin{tikzcd} V' \arrow[r, "\tilde{g}"] \arrow[d, "h'"'] & V \arrow[d, "h"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

such that $\tilde{g}$ is a faithfully flat morphism locally of finite presentation of $\mathbf{QSch}_{/S}$ (just take the diagram corresponding to an étale and surjective morphism $V' \to Y' \times_Y V$ with $V' \in \mathbf{QSch}_{/S}$). As $\tilde{g}$ is open and V is quasi-compact, there exists $V'' \subseteq V'$ open and quasi-compact such that $\tilde{g}|_{V''}$ is again surjective.

Now, $X \times_Y V'' \cong X' \times_{Y'} V''$ is quasi-compact by hypothesis, so that $X \times_Y V$ is quasi-compact by Lemma 8.32 (the natural morphism $X \times_Y V'' \to X \times_Y V$ is surjective because $V'' \to V$ is). $\square$

Definition 8.36. A morphism of algebraic spaces is *of finite type* (respectively *of finite presentation*, respectively *quasi-finite*) if it is locally of finite type (respectively locally of finite presentation, respectively locally quasi-finite) and quasi-compact.

Remark 8.37. By Remark 8.21 and Proposition 8.35 all the above defined properties are stable under composition and base change and are local on the codomain for *fppf*. It is easy to see that the same is true also for the already defined properties of being separated, locally separated and surjective.

Lemma 8.38. *In* $\mathbf{AlgSp}_{/S}$ *every monomorphism (locally of finite type) is separated (and unramified) and every unramified morphism is locally quasi-finite.*

Proof. An unramified morphism of algebraic spaces is locally quasi-finite because the same is true for morphisms of schemes (by Remark 3.117) and both properties are local on the domain and on the codomain for *ét*. On the other hand, since a monomorphism is separated (its diagonal is an isomorphism by Lemma A.2) and the property of being locally of finite type is also local on the domain and on the codomain for *ét*, it is clearly enough to prove the following: given morphisms $U \xrightarrow{\pi} X \xrightarrow{i} V$ of $\mathbf{AlgSp}_{/S}$ with $U, V \in \mathbf{QSch}_{/S}$ and such that π is étale and surjective and i is a monomorphism, then $i \circ \pi$ is formally unramified. By definition, given a commutative diagram

$$\begin{tikzcd} U \arrow[r, "i\circ\pi"] & V \\ W' \arrow[u, "g'"] \arrow[r, "\iota"'] & W \arrow[u, "f"'] \arrow[ul, bend left, "\tilde{g}"'] \arrow[ul, bend right, "g"] \end{tikzcd}$$

with $W \in \mathbf{AffSch}_{/S}$ and ι a closed immersion defined by a nilpotent ideal of $\mathcal{O}_W$, we have to show that $g = \tilde{g}$. As $\pi \circ g = \pi \circ \tilde{g} \colon W \to X$ (because i is a monomorphism), there is a commutative diagram in $\mathbf{QSch}_{/S}$

$$\begin{tikzcd} U \times_X U \arrow[r, "pr_1"] & U \\ W' \arrow[u, "{(g',g')}"] \arrow[r, "\iota"'] & W. \arrow[u, "g"'] \arrow[ul, bend right, "{(g,g)}"'] \arrow[ul, bend left, "{(g,\tilde{g})}"] \end{tikzcd}$$

Then $(g, g) = (g, \tilde{g})$ because pr_1 is étale. □

Proposition 8.39. *Every quasi-finite and separated morphism of* $\mathbf{AlgSp}_{/S}$ *is quasi-affine. In particular, a monomorphism of finite type of* $\mathbf{AlgSp}_{/S}$ *is quasi-affine.*

Proof. The first statement is proved in [15, Thm. A.2], and then the second follows from Lemma 8.38. □

Proposition 8.40. *If* $\delta = (\delta_1, \delta_2) \colon R \to X \times X$ *is an étale equivalence relation in* $\mathbf{AlgSp}_{/S}$ *such that* δ *is quasi-compact and with quotient* $\pi \colon X \to Y$ *in* $\mathbf{Sp}_{/S}$, *then* $Y \in \mathbf{AlgSp}_{/S}$ *and* π *is étale and surjective.*

Proof. Let $\pi' \colon U \to X$ be an étale and surjective morphism with $U \in \mathbf{QSch}_{/S}$. As π and π' (by Corollary 4.60) are epimorphisms of $\mathbf{Sp}_{/S}$, the same is true for $\pi \circ \pi' \colon U \to Y$, which is therefore (by Corollary 4.83) a quotient in $\mathbf{Sp}_{/S}$ of the equivalence relation $\delta' = (\delta'_1, \delta'_2) \colon R' := U \times_Y U \to U \times U$. Note that there is a cartesian diagram in $\mathbf{Sp}_{/S}$

$$\begin{tikzcd} R' \arrow[r] \arrow[d, "\delta'"'] \arrow[dr, phantom, "\square"] & R \arrow[d, "\delta"] \\ U \times U \arrow[r, "\pi' \times \pi'"'] & X \times X \end{tikzcd}$$

(hence R' is an algebraic space). Now, δ is locally of finite type by Lemma A.5 ($\delta_1 = pr_1 \circ \delta$ is étale and Δ_{pr_1} is of finite type by Remark 8.10), and then it is a monomorphism (by Proposition 4.71) of finite type, hence quasi-affine by Proposition 8.39. Thus δ' is also quasi-affine (in particular schematic), so that R' is schematic. Moreover, δ'_1 and δ'_2 are étale and surjective (they are obtained from δ_1 and δ_2 by the base change $\pi' \colon U \to X$), and then the conclusion follows from Theorem 8.11. □

Remark 8.41. It can be proved that every algebraic space is a sheaf also for *fppf* (see [15, Thm. A.4]) and that the above result can be generalized as follows. If $\delta \colon R \to X \times X$ is an *fppf* equivalence relation in $\mathbf{AlgSp}_{/S}$ such that δ is quasi-compact and with quotient $\pi \colon X \to Y$ in $(\mathbf{QSch}_{/S}, \mathit{fppf})^{\sim}$, then $Y \in \mathbf{AlgSp}_{/S}$ (see [15, Cor. 10.4]).

Corollary 8.42. *If $X \to Y$ is a representable, étale and surjective morphism of $\mathbf{Sp}_{/S}$ such that X is an algebraic space and the natural morphism (of $\mathbf{AlgSp}_{/S}$) $X \times_Y X \to X \times X$ is quasi-compact, then Y is an algebraic space, too.*

Proof. Completely similar to that of Corollary 8.13 (using Proposition 8.40 instead of Theorem 8.11). □

Many more notions and results of scheme theory can be extended to algebraic spaces. Here we are going to say something only about a few of them; a rather detailed (although not always correct) treatment of the subject is provided by [14].

Definition 8.43. A *point* of $X \in \mathbf{AlgSp}_{/S}$ is an equivalence class of morphisms $\operatorname{Spec} \mathbb{K} \to X$ of $\mathbf{AlgSp}_{/S}$ (where $\mathbb{K}$ is a field) under the equivalence relation which identifies two such morphisms $\operatorname{Spec} \mathbb{K}_i \to X$ (for $i = 1, 2$) if and only if there is a common field extension $\mathbb{K}$ of $\mathbb{K}_1$ and $\mathbb{K}_2$ such that the two induced morphisms $\operatorname{Spec} \mathbb{K} \to X$ are equal.

A *geometric point* of X is a morphism $\operatorname{Spec} \mathbb{K} \to X$ of $\mathbf{AlgSp}_{/S}$, where $\mathbb{K}$ is an algebraically closed field.

Remark 8.44. In [14] a point of X is defined to be an isomorphism class (in the obvious sense) of monomorphisms $\operatorname{Spec} \mathbb{K} \to X$. The two definitions are equivalent because every morphism $\operatorname{Spec}\mathbb{K} \to X$ factors through a monomorphism $\operatorname{Spec}\mathbb{K}' \to X$ (see [14, II, Prop. 6.2]).

Denoting by $|X|$ the set of points of an algebraic space X, it is clear that every morphism $f\colon X \to Y$ of $\mathbf{AlgSp}_{/S}$ induces (by composition) a map $|f|\colon |X| \to |Y|$. Then there is a natural way to define a topology on $|X|$, namely the open (respectively closed) subsets of $|X|$ are those of the form $\operatorname{im}|i|$ for some open (respectively closed) immersion $i\colon X' \to X$. It is easy to see that in this way $|f|$ is continuous for every $f \in \operatorname{Mor}(\mathbf{AlgSp}_{/S})$, so that we obtain a functor $\mathbf{AlgSp}_{/S} \to \mathbf{Top}$ (whose restriction to $\mathbf{QSch}_{/S}$ clearly coincides with the usual one, up to isomorphism).

Remark 8.45. For every algebraic space X the topological space $|X|$ can be naturally endowed with a sheaf of rings $\mathcal{O}_{|X|}$ as follows. Consider the presheaf of rings $\mathcal{O}\colon \mathbf{QSch}^{\circ}_{/S} \to \mathbf{Rng}$ defined on objects by $U \mapsto \mathcal{O}_U(U)$. Since (as a presheaf of sets) $\mathcal{O} \cong \mathbb{A}^1_S$, it can be extended to a sheaf (for *ét*) of rings on $\mathbf{AlgSp}_{/S}$, denoted again by $\mathcal{O}$. Then, given $U \subseteq |X|$ open (say $U = \operatorname{im}|i|$, where $i\colon X' \to X$ is an open immersion), we set $\mathcal{O}_{|X|}(U) := \mathcal{O}(X')$. One can prove that $\mathcal{O}_{|X|} \cong \mathcal{O}_X$ if X is scheme and that in this way we obtain a functor $\mathbf{AlgSp}_{/S} \to \mathbf{LRngSp}_{/S}$, which is however neither faithful nor full (unlike its restriction to $\mathbf{QSch}_{/S}$, which coincides with the usual one, up to isomorphism). It must be said that there is another way (which we are not going to explain here) to endow an algebraic space X with a structure sheaf of rings $\mathcal{O}_X$.

Definition 8.46. A morphism $f\colon X \to Y$ of $\mathbf{AlgSp}_{/S}$ is *open* (respectively *closed*) if $|f|$ is open (respectively closed). f is *proper* if it is of finite type, separated and universally closed.

Remark 8.47. It can be proved that the new definition of universally open morphism coincides with the old one, and that the same is true for universally closed schematic morphism. For morphisms of $\mathbf{AlgSp}_{/S}$ the property of being universally closed (hence also the property of being proper, by Remark 8.37) is stable under composition and base change and is local on the codomain for *fppf*.

9 Algebraic stacks

9.1 Algebraic and Deligne-Mumford stacks

We fix as usual a base quasi-separated scheme S. We will denote by $\underline{\mathbf{St}}_{/S}$ the 2-category $\underline{\mathbf{St}}^{gpd}_{(\mathbf{QSch}_{/S},ét)}$ and call its objects *stacks over S* (or *S-stacks*, or simply *stacks*); note that (taking into account Example 6.12) $\underline{\mathbf{St}}_{/S} = \underline{\mathbf{St}}^{gpd}_{(\mathbf{QSch}_{/S},sm)}$ and that it can be identified with $\underline{\mathbf{St}}^{gpd}_{(\mathbf{AlgSp}_{/S},ét)} = \underline{\mathbf{St}}^{gpd}_{(\mathbf{AlgSp}_{/S},sm)}$ by Remark 8.22. Generalizing the terminology used for spaces, we give the following definition.

Definition 9.1. An S-space $\mathbf{F}$ is *representable* (respectively *schematic*) if there exists $X \in \mathbf{AlgSp}_{/S}$ (respectively $X \in \mathbf{QSch}_{/S}$) such that $\mathbf{F} \cong X$ in $\underline{\mathbf{St}}_{/S}$. A morphism $P \colon \mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{St}}_{/S}$ is *representable* (respectively *schematic*) if for every morphism $Q \colon \mathbf{H} \to \mathbf{G}$ with $\mathbf{H}$ representable (respectively schematic) the S-stack $\mathbf{F} {}_P{\times}_Q \mathbf{H}$ is representable (respectively schematic).

Recall from Proposition 5.58 and Remark 5.60 that for morphisms of $\underline{\mathbf{St}}_{/S}$ the properties of being representable and schematic are stable under composition and base change, and that properties of morphisms of $\mathbf{AlgSp}_{/S}$ (respectively $\mathbf{QSch}_{/S}$) which are stable under base change can be extended to representable (respectively schematic) morphisms of $\underline{\mathbf{St}}_{/S}$ (such extension coincides with the usual one in case of morphisms of $\mathbf{Sp}_{/S}$).

Remark 9.2. A morphism $\mathbf{F} \to \mathbf{G}$ of $\underline{\mathbf{St}}_{/S}$ is representable or schematic if $\mathbf{F} \times_{\mathbf{G}} U$ has the same property for every morphism $U \to \mathbf{G}$ with $U \in \mathbf{AffSch}_{/S}$. Indeed, if this condition is satisfied, then it is very easy to see that $\mathbf{F} \times_{\mathbf{G}} \mathbf{H}$ is fibred in equivalence relations if the same is true for $\mathbf{H}$, hence $\mathbf{F} \times_{\mathbf{G}} \mathbf{H}$ is isomorphic to a space if $\mathbf{H}$ is representable; then the conclusion follows from Remark 8.3 and Corollary 8.26. Therefore, also in $\underline{\mathbf{St}}_{/S}$ every schematic morphism is representable.

Definition 9.3. $\mathbf{X} \in \underline{\mathbf{St}}_{/S}$ is an *algebraic stack over* S (or an S*-algebraic stack*, or simply an *algebraic stack*) if the following conditions are satisfied:

1. $\Delta_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is representable, quasi-compact and separated;
2. there exist $X \in \mathbf{AlgSp}_{/S}$ and a morphism $\Pi \colon X \to \mathbf{X}$ of $\underline{\mathbf{St}}_{/S}$ (necessarily representable, by Proposition 5.62) which is smooth and surjective (such a morphism is called an *atlas* or a *presentation* of $\mathbf{X}$).

An algebraic stack $\mathbf{X}$ is a *Deligne-Mumford stack* if it admits an étale atlas.[16]

We will denote by $\underline{\mathbf{AlgSt}}_{/S}$ the full 2-subcategory of $\underline{\mathbf{St}}_{/S}$ whose objects are algebraic stacks; it is easy to see that it is a strictly full 2-subcategory.

Remark 9.4. By Corollary 5.67 condition (1) of the above definition is equivalent to the following: for all $U \in \mathbf{AlgSp}_{/S}$ (as usual, one can actually restrict to $U \in \mathbf{AffSch}_{/S}$) and all $\xi_1, \xi_2 \in \mathbf{X}_U$ the presheaf $\mathcal{I}som_U(\xi_1, \xi_2)$ (which is a space because $\mathbf{X}$ is a prestack of groupoids) is an algebraic space and the structure morphism $\mathcal{I}som_U(\xi_1, \xi_2) \to U$ is quasi-compact and separated.

The following results generalize Lemma 8.7 and Lemma 8.8, and can be proved in a similar way.

Lemma 9.5. *Let* P *be a property of morphisms of algebraic spaces which is stable under base change and local on the codomain for ét (hence also for sm, by Remark 4.49). If* $P \colon \mathbf{X} \to \mathbf{Y}$ *is a representable morphism of* $\underline{\mathbf{AlgSt}}_{/S}$ *and there exists an atlas* $Y \to \mathbf{Y}$ *such that the projection morphism* $\mathbf{X} \times_{\mathbf{Y}} Y \to Y$ *satisfies* P*, then* P *satisfies* P*.*

[16] The name comes from [4]. Algebraic stacks are also called *Artin stacks* (from [1]).

Lemma 9.6. *Let* $P\colon \mathbf{F} \to \mathbf{G}$ *and* $Q\colon \mathbf{Y} \to \mathbf{G}$ *be morphisms of* $\underline{\mathbf{St}}_{/S}$ *such that* P *is an epimorphism and* $\mathbf{Y}$ *is an algebraic (respectively Deligne-Mumford) stack. Then there exist an atlas (respectively an étale atlas)* $\Pi\colon Y \to \mathbf{Y}$ *and a 2-commutative diagram in* $\underline{\mathbf{St}}_{/S}$

$$\begin{tikzcd} Y \arrow[r, "\Pi"] \arrow[d] & \mathbf{Y} \arrow[d, "Q"] \arrow[dl, phantom, "\swarrow"] \\ \mathbf{F} \arrow[r, "P"'] & \mathbf{G}. \end{tikzcd}$$

Proposition 9.7. *Given morphisms* $\mathbf{X}_i \to \mathbf{X}$ *(for* $i = 1, 2$*) of* S*-algebraic (respectively Deligne-Mumford) stacks, the* S*-stack* $\mathbf{X}_1 \times_{\mathbf{X}} \mathbf{X}_2$ *is algebraic (respectively Deligne-Mumford).*

Proof. Using Lemma 9.6 instead of Lemma 8.8, it is easy to adapt the proof of Proposition 8.9 (of course, one has to generalize Lemma A.6 to $\underline{\mathbf{Fib}}_{\mathbf{C}}$). □

Remark 9.8. If $\mathbf{X}$ is an algebraic stack and $\Pi\colon X \to \mathbf{X}$ is an atlas, then, in particular, $\mathbf{X} \times \mathbf{X}$ is also an algebraic stack and $\Pi \times \Pi\colon X \times X \to \mathbf{X} \times \mathbf{X}$ is an atlas. Since there is a 2-cartesian diagram in $\underline{\mathbf{AlgSt}}_{/S}$

$$\begin{tikzcd} X \times_{\mathbf{X}} X \arrow[r] \arrow[d, "P"'] \arrow[dr, phantom, "\square"] & \mathbf{X} \arrow[d, "\Delta_{\mathbf{X}}"] \arrow[dl, phantom, "\swarrow"] \\ X \times X \arrow[r, "\Pi\times\Pi"'] & \mathbf{X} \times \mathbf{X}, \end{tikzcd}$$

by Lemma 9.5 $\Delta_{\mathbf{X}}$ satisfies a property P of morphisms of $\mathbf{AlgSp}_{/S}$ which is stable under base change and local on the codomain for *ét* if (and only if) P (which is a morphism of algebraic spaces) satisfies P. So (taking into account Remark 8.21) $\Delta_{\mathbf{X}}$ is locally of finite type (hence of finite type, since it is quasi-compact by hypothesis), and even unramified if $\mathbf{X}$ is a Deligne-Mumford stack (in which case in the above diagram Π can be assumed to be also étale). Indeed, $pr_1 \circ P\colon X \times_{\mathbf{X}} X \to X$ is locally of finite type (and even unramified if Π is étale) because Π has the same property and Δ_{pr_1} is in any case unramified by Remark 8.10, so that the conclusion follows as usual from Lemma A.5. Notice that, if $\mathbf{X}$ is Deligne-Mumford, then (by Lemma 8.38 and Proposition 8.39) $\Delta_{\mathbf{X}}$ is also quasi-finite and quasi-affine, hence schematic. As in Remark 8.10 for algebraic spaces, it is then easy to deduce that, more generally, if $P\colon \mathbf{X} \to \mathbf{Y}$ is a morphism of $\mathbf{AlgSp}_{/S}$, then Δ_P is representable, of finite type and separated (and even unramified, quasi-finite and quasi-affine if $\mathbf{X}$ and $\mathbf{Y}$ are Deligne-Mumford).

Remark 9.9. By what we have just proved $\mathbf{X} \in \underline{\mathbf{St}}_{/S}$ is a Deligne-Mumford stack if and only if $\Delta_{\mathbf{X}}$ is schematic, quasi-compact and separated and there exists $U \in \mathbf{QSch}_{/S}$ and a (schematic) étale and surjective morphism $U \to \mathbf{X}$ (for this last fact, it is enough to compose an étale atlas $X \to \mathbf{X}$ with an étale and surjective morphism $U \to X$ with $U \in \mathbf{QSch}_{/S}$). Thus we see that Deligne-Mumford stacks can be defined without involving algebraic spaces.

Example 9.10. By Remark 8.10 and Remark 9.9 $F \in \mathbf{Sp}_{/S}$ is a Deligne-Mumford stack if and only if it is an algebraic space. Actually it is also true that F is an algebraic space if it is an algebraic stack: as in this case Δ_F is unramified (by Lemma 8.38), this is a consequence of the following result.

Proposition 9.11. *$\mathbf{X} \in \underline{\mathbf{AlgSt}}_{/S}$ is Deligne-Mumford if and only if $\Delta_{\mathbf{X}}$ is unramified.*

Proof. See [15, Thm. 8.1]. □

Theorem 9.12. *If $\mathbf{X} \in \underline{\mathbf{St}}_{/S}$ is algebraic (respectively Deligne-Mumford) and $\Pi\colon X \to \mathbf{X}$ is an atlas (respectively an étale atlas), then*

$$pr_1, pr_2\colon X' := X \times_{\mathbf{X}} X \to X$$

are smooth (respectively étale) and surjective morphisms of $\mathbf{AlgSp}_{/S}$,

$$(pr_1, pr_2)\colon X' \to X \times X$$

is quasi-compact and separated and $\mathbf{X} \cong [X' \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X] \in \underline{\mathbf{St}}_{/S}$.

Conversely, if $X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ is a groupoid in $\mathbf{AlgSp}_{/S}$ such that p_1 and p_2 are smooth (respectively étale) and surjective and $(p_1, p_2)\colon X' \to X \times X$ is quasi-compact and separated, then $\mathbf{X} := [X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X] \in \underline{\mathbf{St}}_{/S}$ is algebraic (respectively Deligne-Mumford), the natural morphism $\Pi\colon X \to \mathbf{X}$ is an atlas (respectively an étale atlas) and $X' \cong X \times_{\mathbf{X}} X$.

Proof. If $\mathbf{X}$ is an algebraic stack and Π is smooth (respectively étale) and surjective, then the same is true for pr_1 and pr_2; moreover, $\mathbf{X} \cong [X' \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X]$ by Corollary 6.35 (Π is an epimorphism of $\underline{\mathbf{St}}_{/S}$ by Lemma 6.29) and (pr_1, pr_2) is quasi-compact and separated by Remark 9.8.

Conversely, note that $X' \cong X \times_{\mathbf{X}} X$ by Corollary 6.35. Then the proof is completely similar to that of Theorem 8.11, using Proposition 5.62 instead of Proposition 4.14 and Lemma 9.6 instead of Lemma 8.8, and taking into account that for morphisms of algebraic spaces the properties of being smooth, étale, surjective, quasi-compact and separated are local on the codomain (hence of effective descent by Corollary 8.27) for *sm*. □

Remark 9.13. It can be proved that every algebraic stack is a stack also for *fppf* (see [15, Cor. 10.7]) and that the second part of Theorem 9.12 admits the following generalization. If $X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ is a groupoid in $\mathbf{AlgSp}_{/S}$ such that p_1 and p_2 are faithfully flat and locally of finite presentation and $(p_1,p_2)\colon X' \to X \times X$ is quasi-compact and separated, then $[X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X] \in \underline{\mathbf{St}}^{gpd}_{(\mathbf{QSch}_{/S},fppf)}$ is an algebraic stack (see [15, Cor. 10.6]). These results are consequences of a theorem by Artin, which says that $\mathbf{X} \in \underline{\mathbf{St}}^{gpd}_{(\mathbf{QSch}_{/S},fppf)}$ is an algebraic stack if $\Delta_{\mathbf{X}}$ is representable, quasi-compact and separated and there exist $X \in \mathbf{AlgSp}_{/S}$ and a (representable) faithfully flat morphism locally of finite presentation $X \to \mathbf{X}$ (see [15, Thm. 10.1]).

Corollary 9.14. $\mathbf{X} \in \underline{\mathbf{St}}_{/S}$ *is an algebraic (respectively Deligne-Mumford) stack if and only if there exists a representable, smooth (respectively étale) and surjective morphism* $\Pi\colon X \to \mathbf{X}$ *with* X *an algebraic space and such that the natural morphism (of* $\mathbf{AlgSp}_{/S}$*)* $(pr_1,pr_2)\colon X' := X \times_{\mathbf{X}} X \to X \times X$ *is quasi-compact and separated.*

Proof. If Π satisfies the hypothesis, then by Lemma 6.29 it is an epimorphism of $\underline{\mathbf{St}}_{/S}$, so that $\mathbf{X} \cong [X' \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X]$ by Corollary 6.35; then $\mathbf{X}$ is an algebraic (respectively Deligne-Mumford) stack (pr_1 and pr_2 are smooth (respectively étale) and surjective because Π is). The other implication is clear, since (pr_1,pr_2) is quasi-compact and separated by Remark 9.8. $\square$

Example 9.15. Let $\varrho\colon X \times G \to X$ be an action in $\mathbf{AlgSp}_{/S}$ such that $G \to S$ is smooth (respectively étale), separated and quasi-compact. Then the quotient stack $[X/G]$ is algebraic (respectively Deligne-Mumford) and the natural morphism $X \to [X/G]$ is an atlas (respectively an étale atlas). Indeed, $pr_1\colon X \times G \to X$ is clearly smooth (respectively étale) and surjective, and then the same is true for $\varrho = pr_1 \circ i$ (here $i\colon X \times G \to X \times G$ is the "inverse" morphism, which is an isomorphism because obviously $i^2 = \mathrm{id}$). Moreover, $(pr_1,\varrho)\colon X \times G \to X \times X$ is separated and quasi-compact by Lemma A.5, since the same is true for $pr_1 = pr_1' \circ (pr_1,\varrho)$ (denoting by $pr_1'\colon X \times X \to X$ the projection) by hypothesis and for $\Delta_{pr_1'}$ by Remark 8.10. Then the conclusion follows from Theorem 9.12 (taking into account Proposition 6.42).

If $G \to S$ is not étale, then $X \to [X/G]$ is not étale, but $[X/G]$ can be Deligne-Mumford just the same (for instance, if the action is free, then $[X/G]$ is a space, hence an algebraic space by Example 9.10): to be precise, $[X/G]$ is Deligne-Mumford if and only if the stabilizers of the geometric points of X are finite and reduced (if $\mathbb{K}$ is an algebraically closed field, the stabilizer of the geometric point $x\colon \operatorname{Spec}\mathbb{K} \to X$ is $G_x := (X \times G)\,{}_{(pr_1,\varrho)}{\times}_{(x,x)} \operatorname{Spec}\mathbb{K}$). Indeed, $[X/G]$ is Deligne-Mumford if and only if $\Delta_{[X/G]}$ is unramified (by Proposition 9.11), if and only if $(pr_1,\varrho)\colon X \times G \to X \times X$ is unramified (by Remark 9.8). Using Proposition 3.115 and the fact that the property of being unramified is local on the domain and on the codomain for *ét*, it is easy to see that (pr_1,ϱ) is unramified if and only if $I(x_1,x_2) := (X \times G)\,{}_{(pr_1,\varrho)}{\times}_{(x_1,x_2)} \operatorname{Spec}\mathbb{K} \to \operatorname{Spec}\mathbb{K}$ is unramified for every morphism $(x_1, x_2)\colon \operatorname{Spec}\mathbb{K} \to X \times X$ (where $\mathbb{K}$ is an algebraically closed field). Since either $I(x_1, x_2) \cong G_{x_1} \cong G_{x_2}$ (if $x_1, x_2 \in X(\operatorname{Spec}\mathbb{K})$ are in the same orbit for the action of $G(\operatorname{Spec}\mathbb{K})$) or $I(x_1, x_2) = \emptyset$ (otherwise), we see that this is the case if and only if $G_x \to \operatorname{Spec}\mathbb{K}$ is unramified for every geometric point $x\colon \operatorname{Spec}\mathbb{K} \to X$. Now, by Remark 3.117 it is clear that if $G_x \to \operatorname{Spec}\mathbb{K}$ is unramified, then G_x is finite (meaning $G_x \to \operatorname{Spec}\mathbb{K}$ if finite) and reduced. Conversely, if G_x is finite and reduced, then $G_x \cong \operatorname{Spec} A$ for some finite $\mathbb{K}$-algebra A; as A is artinian and reduced, it is isomorphic to a finite product of fields, each isomorphic to $\mathbb{K}$ (being a finite extension of $\mathbb{K} = \overline{\mathbb{K}}$), so that $G_x \to \operatorname{Spec}\mathbb{K}$ is unramified.

Example 9.16. It can be proved that $\mathbf{M}_g$ (the moduli stack of curves of genus g, defined in Example 7.26) is a Deligne-Mumford stack (see [4]).

9.2 Extension to algebraic stacks of some properties of algebraic spaces

Definition 9.17. A morphism $P\colon \mathbf{X} \to \mathbf{Y}$ of $\underline{\mathbf{AlgSt}}_{/S}$ is *separated* if $\Delta_P\colon \mathbf{X} \to \mathbf{X} \times_{\mathbf{Y}} \mathbf{X}$ (which is representable, of finite type and separated by Remark 9.8) is univerally closed (hence proper). $\mathbf{X} \in \underline{\mathbf{AlgSt}}_{/S}$ is *separated* if the structure morphism $\mathbf{X} \to S$ is separated.

Remark 9.18. This definition coincides with the usual one for representable morphisms: this follows from the fact that every proper monomorphism of schemes is a closed immersion (by [9, Cor. 18.12.6]). Clearly for morphisms of $\underline{\mathbf{AlgSt}}_{/S}$ the property of being separated is stable under composition and base change.

Definition 9.19. Let P be a property of algebraic spaces which is local for *sm* (respectively for *ét*). Then an S-algebraic (respectively Deligne-Mumford) stack $\mathbf{X}$ satisfies P if there exists an atlas (respectively an étale atlas) $X \to \mathbf{X}$ such that X satisfies P.

Remark 9.20. The same argument of Remark 8.18 shows that, if $\mathbf{X}$ satisfies P and $X' \to \mathbf{X}$ is an atlas (respectively an étale atlas), then X' satisfies P (hence the new definition coincides with the usual one if $\mathbf{X} \in \mathbf{AlgSp}_{/S}$).

Definition 9.21. Let P be a property of morphisms of algebraic spaces which is local on the domain and on the codomain for *sm* (respectively for *ét*) or the property of being surjective. Then a morphism of S-algebraic (respectively Deligne-Mumford) stacks $P\colon \mathbf{X} \to \mathbf{Y}$ satisfies P if there exists a 2-commutative diagram with 2-cartesian square in $\underline{\mathbf{AlgSt}}_{/S}$

$$\begin{tikzcd}
X \arrow[r, "\tilde{\Pi}"] \arrow[dr, "f"'] & \tilde{\mathbf{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{X} \arrow[d, "P"] \arrow[dl, Rightarrow] \\
 & Y \arrow[r, "\Pi"'] & \mathbf{Y}
\end{tikzcd}$$

such that Π and $\tilde{\Pi}$ are atlas (respectively étale atlas) and f satisfies P.

Remark 9.22. As in Remark 8.20 it is easy to see that, if P satisfies P and

$$\begin{tikzcd} X' \arrow[r, "\tilde{\Pi}'"] \arrow[dr, "f'"'] & \tilde{\mathbf{X}}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{X} \arrow[d, "P"] \arrow[dl, phantom, "\swarrow\!\!\!\swarrow"] \\ & Y' \arrow[r, "\Pi'"'] & \mathbf{Y} \end{tikzcd}$$

is another 2-commutative diagram with 2-cartesian square in $\underline{\mathbf{AlgSt}}_{/S}$ such that Π' and $\tilde{\Pi}'$ are atlas (respectively étale atlas), then f' satisfies P. Moreover, the new definition coincides with the usual one for representable morphisms and, if P is stable under base change (and composition) for morphisms of $\mathbf{AlgSp}_{/S}$, then P remains stable under base change (and composition) for morphisms of algebraic (respectively Deligne-Mumford) stacks.

Definition 9.23. $\mathbf{X} \in \underline{\mathbf{AlgSt}}_{/S}$ is *quasi-compact* if there exists an atlas $X \to \mathbf{X}$ with X quasi-compact. A morphism $\mathbf{X} \to \mathbf{Y}$ of $\underline{\mathbf{AlgSt}}_{/S}$ is *quasi-compact* if for every morphism $V \to \mathbf{Y}$ with $V \in \mathbf{QSch}_{/S}$ quasi-compact, the algebraic stack $\mathbf{X} \times_{\mathbf{Y}} V$ is quasi-compact.

Remark 9.24. Using Lemma 8.32 it is easy to see that these new definitions coincide with the usual ones for algebraic spaces and for representable morphisms of algebraic stacks. Moreover, with a similar proof one can show more generally that, if $\mathbf{X} \to \mathbf{X}'$ is a surjective morphism of $\underline{\mathbf{AlgSt}}_{/S}$ with $\mathbf{X}$ quasi-compact, then $\mathbf{X}'$ is quasi-compact, too. It follows that, if $\mathbf{X} \to \mathbf{Y}$ is a quasi-compact morphism of $\underline{\mathbf{AlgSt}}_{/S}$, then $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Y}'$ is quasi-compact for every morphism of algebraic stacks $\mathbf{Y}' \to \mathbf{Y}$ with $\mathbf{Y}'$ quasi-compact; from this we obtain that for morphisms of $\underline{\mathbf{AlgSt}}_{/S}$ the property of being quasi-compact is stable under composition and base change.

Definition 9.25. An algebraic stack is *noetherian* if it is locally noetherian and quasi-compact.

A morphism of algebraic stacks is *of finite type* (respectively *of finite presentation*, respectively *quasi-finite*) if it is locally of finite type (respectively locally of finite presentation, respectively locally quasi-finite) and quasi-compact.

Remark 9.26. By Remark 9.22 and Remark 9.24 the above defined properties of morphisms of $\underline{\mathbf{AlgSt}}_{/S}$ are stable under composition and base change.

Definition 9.27. A *point* of $\mathbf{X} \in \underline{\mathbf{AlgSt}}_{/S}$ is an equivalence class of morphisms $\operatorname{Spec} \mathbb{K} \to \mathbf{X}$ of $\underline{\mathbf{AlgSt}}_{/S}$ (where $\mathbb{K}$ is a field) under the equivalence relation which identifies two such morphisms $\operatorname{Spec} \mathbb{K}_i \to \mathbf{X}$ (for $i = 1, 2$) if and only if there is a common field extension $\mathbb{K}$ of $\mathbb{K}_1$ and $\mathbb{K}_2$ such that the two induced morphisms $\operatorname{Spec} \mathbb{K} \to \mathbf{X}$ are 2-isomorphic.

Denoting by $|\mathbf{X}|$ the set of points of an algebraic stack $\mathbf{X}$, every morphism $P\colon \mathbf{X} \to \mathbf{Y}$ of $\underline{\mathbf{AlgSt}}_{/S}$ induces a map $|P|\colon |\mathbf{X}| \to |\mathbf{Y}|$. Then there is a natural way to define a topology on $|\mathbf{X}|$, namely the open (respectively closed) subsets of $|\mathbf{X}|$ are those of the form $\mathrm{im}|I|$ for some representable open (respectively closed) immersion $I\colon \mathbf{X}' \to \mathbf{X}$. It is easy to see that in this way $|P|$ is continuous for every $P \in \mathrm{Mor}(\underline{\mathbf{AlgSt}}_{/S})$ and that the restrictions of these new definitions to $\mathbf{AlgSp}_{/S}$ coincide with the old ones.

Definition 9.28. A morphism $P\colon \mathbf{X} \to \mathbf{Y}$ of $\underline{\mathbf{AlgSt}}_{/S}$ is *open* (respectively *closed*) if $|P|$ is open (respectively closed). P is *proper* if it is of finite type, separated and universally closed.

Remark 9.29. As usual, one can check that these new definitions are compatible with the old ones and that for morphisms of $\underline{\mathbf{AlgSt}}_{/S}$ the property of being universally closed (hence also the property of being proper) is stable under composition and base change.

A Auxiliary results

A.1 Some categorical lemmas

We fix a category $\mathbf{C}$ with fibred products.

Lemma A.1. *Given a commutative diagram in* $\mathbf{C}$

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "k"'] & V' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\square"] & W' \arrow[d, "h"] \\ U \arrow[r, "f"'] & V \arrow[r, "g"'] & W \end{tikzcd}$$

such that the square on the right is cartesian, then the square on the left is cartesian if and only if the composition

$$\begin{tikzcd} U' \arrow[r, "g' \circ f'"] \arrow[d, "k"'] & W' \arrow[d, "h"] \\ U \arrow[r, "g \circ f"'] & W \end{tikzcd}$$

is cartesian.

Proof. Easy exercise on the definition of cartesian diagram. Notice also that, in view of Example 5.3, this is a particular case of Lemma 5.5. $\square$

Lemma A.2. *Let $f\colon U \to V$ be a morphism of $\mathbf{C}$. Then $\Delta_f\colon U \to U \times_V U$ is a monomorphism, and it is an isomorphism if and only if f is a monomorphism.*

Proof. It is clear that the map $\mathrm{Hom}_{\mathbf{C}}(W,U) \xrightarrow{\Delta_f \circ} \mathrm{Hom}_{\mathbf{C}}(W, U \times_V U)$ is always injective (hence Δ_f is a monomorphism), and it is bijective for every $W \in \mathbf{C}$ if and only if f is a monomorphism. The conclusion then follows from Yoneda's lemma (see Corollary 4.4). $\square$

Lemma A.3. *Every cartesian diagram in* $\mathbf{C}$

$$\begin{tikzcd} V' \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & U' \arrow[d, "g"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

induces a commutative diagram with cartesian squares (p and p' denoting the natural morphisms)

$$\begin{tikzcd} V' \arrow[r, "\Delta_{f'}"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & V' \times_{U'} V' \arrow[r, "p'"] \arrow[d, "h \times h"'] \arrow[dr, phantom, "\square"] & U' \arrow[d, "g"] \\ V \arrow[r, "\Delta_f"'] & V \times_U V \arrow[r, "p"'] & U. \end{tikzcd}$$

Moreover, $\Delta_{h\times h}$ can be identified with $\mathrm{id} \times \mathrm{id}\colon V' \times_{U'} V' \to V' \times_U V'$.

Proof. As for the first statement, by Lemma A.1 it is enough to prove that the square on the right is cartesian, which is straightforward. Then, applying what we have just proved to the cartesian diagram on the right, and taking into account that in any case there is a natural commutative diagram with cartesian squares

$$\begin{tikzcd} V' \times_{U'} V' \arrow[r, "\mathrm{id}\times\mathrm{id}"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & V' \times_U V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \times_U V \arrow[d, "p"] \\ U' \arrow[r, "\Delta_g"'] & U' \times_U U' \arrow[r] & U \end{tikzcd}$$

(again by Lemma A.1, it is enough to check that for the square on the right, which is straightforward), we conclude that $(V' \times_{U'} V') \times_{(V \times_U V)} (V' \times_{U'} V') \cong V' \times_U V'$ and that (with this identification) $\Delta_{h\times h}$ coincides with $\mathrm{id} \times \mathrm{id}$. $\square$

Lemma A.4. *Let P and P' be properties of morphisms of $\mathbf{C}$ such that $f \in \mathrm{Mor}(\mathbf{C})$ satisfies P' if and only if Δ_f satisfies P. If P is stable under base change (and composition), then also P' is stable under base change (and composition).*

Proof. The fact that P' is stable under base change follows immediately from Lemma A.3. In order to prove that P' is also stable under composition if the same is true for P, it is enough to observe that, if $U \xrightarrow{f} V \xrightarrow{g} W$ are morphisms of $\mathbf{C}$, then in the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Delta_f"] & U \times_V U \arrow[r, "i"] \arrow[d, "p"'] & U \times_W U \arrow[d, "f \times f"] \\ & V \arrow[r, "\Delta_g"'] & V \times_W V \end{tikzcd}$$

(where i and p are the natural morphisms) the square is cartesian and $\Delta_{g \circ f} = i \circ \Delta_f$. □

Lemma A.5. *Let* P *be a property of morphisms of* $\mathbf{C}$ *which is stable under composition and base change. If* $U \xrightarrow{f} V \xrightarrow{g} W$ *are morphisms of* $\mathbf{C}$ *such that* $g \circ f$ *and* $\Delta_g \colon V \to V \times_W V$ *satisfy* P*, then* f *satisfies* P*, too.*

Proof. Since f factors as

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "(\mathrm{id}, f)"'] & & V \\ & U \times_W V \arrow[ur, "pr_2"'] & \end{tikzcd}$$

it is enough to show that pr_2 and (id, f) satisfy P. As $g \circ f$ and Δ_g satisfy P, the cartesian diagrams

$$\begin{tikzcd} U \times_W V \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "g"] \\ U \arrow[r, "g \circ f"'] & W \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "(\mathrm{id}, f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U \times_W V \arrow[d, "f \times \mathrm{id}"] \\ V \arrow[r, "\Delta_g"'] & V \times_W V \end{tikzcd}$$

imply that pr_2 and (id, f) satisfy P, too. □

Lemma A.6. *Let* P *be a property of morphisms of* $\mathbf{C}$ *which is stable under composition and base change. Given a commutative diagram in* $\mathbf{C}$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] & W \arrow[d, "h"] & V \arrow[l] \arrow[d, "g"] \\ U' \arrow[r] & W' & V' \arrow[l] \end{tikzcd}$$

such that f*,* g *and* h *satisfy* P*, then* $f \times g \colon U \times_W V \to U' \times_{W'} V'$ *satisfies* P*, too.*

Proof. Just notice that there are natural cartesian diagrams

$$
\begin{tikzcd}
U \times_W V \arrow[r] \arrow[d, "\mathrm{id}\times\mathrm{id}"'] \arrow[dr, phantom, "\square"] & W \arrow[d, "h"] \\
U \times_{W'} V \arrow[r] & W'
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times_{W'} V \arrow[r] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\
U' \times_{W'} V \arrow[r] & U'
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \times_{W'} V \arrow[r] \arrow[d, "\mathrm{id}\times g"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "g"] \\
U' \times_{W'} V' \arrow[r] & V'
\end{tikzcd}
$$

and that $f \times g = (\mathrm{id} \times g) \circ (f \times \mathrm{id}) \circ (\mathrm{id} \times \mathrm{id})$. □

A.2 2-categories and 2-functors

This sections contains what is needed in the paper about (strict) 2-categories and 2-functors. A more thorough treatment can be found, for instance, in [3] or [13].

Definition A.7. A *(strict) 2-category* $\underline{\mathbf{C}}$ is given by:

- a set $\mathrm{Ob}(\underline{\mathbf{C}})$ of *objects* of $\underline{\mathbf{C}}$ (as in the case of categories, we will usually write $U \in \underline{\mathbf{C}}$ instead of $U \in \mathrm{Ob}(\underline{\mathbf{C}})$ if U is an object of $\underline{\mathbf{C}}$);
- a set 1-$\mathrm{Mor}(\underline{\mathbf{C}})$ (or simply $\mathrm{Mor}(\underline{\mathbf{C}})$) of 1-*morphisms* (or simply *morphisms*) of $\underline{\mathbf{C}}$, each having a *source* and a *target* given by objects of $\underline{\mathbf{C}}$ (for $U, V \in \underline{\mathbf{C}}$, the set of morphisms with source U and target V will be denoted by 1-$\mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$, or simply by $\mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$);
- a set 2-$\mathrm{Mor}(\underline{\mathbf{C}})$ of 2-*morphisms* of $\underline{\mathbf{C}}$, each having a *source* and a *target* given by 1-morphisms of $\underline{\mathbf{C}}$ with same source and target (for $U, V \in \underline{\mathbf{C}}$ and $f, g \in \mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$, the set of 2-morphisms with source f and target g will be denoted by 2-$\mathrm{Hom}_{\underline{\mathbf{C}}}(f, g)$);

together with:

- *composition* of 1-morphisms, i.e. for all $U, V, W \in \underline{\mathbf{C}}$, a map
$$\begin{aligned}\mathrm{Hom}_{\underline{\mathbf{C}}}(U, V) \times \mathrm{Hom}_{\underline{\mathbf{C}}}(V, W) &\to \mathrm{Hom}_{\underline{\mathbf{C}}}(U, W);\\ (f, g) &\mapsto g \circ f\end{aligned}$$
- *vertical composition* of 2-morphisms, i.e. for all $U, V \in \underline{\mathbf{C}}$ and all $f, g, h \in \mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$, a map
$$\begin{aligned}2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f, g) \times 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(g, h) &\to 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f, h);\\ (\mu, \nu) &\mapsto \nu \circ \mu\end{aligned}$$
- *horizontal composition* of 2-morphisms, i.e. for all $U, V, W \in \underline{\mathbf{C}}$, all $f, g \in \mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$ and all $h, k \in \mathrm{Hom}_{\underline{\mathbf{C}}}(V, W)$, a map
$$\begin{aligned}2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f, g) \times 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(h, k) &\to 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(h \circ f, k \circ g);\\ (\mu, \nu) &\mapsto \nu \star \mu\end{aligned}$$

such that the following axioms are satisfied:

1. composition of 1-morphisms is associative and for every $U \in \underline{\mathbf{C}}$ there is an identity morphism id_U (i.e., the objects and 1-morphisms of $\underline{\mathbf{C}}$, together with composition of 1-morphisms, form an ordinary category, called the *underlying category of* $\underline{\mathbf{C}}$);

2. vertical composition of 2-morphisms is associative and there is an identity 2-morphism id_f for every $f \in 1\text{-}\mathrm{Mor}(\underline{\mathbf{C}})$;

3. horizontal composition of 2-morphisms is associative, $\mathrm{id}_g \star \mathrm{id}_f = \mathrm{id}_{g \circ f}$ if f and g are two composable morphisms and $\mathrm{id}_{\mathrm{id}_V} \star \mu = \mu = \mu \star \mathrm{id}_{\mathrm{id}_U}$ if μ is a 2-morphisms between two 1-morphisms with same source U and same target V;

4. horizontal and vertical composition are compatible, meaning that, given $U_1, U_2, U_3 \in \underline{\mathbf{C}}$, $f_i, g_i, h_i \in \mathrm{Hom}_{\underline{\mathbf{C}}}(U_i, U_{i+1})$, $\mu_i \in 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f_i, g_i)$ and $\nu_i \in 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(g_i, h_i)$ for $i = 1, 2$, there is the equality
$$(\nu_2 \circ \mu_2) \star (\nu_1 \circ \mu_1) = (\nu_2 \star \nu_1) \circ (\mu_2 \star \mu_1) \in 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f_2 \circ f_1, h_2 \circ h_1).$$

Remark A.8. In a diagram in a 2-category we will represent a 2-morphism with a double arrow $\Rightarrow$. For instance, the situation of the last axiom would look as follows:

$$\begin{tikzcd}
U_1 \arrow[r, bend left=70, "f_1"] \arrow[r, "g_1"] \arrow[r, bend right=70, "h_1"'] \arrow[r, phantom, shift left=4ex, "\mu_1\Downarrow"] \arrow[r, phantom, shift right=4ex, "\nu_1\Downarrow"] & U_2 \arrow[r, bend left=70, "f_2"] \arrow[r, "g_2"] \arrow[r, bend right=70, "h_2"'] \arrow[r, phantom, shift left=4ex, "\mu_2\Downarrow"] \arrow[r, phantom, shift right=4ex, "\nu_2\Downarrow"] & U_3.
\end{tikzcd}$$

Remark A.9. Given objects U, V of a 2-category $\underline{\mathbf{C}}$, we can define a category $\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)$ by setting
$$\mathrm{Ob}(\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)) := \mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$$
$$\mathrm{Hom}_{\mathbf{Hom}_{\underline{\mathbf{C}}}(U,V)}(f, g) := 2\text{-}\mathrm{Hom}_{\underline{\mathbf{C}}}(f, g)$$

for all $f, g \in \mathrm{Hom}_{\underline{\mathbf{C}}}(U, V)$ (composition of morphisms in $\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)$ is of course given by vertical composition of 2-morphisms in $\underline{\mathbf{C}}$): then axiom (2) precisely says that $\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)$ is indeed a category. Setting for brevity $\mathbf{C}_{U,V} := \mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)$, the other axioms also imply that for all $U, V, W \in \underline{\mathbf{C}}$ there is a functor
$$F_{U,V,W} \colon \mathbf{C}_{U,V} \times \mathbf{C}_{V,W} \to \mathbf{C}_{U,W}$$

(defined on objects by composition of morphisms in $\underline{\mathbf{C}}$ and on morphisms by horizontal composition of 2-morphisms) such that for all $U, V, W, X \in \underline{\mathbf{C}}$ the diagram

$$\begin{tikzcd} \mathbf{C}_{U,V} \times \mathbf{C}_{V,W} \times \mathbf{C}_{W,X} \arrow[r, "\mathrm{id}\times F_{V,W,X}"] \arrow[d, "F_{U,V,W}\times\mathrm{id}"'] & \mathbf{C}_{U,V} \times \mathbf{C}_{V,X} \arrow[d, "F_{U,V,X}"] \\ \mathbf{C}_{U,W} \times \mathbf{C}_{W,X} \arrow[r, "F_{U,W,X}"'] & \mathbf{C}_{U,X} \end{tikzcd}$$

commutes (meaning equality, not just isomorphism of functors) and, denoting (for every category $\mathbf{D}$ and for every $T \in \mathbf{D}$) by $K_T \colon \{*\} \to \mathbf{D}$ the functor which sends $*$ to T, the diagrams

$$\begin{tikzcd} \mathbf{C}_{U,U} \times \mathbf{C}_{U,V} \arrow[dr, "F_{U,U,V}"] & \\ \{*\} \times \mathbf{C}_{U,V} \arrow[u, "K_{\mathrm{id}_U}\times\mathrm{id}"] \arrow[r, "\sim"'] & \mathbf{C}_{U,V} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}_{U,V} \times \mathbf{C}_{V,V} \arrow[dr, "F_{U,V,V}"] & \\ \mathbf{C}_{U,V} \times \{*\} \arrow[u, "\mathrm{id}\times K_{\mathrm{id}_V}"] \arrow[r, "\sim"'] & \mathbf{C}_{U,V} \end{tikzcd}$$

also commute (the horizontal maps are the natural isomorphisms).

Conversely, it is easy to prove that the data of a set of objects $\mathrm{Ob}(\underline{\mathbf{C}})$ and, for all $U, V, W \in \mathrm{Ob}(\underline{\mathbf{C}})$, of a category $\mathbf{C}_{U,V}$, of an object $\mathrm{id}_U \in \mathbf{C}_{U,U}$ and of a functor $F_{U,V,W} \colon \mathbf{C}_{U,V} \times \mathbf{C}_{V,W} \to \mathbf{C}_{U,W}$ such that the above diagrams commute, determine a 2-category $\underline{\mathbf{C}}$ with the given set of objects, with $\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V) = \mathbf{C}_{U,V}$ and with composition of morphisms and horizontal composition of 2-morphisms induced by the functors $F_{U,V,W}$.

Example A.10. Every ordinary category $\mathbf{C}$ naturally defines a 2-category (which we will denote again by $\mathbf{C}$), with the same objects and morphisms (and also the same composition of morphisms, of course), and having as 2-morphisms only the identities id_f for $f \in \mathrm{Mor}(\mathbf{C})$. When necessary, we will regard every category as a 2-category in this way. Conversely, it is clear that every 2-category having only the identities as 2-morphisms comes from a category as above. Therefore, we will say that such a 2-category is a category.

Example A.11. The prototype of 2-category is $\underline{\mathbf{Cat}}$ (the 2-category of all categories): its objects are categories, its morphisms are functors and its 2-morphisms are natural transformations of functors. As composition of functors and vertical composition of natural transformations are the obvious ones, we only recall how horizontal composition of natural transformations is defined. Given a diagram in $\underline{\mathbf{Cat}}$

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\alpha"'] & \mathbf{D} \arrow[r, bend left, "H"{name=H}] \arrow[r, bend right, "K"'{name=K}] \arrow[from=H, to=K, Rightarrow, "\beta"'] & \mathbf{E} \end{tikzcd}$$

the natural transformation

$$\beta \star \alpha \in 2\text{-Hom}_{\underline{\mathbf{Cat}}}(H \circ F, K \circ G) = \mathrm{Hom}_{\mathbf{Fun}(\mathbf{C},\mathbf{E})}(H \circ F, K \circ G)$$

is defined as follows. For every $U \in \mathbf{C}$ the diagram in $\mathbf{E}$

$$\begin{tikzcd} H(F(U)) \arrow[r, "H(\alpha(U))"] \arrow[d, "\beta(F(U))"'] & H(G(U)) \arrow[d, "\beta(G(U))"] \\ K(F(U)) \arrow[r, "K(\alpha(U))"'] & K(G(U)) \end{tikzcd}$$

commutes (because β is a natural transformation), and we define

$$(\beta \star \alpha)(U) := \beta(G(U)) \circ H(\alpha(U)) = K(\alpha(U)) \circ \beta(F(U)) \\ \in \mathrm{Hom}_{\mathbf{E}}((H \circ F)(U), (K \circ G)(U)).$$

It is easy to see (using the fact that also α is a natural transformation) that $\beta \star \alpha$ is indeed a natural transformation and then that $\underline{\mathbf{Cat}}$ actually satisfies the axioms of 2-category. Notice that $\mathbf{Hom}_{\underline{\mathbf{Cat}}}(\mathbf{C}, \mathbf{D}) = \mathbf{Fun}(\mathbf{C}, \mathbf{D})$.

Definition A.12. A 2-morphism in a 2-category is a 2-*isomorphism* if it is invertible with respect to vertical composition of 2-morphisms.

Remark A.13. As for 1-morphisms, it would be natural (in analogy with the standard definitions for $\underline{\mathbf{Cat}}$) to call a morphism $f\colon U \to V$ in a 2-category an *isomorphism* (respectively an *equivalence*) if it is invertible with respect to composition of morphisms (respectively, if there exists a morphism $g\colon V \to U$ together with 2-isomorphisms $f \circ g \cong \mathrm{id}_V$ and $g \circ f \cong \mathrm{id}_U$). However, we will use these definitions only in this appendix, because essentially the only 2-categories we are interested in are given by families of fibred categories, and for them another terminology will be used (see Definition 5.22 and Remark 5.23).

Lemma A.14. *A morphism $f\colon U \to V$ in a 2-category $\underline{\mathbf{C}}$ is an equivalence if and only if the natural functor*

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f)\colon \mathbf{Hom}_{\underline{\mathbf{C}}}(U', U) \to \mathbf{Hom}_{\underline{\mathbf{C}}}(U', V)$$

(defined on objects by $g \mapsto f \circ g$ and on morphisms by $\alpha \mapsto \mathrm{id}_f \star \alpha$, and often denoted simply by $f\circ$) is an equivalence of categories for every $U' \in \underline{\mathbf{C}}$. Similarly, f is an equivalence if and only if the natural functor

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(f, V')\colon \mathbf{Hom}_{\underline{\mathbf{C}}}(V, V') \to \mathbf{Hom}_{\underline{\mathbf{C}}}(U, V')$$

(defined on objects by $g \mapsto g \circ f$ and on morphisms by $\alpha \mapsto \alpha \star \mathrm{id}_f$, and often denoted simply by $\circ f$) is an equivalence of categories for every $V' \in \underline{\mathbf{C}}$.

Proof. If f is an equivalence, say $g\colon V \to U$ is such that $g \circ f \cong \mathrm{id}_U$ and $f \circ g \cong \mathrm{id}_V$, then for every $U' \in \underline{\mathbf{C}}$ the functor $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', g)\colon \mathbf{Hom}_{\underline{\mathbf{C}}}(U', V) \to \mathbf{Hom}_{\underline{\mathbf{C}}}(U', U)$ is a quasi-inverse of $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f)$ (because

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(U', g) \circ \mathrm{Hom}_{\underline{\mathbf{C}}}(U', f) = \mathrm{Hom}_{\underline{\mathbf{C}}}(U', g \circ f) \cong \mathrm{Hom}_{\underline{\mathbf{C}}}(U', \mathrm{id}_U) = \mathrm{id}_{\mathbf{Hom}_{\underline{\mathbf{C}}}(U', U)}$$

and similarly $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f) \circ \mathrm{Hom}_{\underline{\mathbf{C}}}(U', g) \cong \mathrm{id}_{\mathbf{Hom}_{\underline{\mathbf{C}}}(U', V)}$).

Assume conversely that $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f)$ is an equivalence of categories for every $U' \in \underline{\mathbf{C}}$: taking $U' = V$, we see that there exists $g\colon V \to U$ such that $f \circ g \cong \mathrm{id}_V$. Since both $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f)$ and

$$\mathrm{Hom}_{\underline{\mathbf{C}}}(U', f) \circ \mathrm{Hom}_{\underline{\mathbf{C}}}(U', g) = \mathrm{Hom}_{\underline{\mathbf{C}}}(U', f \circ g) \cong \mathrm{id}_{\mathbf{Hom}_{\underline{\mathbf{C}}}(U', V)}$$

are equivalences of categories, it follows that $\mathrm{Hom}_{\underline{\mathbf{C}}}(U', g)$ is an equivalence for every $U' \in \underline{\mathbf{C}}$, too. Therefore, by the same argument, there exists $f'\colon U \to V$ such that $g \circ f' \cong \mathrm{id}_U$. Since $f \cong f \circ g \circ f' \cong f'$, we have also $g \circ f \cong \mathrm{id}_U$, which proves that f is an equivalence.

The proof of the second statement is completely similar. □

In a 2-category the usual notion of commutative diagram (where one requires equality of the compositions of morphisms obtained following all possible "paths" between two objects) is often too restrictive, and it is usually more useful to consider the weaker notion of 2-*commutative diagram* (where equalities are replaced by 2-isomorphisms). So, for instance, we will say that the diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "k"'] & V \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow\mu"] \\ W \arrow[r, "h"'] & X \end{tikzcd}$$

is 2-commutative if $\mu\colon f \circ g \to h \circ k$ is a 2-isomorphism.

Definition A.15. A 2-*subcategory* $\underline{\mathbf{C}}'$ of a 2-category $\underline{\mathbf{C}}$ is given by subsets of objects, 1-morphisms and 2-morphisms of $\underline{\mathbf{C}}$, which form a 2-category with compositions (of 1-morphisms and 2-morphisms) defined as in $\underline{\mathbf{C}}$. A 2-subcategory $\underline{\mathbf{C}}' \subseteq \underline{\mathbf{C}}$ is *full* if $\mathbf{Hom}_{\underline{\mathbf{C}}'}(U, V) = \mathbf{Hom}_{\underline{\mathbf{C}}}(U, V)$ for all $U, V \in \underline{\mathbf{C}}'$. $\underline{\mathbf{C}}'$ is a *strictly full* 2-subcategory of $\underline{\mathbf{C}}$ if it is full and every object of $\underline{\mathbf{C}}$ equivalent to an object of $\underline{\mathbf{C}}'$ is in $\underline{\mathbf{C}}'$.

Example A.16. We will denote by $\underline{\mathbf{Gpd}}$ the (strictly) full 2-subcategory of $\underline{\mathbf{Cat}}$ whose objects are groupoids. Notice that the (not strictly) full 2-subcategory of $\underline{\mathbf{Cat}}$ whose objects are sets can be identified with $\mathbf{Set}$.

Definition A.17. Let $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ be 2-categories. A *(strict)* 2-*functor* $\underline{F}\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is given by maps $\mathrm{Ob}(\underline{\mathbf{C}}) \to \mathrm{Ob}(\underline{\mathbf{D}})$, $1\text{-}\mathrm{Mor}(\underline{\mathbf{C}}) \to 1\text{-}\mathrm{Mor}(\underline{\mathbf{D}})$ and $2\text{-}\mathrm{Mor}(\underline{\mathbf{C}}) \to 2\text{-}\mathrm{Mor}(\underline{\mathbf{D}})$ which are compatible with sources and targets and which preserve compositions and identities (in particular, $\underline{F}$ induces an ordinary functor between the underlying categories of $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$, which is called the *underlying functor of* $\underline{F}$).

Definition A.18. Let $\underline{F}, \underline{G} \colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ be 2-functors. A *(strict)* 2*-natural transformation* $\underline{\alpha} \colon \underline{F} \to \underline{G}$ is given by morphisms (of $\underline{\mathbf{D}}$) $\underline{\alpha}(U) \colon \underline{F}(U) \to \underline{G}(U)$ for every $U \in \underline{\mathbf{C}}$ such that the following conditions are satisfied:

1. if $f \colon U \to V$ is a morphism of $\underline{\mathbf{C}}$,

$$\underline{\alpha}(V) \circ \underline{F}(f) = \underline{G}(f) \circ \underline{\alpha}(U) \colon \underline{F}(U) \to \underline{G}(V)$$

   (in particular, $\underline{\alpha}$ induces an ordinary natural transformation between the underlying functors of $\underline{F}$ and $\underline{G}$);

2. if $\mu \colon f \to g$ is a 2-morphism of $\underline{\mathbf{C}}$ (with $f, g \colon U \to V$),

$$\begin{aligned} &\mathrm{id}_{\underline{\alpha}(V)} \star \underline{F}(\mu) = \underline{G}(\mu) \star \mathrm{id}_{\underline{\alpha}(U)} \\ &\qquad : \underline{\alpha}(V) \circ \underline{F}(f) = \underline{G}(f) \circ \underline{\alpha}(U) \to \underline{\alpha}(V) \circ \underline{F}(g) = \underline{G}(g) \circ \underline{\alpha}(U). \end{aligned}$$

Definition A.19. Let $\underline{F}, \underline{G} \colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ be 2-functors and $\underline{\alpha}, \underline{\beta} \colon \underline{F} \to \underline{G}$ 2-natural transformations. A *modification* $\underline{\epsilon} \colon \underline{\alpha} \to \underline{\beta}$ is given by 2-morphisms (of $\underline{\mathbf{D}}$)

$$\underline{\epsilon}(U) \colon \underline{\alpha}(U) \to \underline{\beta}(U)$$

for every $U \in \underline{\mathbf{C}}$ such that

$$\begin{aligned} &\underline{\epsilon}(V) \star \mathrm{id}_{\underline{F}(f)} = \mathrm{id}_{\underline{G}(f)} \star \underline{\epsilon}(U) \\ &\qquad : \underline{\alpha}(V) \circ \underline{F}(f) = \underline{G}(f) \circ \underline{\alpha}(U) \to \underline{\beta}(V) \circ \underline{F}(f) = \underline{G}(f) \circ \underline{\beta}(U). \end{aligned}$$

for every morphism $f \colon U \to V$ of $\underline{\mathbf{C}}$.

Remark A.20. The above definitions suggest that there should be a notion of 3-category such that 2-categories form a 3-category. This is actually true, but fortunately we will not need it. For our purposes it is enough to point out the following fact, whose proof is straightforward.

Proposition A.21. *Let $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ be* 2*-categories.* 2*-functors from $\underline{\mathbf{C}}$ to $\underline{\mathbf{D}}$ form the objects of a* 2*-category $\underline{\mathbf{Fun}}(\underline{\mathbf{C}}, \underline{\mathbf{D}})$, with morphisms given by* 2*-natural transformations and* 2*-morphisms by modifications (compositions of morphisms and* 2*-morphisms are induced by those of $\underline{\mathbf{D}}$).*

Definition A.22. A 2-functor $\underline{F} \colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is 2*-faithful* (respectively 2*-full*, respectively *essentially full*) if for all $U, V \in \underline{\mathbf{C}}$ the induced functor

$$\mathbf{Hom}_{\underline{\mathbf{C}}}(U, V) \to \mathbf{Hom}_{\underline{\mathbf{D}}}(\underline{F}(U), \underline{F}(V))$$

is faithful (respectively full, respectively essentially surjective). $\underline{F}$ is *essentially surjective* if every object V of $\underline{\mathbf{D}}$ is equivalent to $\underline{F}(U)$ for some object U of $\underline{\mathbf{C}}$.

Definition A.23. A 2-functor $\underline{F}\colon \underline{\mathbf{C}} \to \underline{\mathbf{D}}$ is a *(strict) 2-equivalence* if there exists a 2-functor $\underline{G}\colon \underline{\mathbf{D}} \to \underline{\mathbf{C}}$ (called a *2-quasi-inverse* of $\underline{F}$) such that $\underline{G} \circ \underline{F}$ is equivalent to $\mathrm{id}_{\underline{\mathbf{C}}}$ in $\underline{\mathbf{Fun}}(\underline{\mathbf{C}}, \underline{\mathbf{C}})$ and $\underline{F} \circ \underline{G}$ is equivalent to $\mathrm{id}_{\underline{\mathbf{D}}}$ in $\underline{\mathbf{Fun}}(\underline{\mathbf{D}}, \underline{\mathbf{D}})$.

A 2-functor is a *lax 2-equivalence* if it is 2-fully faithful, essentially full and essentially surjective.

Remark A.24. Of course a strict 2-equivalence is also a lax one, but the converse is false in general. In order to explain the latter definition we have to say that there are also (various) notions of *lax 2-category*, of *lax 2-functor* (between strict or lax 2-categories) and of *lax 2-natural transformation* (between strict or lax 2-functors), which we are not going to define in general, since we will use only lax 2-functors in a particular case, so that we give in section Section 5.3 the definitions we need. Here it is enough to say that in each case the main difference is that some equalities between 1-morphisms are replaced by 2-(iso)morphisms. So, for instance, in a lax 2-category composition of 1-morphisms, instead of being associative, is only associative up to 2-(iso)morphisms (subject to suitable compatibilities). One can prove that the above definition of lax 2-equivalence corresponds to the natural one to use when dealing with lax 2-functors (between strict 2-categories).

A.3 Miscellaneous results

Lemma A.25. *Given a commutative diagram of rings*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"'] & B \arrow[d, "\psi'"] \arrow[dl, "\psi"'] \\ C \arrow[r, two heads] & C/\mathtt{I} \end{tikzcd}$$

(where $\mathtt{I} \subset C$ is an ideal such that $\mathtt{I}^2 = 0$), the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\rho"'] & B \arrow[d, "\psi'"] \arrow[dl, "\tilde{\psi}"'] \\ C \arrow[r, two heads] & C/\mathtt{I} \end{tikzcd}$$

is also commutative if and only if $\tilde{\psi} - \psi\colon B \to \mathtt{I} \subset C$ is an A-derivation, where $\mathtt{I}$ is viewed as a B-module via ψ' (the fact that $\mathtt{I}^2 = 0$ implies that $\mathtt{I}$ is in a natural way a $C/\mathtt{I}$-module).

Proof. Exercise, or see [17, Prop. 2.8]. $\square$

Lemma A.26. *Let $A \to A'$ be a morphism of rings and $\mathtt{I} \subset A'$ an ideal. Then $d^{\mathtt{I}}_{A'/A}$ is left invertible (i.e., the sequence*

$$0 \to \mathtt{I}/\mathtt{I}^2 \xrightarrow{d^{\mathtt{I}}_{A'/A}} \Omega_{A'/A} \otimes_{A'} (A'/\mathtt{I}) \to \Omega_{(A'/\mathtt{I})/A} \to 0$$

is exact and splits) if and only if there is a morphism of A-algebras $\phi\colon A'/\mathtt{I} \to A'/\mathtt{I}^2$ such that $\pi \circ \phi = \mathrm{id}_{A'/\mathtt{I}}$ (where $\pi\colon A'/\mathtt{I}^2 \to A'/\mathtt{I}$ is the projection).

Proof. [17, lemma 2.16]. □

Lemma A.27. *Let B be a ring and $f\colon M \to N$ a morphism of B-modules with M finitely generated and N projective. For every $\mathtt{q} \in \operatorname{Spec} B$ the following conditions are equivalent:*

1. *$f_{\mathtt{q}}\colon M_{\mathtt{q}} \to N_{\mathtt{q}}$ is left invertible;*
2. *there exists $b \in B \setminus \mathtt{q}$ such that $f_b\colon M_b \to N_b$ is left invertible;*
3. *there exist $x_1, \ldots, x_m \in M$ and $\varphi_1, \ldots \varphi_m \in B^\vee := \operatorname{Hom}_B(N, B)$ (for some $m \in \mathbb{N}$) such that $(x_1, \ldots, x_m)_{\mathtt{q}} = M_{\mathtt{q}}$ and $\det\bigl(\varphi_i(f(x_j))\bigr)_{1\le i,j\le m} \notin \mathtt{q}$.*

Proof. [9, Chap. 0, Cor. 19.1.12]. □

Definition A.28. Let X be a topological space, $\mathcal{G}$ a sheaf of groups and $\mathcal{P}$ a sheaf of sets on X. Assume that $\varrho\colon \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ is an action of $\mathcal{G}$ on $\mathcal{P}$ (i.e., ϱ is a morphism of sheaves such that $\varrho(U)$ is an action of $\mathcal{G}(U)$ on $\mathcal{P}(U)$ for every open subset U of X). $(\mathcal{P}, \varrho)$ (or simply $\mathcal{P}$, by abuse of notation) is a *pseudo-torsor under* $\mathcal{G}$ (or a $\mathcal{G}$*-pseudo-torsor*) if for every $U \subseteq X$ open either $\mathcal{P}(U) = \emptyset$ or $\varrho(U)$ is free and transitive. $\mathcal{P}$ is a *torsor* under $\mathcal{G}$ (or a $\mathcal{G}$*-torsor*) if moreover there exists a base $\{U_i\}_{i\in I}$ of the topology of X such that $\mathcal{P}(U_i) \neq \emptyset$ for every $i \in I$.

A morphism of (pseudo)torsors is just a morphism of sheaves which is compatible with the actions (in the obvious sense).

A $\mathcal{G}$-torsor is *trivial* if it is isomorphic to $\mathcal{G}$ with action $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ given by multiplication.

Proposition A.29. *Let X be a topological space and $\mathcal{G}$ a sheaf of groups on X. There is a natural bijection between the set of isomorphism classes of $\mathcal{G}$-torsors on X and $H^1(X, \mathcal{G})$ (such that the class of trivial $\mathcal{G}$-torsors corresponds to 0).*

Proof. [9, 16.5.15]. □

Lemma A.30. *Let X be a ringed space and $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(X)$. If $\mathcal{F}$ is of finite presentation, then the natural morphism of $\mathcal{O}_{X,x}$-modules $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})_x \to \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is an isomorphism. If moreover X is a scheme and $\mathcal{G}$ is quasi-coherent, then $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ is quasi-coherent, too.*

Proof. The question being local, we can assume that there is an exact sequence $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$ for some $n, m \in \mathbb{N}$. Applying $\mathcal{H}om_X(-, \mathcal{G})$ yields the exact sequence

$$0 \to \mathcal{H}om_X(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_X(\mathcal{O}_X^m, \mathcal{G}) \cong \mathcal{G}^m \to \mathcal{H}om_X(\mathcal{O}_X^n, \mathcal{G}) \cong \mathcal{G}^n,$$

so that, if $\mathcal{G} \in \mathbf{QCoh}(X)$ and X is a scheme, $\mathcal{H}om_X(\mathcal{F},\mathcal{G}) \in \mathbf{QCoh}(X)$ because kernel of a morphism in the abelian category $\mathbf{QCoh}(X)$. As for the first statement, from the same sequence we also get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}om_X(\mathcal{F},\mathcal{G})_x \arrow[r] \arrow[d] & \mathcal{H}om_X(\mathcal{O}_X^n,\mathcal{G})_x \arrow[r] \arrow[d] & \mathcal{H}om_X(\mathcal{O}_X^m,\mathcal{G})_x \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^n,\mathcal{G}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^m,\mathcal{G}_x)
\end{tikzcd}$$

with exact rows. Since the vertical maps in the middle and on the right are clearly isomorphisms, it follows from the five lemma that the map on the left is an isomorphism, too. □

Corollary A.31. *Let X be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation. If $x \in X$ is such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^n$ for some $n \in \mathbb{N}$, then there is an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U^n$.*

Proof. For $\mathcal{G},\mathcal{H} \in \mathbf{Mod}(X)$ let $\varphi_{\mathcal{G},\mathcal{H}}\colon \mathcal{H}om_X(\mathcal{G},\mathcal{H})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x,\mathcal{H}_x)$ be the natural map, and fix an isomorphism $a\colon \mathcal{F}_x \xrightarrow{\sim} \mathcal{O}_{X,x}^n$. Surjectivity of $\varphi_{\mathcal{F},\mathcal{O}_X^n}$ and $\varphi_{\mathcal{O}_X^n,\mathcal{F}}$ implies that there exist $\alpha \in \mathrm{Hom}_V(\mathcal{F}|_V,\mathcal{O}_V^n)$ and $\alpha' \in \mathrm{Hom}_V(\mathcal{O}_V^n,\mathcal{F}|_V)$ (for some open neighbourhood V of x) such that $\varphi_{\mathcal{F},\mathcal{O}_X^n}(\alpha_x) = a$ and $\varphi_{\mathcal{O}_X^n,\mathcal{F}}(\alpha'_x) = a^{-1}$. As $\varphi_{\mathcal{O}_X^n,\mathcal{O}_X^n}((\alpha \circ \alpha')_x) = \mathrm{id}_{\mathcal{O}_{X,x}^n}$ and $\varphi_{\mathcal{F},\mathcal{F}}((\alpha' \circ \alpha)_x) = \mathrm{id}_{\mathcal{F}_x}$, injectivity of $\varphi_{\mathcal{O}_X^n,\mathcal{O}_X^n}$ and $\varphi_{\mathcal{F},\mathcal{F}}$ implies that there is an open neighbourhood $x \in U \subseteq V$ such that $(\alpha \circ \alpha')|_U = \mathrm{id}_{\mathcal{O}_U^n}$ and $(\alpha' \circ \alpha)|_U = \mathrm{id}_{\mathcal{F}|_U}$ (hence $\alpha|_U\colon \mathcal{F}|_U \to \mathcal{O}_U^n$ is an isomorphism). □

B Some complements

B.1 Limits

In this section we give the definition and main properties of limits; more details and proofs can be found in [2] or in many books on category theory, like [16].

Having fixed two categories $\mathbf{I}$ (the "index" category) and $\mathbf{C}$ (the category where we want to take limits), for every $U \in \mathbf{C}$ we will denote by $K_U \in \mathbf{Fun}(\mathbf{I},\mathbf{C})$ the constant functor sending each object of $\mathbf{I}$ to U and each morphism of $\mathbf{I}$ to id_U; similarly, if $f\colon U \to V$ is a morphism of $\mathbf{C}$, $K_f \in \mathrm{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(K_U,K_V)$ will be the natural transformation defined by $K_f(X) = f$ for every $X \in \mathbf{I}$.

Definition B.1. Let $H\colon \mathbf{I} \to \mathbf{C}$ be a functor. The *direct limit* (or *inductive limit*, or *colimit*) of H is the functor $\varinjlim H \in \mathbf{Fun}(\mathbf{C},\mathbf{Set}) = \widehat{\mathbf{C}^\circ}$ defined on every object U of $\mathbf{C}$ by

$$(\varinjlim H)(U) := \mathrm{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(H,K_U)$$

and on every morphism $f\colon U \to V$ of $\mathbf{C}$ by

$$\begin{aligned}(\varinjlim H)(f)\colon \operatorname{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(H, K_U) &\to \operatorname{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(H, K_V).\\ \alpha &\mapsto K_f \circ \alpha\end{aligned}$$

The *inverse limit* (or *projective limit*, or *limit*) of H is the functor $\varprojlim H \in \mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Set}) = \widehat{\mathbf{C}}$ defined on every object U of $\mathbf{C}$ by

$$(\varprojlim H)(U) := \operatorname{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(K_U, H)$$

and on every morphism $f\colon U \to V$ of $\mathbf{C}$ by

$$\begin{aligned}(\varprojlim H)(f)\colon \operatorname{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(K_V, H) &\to \operatorname{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(K_U, H).\\ \alpha &\mapsto \alpha \circ K_f\end{aligned}$$

In case $\varinjlim H$ (as a presheaf on $\mathbf{C}^\circ$) or $\varprojlim H$ (as a presheaf on $\mathbf{C}$) is representable (in both cases, by an object of $\mathbf{C}$, since $\mathrm{Ob}(\mathbf{C}^\circ) = \mathrm{Ob}(\mathbf{C})$), we will call the corresponding limit *representable*, or we will say that the limit exists (in $\mathbf{C}$).

Remark B.2. Using Corollary 4.6 we see that $\varprojlim H$ is representable (similar considerations hold, of course, also for $\varinjlim H$) if and only if there is an object of $\mathbf{C}$, necessarily unique up to isomorphism and usually denoted again by $\varprojlim H$, together with a natural transformation $\alpha\colon K_{\varprojlim H} \to H$ which satisfies the following universal property: for every $V \in \mathbf{C}$ and every $\beta\colon K_V \to H$, there exists a unique $f\colon V \to \varprojlim H$ in $\mathbf{C}$ such that $\beta = \alpha \circ K_f$.

Example B.3. If the index category $\mathbf{I}$ is just a set I (so that $H\colon I \to \mathbf{C}$ is completely determined by the set of objects $\{U_i := H(i)\}_{i\in I}$), then, by definition of natural transformation, $\varprojlim H$ is representable if and only if there is an object $\varprojlim H \in \mathbf{C}$ together with morphisms $p_i\colon \varprojlim H \to U_i$ (for $i \in I$) satisfying the following property: given morphisms $f_i\colon V \to U_i$ (for $i \in I$), there exists a unique $f\colon V \to \varprojlim H$ such that $f_i = p_i \circ f$ for every $i \in I$. In other words, $\varprojlim H$ is representable if and only if $\prod_{i\in I} U_i$ exists in $\mathbf{C}$, and in this case $\varprojlim H \cong \prod_{i\in I} U_i$. In a similar way, one can see that $\varinjlim H$ is representable if and only if $\coprod_{i\in I} U_i$ exists in $\mathbf{C}$, and that in this case $\varinjlim H \cong \coprod_{i\in I} U_i$. It is also easy to prove that existence of (co)kernels and fibred (co)products corresponds to representability of some suitable (co)limits. Specifically, if we represent a category using dots for objects and arrows for non identity morphisms, when $\mathbf{I} = (\cdot \rightrightarrows \cdot)$ representable (co)limits yield (co)kernels, whereas when $\mathbf{I} = (\cdot \to \cdot \leftarrow \cdot)$ (respectively $\mathbf{I} = (\cdot \leftarrow \cdot \to \cdot)$) representable inverse (respectively direct) limits yield fibred products (respectively coproducts).

Example B.4. The usual notion of limit over a filtered set is also a particular case of Definition B.1. In fact, every preordered set $(I, \leq)$ determines a category $\mathbf{I}$ with $\mathrm{Ob}(\mathbf{I}) := I$ and such that $\mathrm{Hom}_{\mathbf{I}}(i,j)$ contains exactly one morphism if $i \leq j$ and is empty otherwise (conversely, every category such that there is at most one morphism between any two objects comes from a preordered set in this way).

Definition B.5. A limit is *finite* if the index category $\mathbf{I}$ is such that $\mathrm{Ob}(\mathbf{I})$ and $\mathrm{Mor}(\mathbf{I})$ are finite sets.

Proposition B.6. *All (finite) (co)limits are representable in a category* $\mathbf{C}$ *if and only if all (finite) (co)products and either (co)kernels or fibred (co)products exist in* $\mathbf{C}$*.*

Corollary B.7. *All (direct and inverse) limits are representable in* $\mathbf{Set}$*.*

Definition B.8. Let $\mathbf{I}$ and $\mathbf{C}$ be categories such that $\varinjlim H$ (respectively $\varprojlim H$) is representable for every functor $H\colon \mathbf{I} \to \mathbf{C}$. Then a functor $F\colon \mathbf{C} \to \mathbf{D}$ *preserves* (or *commutes with*) direct (respectively inverse) limits from $\mathbf{I}$ if for every functor $H\colon \mathbf{I} \to \mathbf{C}$, denoting by $\alpha\colon H \to K_{\varinjlim H}$ (respectively $\alpha\colon K_{\varprojlim H} \to H$) a representation of $\varinjlim H$ (respectively $\varprojlim H$),

$$\mathrm{id}_F \star \alpha\colon F \circ H \to F \circ K_{\varinjlim H} = K_{F(\varinjlim H)}$$

(respectively

$$\mathrm{id}_F \star \alpha\colon F \circ K_{\varprojlim H} = K_{F(\varprojlim H)} \to F \circ H)$$

represents $\varinjlim(F \circ H)$ (respectively $\varprojlim(F \circ H)$).

The result of Proposition 4.8 can be generalized as follows.

Proposition B.9. *For every category* $\mathbf{C}$*, all limits are representable in the category of presheaves* $\widehat{\mathbf{C}}$*, and they can be computed "componentwise", in the sense that for every* $U \in \mathbf{C}$ *the functor* $\mathrm{Hom}_{\widehat{\mathbf{C}}}(U, -)\colon \widehat{\mathbf{C}} \to \mathbf{Set}$ *(which sends* $F \in \widehat{\mathbf{C}}$ *to* $F(U)$*) commutes with all limits.*

Moreover, the functor $\mathrm{h}\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *preserves all inverse limits which are representable in* $\mathbf{C}$*.*

Definition B.10. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *left* (respectively *right*) *exact* if all finite inverse (respectively direct) limits are representable in $\mathbf{C}$ and F preserves them. F is *exact* if it is both left and right exact.

Remark B.11. In the particular case of an additive functor between two abelian categories $F\colon \mathbf{A} \to \mathbf{B}$, this general notion of (left/right) exactness coincides with the usual one. Indeed, notice first that all finite limits are representable in $\mathbf{A}$ by Proposition B.6; moreover, F preserves finite products and coproducts because it is additive. Therefore, the same argument of Proposition B.6 implies that F is left exact (completely analogous considerations can be made for right exactness) if and only if it preserves kernels (of double arrows). Now, for an additive functor this is equivalent to preserving kernels of single arrows, which can be reformulated by saying that F preserves exact sequences of the form $0 \to X \to Y \to Z$, and this is easily seen to be equivalent to the fact that for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathbf{A}$, $0 \to F(X) \to F(Y) \to F(Z)$ is exact in $\mathbf{B}$ (the usual definition of left exactness).

Proposition B.12. *If a functor $F\colon \mathbf{C} \to \mathbf{D}$ is a left (respectively right) adjoint, then it preserves all representable direct (respectively inverse) limits. In particular, if all finite direct (respectively inverse) limits exist in $\mathbf{C}$, then F is right (respectively left) exact.*

B.2 Grothendieck topologies

All proofs of the results stated in this section can be found in [2].

Definition B.13. Let U be an object of some category $\mathbf{C}$. A *sieve* of U is a subfunctor of U: more precisely, a sieve of U is an equivalence class of monomorphisms $R \hookrightarrow U$ of $\widehat{\mathbf{C}}$ (under the equivalence relation $(\iota\colon R \hookrightarrow U) \sim (\iota'\colon R' \hookrightarrow U)$ if and only if there is an isomorphism $\alpha\colon R \xrightarrow{\sim} R'$ such that $\iota = \iota' \circ \alpha$); by abuse of notation, we will denote such a sieve simply by R.

Remark B.14. The set $\mathrm{S}(U)$ of sieves of $U \in \mathbf{C}$ is ordered by $R \leq R'$ if and only if $R' \subseteq R$, and it is stable under unions and intersections. Notice also that, since the property of being a monomorphism is stable under base change, if $R \in \mathrm{S}(U)$, then $R \times_U V \in \mathrm{S}(V)$ for every morphism $V \to U$ of $\mathbf{C}$.

Definition B.15. A *(Grothendieck) topology* T on a category $\mathbf{C}$ consists of the datum, for each object U of $\mathbf{C}$, of a subset $\mathrm{J}(U) = \mathrm{J}^{\mathrm{T}}(U) \subseteq \mathrm{S}(U)$ (whose elements are called *covering sieves* of U for T), such that the following axioms are satisfied.

T1 If $U \in \mathbf{C}$ and $R \in \mathrm{J}(U)$, then $R \times_U V \in \mathrm{J}(V)$ for every morphism $V \to U$.

T2 If $U \in \mathbf{C}$ and $R, R' \in \mathrm{S}(U)$ are such that $R \in \mathrm{J}(U)$ and $R' \times_U V \in \mathrm{J}(V)$ for every $V \in \mathbf{C}$ and every morphism $V \to U$ which is in $R(V) \subseteq U(V)$, then $R' \in \mathrm{J}(U)$, too.

T3 $U \in \mathrm{J}(U)$ for every object U of $\mathbf{C}$.

Definition B.16. A *site* is a couple $(\mathbf{C}, \mathrm{T})$, where T is a topology on the category $\mathbf{C}$.

Remark B.17. Let $(\mathbf{C}, \mathrm{T})$ be a site and U an object of $\mathbf{C}$. If $R \in \mathrm{J}(U)$ and $R \subseteq R' \in \mathrm{S}(U)$, then $R' \in \mathrm{J}(U)$, too (this follows from **T2**, since, for every morphism $V \to U$ in $R(V)$, $R' \times_U V = R \times_U V \in \mathrm{J}(V)$ by **T1**). Note also that, if $R_1, R_2 \in \mathrm{J}(U)$, then $R_1 \cap R_2 \in \mathrm{J}(U)$, too (again, this follows from **T2** applied to $R = R_1$ and $R' = R_1 \cap R_2$, since, for every morphism $V \to U$ in $R_1(V)$, $(R_1 \cap R_2) \times_U V = R_2 \times_U V \in \mathrm{J}(V)$ by **T1**); this implies that $(\mathrm{J}(U), \leq)$ is a filtered set.

Definition B.18. Let T and T′ be two topologies on a category **C**. We will say that T′ is *finer* than T (and write $\mathrm{T} \leq \mathrm{T}'$) if $\mathrm{J}^{\mathrm{T}}(U) \subseteq \mathrm{J}^{\mathrm{T}'}(U)$ for every $U \in \mathbf{C}$.

Example B.19. On every category **C** the *chaotic* (respectively *discrete*) topology is defined by $\mathrm{J}(U) := \{U\}$ (respectively $\mathrm{J}(U) := \mathrm{S}(U)$) for every $U \in \mathbf{C}$; clearly it is the least fine (respectively the finest) topology on **C**.

If $\{\mathrm{T_i}\}_{\mathrm{i} \in \mathrm{I}}$ is a family of topologies on **C**, it is obvious that the intersection $\mathrm{T} := \bigcap_{\mathrm{i} \in \mathrm{I}} \mathrm{T_i}$ (defined by $\mathrm{J}^{\mathrm{T}}(U) := \bigcap_{i \in I} \mathrm{J}^{\mathrm{T_i}}(U)$ for every $U \in \mathbf{C}$) is again a topology (it is the finest topology among those which are less fine than each $\mathrm{T_i}$).

Definition B.20. Let U be an object of a category **C**. Given a family $\mathcal{U} = \{f_i \colon U_i \to U\}_{i \in I} \in \mathrm{Tar}(U)$, the sieve $R_{\mathcal{U}} \in \mathrm{S}(U)$ *generated* by $\mathcal{U}$ is defined by

$$R_{\mathcal{U}}(V) := \{g \in U(V) \mid \exists i \in I,\ \exists g_i \in U_i(V) \text{ such that } g = f_i \circ g_i\}$$

for every $V \in \mathbf{C}$ (and in the obvious way on morphisms).

If T is a topology on **C**, $\mathcal{U} \in \mathrm{Tar}(U)$ is a *covering family* for T if $R_{\mathcal{U}} \in \mathrm{J}^{\mathrm{T}}(U)$.

Definition B.21. Given a category **C** and, for $U \in \mathbf{C}$, a subset $\mathrm{Cov}(U)$ of $\mathrm{Tar}(U)$, the topology *generated* by $\{\mathrm{Cov}(U)\}_{U \in \mathbf{C}}$ is the intersection of all the topologies T such that all the elements of each $\mathrm{Cov}(U)$ are covering families for T.

Remark B.22. This definition applies in particular when $\mathrm{Cov}(U) = \mathrm{Cov}^{\tau}(U)$ for some pretopology τ on **C**. In this case the topology generated by a pretopology is usually given the same name (so, for instance, the topologies *Zar*, *ét*, *sm* and *fppf* on **Sch** are those generated by the corresponding pretopologies).

In general, the topologies generated by arbitrary collections of covering families can be complicated to describe, but for those generated by pretopologies we have the following result.

Proposition B.23. *Let* T *be the topology generated by a pretopology* τ *on* **C**. *Then* $R \in \mathrm{S}(U)$ *is in* $\mathrm{J}^{\mathrm{T}}(U)$ *if and only if there exists* $\mathcal{U} \in \mathrm{Cov}^{\tau}(U)$ *such that* $R_{\mathcal{U}} \subseteq R$.

Corollary B.24. *Let* τ, τ' *be two pretopologies on* **C** *and let* T, T′ *be the topologies they generate. If for every* $U \in \mathbf{C}$ *and every* $\mathcal{U} \in \mathrm{Cov}^{\tau}(U)$ *there exists* $\mathcal{U}' \in \mathrm{Cov}^{\tau'}(U)$ *such that* $\mathcal{U} \leq \mathcal{U}'$, *then* $\mathrm{T} \leq \mathrm{T}'$.

Example B.25. The argument used in Example 4.36 implies that *sm* = *ét* as topologies on **Sch**.

The above example shows that different pretopologies can generate the same topology on a category. However, in general not every topology is generated by some pretopology, but the following result shows that this is true if the category has fibred products.

Proposition B.26. *Let* $(\mathbf{C}, \mathrm{T})$ *be a site and assume that* $\mathbf{C}$ *has fibred products. Then* $\mathrm{Cov}^{\tau}(U) := \{\mathcal{U} \in \mathrm{Tar}(U) \mid R_{\mathcal{U}} \in \mathrm{J}^{\mathrm{T}}(U)\}$ *for* $U \in \mathbf{C}$ *defines a pretopology* τ *on* $\mathbf{C}$*, and the topology generated by* τ *is* T.

Nevertheless, even when the category has fibred products, in some cases it is simpler to describe a topology as the one generated by collections of covering families which do not form a pretopology.

Example B.27. On **Sch** the *fpqc* (faithfully flat and quasi-compact) topology is generated by $\{\mathrm{Cov}(U)\}_{U \in \mathbf{Sch}}$, where

$$\mathrm{Cov}(U) := \mathrm{Cov}^{Zar}(U) \cup \{f \colon V \to U \mid f \text{ faithfully flat and quasi-compact}\}.$$

We claim that *fppf* $\leq$ *fpqc* (we will see later that they are not equal): by Lemma 7.5, it is enough to prove that $R_{\mathcal{V}} \in \mathrm{J}^{fpqc}(U)$ if $\mathcal{V} = \{f_i \colon V_i \to U\}_{i \in I} \in \mathrm{Tar}(U)$ is such that, for every $i \in I$, f_i is the composition of a faithfully flat morphism of affine schemes $f_i' \colon V_i \to U_i$ and of an open immersion $U_i \subseteq U$. Now, for every $i \in I$ and every morphism $U' \to U_i$, we have $R_{\mathcal{V}} \times_U U' = R_{\{f_i'\}} \times_{U_i} U' \in \mathrm{J}^{fpqc}(U')$ by **T1** (recall that f_i' is quasi-compact, so that $R_{\{f_i'\}} \in \mathrm{J}^{fpqc}(U_i)$ by definition), whence $R_{\mathcal{V}} \in \mathrm{J}^{fpqc}(U)$ by **T2** applied to $R = \{U_i \subseteq U\}_{i \in I} \in \mathrm{Cov}^{Zar}(U)$ and $R' = R_{\mathcal{V}}$.

Definition B.28. Let $(\mathbf{C}, \mathrm{T})$ be a site. A presheaf $F \in \widehat{\mathbf{C}}$ is *separated* (respectively a *sheaf*) for T if for every $U \in \mathbf{C}$ and every $R \in \mathrm{J}^{\mathrm{T}}(U)$ the natural map

$$F(U) \cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(U, F) \to \mathrm{Hom}_{\widehat{\mathbf{C}}}(R, F) := F(R)$$

is injective (respectively bijective).

As usual, if $(\mathbf{C}, \mathrm{T})$ is a site, we will denote by $(\mathbf{C}, \mathrm{T})^{\sim}$ (or simply by $\mathbf{C}^{\sim}$) the full subcategory of $\widehat{\mathbf{C}}$ whose objects are the sheaves for T.

Definition B.29. A *topos* is a category which is equivalent to $(\mathbf{C}, \mathrm{T})^{\sim}$ for some site $(\mathbf{C}, \mathrm{T})$.

Proposition B.30. *Let* $\{F_k\}_{k \in K}$ *be a family of presheaves on* $\mathbf{C}$*. Then the finest topology* T *with the property that* F_k *is separated (respectively a sheaf) for every* $k \in K$ *is defined as follows. For every* $U \in \mathbf{C}$ *the set of covering sieves* $\mathrm{J}^{\mathrm{T}}(U)$ *is formed by those* $R \in \mathrm{S}(U)$ *such that for every morphism* $V \to U$ *of* $\mathbf{C}$ *the natural map* $F_k(V) \to F_k(R \times_U V)$ *is injective (respectively bijective) for every* $k \in K$.

Corollary B.31. *For topologies generated by pretopologies, the definitions of separated presheaf and of sheaf given in Definition B.28 and in Definition 4.30 coincide.*

Proof. Let τ be a pretopology on $\mathbf{C}$, and let T be the topology generated by τ. First we claim that $F \in \widehat{\mathbf{C}}$ is separated (respectively a sheaf) according to Definition B.28 if and only if for every $U \in \mathbf{C}$ and every $\mathcal{U} \in \mathrm{Cov}^{\tau}(U)$ the natural map $F(U) \to F(R_{\mathcal{U}})$ is injective (respectively bijective). The other implication being trivial, we can assume that F satisfies this condition. Let $\{F_k\}_{k \in K}$ be the family of presheaves on $\mathbf{C}$ consisting of all separated presheaves (respectively sheaves) for T together with F. Then by Proposition B.30 there is a topology T' on $\mathbf{C}$ such that F is separated (respectively a sheaf) for T' and whose set of covering sieves $\mathrm{J}^{\mathrm{T}'}(U)$ (for $U \in \mathbf{C}$) is formed by those $R \in \mathrm{S}(U)$ such that for every morphism $V \to U$ of $\mathbf{C}$ the natural map $F_k(V) \to F_k(R \times_U V)$ is injective (respectively bijective) for every $k \in K$. Since clearly $R_{\mathcal{U}} \in \mathrm{J}^{\mathrm{T}'}(U)$ for every $\mathcal{U} \in \mathrm{Cov}^{\tau}(U)$, Remark B.17 implies that $\mathrm{J}^{\mathrm{T}}(U) \subseteq \mathrm{J}^{\mathrm{T}'}(U)$ for every $U \in \mathbf{C}$. Therefore $\mathrm{T} \leq \mathrm{T}'$, and the claim follows. To conclude, in view of Remark 4.31, it is enough to prove that for every $\mathcal{U} \in \mathrm{Cov}^{\tau}(U)$ there is a natural isomorphism $F(\mathcal{U}) \cong F(R_{\mathcal{U}})$. Assume $\mathcal{U} = \{f_i \colon U_i \to U\}_{i \in I}$: then to every $\xi = (\xi_i)_{i \in I} \in F(\mathcal{U}) \subseteq \prod_{i \in I} F(U_i)$ we can associate $\tilde{\xi} \in F(R_{\mathcal{U}}) = \mathrm{Hom}_{\widehat{\mathbf{C}}}(R_{\mathcal{U}}, F)$ defined as follows. Given $V \in \mathbf{C}$ and $g \in R_{\mathcal{U}}(V) \subseteq U(V)$, by definition of $R_{\mathcal{U}}$ there exist $i \in I$ and $g_i \in U_i(V)$ such that $g = f_i \circ g_i$. It is then easy to see that the map

$$\begin{aligned} \tilde{\xi}(V) \colon R_{\mathcal{U}}(V) &\to F(V) \\ g &\mapsto g_i^*(\xi_i) \end{aligned}$$

is well defined, that $\tilde{\xi}$ is a natural transformation and that the map $F(\mathcal{U}) \to F(R_{\mathcal{U}})$ defined by $\xi \mapsto \tilde{\xi}$ is bijective. □

Remark B.32. One can extend Definition B.28 to a definition of (pre)stack on an arbitrary site. Namely, $\mathbf{F} \in \underline{\mathbf{Fib}}_{\mathbf{C}}$ is a *prestack* (respectively a *stack*) for T if for every $U \in \mathbf{C}$ and every $R \in \mathrm{J}^{\mathrm{T}}(U)$ the natural functor

$$\mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(U, \mathbf{F}) \to \mathbf{Hom}_{\underline{\mathbf{Fib}}_{\mathbf{C}}}(R, \mathbf{F})$$

is fully faithful (respectively an equivalence). Although we are not going to do it here, much of what we are going to say about sheaves in general can be naturally extended also to stacks. In particular, adapting the proof of Corollary B.31, it can be shown that the new definition of (pre)stack coincides with the usual one if the topology is generated by a pretopology.

Corollary B.33. *On every category there exists a unique topology which is the finest among those with the property that all representable presheaves are sheaves. Such a topology is called* canonical *and is denoted by can.*

Extending the definition given for pretopologies in Definition 4.37, we will say that a topology T is *subcanonical* if $\mathrm{T} \leq$ *can*.

Definition B.34. Let $(\mathbf{C}, \mathrm{T})$ and $(\mathbf{C}', \mathrm{T}')$ be two sites. A functor $F\colon \mathbf{C}' \to \mathbf{C}$ is *continuous* if the natural functor $\circ F\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}'}$ restricts to a functor $\circ F\colon \mathbf{C}^{\sim} \to \mathbf{C}'^{\sim}$.

Definition B.35. Let $(\mathbf{C}, \mathrm{T})$ be a site and let $F\colon \mathbf{C}' \to \mathbf{C}$ be a functor. Then the topology *induced* by F on $\mathbf{C}'$ is the finest topology on $\mathbf{C}'$ with the property that F is a continuous functor (notice that such a topology exists by Proposition B.30).

Remark B.36. The above definition is compatible with the notion of induced pretopology introduced in Proposition 4.41.

Example B.37. The *fpqc* topology on $\mathbf{Sch}_{/S}$ is subcanonical and most of the results of faithfully flat descent theory stated for *fppf* actually hold also for *fpqc* (see Remark 7.13). The inclusions $\mathbf{Sch}_{/S} \subset (\mathbf{Sch}_{/S}, \mathit{fpqc})^{\sim} \subset (\mathbf{Sch}_{/S}, \mathit{fppf})^{\sim}$ are strict (in particular, $\mathit{fpqc} \neq \mathit{fppf}$). Indeed, the non representable presheaf P of Example 7.9 is clearly a sheaf also for *fpqc*, whereas, in the notation of Example 7.10, it is easy to see that $F_{f,\mathit{fppf}}$ is a sheaf for *fppf* but not for *fpqc* if f is the morphism of $\mathbf{Sch}_{/S}$ induced by a non finite extension of fields (e.g., $\kappa(s) \subset \kappa(s)(t)$ for some $s \in S$).

Proposition 4.44 admits the following generalization.

Proposition B.38. *Let $\mathbf{C}' \subseteq \mathbf{C}$ be the inclusion of a full subcategory, let* T *be a pretopology on $\mathbf{C}$ and endow $\mathbf{C}'$ with the induced pretopology. If for every $U \in \mathbf{C}$ there exists a covering family (for* T*) $\{U_i \to U\}_{i \in I}$ such that $U_i \in \mathbf{C}'$ for every $i \in I$, then the natural restriction functor $\mathbf{C}^{\sim} \to \mathbf{C}'^{\sim}$ is an equivalence of categories. The viceversa holds if* T *is subcanonical.*

Let $(\mathbf{C}, \mathrm{T})$ be a site: we are going to see how the sheaf associated to a presheaf is defined in general. Given $F \in \widehat{\mathbf{C}}$, for every $U \in \mathbf{C}$ we set $L(F)(U) := \varinjlim F(R)$, where the limit is taken over the filtered set $\mathrm{J}^{\mathrm{T}}(U)$. It is easy to see that this extends naturally to a functor $L(F) \in \widehat{\mathbf{C}}$, that there is a natural morphism $\epsilon_F\colon F \to L(F)$ in $\widehat{\mathbf{C}}$, and that in this way we obtain a functor $L\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ together with a natural transformation $\epsilon\colon \mathrm{id}_{\widehat{\mathbf{C}}} \to L$.

Proposition B.39. *1. $L(F)$ is a separated presheaf for every $F \in \widehat{\mathbf{C}}$;*

2. $F \in \widehat{\mathbf{C}}$ is separated if and only if $\epsilon_F\colon F \to L(F)$ is a monomorphism of $\widehat{\mathbf{C}}$, and in this case $L(F)$ is a sheaf;

3. $F \in \widehat{\mathbf{C}}$ is sheaf if and only if $\epsilon_F\colon F \to L(F)$ is an isomorphism of $\widehat{\mathbf{C}}$;

4. $L\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is left exact.

It is then clear that $F^a := L(L(F)) \in \mathbf{C}^{\sim}$ (it is called the *sheaf associated to* F); we also set $\rho_F := \epsilon_{L(F)} \circ \epsilon_F\colon F \to F^a$. One can prove that these definitions of F^a and of ρ_F coincide (up to isomorphism) with those given in Section 4.3 for pretopologies (see Remark 4.53) and that Proposition 4.52 remains true in general. Also the other results stated in Section 4.3 (like Proposition 4.57 and Corollary 4.58) are true for arbitrary sites, while Proposition 4.54 admits the following generalization.

Proposition B.40. *Let* $(\mathbf{C}, \mathrm{T})$ *be a site. Then all (direct and inverse) limits are representable in* $\mathbf{C}^{\sim} = (\mathbf{C}, \mathrm{T})^{\sim}$. *Moreover, the inclusion functor* $\mathbf{C}^{\sim} \subseteq \widehat{\mathbf{C}}$ *preserves inverse limits, whereas* $-^a \colon \widehat{\mathbf{C}} \to \mathbf{C}^{\sim}$ *preserves direct limits and finite inverse limits (in particular, it is exact).*

Remark B.41. By Proposition B.12 the fact that $\mathbf{C}^{\sim} \subseteq \widehat{\mathbf{C}}$ (respectively $-^a \colon \widehat{\mathbf{C}} \to \mathbf{C}^{\sim}$) preserves inverse (respectively direct) limits is a formal consequence of Proposition 4.52. The fact that $-^a$ also preserves finite inverse limits follows from Proposition B.39.

Corollary B.42. *All inverse limits in* $\mathbf{C}^{\sim}$ *can be computed as in* $\widehat{\mathbf{C}}$, *whereas all direct limits in* $\mathbf{C}^{\sim}$ *can be computed by first computing them in* $\widehat{\mathbf{C}}$ *and then applying* $-^a$.

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Received: 20 September 2017.
Accepted: 13 November 2017.

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