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On the abstract Bogomolov-Tian-Todorov Theorem

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Abstract. *We describe an abstract version of the Theorem of Bogomolov-Tian-Todorov, whose underlying idea is already contained in various papers by Bandiera, Fiorenza, Iacono, Manetti. More explicitly, we prove an algebraic criterion for a differential graded Lie algebra to be homotopy abelian. Then, we collect together many examples and applications in deformation theory and other settings.*

1 Introduction

Let X be a compact Kähler manifold. If X has trivial canonical bundle, then, the so called Bogomolov-Tian-Todorov (BTT) Theorem states that the deformations of X are unobstructed. In [5], Bogolomov proved it in the particular case of complex hamiltonian manifolds; later, Tian [38] and Todorov [39] proved independently the theorem for compact Kähler manifolds with trivial canonical bundle. More algebraic proofs of BTT theorem, based on T^1-lifting Theorem and degeneration of the Hodge spectral sequence, were given in [35] for $\mathbb{K} = \mathbb{C}$ and in [25, 9] for any $\mathbb{K}$ as above.

For compact Kähler manifolds with torsion canonical bundle, the theorem follows from the more general fact that the derived infinitesimal deformations are unobstructed. The guiding principle is that in characteristic zero any deformation problem is controlled by a differential graded Lie algebra (DG-Lie algebra), with quasi-isomorphic DG-Lie algebras control the same deformation problem [15, 19]. More precisely, the deformation functor associated with the geometric problem is isomorphic to the deformation functor Def_L associated with a DG-Lie algebra L via Maurer-Cartan equation up to gauge equivalence.

Therefore, it is worth to have an explicit description of a DG-Lie algebra associated with the problem. It turns out that the obstructions to the smoothness of the functor Def_L are contained in the cohomology vector space $H^2(L)$. However, if L is an abelian DG-Lie algebra then, even if $H^2(L)$ is not zero, the functor Def_L is smooth, i.e., it has no obtructions. In particular, it is actually enough to prove that the DG-Lie algebra is homotopy abelian, i.e., quasi-isomorphic to an abelian DG-Lie algebra, to assure that the associated deformation functor is smooth.

For a compact complex manifold X, the infinitesiaml deformation are controlled by the Kodaira-Spencer DG-Lie algebras KS_X (Example 2.4). If X is a

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compact Kähler manifold with trivial or torsion canonical bundle, then the Lie version of BTT theorem asserts that KS_X is homotopy abelian (Section 4.1). For $\mathbb{K} = \mathbb{C}$, this was first proved in the seminal paper by W.M. Goldman and J.J. Millson [16], see also [27]. For any algebraically closed field $\mathbb{K}$ of characteristic 0, this was proved in a completely algebraic way in [22], using the degeneration of the Hodge-to-de Rham spectral sequence and the notion of Cartan homotopy.

In [22], the proof involves L_∞-algebras and L_∞-morphisms and it is based on a series of algebraic results that were also applied in other context [23, 21, 4, 13]. In this paper, we review some of these ideas of independent interest to establish the following criterion, that we call Abstract Bomolov-Tian-Todorov Theorem, for homotopy abelianity of a DG-Lie algebra (Theorem 3.3).

Theorem 1.1 (Abstract BTT Theorem)**.** *Let L and M be DG-Lie algebras over a field $\mathbb{K}$ of characteristic 0 and $H \subseteq M$ a DG-Lie subalgebra. Assume that there exists a linear map*

$$\boldsymbol{i} \in \operatorname{Hom}^{-1}_{\mathbb{K}}(L, M), \qquad a \mapsto \boldsymbol{i}_a,$$

of degree -1 such that:

1. *$[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$ and $\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, d\boldsymbol{i}_b]$ for every $a, b \in L$;*
2. *$d\boldsymbol{i}_a + \boldsymbol{i}_{da} \in H$ for every $a \in L$;*
3. *the inclusion $H \hookrightarrow M$ is injective in cohomology;*
4. *the induced morphism of complexes $\boldsymbol{i}\colon L \to (M/H)[-1]$ is injective in cohomology.*

Then, the DG-Lie algebra L is homotopy abelian.

According to [11, 22], every linear map $\boldsymbol{i}\colon L \to M$ of degree -1 that satisfies the previous condition (1) is called Cartan homotopy (Section 3). According to [13], any pair $(\boldsymbol{i}, H)$ satisfying the above conditions (1) and (2) is called Cartan calculus.

The previous theorem is already implicitly proved in [10, 22], using L_∞-algebras and L_∞-morphisms. Here, we show an alternative self contained proof based on the same ideas but involving only DG-Lie algebras. The main motivation of this paper is to gather together various ideas in one place and to provide an easier and more accessible proof (only DG-Lie algebras). Moreover, we collect many examples and applications of the Abstract BTT Theorem in deformation theory and in other derived settings (Section 4 and Section 5). Among the others, we include the classical BTT Theorem for Calabi-Yau manifolds of [22], the logarithmic version for log Calaby-Yau pairs (X, D) of [21], the DG-Lie algebra associated with a Batalin-Vilkovisky algebra with the degeneration property [12, 21], and the DG-Lie algebra whose associated coderivation DG-Lie algebra has the splitting principle [4, 2].

The notion of homotopy abelianity is closely related to the notion of formality. A DG-Lie algebra is formal if it is quasi-isomorphic to its cohomology and so a DG-Lie algebra is homotopy abelian if and only if it is formal and $H^*(L)$ is abelian. Then, homotopy abelianity condition is stronger than formality and this could explain why it is easier to provide a criterion for it. Indeed, there are not so many analog of the previous theorem that guarantees the formality of a DG-Lie algebra. We refer to [17, 26, 12, 31, 2] and reference therein for formality criterion.

From the geometric point of view, homotopy abelianity assures the smoothness of the associated moduli problem while formality only implies that the singularity are not too bad (see [15, 33, 31, 32] for more details). If we are only interested in the smoothness of the problem, then instead of Hypothesis (4) it is enough to require that only $H^2(\boldsymbol{i})$ is injective (Remark 3.4).

The paper goes as follows: Section 2 is included for the non expert readers and it contains the relevant definitions and properties about DG-Lie algebras. Section 3 is devoted to the proof of the Abstract BTT Theorem. Some applications and examples are collected in Section 4, while Section 5 contains some generalizations and further applications.

Throughout the paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0, if it is not differently specified.

2 Background on DG-Lie algebras

A *differential graded Lie algebra* (DG-Lie algebra) is the data of a triple $(L, d, [\ ,\ ])$, where (L, d) is a differential graded vector space (DG-vector space) and $[\ ,\ ]\colon L \times L \to L$ is a bilinear map of degree 0 (called bracket), such that the following conditions are satisfied:

1. (graded skewsymmetry) $[x, y] = -(-1)^{ij}[y, x] \in L^{i+j}$, for every $x \in L^i$ and $y \in L^j$;
2. (graded Jacobi identity) $[x, [y, z]] = [[x, y], z] + (-1)^{ij}[y, [x, z]]$, for every $x \in L^i$, $y \in L^j$ and $z \in L$;
3. (graded Leibniz rule) $d[x, y] = [dx, y] + (-1)^i[x, dy]$, for every $x \in L^i$ and $y \in L$.

In particular, the Leibniz rule implies that the bracket of a DG-Lie algebra L induces a structure of graded Lie algebra on its cohomology $H^*(L) = \bigoplus_i H^i(L)$. A DG-Lie algebra is called *contractible* if $H^*(L) = 0$. A DG-Lie algebra is called *abelian* if its bracket is trivial.

Example 2.1. If $L = \bigoplus_i L^i$ is a DG-Lie algebra, then L^0 is a Lie algebra in the usual sense; vice-versa, every Lie algebra is a differential graded Lie algebra concentrated in degree 0.

Example 2.2. Let (V, d_V) be a differential graded vector space over $\mathbb{K}$ and $\mathrm{Hom}^i_{\mathbb{K}}(V, V)$ the space of the linear map $V \to V$ of degree i. Then, $\mathrm{Hom}^*_{\mathbb{K}}(V, V) = \bigoplus_i \mathrm{Hom}^i_{\mathbb{K}}(V, V)$ is a DG-Lie algebra with bracket

$$[f, g] = fg - (-1)^{\deg(f)\deg(g)} gf,$$

and differential d given by

$$d(f) = [d_V, f] = d_V f - (-1)^{\deg(f)} f d_V.$$

For later use, we point out that by Künneth formula, there exists a natural isomorphism

$$H^*(\mathrm{Hom}^*_{\mathbb{K}}(V, V)) \xrightarrow{\simeq} \mathrm{Hom}^*_{\mathbb{K}}(H^*(V), H^*(V)).$$

Example 2.3. Let M be a DG-Lie algebra and $\mathbb{K}[t, dt]$ the differential graded algebra of polynomial differential forms over the affine line. More precisely, $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$, where t has degree 0 and dt has degree 1. Then, $M[t, dt] = M \otimes \mathbb{K}[t, dt]$ is a DG-Lie algebra. As vector space $M[t, dt]$ is generated by elements of the form $mp(t) + nq(t)dt$, with $m, n \in M$ and $p(t), q(t) \in \mathbb{K}[t]$. The differential and the bracket on $M[t, dt]$ are defined as follows:

$$d(mp(t) + nq(t)dt) = (dm)p(t) + (-1)^{\deg(m)} mp'(t)dt + (dn)q(t)dt,$$

$$[mp(t), nq(t)] = [m, n]p(t)q(t), \quad [mp(t), nq(t)dt] = [m, n]p(t)q(t)dt.$$

Note that $[mdt, ndt] = 0$, for every $m, n \in M$.

Example 2.4. Let Θ_X be the holomorphic tangent bundle of a complex manifold X. The *Kodaira-Spencer* DG-Lie algebra of X is

$$KS_X = (\bigoplus_i \Gamma(X, \mathcal{A}^{0,i}_X(\Theta_X)) = \bigoplus_i A^{0,i}_X(\Theta_X), d, [\ ,\ ]),$$

where $KS^i_X = A^{0,i}_X(\Theta_X)$ is the vector space of the global sections of the sheaf of germs of the differential $(0, i)$-forms with coefficients in Θ_X, d is the opposite of Dolbeault's differential and the bracket is the extension of the usual bracket of vector fields. Explicitly, if $z_1, \ldots, z_n$ are local holomorphic coordinates on X, we have

$$d\left(f d\bar{z}_I \frac{\partial}{\partial z_i}\right) = -\overline{\partial}(f) \wedge d\bar{z}_I \frac{\partial}{\partial z_i},$$

$$\left[f \frac{\partial}{\partial z_i} d\bar{z}_I, g \frac{\partial}{\partial z_j} d\bar{z}_J\right] = \left(f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i}\right) d\bar{z}_I \wedge d\bar{z}_J, \qquad \forall\ f, g \in \mathcal{A}^{0,0}_X.$$

Example 2.5. Let D be a submanifold of a complex manifold X of codimension 1. We denote by $\Theta_X(-\log D)$ the sheaf of germs of the tangent vectors to X which are tangent to D [36, Section 3.4.4]. Denoting by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of D in

X, then $\Theta_X(-\log D)$ is the subsheaf of the derivations of the sheaf $\mathcal{O}_X$ preserving the ideal sheaf $\mathcal{I}$ of D. Moreover, we have the following short exact sequence

$$0 \to \Theta_X(-\log D) \to \Theta_X \to N_{D/X} \to 0.$$

Since we are in codimension 1, the sheaf $\Theta_X(-\log D)$ is dual to the sheaf $\Omega^1_X(\log D)$ of logarithmic differentials, so it is in particular locally free, see for instance [7, p. 72], [8, Chapter 2] or [40, Chapter 8].

Then, we can define the DG-Lie algebra of the pair (X, D)

$$KS_{(X,D)} = (\bigoplus_i \Gamma(X, \mathcal{A}^{0,i}_X(\Theta_X(-\log D))), d, [\ ,\ ]).$$

Note that $KS_{(X,D)}$ is a DG-Lie subalgebra of KS_X.

2.1 Morphisms of DG-Lie algebras

A *morphism* of DG-Lie algebras is a linear map $\varphi\colon L \to M$ that preserves degrees and commutes with brackets and differentials. A *quasi-isomorphism* of DG-Lie algebras is a morphism that induces an isomorphism in cohomology.

Two DG-Lie algebras L and M are said to be *quasi-isomorphic* if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

Example 2.6. Let M be a DG-Lie algebra and $M[t, dt]$ the DG-Lie algebra introduced in Example 2.3. Then, for every $a \in \mathbb{K}$, we can define the evaluation morphism

$$e_a : M[t, dt] \to M,$$

$$e_a(\sum m_i t^i + n_i t^i dt) = \sum m_i a^i.$$

Note that, every e_a is a morphism of DG-Lie algebras which is a left inverse of the inclusion $i : M \to M[t, dt]$, i.e., $e_a \circ i = \mathrm{Id}_M$. In particular, e_a is also a surjective quasi-isomorphism. We often use the short notation $m(a) = e_a(m(t, dt))$, for every $m(t, dt) \in M[t, dt]$.

For every two morphisms of DG-Lie algebras $f : L \to N$ and $g : M \to N$, we can consider the pull-back:

$$\begin{tikzcd} L \times_N M \arrow[r, "f'"] \arrow[d, "g'"] & M \arrow[d, "g"] \\ L \arrow[r, "f"'] & N. \end{tikzcd}$$

Note that if g (or f) is a surjective quasi-isomorphism then g' (or f') is also a surjective quasi-isomorphism.

Lemma 2.7. *(Factorisation Lemma) Let $f : L \to M$ be a morphism of DG-Lie algebras, then there exists a DG-Lie algebra P and a factorisation*

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "i"'] & & M \\ & P, \arrow[ur, "g"'] & \end{tikzcd}$$

such that $g : P \to M$ is a surjective morphism and $i : L \to P$ is an injective quasi-isomorphism (which is a right inverse of a surjective quasi-isomorphism).

Proof. An explicit factorisation can be defined as follow. Let P_f be the DG-Lie algebra

$$P_f = \{(x, m(t, dt)) \in L \times M[t, dt] \mid m(1) = f(x)\};$$

note that P_f is given by the pull back diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "p"'] & M[t, dt] \arrow[d, "e_1"] \\ L \arrow[r, "f"'] & M, \end{tikzcd}$$

where p is the projection on the first factor. In particular, since e_1 is a surjective quasi-isomorphism, p is also a surjective quasi-isomorphism. Next, define

$$i : L \to P_f \qquad i(x) = (x, f(x)), \qquad \forall\ x \in L;$$

and

$$g \colon P_f \to M \qquad g(x, m(t, dt)) = m(0), \qquad \forall\ (x, m(t, dt)) \in P_f.$$

The morphism g is surjective: for any $m \in M$, there exists $(0, (1-t)m) \in P_f$ such that $g(0, (1-t)m) = e_0((1-t)m) = m$. As regard the morphism i, it is a right inverse of p, i.e., $pi = \mathrm{Id}_L$ and so it is an injective quasi-isomorphism. Finally, we have

$$(g \circ i)(x) = g(x, f(x)) = f(x), \qquad \forall\ x \in L.$$

$\square$

Corollary 2.8. *Let L and M be DG-Lie algebras. Then, L and M are quasi-isomorphic if and only if there exist a DG-Lie algebra P and two surjective quasi-isomorphisms $p : P \to L$ and $q : P \to M$.*

Proof. The DG-Lie algebras L and M are quasi-isomorphic if and only if there exists a sequence of quasi-isomorphisms

$$\begin{tikzcd} & K_1 \arrow[dl] \arrow[dr] & & K_2 \arrow[dl] \arrow[dr] & & & K_n \arrow[dl] \arrow[dr] & \\ L & & H_1 & & H_2 & \cdots\ H_{n-1} & & M. \end{tikzcd}$$

By Factorisation Lemma 2.7, we can assume that all the quasi-isomoprhisms in the sequence are surjective. Indeed, consider the following diagram

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "t"] & \\ L & & N \end{tikzcd}$$

and apply Factorisation Lemma 2.7 to the morphism $(f,t)\colon K \to L \times N$ in order to obtain a diagram of quasi-isomorphisms

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[d] \arrow[dr, "t"] & \\ L & P \arrow[l] \arrow[r] & N, \end{tikzcd}$$

where the two horizontal arrows are surjective. Finally, any sequence of surjective quasi-isomorphisms can be replaced by two surjective quasi-isomorphisms using fibre product and the fact that surjective quasi-isomorphisms are stable under pull backs

$$\begin{tikzcd} & & K_1 \times_{H_1} K_2 \arrow[dl] \arrow[dr] & & \\ & K_1 \arrow[dl] \arrow[dr] & & K_2 \arrow[dl] \arrow[dr] & \\ L & & H_1 & & M. \end{tikzcd}$$

□

Definition 2.9. A DG-Lie algebra L is called *formal* if it quasi-isomorphic to $H^*(L)$ (intended as a DG-Lie algebra with trivial differential).

A DG-Lie algebra L is called *homotopy abelian* if it is quasi-isomorphic to an abelian DG-Lie algebra.

Example 2.10. Any DG-vector space is formal (and abelian as DG-Lie algebra). Let (V, d_V) be a DG-vector space, then the DG-Lie algebra $\mathrm{Hom}^*_{\mathbb{K}}(V, V)$ is formal.

Lemma 2.11. *(Transfer Lemma) Let $f : L \to M$ be a morphism of DG-Lie algebras and denote by $H^*(f) : H^*(L) \to H^*(M)$ the induced morphism in cohomology.*

1. *If M is homotopy abelian and $H^*(f)$ is injective, then L is also homotopy abelian.*

2. *If L is homotopy abelian and $H^*(f)$ is surjective, then M is also homotopy abelian.*

Proof. There are various proofs of this fact, see for instance [18, Proposition 4.11], [22, Lemma 1.10]. We follow the proof given in [13, Lemma 2.7]. As regard (1), since M is homotopy abelian, Corollary 2.8 implies the existence of an abelian DG-Lie algebra A and two surjective quasi-isomorphisms

$$\begin{tikzcd} & K \arrow[dl, "m"'] \arrow[dr, "a"] & \\ M & & A. \end{tikzcd}$$

Applying the pull back we obtain the diagram

$$\begin{tikzcd} & L \times_M K \arrow[dl, "m'"'] \arrow[r, "f'"] & K \arrow[dl, "m"'] \arrow[dr, "a"] & \\ L \arrow[r, "f"] & M & & A, \end{tikzcd}$$

where m, m' and a are surjective quasi-isomorphisms and the morphisms f and f' induce injective morphisms in cohomology.

To conclude the proof it is enough to consider a graded vector space E with a projection $e : A \to E$, such that eaf' is a quasi-isomorphism:

$$\begin{tikzcd} & L \times_M K \arrow[dl, "m'"'] \arrow[r, "f'"] & K \arrow[dr, "a"] & & \\ L & & & A \arrow[r, "e"] & E. \end{tikzcd}$$

As regard (2), since L is homotopy abelian, Corollary 2.8 implies the existence of an abelian DG-Lie algebra A with trivial differential and two surjective quasi-isomorphisms

$$\begin{tikzcd} & K \arrow[dl, "a"'] \arrow[dr, "l"] & \\ A & & L. \end{tikzcd}$$

Then, we can choose a graded Lie algebra H together with a morphism $h : H \to A$ such that the composition

$$H \xrightarrow{h} A \xrightarrow{H^*(a)^{-1}} H^*(K) \xrightarrow{H^*(l)} H^*(L) \xrightarrow{H^*(f)} H^*(M),$$

is an isomorphism. Finally, taking the fibre product of h and a, we obtain a commutative diagram

$$\begin{tikzcd} & H \times_A \times K \arrow[r, "h'"] \arrow[dl, "a'"'] & K \arrow[dl, "a"'] \arrow[dr, "l"] & & \\ H \arrow[r, "h"] & A & & L \arrow[r, "f"] & M, \end{tikzcd}$$

where a, a' and flh' are quasi-isomorphisms.

□

Remark 2.12. From the point of view of deformation theory, we could be only interested in analysing the obstruction problem. We already noticed that the obstructions of the deformation functor Def_L associated with a DG-Lie algebra L are contained in the vector space $H^2(L)$. Moreover, any morphism of DG-Lie algebras $f : L \to M$ induces a morphism $f : \mathrm{Def}_L \to \mathrm{Def}_M$ of the associated deformation functors, that behaves well with respect to the obstructions. If M is homotopy abelian, then Def_M is smooth. Therefore, it is enough that the morphism $H^2(f) : H^2(L) \to H^2(M)$ is injective for the smoothness of the functor Def_L.

Definition 2.13. Let $f : L \to M$ be a morphism of DG-Lie algebras, the homotopy fibre of f is defined as the DG-Lie algebra

$$TW(f) = \{(x, m(t, dt)) \in L \times M[t, dt] \ \mid \ m(0) = 0,\, m(1) = f(x)\}.$$

Note that the projection $TW(f) \to L$ is a morphism of DG-Lie algebras.

Remark 2.14. Let $f : L \to M$ be a morphism of DG-Lie algebras and $L \xrightarrow{i} P_f \xrightarrow{g} M$ the explicit factorisation, given in the proof of Factorisation Lemma 2.7. Then, $TW(f) = \ker g$. It can be proved that for any other factorisation $L \xrightarrow{i'} P' \xrightarrow{g'} M$, the kernel $\ker g'$ is quasi-isomorphic to $TW(f)$ [32, Section 6.1]. Moreover, every commutative diagram of morphisms of DG-Lie algebras:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d] & M \arrow[d] \\ L' \arrow[r, "f'"] & M', \end{tikzcd}$$

induces a morphism of the homotopy fibres $TW(f) \to TW(f')$.

Remark 2.15. If $f \colon L \to M$ is an injective morphism of DG-Lie algebras, then its cokernel $M/f(L)$ is a DG vector space and the map

$$TW(f) \to M/f(L)[-1]$$

$$(x, p(t)m_0 + q(t)dtm_1) \mapsto \left(\int_0^1 q(t)dt\right) m_1 \quad (\text{mod } f(L)),$$

is a surjective quasi-isomorphism.

Lemma 2.16. *(Homotopy fibre) Let $f : L \to M$ be a morphism of DG-Lie algebras.*

1. *If the induced morphism $H^*(f) : H^*(L) \to H^*(M)$ is injective, then the $TW(f)$ is homotopy abelian.*

2. *If the induced morphism $H^1(f) : H^1(L) \to H^1(M)$ is injective, then $\mathrm{Def}_{TW(f)}$ is unobstructed.*

Proof. [22, Proposition 3.4] , [23, Lemma 2.1] or [13, Corollary 2.8]. As regard (1), consider the DG vector space $M[-1]$ as an abelian DG-Lie algebra and the morphism of DG-Lie algebras

$$\rho : M[-1] \to TW(f) \qquad \forall\, m \in M \qquad \rho(m) = (0, dtm).$$

According to (2) of Lemma 2.11, it is enough to show that ρ induces a surjective map in cohomology but this follows from the exact sequence

$$\cdots \to H^{i-1}(M) \to H^i(TW(f)) \to H^i(L) \xrightarrow{H^i(f)} H^i(M) \to \cdots,$$

since the morphisms $H^i(f)$ are all injective by hypothesis.

As regard (2), the morphism of DG-Lie algebras $\rho : M[-1] \to TW(f)$ induces a morphism of deformation functors $\rho : \mathrm{Def}_{M[-1]} \to \mathrm{Def}_{TW(f)}$, with $\mathrm{Def}_{M[-1]}$ a smooth functor. Then, by the Standard Smoothness Criterion [9], [30, Theorem 4.11], since ρ is injective on obstructions, if $H^1(\rho)$ is surjective then $\mathrm{Def}_{TW(f)}$ is smooth. By the above exact sequence, if $H^1(f) : H^1(L) \to H^1(M)$ injective then $H^1(\rho)$ is surjective.

□

Example 2.17. Let W be a differential graded vector space $U \subset W$ be a DG subspace. If the induced morphism in cohomology $H^*(U) \to H^*(W)$ is injective, then the inclusion of DG-Lie algebras

$$f \colon \{f \in \mathrm{Hom}^*_{\mathbb{K}}(W, W) \mid f(U) \subset U\} \to \mathrm{Hom}^*_{\mathbb{K}}(W, W)$$

satisfies the hypothesis of Lemma 2.16 and so the DG-Lie algebra $TW(f)$ is homotopy abelian [22, Example 3.5] and [13, Proposition 5.10].

The deformation functor associated with the DG-Lie algebra $TW(f)$ has a natural interpretation as the local structure of the derived Grassmannian of W at the point U. Therefore, the derived Grassmannian of W is smooth at the points corresponding to subspaces U such that $H^*(U) \to H^*(W)$ is injective [13].

3 Cartan homotopies and Main Theorem

Let L and M be two DG-Lie algebras. A *Cartan homotopy* is a linear map of degree -1

$$\boldsymbol{i}\colon L \to M$$

such that, for every $a, b \in L$, we have:

$$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, d_M \boldsymbol{i}_b] \qquad \text{and} \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$$

For every Cartan homotopy $\boldsymbol{i}$, it is defined the Lie derivative map

$$\boldsymbol{l}\colon L \to M, \qquad \boldsymbol{l}_a = d_M \boldsymbol{i}_a + \boldsymbol{i}_{d_L a}.$$

It follows from the definition of $\boldsymbol{i}$ that $\boldsymbol{l}$ is a morphism of DG-Lie algebras and we can write the conditions of being a Cartan homotopy as

$$\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, \boldsymbol{l}_b] \qquad \text{and} \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b] = 0.$$

Note that, as a morphism of complexes, $\boldsymbol{l}$ is homotopic to 0 (with homotopy $\boldsymbol{i}$).

Example 3.1. Let X be a smooth variety. Denote by Θ_X the tangent sheaf and by (Ω^*_X, d) the algebraic de Rham complex. Then, for every open subset $U \subset X$, the contraction of a vector with a differential form

$$\Theta_X(U) \otimes \Omega^k_X(U) \xrightarrow{\;\lrcorner\;} \Omega^{k-1}_X(U)$$

induces a linear map of degree -1

$$\boldsymbol{i}\colon \Theta_X(U) \to \operatorname{Hom}^*(\Omega^*_X(U), \Omega^*_X(U)), \qquad \boldsymbol{i}_\xi(\omega) = \xi \lrcorner \omega$$

that is a Cartan homotopy. Indeed, the above conditions coincide with the classical Cartan's homotopy formulas.

Example 3.2. Let D be a smooth subvariety of codimension 1 of a smooth variety X. Let $(\Omega^*_X(\log D), d)$ be the logarithmic differential complex and $\Theta_X(-\log D)$ the logarithmic tangent sheaf. It is easy to prove explicitly that for every open subset $U \subset X$, we have

$$(\,\Theta_X(-\log D)(U) \lrcorner \Omega^k_X(\log D)(U)\,) \subset \Omega^{k-1}_X(\log D)(U).$$

Then, as above, the induced linear map of degree -1

$$\boldsymbol{i}\colon \Theta_X(-\log D)(U) \to \operatorname{Hom}^*(\Omega^*_X(\log D)(U), \Omega^*_X(\log D)(U)), \qquad \boldsymbol{i}_\xi(\omega) = \xi \lrcorner \omega$$

is a Cartan homotopy.

We are now ready to prove the main theorem.

Theorem 3.3. *(Abstract BTT Theorem) Let L and M be DG-Lie algebras over a field $\mathbb{K}$ of characteristic 0 and $H \subseteq M$ a DG-Lie subalgebra. Assume that there exists a linear map*

$$\boldsymbol{i} \in \operatorname{Hom}_{\mathbb{K}}^{-1}(L, M), \qquad a \mapsto \boldsymbol{i}_a,$$

of degree -1 such that:

1. $[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$ *and* $\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, d\boldsymbol{i}_b]$ *for every* $a, b \in L$;
2. $d\boldsymbol{i}_a + \boldsymbol{i}_{da} \in H$ *for every* $a \in L$;
3. *the inclusion* $H \hookrightarrow M$ *is injective in cohomology;*
4. *the induced morphism of complexes* $\boldsymbol{i} \colon L \to (M/H)[-1]$ *is injective in cohomology.*

Then, the DG-Lie algebra L is homotopy abelian.

Proof. Let s be a formal variable of degree -1 and consider the commutative DG-algebra $\mathbb{K}[s]$; note that $s^2 = 0$ and the differential d on $\mathbb{K}[s]$ is defined as

$$d : \mathbb{K}\, s \to \mathbb{K}\,, \qquad d(s) = 1.$$

In particular, $\mathbb{K}[s]$ is a contractible DG algebra and the inclusion $\mathbb{K} \hookrightarrow \mathbb{K}[s]$ is a morphism of DG algebras.

Next, we consider the DG-Lie algebra $\mathbb{K}[s] \otimes L$. For all $s \otimes a \in \mathbb{K}[s] \otimes L$, we have $deg(s \otimes a) = deg(a) - 1$ and

$$d(s \otimes a) = 1 \otimes a - s \otimes da.$$

Then, we can define a morphism of DG-Lie algebras by

$$\varphi : \mathbb{K}[s] \otimes L \to M, \qquad \varphi(s \otimes a) = \boldsymbol{i}_a;$$

in particular, $\varphi(1 \otimes a) = \varphi(d(s \otimes a) + s \otimes da) = d(\varphi(s \otimes a)) + \boldsymbol{i}_{da} = d\boldsymbol{i}_a + \boldsymbol{i}_{da} = \boldsymbol{l}_a$, for any $a \in L$,and it is contained in H by Hypothesis (2).

Thus, we can construct a commutative diagram of morphisms of DG-Lie algebras

$$\begin{tikzcd} L \arrow[r, "\psi"] \arrow[d, "\alpha"'] & H \arrow[d, hook, "\chi"] \\ \mathbb{K}[s] \otimes L \arrow[r, "\varphi"] & M, \end{tikzcd}$$

where $\alpha(a) = 1 \otimes a$ and $\psi(a) = \boldsymbol{l}_a$, for any $a \in L$.

According to Remark 2.14, this diagram induces a morphisms of DG-Lie algebras

$$\phi : TW(\alpha) \to TW(\chi).$$

Hypothesis (3) and Lemma 2.16 applied to the inclusion $\chi : H \to M$ imply that the DG-Lie algebra $TW(\chi)$ is homotopy abelian. Moreover, according to Remark 2.15, $TW(\chi)$ is quasi-isomorphic as a DG vector spaces to $(M/H)[-1]$. Since $\mathbb{K}[s] \otimes L$ is contractible, then $TW(\alpha) \to L$ is a quasi-isomorphism. Finally, Hypothesis (4) implies that ϕ is injective in cohomology and so Lemma 2.16 implies that $TW(\alpha)$ is homotopy abelian. It follows that L is also homotopy abelian.

□

Remark 3.4. In the above notation and using the first three hypothesis, we have constructed a diagram:

$$\begin{tikzcd} TW(\alpha) \arrow[r, "\phi"] \arrow[d] & TW(\chi) \\ L, & \end{tikzcd}$$

where the vertical map is a quasi-isomorphism. Then, Hypothesis (4) implies that the horizontal map is injective in cohomology. If we are only interested in the analysis of the obstruction of Def_L, then Hypothesis (3) can be relaxed to the condition that only $H^1(\chi)$ is injective and Hypothesis (4) can be relaxed to the condition that only $H^2(\boldsymbol{i})$ is injective. Indeed, $\mathrm{Def}_{TW(\chi)}$ is unobstructed, and so for the vanishing of the obstructions of Def_L it is enough that $H^2(\phi)$ is injective (see Section 5 for further generalizations).

4 Examples and Applications

In this section, we collect some applications of the main Theorem 3.3.

4.1 Deformations of compact manifolds

Let X be a holomorphic compact manifold and denote by Θ_X its holomorphic tangent bundle. We introduced the Kodaira Spencer DG-Lie algebra KS_X in Example 2.4. Let $(A_X^*, d) = (\bigoplus_{p,q} \Gamma(X, \mathcal{A}_X^{p,q}), d = \partial + \overline{\partial})$ be the De Rham complex of X, where $\mathcal{A}_X^{p,q}$ denotes the sheaf of (p,q)-differential forms on X and consider the DG-Lie Algebra $M = \mathrm{Hom}^*_{\mathbb{C}}(A_X^*, A_X^*)$.

Corollary 4.1. *Let X be a compact manifold with torsion canonical bundle such that the Hodge-de Rham spectral sequence degenerates at E_1-level. Then, KS_X is homotopy abelian.*

Proof. [35, Corollary 2], [28, Corollary B], [22, Corally 6.5]. Let us first consider the case in which the canonical bundle is trivial. The contraction of vector fields and differential forms together with the cup product defines a Cartan homotopy [30, Section 6]

$$\boldsymbol{i}\colon KS_X \to M = \mathrm{Hom}^*(A_X^*, A_X^*), \quad \boldsymbol{i}_\eta(\omega) = \eta \lrcorner \omega, \qquad \forall\, \eta \in KS_X, \ \forall\, \omega \in A_X.$$

Moreover, for any $\eta \in KS_X$, $\boldsymbol{l}_\eta \in H = \{\varphi \in M \mid \varphi(A_X^{n,*}) \subset A_X^{n,*}\}$, where n is the dimension of X. By the degeneration property the inclusion $H \hookrightarrow M$ is injective in cohomology. The canonical bundle is trivial and so the cup product with a non trivial section of it, gives the isomorphisms $H^i(\Theta_X) \cong H^i(\Omega_X^{n-1})$, where Ω_X^{n-1} denotes the sheaf of holomorphic differential $n-1$ forms. Then, $H^*(KS_X) \to \mathrm{Hom}^*(H^0(\Omega_X^n), H^*(\Omega_X^{n-1}))$ is injective and this implies that $KS_X \to M/H[-1]$ is also injective in cohomology. Therefore, the hypothesis of Theorem 3.3 are satisfied.

In the case of torsion canonical bundle, there exists $m > 0$ such that $K_X^m = \mathcal{O}_X$, where K_X denote the canonical bundle. Next, consider the unramified m-cyclic cover, defined by K_X, i.e., $\pi : Y = \mathrm{Spec}(\bigoplus_{i=0}^{m-1} L^{-i}) \to X$ (see [34] for full details on abelian covers). Then, $\pi : Y \to X$ is a finite flat map of degree m and Y is a compact manifold with trivial canonical bundle, since $K_Y \cong \pi^* K_X \cong \mathcal{O}_Y$. Therefore, KS_Y is homotopy abelian. Finally, it is enough to consider the morphism of DG-Lie algebras $KS_X \to KS_Y$ induced by pull back. This morphism is injective in cohomology and so (1) of Lemma 2.11 implies that KS_X is also homotopy abelian.

□

Remark 4.2. The degeneration hypothesis is satisfied if the $\partial\overline{\partial}$-Lemma holds, for instance for Kähler manifolds.

Remark 4.3. It is well known that the Kodaira Spencer DG-Lie algebra KS_X controls the infinitesimal deformations of X. Then, the infinitesimal deformations of a compact Kähler manifold with torsion canonical bundle X are unobstructed. This is the classical Bogomolov-Tian-Todorov Theorem.

Remark 4.4. Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic 0. Then, the analogous of Corollary 4.1 holds.

In this case, we can replace the Kodaira-Spencer DG-Lie algebra with the DG-Lie algebra $\mathrm{Tot}(\Theta_X(\mathcal{U}))$, obtained applying the Thom-Withney totalization to the semicosimplicial DG-Lie algebras $\Theta_X(\mathcal{U})$, for any affine open cover $\mathcal{U}$ of X [22, Theorem 5.3]. Also in this case, if the canonical bundle of X is torsion, then $\mathrm{Tot}(\Theta_X(\mathcal{U}))$ is homotopy-abelian and so X has unobstructed deformations [22, Theorem 6.2 and Corollary 6.5].

4.2 Deformations of pairs (divisor, manifold)

Let D be a smooth divisor in a compact manifold X. We introduced the Kodaira Spencer DG-Lie algebra of the pair $KS_{(X,D)}$ in Example 2.5. Then, considering $\Theta_X(-\log D)$ instead of Θ_X, we can proceed as in the case of KS_X of Corollary 4.1.

Corollary 4.5. *Let D be a smooth divisor in a compact manifold X, such that the logarithmic canonical bundle $\Omega_X^n(\log D)$ is trivial and the logarithmic Hodge-*

de Rham spectral sequence degenerates at E_1-level. Then, $KS_{(X,D)}$ is homotopy abelian.

Proof. [21, Theorem 5.1 and Corollary 5.4]. The proof is analogous to the one of Corollary 4.1. Here, the Cartan homotopy is given by the contraction of logarithmic tangent vector and logarithmic differentials:

$$\boldsymbol{i}\colon KS_{(X,D)} \to M = \operatorname{Hom}^*_{\mathbb{C}}(\mathcal{A}^{*,*}_X(\log D), \mathcal{A}^{*,*}_X(\log D)).$$

Then, for any $\eta \in KS_{(X,D)}$, $\boldsymbol{l}_\eta \in H = \{\varphi \in M \mid \varphi(\mathcal{A}^{n,*}_X(\log D)) \subset \mathcal{A}^{n,*}_X(\log D)\}$, where n is the dimension of X.

Next, the degeneration property implies that the inclusion $H \hookrightarrow M$ is injective in cohomology. Since the logarithmic canonical bundle $\Omega^n_X(\log D)$ is trivial, the cup product with a non trivial section of it, gives the isomorphisms $H^i(\Theta_X(\log D)) \cong H^i(\Omega^{n-1}_X(\log D))$.

This implies that $H^*(KS_{(X,D)}) \to \operatorname{Hom}^*(H^0(\Omega^n_X(\log D)), H^*(\Omega^{n-1}_X(\log D)))$ is injective and so $KS_{(X,D)} \to M/H[-1]$ is also injective in cohomology. Therefore, the hypothesis of Theorem 3.3 are satisfied.

□

Remark 4.6. The degeneration hypothesis is satisfied for any globally normal crossing divisor D in a compact Kähler manifold [40, Theorem 8.35].

Remark 4.7. It can be also proved that $KS_{(X,D)}$ is homotopy abelian for a smooth divisor D in a Calabi-Yau manifold X [21, Theorem 4.7]. In this case, the relevant spectral sequence is the one associated with the Hodge filtration

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))$$

and it degenerates at the E_1-level [14, Section 4.3].

Remark 4.8. It is well known that the Kodaira Spencer DG-Lie algebra $KS_{(X,D)}$ controls the infinitesimal deformations of the (X, D). Then, the infinitesimal deformations of the pair (X, D) are unobstructed when D is a smooth divisor in a compact Kähler manifold X such that $\Omega^n_X(\log D)$ is trivial [21, Corollary 4.5] and when D is a smooth divisor in a compact Calabi-Yau manifold X [21, Corollary 4.8].

Remark 4.9. In general, if the ground field is an algebraically closed field of characteristic 0, we can replace the DG-Lie algebra $KS_{(X,D)}$ with the DG-Lie algebra $TW(\Theta_X(-\log D)(\mathcal{U}))$ obtained applying the Thom-Withney realisation to the semicosimplicial DG-Lie algebras $\Theta_X(-\log D)(\mathcal{U})$, for any affine open cover $\mathcal{U}$ of X [21, Theorem 4.3]. Also in this case, $TW(\Theta_X(-\log D)(\mathcal{U}))$ is homotopy abelian, for a smooth divisor D in a smooth projective variety X such that $\Omega^n_X(\log D)$ is trivial and when D is a smooth divisor in a smooth projective Calabi-Yau variety.

If D is not smooth but only a simple normal crossing divisor, then $KS_{(X,D)}$ (or $TW(\Theta_X(-\log D)(\mathcal{U}))$) controls the locally trivial infinitesimal deformations of the pair (X, D). Then, the computations above shows that the locally trivial infinitesimal deformations are unobstructed.

4.3 Differential Batalin-Vilkovisky algebras

The main references for this example are [37, 18] and [21, Section 7].

Definition 4.10. Let k be a fixed odd integer. A *differential Batalin-Vilkovisky algebra* (dBV for short) of degree k over $\mathbb{K}$ is the data (A, d, Δ), where (A, d) is a differential $\mathbb{Z}$-graded commutative algebra with unit $1 \in A$, and Δ is an operator of degree $-k$, such that $\Delta^2 = 0$, $\Delta(1) = 0$ and

$$\Delta(abc) + \Delta(a)bc + (-1)^{\overline{a}\,\overline{b}}\Delta(b)ac + (-1)^{\overline{c}(\overline{a}+\overline{b})}\Delta(c)ab = \\ = \Delta(ab)c + (-1)^{\overline{a}(\overline{b}+\overline{c})}\Delta(bc)a + (-1)^{\overline{b}\overline{c}}\Delta(ac)b.$$

For any graded dBV algebra (A, d, Δ) of degree k, it is canonically defined a DG-Lie algebra $(L, d, [-,-])$, where: $L = A[k]$, $d_L = -d_A$ and,

$$[a, b] = (-1)^p(\Delta(ab) - \Delta(a)b) - a\Delta(b), \qquad \forall\ a \in A^p.$$

Definition 4.11. A dBV algebra (A, d, Δ) of degree k has the *degeneration property* if for every $a_0 \in A$, such that $da_0 = 0$, there exists a sequence a_i, $i \geq 0$, with $\deg(a_i) = \deg(a_{i-1}) - k - 1$ and such that

$$\Delta a_i = da_{i+1}, \qquad i \geq 0.$$

Example 4.12. Let (V, d, Δ) be a $(1, k)$-bicomplex, where k is an odd integer, i.e., (V, d) is a DG vector space and $\Delta \in \operatorname{Hom}^k_{\mathbb{K}}(V, V)$ such that

$$\Delta^2 = 0 \quad [d, \Delta] = d\Delta + \Delta d = 0.$$

If the $d\Delta$-lemma holds, i.e.,

$$\ker d \cap \Delta(V) = \ker \Delta \cap d(V) = d\Delta(V),$$

then (V, d, Δ) has the degeneration property. Indeed, if $da_0 = 0$ we have $\Delta a_0 \in d\Delta(V)$ and then there exists $b \in V$ such that $d\Delta(b) = \Delta a_0$. It is sufficient to take $a_1 = \Delta(b)$ and $a_i = 0$ for every $i \geq 2$. Note that the converse is not true in general. For instance, if $d = \Delta$, then (V, d, Δ) has the degeneration property, while $\ker \Delta \cap d(V) = d\Delta(V)$ if and only if $d = \Delta = 0$.

Example 4.13. Let (V, d, Δ) be a $(1, k)$-bicomplex as in Example 4.12 and suppose the existence of an operator $f \in \operatorname{Hom}^{k-1}_{\mathbb{K}}(V, V)$ such that

$$\Delta = [d, f], \qquad [f, \Delta] = 0.$$

Then (V, d, Δ) has the degeneration property. Indeed, consider a formal parameter t and the associative graded algebra $\operatorname{Hom}^*_{\mathbb{K}}(V, V)[[t]]$. Then, we have

$$e^{tf}de^{-tf} = e^{[tf,-]}d = d + t[f, d] = d - t\Delta$$

and therefore

$$t\Delta e^{tf} = -e^{tf}d + de^{tf} = [d, e^{tf}].$$

Let $a \in V$ be such that $da = 0$ and define the sequence a_i by the rule

$$\sum_{i\geq 0} a_i t^i = e^{tf}(a).$$

It implies $a_0 = a$ and

$$\sum_{i\geq 0} t^{i+1}\Delta a_i = t\Delta e^{tf}a = de^{tf}a = \sum_{i\geq 0} t^i da_i$$

and then $da_{i+1} = \Delta a_i$ for every i.

Theorem 4.14. *Let (A, d, Δ) be a dBV algebra with the degeneration property. Then, the associated DG-Lie algebra $L = A[k]$ is homotopy abelian.*

Proof. Here we only give a sketch of the proof. The original proof can be found in [37, Theorem 1] or [18, Theorem 4.14] for $k = 1$ and in [21, Theorem 7.6] for any odd k.

Given a dBV algebra (A, d, Δ) and t a formal central variable of (even) degree $1 + k$, we can define the graded vector space $A[[t]]$ of formal power series with coefficients in A and the graded vector space $A((t)) = \bigcup_{p\in\mathbb{Z}} t^p A[[t]]$ of formal Laurent power series. Note that $d(t) = \Delta(t) = 0$ and $d - t\Delta$ is a well-defined differential on $A((t))$.

Let $F^\bullet$ be the filtration on the complex $(A((t)), d - t\Delta)$ defined by $F^p = t^p A[[t]]$, for every $p \in \mathbb{Z}$. Note that $A((t)) = \bigcup_{p\in\mathbb{Z}} F^p$ and $F^0 = A[[t]]$ and the map $a \mapsto \dfrac{a}{t}$ induces an isomorphism of DG-vector spaces $A \to F^{-1}/F^0$.

The degeneration property implies that the inclusion $F^p \to A((t))$ is injective in cohomology, for every p, and, in particular, $F^0 = A[[t]] \to A((t))$ is injective in cohomology.

Consider $M = \operatorname{Hom}^*_{\mathbb{K}}(A((t)), A((t)))$ and $H = \{\varphi \in M \mid \varphi(A[[t]]) \subset A[[t]]\}$; then, the degeneration property implies that $H \hookrightarrow M$ is injective in cohomology (Hypotheis (3) of Theorem 3.3).

A tedious but straightforward computation shows that the map

$$\boldsymbol{i}\colon L \to M = \operatorname{Hom}^*_{\mathbb{K}}(A((t)), A((t))), \qquad a \longmapsto \boldsymbol{i}_a(b) = \frac{1}{t}ab$$

is a Cartan homotopy [21, Lemma 7.2] and, for any $a \in L$, $\boldsymbol{l}_a \in H$, i.e., Hypotheis (1) and (2) of Theorem 3.3 are satisfied. Therefore, we only need to show that the morphism of DG-vector spaces $\boldsymbol{i} : L \to M/H[-1]$ is injective in cohomology and this follows again by degeneration property. Indeed, $M/H[-1] = \operatorname{Hom}^*_{\mathbb{K}}\left(A[[t]], \frac{A((t))}{A[[t]]}\right)[-1]$ and so we need to prove that:

$$\boldsymbol{i}\colon A \to \operatorname{Hom}^*_{\mathbb{K}}\left(A[[t]], \frac{A((t))}{A[[t]]}\right)[-k-1]$$

is injective in cohomology. It suffices to show the injectivity for the composition with the evaluation at $1 \in A[[t]]$, i.e., the injectivity in cohomology of the morphism

$$A \to \frac{A((t))}{A[[t]]}, \qquad a \mapsto \frac{a}{t},$$

or equivalently of the morphism

$$\beta : \frac{F^{-1}}{F^0} \to \frac{A((t))}{F^0}.$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & F^0 \arrow[r] \arrow[d, equal] & F^{-1} \arrow[r] \arrow[d, "\alpha"'] & \frac{F^{-1}}{F^0} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & F^0 \arrow[r] & A((t)) \arrow[r] & \frac{A((t))}{F^0} \arrow[r] & 0;
\end{tikzcd}$$

the degeneration property implies that α is injective in cohomology and so β is also injective in cohomology.

$\square$

Remark 4.15. In [1], the author uses similar techniques to extend this result to the context of commutative BV_∞-algebras. In this case, the degeneration property implies that the associated $L_\infty[1]$-algebras is homotopy abelian [1, Theorem 6.16], see also [6].

Example 4.16. [12, Section 5] and [4, Section 6]. Let X be a complex manifold and Θ_X it tangent bundle. An holomorphic Poisson bi-vector on X is an element $\pi \in H^0(X, \wedge^2\Theta_X)$, such that $[\pi,\pi]_{SN} = 0$, where $[\ ,\ ]_{SN}$ denotes the Schouten-Nijenhuis bracket of polyvectorfields [32]. In this case, we say that the pair (X,π) is a holomorphic Poisson manifold. Then, let (A^*_X, d) be the de Rham complex, where $A^i_X = \bigoplus_{p+q=i} A^{p,q}_X$ and consider the map

$$\boldsymbol{i}_\pi \in \operatorname{Hom}^{-2}(A^*_X, A^*_X) \qquad \boldsymbol{i}_\pi(\alpha) = \pi \lrcorner \alpha.$$

Note that $[\boldsymbol{i}_\pi, \boldsymbol{i}_\pi] = 0$ and, setting $\Delta = [d, \boldsymbol{i}_\pi]$, we have

$$[\boldsymbol{i}_\pi, \Delta] = [\boldsymbol{i}_\pi, [d, \boldsymbol{i}_\pi]] = \boldsymbol{i}_{[\pi,\pi]_{SN}} = 0.$$

According to Example 4.13, (A^*_X, d, Δ) has the degeneration property and so Theorem 4.14 implies that the associated DG-Lie algebra $(A^*_X[1], -d, [,]_\pi)$ is homotopy abelian (see [12, Theorem 5.3] and [4, Section 6]).

4.4 The splitting property

Let $(L, d, [\ ,\ ])$ be a DG-Lie algebra. Denote by $L[1]$ the shifted graded vector space: $L[1]^i = L^{i+1}$ and by $L[1]^{\odot n}$ its n-th symmetric power. Note that $1 \in L[1]^{\odot 0} = \mathbb{K}$. Then, we can consider the associated differential graded cocommutative coalgebra $(SL[1], \Delta, Q)$, where $SL[1] = \bigoplus_{n\geq 0} L[1]^{\odot n}$, Δ is the usual coproduct and Q the coderivation associated with d and $[\ ,\ ]$.

More explicitly, let $q_1(x) = -d(x)$ and $q_2(x \odot y) = -(-1)^i[x, y]$ for any $x \in L[1]^i$ and $y \in L[1]$, then

$$Q(v_1 \odot \ldots \odot v_n) = \sum_{k=1}^{2} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)},$$

where $S(p, q)$ denotes the set of unshuffles of type (p, q) and $\epsilon(\sigma)$ is the Koszul sign. It turns out that $Q^2 = 0$.

Note that, any morphism $F : SL[1] \to SL[1]$ is uniquely determined by the corestriction $pF = (f_0, f_1, f_2, \ldots) : SL[1] \to SL[1] \to L[1]$, where $p : SL[1] \to L[1]^{\odot 1} = L[1]$ is the natural projection and $f_i : L[1]^{\odot i} \to L[1]$ are the components of pF.

Next, denote by $(\mathrm{Coder}^*(SL[1]), [Q,\ ], [\ ,\ ])$ the DG-Lie algebra of coderivation of $SL[1]$ and consider the surjective morphism of DG-vector spaces

$$\mathrm{Coder}^*(SL[1]) \xrightarrow{b} L[1], \qquad b(\alpha) = p\alpha(1), \quad \forall \alpha.$$

Corollary 4.17 ([3]). *In the above assumption, if b: $\mathrm{Coder}^*(SL[1]) \to L[1]$ induces a surjective morphism in cohomology, then L is homotopy abelian.*

Proof. Here we only give a sketch of the proof as a consequence of the abstract BTT theorem, while the original proof contained in [3] uses the theory of derived brackets. Let $M = \mathrm{Coder}^*(SL[1])$ and $H = \ker b \subset M$. The hypothesis implies that the inclusion $H \hookrightarrow M$ is injective in cohomology, i.e., Hypothesis (3) of Theorem 3.3 is satisfied. The map

$$\boldsymbol{i}\colon L \to \mathrm{Coder}^*(SL[1]), \qquad a \longmapsto \boldsymbol{i}_a(w) = x \odot w$$

is a properly defined Cartan homotopy. Moreover, for any $a \in L$, $\boldsymbol{l}_a \in H$, and so Hypothesis (1) and (2) are fulfilled. This also implies that it is well defined the morphism of DG vector spaces $\boldsymbol{i}\colon L \to (M/H)[-1]$, that is injective in cohomology. Therefore, by Theorem 3.3, L is homotopy abelian.

□

Remark 4.18. According to [3], we say that L has the splitting property if it satisfies the hypothesis of the previous theorem. We refer to [3, Proposition 2.2] for equivalent conditions. In this paper, the author proves this result for $L_\infty[1]$-algebras, analysing the spectral sequence computing the Chevalley-Eilenberg cohomology and showing the equivalence between the degeneration of the spectral sequence at the first page and the homotopy abelianity property. In [31], the author also analyses the spectral sequence computing the Chevalley-Eilenberg cohomology of a DG-Lie algebra. In particular, he proved the equivalence between degeneration of the spectral sequence at the second page and formality property.

Remark 4.19. It can be also proved that the splitting property implies that a DG-Lie algebra L is homotopy abelian if the adjoint morphism

$$ad\colon L \to \mathrm{Der}^*_{\mathbb{K}}(L, L), \qquad ad_x(y) = [x, y],$$

is trivial in cohomology: a detailed proof will appear in the forthcoming paper [24].

4.5 Derived brackets of Lie type

Here we follow [32, Section 5.6]. Let $(M, [,], d)$ be a DG-Lie algebra, such that there exist a DG-Lie subalgebra L and a graded vector space A satisfying the following conditions:

(i) $M = L \oplus A$ as graded vector space;

(ii) $[a, b] = 0$ for every $a, b \in A$;

(iii) $[da, b] \in A$, for every $a, b \in A$.

Consider the projection $p : M \to A$ and the operators:

$$\delta : A^i \to A^{i+1}, \qquad \delta a = -pda,$$

$$\{- \ , \ -\} : A^{i-1} \times A^{j-1} \to A^{i+j-1}, \qquad \{a, \ b\} := -(-1)^i [da, b].$$

A straightforward computation shows that $(A[-1], \delta, \{- \ , \ -\})$ is a DG-Lie algebra [32, Proposition 5.6.6].

Corollary 4.20. *If $L \to M$ is injective in cohomology, then the DG-Lie algebra $A[-1] = (A[-1], \delta, \{- \ , \ -\})$ is homotopy abelian.*

Proof. The linear map:

$$\boldsymbol{i} : A[-1] \to M \qquad \boldsymbol{i}_a = \boldsymbol{i}(a) = a$$

is a Cartan homotopy. Indeed, for every $a, b \in A[-1]$, we have $[\boldsymbol{i}_a, \boldsymbol{i}_b] = [a, b] = 0$, by Condition (ii) above. Moreover, for every $a \in A[-1]^i$ and $b \in A[-1]$, we have $0 = d([a, b]) = [da, b] + (-1)^{i-1}[a, db]$ and so

$$\boldsymbol{i}_{\{a,b\}} = \{a, \ b\} = -(-1)^i [da, b] = [a, db] = [\boldsymbol{i}_a, d\boldsymbol{i}_b].$$

Note that, for every $a \in A[-1]$, we have

$$p(\boldsymbol{l}_a) = p(d\boldsymbol{i}_a + \boldsymbol{i}_{\delta a}) = p(da - pda) = 0$$

and so for every $a \in A[-1]$, we have $\boldsymbol{l}_a \in L$, i.e., Hypothesis (2) of Theorem 3.3 is satisfied. The last two hypotheses of Theorem 3.3 are satisfied by assumption, and so we conclude that $A[-1] = (A[-1], \delta, \{-\ ,\ -\})$ is homotopy abelian. $\square$

Example 4.21. [32, Section 5.6]. Let V and W be two graded vector spaces over $\mathbb{K}$ and $\pi \in \mathrm{Hom}^1_{\mathbb{K}}(W, V)$, such that $0 = d_{\mathrm{Hom}}\pi = d_W\pi + \pi d_V$. In [32, Section 5.6], the author introduced the notion of the derived bracket of π on $\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1]$ defined as:

$$[-\ ,-\ ]_\pi : (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1])^i \times (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1])^j \to (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1])^{i+j}$$

$$[f, g]_\pi = f\pi g - (-1)^{ij} g\pi f.$$

On the graded vector space $\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1]$, it is also defined a differential δ as:

$$\delta(f) = -d_W f - (-1)^i f d_V, \qquad \forall\ f \in (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1])^i.$$

Then, $A[-1] = (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1], [-\ ,-\ ]_\pi, \delta)$ is a DG-Lie algebra. To view this example in the above setting, it is enough to consider the DG-Lie algebra $M = \mathrm{Hom}^*_{\mathbb{K}}(V \oplus W, V \oplus W)$ with differential given by $[D,\ -]$ where

$$D = \begin{pmatrix} d_V & -\pi \\ 0 & d_W \end{pmatrix} : V \oplus W \to V \oplus W.$$

It turns out that $\{-\ ,\ -\} = [-\ ,-\ ]_\pi$; indeed, for every $a \in A[-1]^i$ and $b \in A[-1]^j$, we have

$$\{a, b\} = -(-1)^i [Da, b] = -(-1)^i [[D, a], b].$$

By definition,

$$\begin{aligned} [[D, a], b] &= [Da - (-1)^{i-1} aD, b] \\ &= Dab - (-1)^{i-1} aDb - (-1)^{i(j-1)}(bDa - (-1)^{i-1} baD) \\ &= Dab - (-1)^{i-1} aDb - (-1)^{i(j-1)} bDa - (-1)^{ij} baD. \end{aligned}$$

Viewing a and b as

$$a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : V \oplus W \to V \oplus W,$$

we have

$$Da = \begin{pmatrix} -\pi a & 0 \\ d_W a & 0 \end{pmatrix};$$

thus $Dab = abD = 0$, and

$$bDa = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \cdot \begin{pmatrix} -\pi a & 0 \\ d_w a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b\pi a & 0 \end{pmatrix}, \qquad aDb = \begin{pmatrix} 0 & 0 \\ -a\pi b & 0 \end{pmatrix}.$$

Therefore,

$$\{a, b\} = -(-1)^i [Da, b] = -aDb + (-1)^{ij} bDa = a\pi b - (-1)^{ij} b\pi a = [a, b]_\pi.$$

Finally, if the inclusion $L \to M$ is injective in cohomology, then the DG-Lie algebra $A[-1] = (\mathrm{Hom}^*_{\mathbb{K}}(V, W)[-1], [-\ ,-\ ]_\pi, \delta)$ is homotopy abelian.

5 Further applications

According to Remark 3.4, if L, M and $H \subseteq M$ are DG-Lie algebras, such that $\chi : H \hookrightarrow M$ is injective in cohomology and $\boldsymbol{l}_a \in H$ for every $a \in L$, we have a diagram:

$$\begin{array}{ccc} TW(\alpha) & \xrightarrow{\phi} & TW(\chi) \\ \downarrow{\scriptstyle p} & & \\ L, & & \end{array}$$

where $TW(\chi)$ is an homotopy abelian DG-Lie algebras and the vertical arrow is a quasi isomorphism. In particular, we have a morphism:

$$s : H^2(L) \xrightarrow{H^2(p)^{-1}} H^2(TW(\alpha)) \xrightarrow{H^2(\phi)} H^2(TW(\chi))$$

Since $\mathrm{Def}_{TW(\chi)}$ is smooth, s annihilates all the obstructions of Def_L. This idea has been applied in various deformation cases. For instance, in [11, Proposition 4.6], the authors consider the deformations of compact Kähler manifolds and they prove the Kodaira's principle, ambient cohomology annihilates obstructions. A semiregularity map annihilating all the obstructions to the infinitesimal deformations of holomorphic maps of compact Kähler manifolds with fixed codomain were analysed in [20, Corollary 4.14]. In [23, Theorem 11.1], it is proved that the Bloch's semiregularity map annihilates all the obstructions to embedded deformations of a local complete intersection, extending the proof given in [29, Theorem 9.1] for the embedded deformations of a smooth submanifold.

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