

# Neutral linear series and Brill-Noether theory of singular curves

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*To Edoardo Sernesi on occasion of his 70-th birthday*

**Abstract.** *On a projective nonsingular curve  $C$  of genus  $g$  the variety  $N_n^r(A)$  of neutral linear series parameterizes linear series of degree  $n$  and dimension  $r$  to which a given effective divisor  $A$  of degree  $s$  does not impose independent conditions. Edoardo Sernesi proved that, in the appropriate ranges of  $g, r, n, s$ , if  $C$  is a Brill-Noether curve then  $N_n^r(A)$  is non-empty. As an application, here we deduce a Brill-Noether existence theorem for irreducible curves obtained from a Brill-Noether curve by identifying a pair of points to an ordinary node.*

## 1 Introduction

The notion of neutral linear series belongs to classical algebraic geometry and in particular to Francesco Severi (see [6] for a systematic investigation and [5] for a modern reappraisal).

Let  $C$  be a smooth projective irreducible curve of genus  $g$ . Given nonnegative integers  $r, n$  and denoting by  $G_n^r$  the scheme of linear series of degree  $n$  and dimension  $r$  on  $C$ , for every effective divisor  $A = p_1 + \cdots + p_s$  of degree  $s$  such that  $2 \leq s \leq r + 1$  there is a closed subscheme  $N_n^r(A)$  of  $G_n^r$  which is supported on the series to which  $A$  imposes at most  $s - 1$  conditions. Recall that every irreducible component of  $G_n^r$  has dimension at least  $\rho(g, r, n) := g - (r + 1)(g - n + r)$ . The curve  $C$  is said to be a *Brill-Noether curve* if  $G_n^r$  has pure dimension  $\rho(g, r, n)$  for every  $n, r \geq 0$ .

In [7] Edoardo Sernesi proved the following result:

**Theorem 1.1** ([7]). *Let  $C$  be a Brill-Noether curve of genus  $g$  and let  $A$  be an effective divisor of degree  $s$ , with  $2 \leq s \leq r + 1$ , on  $C$ . If  $\rho(g, r, n) \geq r - s + 2$  then  $N_n^r(A) \neq \emptyset$  and every component of  $N_n^r(A)$  has codimension at most  $r - s + 2$  in  $G_n^r$ .*

This is part (a) of the main theorem in [7], which is completely independent from part (b). Indeed, Theorem 1.1 is a consequence of the definition of  $N_n^r(A)$  as a determinantal scheme and of an argument of Fulton and Lazarsfeld in [4]. We also mention a related nonemptiness result, Theorem 0.5 in [3], obtained by the theory of limit linear series and stating that, under suitable numerical assumptions, on a general smooth curve there exists a pair  $(V, A)$  such that  $V \in N_n^r(A)$ .

In the present note, Theorem 1.1 is applied to a Brill-Noether existence theorem in the singular case. For every integral projective curve  $X$ , let  $\overline{W}_n^r(X)$  be the variety of torsion-free sheaves  $F$  on  $X$  such that  $\deg(F) = n$  and  $h^0(X, F) \geq r + 1$  and let  $W_n^r(X)$  denote its open subvariety consisting of line bundles  $L$  on  $X$  such that  $\deg(L) = n$  and  $h^0(X, L) \geq r + 1$ . Since  $X$  is integral, as in the smooth case it is possible to define a scheme  $G_n^r(X)$  parameterizing pairs  $(L, V)$ , where  $L$  is a line bundle of degree  $d$  on  $X$  and  $V$  is a vector subspace of  $H^0(X, L)$  with  $\dim(V) = r + 1$ .

**Theorem 1.2.** *Let  $X$  be an integral projective curve of arithmetic genus  $g$  obtained from a Brill-Noether curve by identifying a pair of distinct points to an ordinary node  $z$ . Assume  $\rho(g, r, n) \geq 0$ .*

- (i) *The general element of every irreducible component of  $\overline{W}_n^r(X)$  is a line bundle, i.e.  $W_n^r(X)$  is dense in  $\overline{W}_n^r(X)$ .*
- (ii) *The open subset  $G_n^r(X, z)$  of  $G_n^r(X)$  corresponding to linear series for which  $z$  is not a base point is non-empty.*

We point out that the existence of line bundles in  $W_n^r(X)$  on a singular  $X$  is not trivial even though  $X$  is smoothable, since the variety of line bundles on  $X$  is not complete (see [5], especially Section 4). Indeed, it is easy to construct examples of both nodal and cuspidal curves with hyperelliptic normalization which carry no line bundle with prescribed dimension of global sections and degree, even if the corresponding Brill-Noether number is positive (see for instance [1], Example 2.2 and Remark 2.3).

We work over the field  $\mathbb{C}$  of complex numbers.

## 2 The proofs

For completeness sake, let us quote from the unpublished manuscript [7] a sketch of the proof of Theorem 1.1. Since the scheme  $N_n^r(A)$  is obviously defined by a determinantal condition, if it is not empty then each of its components has codimension at most  $r - s + 2$  in  $G_n^r$ .

Assume first  $\rho(g, r, n) \geq r + 1$ . In this case,  $G_{n-1}^r \neq \emptyset$  because

$$\dim(G_{n-1}^r) = \rho(g, r, n - 1) = \rho(g, r, n) - (r + 1) \geq 0$$

and  $C$  is a Brill-Noether curve. Let  $|V| \in G_{n-1}^r$ . For every  $p \in C$  the linear series  $|V| + p$  belongs to  $G_n^r$  and  $|V| + p \in N_n^r(A)$  for every  $A = p + p_2 + \cdots + p_s$ . Therefore  $N_n^r(A) \neq \emptyset$  for all  $A \in C^{(s)}$ .

Assume now  $\rho(g, r, n) \leq r$ . One possibility is that  $g - n + r < 0$ , meaning that all the series  $|V| \in G_n^r$  are incomplete. This implies that  $G_{n-1}^r \neq \emptyset$  and we can conclude as before that  $N_n^r(A) \neq \emptyset$  for all  $A \in C^{(s)}$ .

The second possibility is that  $g - n + r \geq 0$ : then we have

$$\rho(g, r + 1, n) = \rho(g, r, n) - (r + 1) - (g - n + r) - 1 < 0$$

and  $C$  is a Brill-Noether curve. Now, if  $\text{Pic}_n(C)$  denotes the subvariety of  $\text{Pic}(C)$  parametrizing isomorphism classes of invertible sheaves  $L$  of degree  $n$ , and  $W_n^r$  the closed subscheme of  $\text{Pic}_n(C)$  defined by the condition  $h^0(C, L) \geq r + 1$ , it follows that  $W_n^{r+1} = \emptyset$  and  $G_n^r = W_n^r$ . Fixing a point  $b \in C$  and an integer  $m \gg 0$  we may embed  $\text{Pic}_n(C) \subset \text{Pic}_m(C)$  tensoring by  $\mathcal{O}((m - n)b)$ . We obtain the image of  $W_n^r$  as the determinantal scheme defined by the condition  $\text{rank}(\tau) \leq m - g - r$ , where

$$\tau : \pi_*\mathcal{P} \longrightarrow \pi_*[\mathcal{P} \otimes \alpha^*\mathcal{O}_{(m-n)b}]$$

is the restriction map coming from the diagram:

$$\begin{array}{ccc} & C \times \text{Pic}_m(C) & \\ \alpha \swarrow & \downarrow \pi & \\ C & & \text{Pic}_m(C) \end{array}$$

and  $\mathcal{P}$  is a Poincaré bundle on  $C \times \text{Pic}_m(C)$ . We have  $E_n^r = \ker(\tau|_{W_n^r})$ , which is a subsheaf of the locally free  $\mathcal{F} := (\pi_*\mathcal{P})|_{W_n^r}$ . Therefore the dual  $E_n^{r\vee}$  is a quotient of  $\mathcal{F}^\vee$  which is ample because it is the restriction to  $W_n^r$  of the ample sheaf  $(\pi_*\mathcal{P})^\vee$  (compare [4], §2). It follows that  $E_n^{r\vee}$  is ample: from this fact one deduces as in [4] that  $N_n^r(A) \neq \emptyset$  for all  $A \in C^{(s)}$ . This completes the proof of Theorem 1.1.

Next we turn to Theorem 1.2. Let  $\nu : \tilde{X} \rightarrow X$  be the normalization of  $X$  and let  $\nu^{-1}(z) = \{p, q\}$ . Since by assumption  $\rho(g, r, n) \geq 0$ , from the smooth case it follows by semicontinuity that  $\overline{W}_n^r(X) \neq \emptyset$  and that  $\dim(Z) \geq \rho(g, r, n)$  for every irreducible component of  $\overline{W}_n^r(X)$  (see for instance [1], Proposition 2.1). If all torsion-free sheaves in  $Z$  are not locally free, then by [5], Theorem 4.6 (ii), we have  $Z \subseteq \nu_*(W_{n-1}^r(\tilde{X}))$ . Since  $\tilde{X}$  is a Brill-Noether curve, we obtain the contradiction

$$\dim Z \leq \dim(W_{n-1}^r(\tilde{X})) = \rho(g, r, n - 1) < \rho(g, r, n),$$

hence claim (i) holds true.

For every  $(r + 1)$ -dimensional  $V \subset H^0(X, L)$ , where  $L$  is an invertible sheaf of degree  $n$ , let  $\pi^*V$  the corresponding  $(r + 1)$ -dimensional subspace of  $H^0(C, \pi^*L)$ . The following auxiliary result is inspired by [2], proof of Theorem 4.1, and is essentially contained in [7], Proposition 3.2:

**Proposition 2.1.** *By associating to the pair  $(V, L) \in G_n^r(X, z)$  the pair  $(\pi^*V, \pi^*L)$ , we obtain an open embedding*

$$\Phi_0 : G_n^r(X, z) \cong N_n^r(p + q)_0 \subset N_n^r(p + q)$$

where  $N_n^r(p + q)_0$  denotes the open subset of  $N_n^r(p + q)$  consisting of neutral linear series without base points at  $p$  and  $q$ .

We are now in the position to complete the proof of Theorem 1.2. Since by assumption  $\rho(g, r, n) \geq 0$ , we have  $\rho(g - 1, r, n) = \rho(g, r, n) + r \geq r$ , hence from

Theorem 1.1 it follows that the scheme of neutral linear series  $N_n^r(p+q)$  on the Brill-Noether curve  $\tilde{X}$  is non-empty and  $\Sigma$  has codimension at most  $r$  in  $G_n^r(\tilde{X})$  for every irreducible component  $\Sigma$  of  $N_n^r(p+q)$ . The set of  $g_n^r$ 's with a fixed point has codimension  $r+1$  in  $G_n^r(\tilde{X})$ , hence the general element  $V$  of  $\Sigma$  is base-point free. In particular,  $N_n^r(p+q)_0$  is non-empty and Proposition 2.1 implies that  $G_n^r(X, z)$  is non-empty as well, hence also claim (ii) follows.

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