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Normal bicanonical and tricanonical threefolds

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Abstract. *The first author to construct a non-normal bicanonical threefold in* $\mathbb{P}^4$ *was L. Godeaux in* 1936 *[5]. This threefold has degree* 8*. In the first part of the present paper, starting from a normal threefold of general type where* $q_1 = q_2 = p_g = 0, P_2 = P_3 = 5$*, of degree* 6 *(cf. [11]), we construct Godeaux's example and two examples of tricanonical threefolds in* $\mathbb{P}^4$*. One of the tricanonical threefolds is normal. In the second part of the paper, we construct (starting from the beginning) a normal bicanonical threefold of degree* 8 *that has the birational invariants given by* $q_1 = q_2 = p_g = 0$ *and* $P_2 = 5$*. No other examples of bicanonical and tricanonical threefolds in* $\mathbb{P}^4$ *are known.*

Sunto. *Il primo autore che ha costruito un'ipersuperficie bicanonica non normale in* $\mathbb{P}^4$ *è stato L. Godeaux nel* 1936 *[5]. Tale ipersuperficie ha ordine* 8*. Nella prima parte del presente lavoro, partendo da una varietà tridimensionale, normale e di tipo generale avente* $q_1 = q_2 = p_g = 0, P_2 = P_3 = 5$*, di ordine* 6 *(cfr. [11]), si costruisce l'esempio di Godeaux e due esempi di ipersuperficie tridimensionali tricanoniche in* $\mathbb{P}^4$*. Una delle ipersuperficie tricanoniche è normale. Nella seconda parte del lavoro si costruisce, partendo dall'inizio, una varietà tridimensionale, normale e bicanonica, di ordine* 8 *avente gli invarianti birazionali* $q_1 = q_2 = p_g = 0$ *e* $P_2 = 5$*. Non si conoscono altri esempi di ipersuperficie bicanoniche e tricanoniche in* $\mathbb{P}^4$*.*

1 Introduction

In a previous work [12], we constructed normal canonical hypersurfaces in the projective space $\mathbb{P}^d$ for any $d \geq 4$, following an introduction with a concise historical note on canonical surfaces in $\mathbb{P}^3$. This historical note began with some considerations on Chapter VIII of Enriques's book "Le superficie algebriche" (cf. [3] and also [4, Ch. V]); the English translation of the chapters title is: *Regular canonical and pluricanonical surfaces.*

While there is an abundance of literature on canonical surfaces in $\mathbb{P}^3$, we have found few works on pluricanonical (bicanonical and tricanonical) surfaces in $\mathbb{P}^3$. In truth, numerous publications concern studies on either the birationality or the non-birationality of the bicanonical (or tricanonical) transformation (improperly called a map) of a surface of general type, whereas bicanonical (or tricanonical) surfaces in $\mathbb{P}^3$ require not only that the bicanonical (or tricanonical) transformation

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be birational, but also that the bigenus $P_2 = 4$ (or the trigenus $P_3 = 4$) (see the properties due to the definition that we have adopted in Section 2 below).

In his book, Ch. VIII, Sections 14 - 21, Enriques provides the first published explanation for the theory of bicanonical and tricanonical surfaces as images of the bicanonical and tricanonical transformations. He also provides a detailed account of how to construct the first tricanonical surface in $\mathbb{P}^3$ starting from a double plane with a branch curve of degree 10 having five [3,3] points and one ordinary 4-ple point that are not on a conic. Unfortunately, Enriques considers the Campedelli construction, which is incorrect (cf. [1, 2]). So, although we cannot accept Enriques's statement proving that the tricanonical transformation of a desingularization of the Campedelli double plane is birational having as its image a tricanonical surface in $\mathbb{P}^3$ (cf. [3, pp. 308-309]), it is important to bear in mind that - after Campedelli - many curves of degree 10 were constructed with five [3,3] points and one ordinary 4-ple point, that are not on a conic (cf. [2, 8, 10, 13]). The double planes, having the above curves of degree 10 as a branch locus, are called numerical Godeaux surfaces. We now know that any numerical Godeaux surface has $P_3 = 4$ and a birational tricanonical transformation (cf. [7]). In short, we consider Enriques's "justifications of the claim" (in his own words: "giustificazioni dell'asserto") as the first incredible intuition of the result, even though they were not correct.

Concerning surfaces, it has to be said that m-canonical surfaces in $\mathbb{P}^3$ do not exist for $m > 3$. This follows from the m-genus of a minimal model surface of general type: $P_m = \frac{m(m-1)}{2}(K^2) + 1 - q + p_g$, where $q \leq p_g$ (cf. [1, 2]). We do not know whether m-canonical hypersurfaces in $\mathbb{P}^d$, $d \geq 4$, exist for $m > 3$.

We note explicitly that the canonical (or bicanonical, or tricanonical) transformations that Enriques considers can be generically $n : 1$; if they are birational, i.e. generically $1 : 1$, and Enriques uses the adjective "simple" [Italian: semplice] to describe this situation, calling each of the above surfaces "simple canonical (or simple bicanonical, or simple tricanonical)" surfaces. In the present paper we omit this adjective, however.

We do not know of any bicanonical (or tricanonical) surfaces in $\mathbb{P}^3$ *that are normal*, i.e. nonsingular in codimension 1. This naturally prompts us to seek any bicanonical, or tricanonical hypersurfaces that are normal in $\mathbb{P}^d$ for $d \geq 4$, as we did in the case of normal canonical hypersurfaces in $\mathbb{P}^d$, $d \geq 4$ [12]. A good tool for this investigation is the theory of pluricanonical adjoints to normal hypersurfaces, which allows us to compute the pluricanonical transformations of their desingularizations without further ado. Said theory is revisited and developed in [11], based on the assumption of normality for the hypersurfaces in $\mathbb{P}^d$, and also assuming that the singularities are locally given by straight lines and planes.

As a first approach to the problem, we consider threefolds, i.e. $d = 4$, because we know of examples that attract our attention as the most natural examples of bicanonical and tricanonical threefolds in $\mathbb{P}^4$, even if they were constructed in a different context. In addition, there is Godeaux's example of a bicanonical threefold and we find the same equation of that threefold. It is worth adding that Godeaux's paper [5] is quite difficult to find: it never seems to be quoted, and we initially only chanced upon a review of it in Zentralblatt.

The equations of these threefolds $V \subset \mathbb{P}^4$ are given by a large number of monomials and we have to write them all because, for our purposes, we need to kill five coefficients (Cf. Remark 4.1, Section 4 and Remark 6.1, Section 6). The equations of the linear systems given by bicanonical and tricanonical adjoints to V are very straightforward, however. Since the rational transformations associated with these linear systems can be identified with the bicanonical and tricanonical transformations $\varphi_{|2K_X|}$ and $\varphi_{|3K_X|}$ of a desingularization $X \to V$, we have very simple equations of the bicanonical and tricanonical images of X and we can easily check whether they are normal or not. The search for the normal images is the main purpose of the present paper.

In the first part of the paper we present bicanonical and tricanonical threefolds in $\mathbb{P}^4$. One of the tricanonical threefolds is normal. All these facts are deduced from previous papers, one by the present author [11], and one by M.C. Ronconi [9].

In the second part of the paper (Section 7), we construct a normal bicanonical threefold $V \subset \mathbb{P}^4$ of degree 8, starting right from the beginning.

The irregularities $q_i(X) = \dim_{\mathbf{k}} H^i(X, \mathcal{O}_X)$, for $i = 1, 2$, of a desingularization $X \to V$ are also taken into consideration, and we show that X is *totally regular*, i.e. $q_i(X) = 0$ for $i = 1, 2$.

These threefolds have no analogous surfaces.

We tried without success to generalize the constructions of the above-mentioned threefolds in a higher dimension. This is probably due to the many ad hoc properties of threefolds that other varieties of different dimensions do not have. The fact that we find the same equation as Godeaux's threefold also confirms these ad hoc properties.

The varieties that we present are defined over the ground field $\mathbf{k}$, which is an algebraically closed field of characteristic zero, that we can assume to be the field of complex numbers.

2 m-canonical hypersurfaces in $\mathbb{P}^d$

Here, we report the definition of an m-canonical hypersurface V in the projective space $\mathbb{P}^d$, according to the definition used nowadays.

A degree $n \geq d+2$ algebraic hypersurface $V \subset \mathbb{P}^d$, $d \geq 2$, is called *m-canonical* if the linear system of the m-canonical adjoints in $\mathbb{P}^d$ to V (cf., for example, [11]) is given by degree $m(n-(d+1))$ hypersurfaces of the type $\overline{\Phi} + H$, where $\overline{\Phi}$ is a fixed hypersurface of degree $m(n-(d+1))-1$ in $\mathbb{P}^d$ and H is the complete linear

system of the hyperplanes in $\mathbb{P}^d$.
1-canonical = canonical; 2-canonical = bicanonical; 3-canonical = tricanonical.

The above definition can be reformulated as follows (loc. cit.). Let $\sigma : X \to V$ be a sequence of blow-ups resolving the singularities of $V \subset \mathbb{P}^d$. The hypersurface $V \subset \mathbb{P}^d$ is called *m-canonical* if the (complete) m-canonical system $|mK_X|$ on X is given by $|mK_X| = |M| + F$, where $|M|$ is the moving part of $|mK_X|$, F the fixed part of $|mK_X|$, and the moving part $|M|$ of $|mK_X|$ is cut out by the pull-back, with respect to σ, of the linear system of the hyperplane sections on V.

Based on this definition, from the normality of the hypersurfaces, the m-genus of X is consequently $P_m(X) = P_m = d + 1$ and the m-canonical transformation $\varphi_{|mK_X|} : X \dashrightarrow \mathbb{P}^{P_m-1} = \mathbb{P}^d$ is birational (to its image). In particular, X is a $(d-1)$-dimensional variety of general type.

In the case where V is a (hyper)surface in $\mathbb{P}^3$, cf. [3, Ch. VIII] and also [4, Ch. V].

One way to construct m-canonical hypersurfaces in $\mathbb{P}^d$ indirectly is to consider a nonsingular variety Y of dimension $d-1$ such that the m-genus of Y is $P_m = d+1$ and the canonical transformation $\varphi_{|mK_Y|} : Y \dashrightarrow \mathbb{P}^{P_m-1} = \mathbb{P}^d$ is birational (to its image). The image $\varphi_{|mK_Y|}(Y)$ of Y under $\varphi_{|mK_Y|}$ is thus an m-canonical hypersurface in $\mathbb{P}^d$.

The normality and non-normality of m-canonical hypersurfaces in $\mathbb{P}^d$ are taken into special consideration, bearing in mind that the normality is not a birational invariant. (Cf. [11] for historical notes on the normality of m-canonical surfaces.)

3 A construction of a bicanonical threefold in $\mathbb{P}^4$, which coincides with Godeaux threefold

In [11], we constructed the following threefold V_1 in $\mathbb{P}^4$, of homogeneous coordinates X_0, X_1, X_2, X_3, X_4. The equation of V_1 is given by

$$
\begin{aligned}
V_1 :\ & X_0^3(a_{33000}X_1^3 + a_{32100}X_1^2X_2 + a_{31200}X_1X_2^2 + a_{30300}X_2^3)+\\
& X_1^3(a_{23010}X_0^2X_3 + a_{13020}X_0X_3^2 + a_{03030}X_3^3)+\\
& X_2^3(a_{20301}X_0^2X_4 + a_{10302}X_0X_4^2 + a_{00303}X_4^3)+\\
& X_3^3(a_{02031}X_1^2X_4 + a_{01032}X_1X_4^2 + a_{00033}X_4^3)+\\
& X_4^3(a_{00213}X_2^2X_3 + a_{00123}X_2X_3^2)+\\
& a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + a_{22101}X_0^2X_1^2X_2X_4+\\
& a_{22020}X_0^2X_1^2X_3^2 + a_{22011}X_0^2X_1^2X_3X_4 + a_{21210}X_0^2X_1X_2^2X_3+\\
& a_{21201}X_0^2X_1X_2^2X_4 + a_{21111}X_0^2X_1X_2X_3X_4 + a_{20211}X_0^2X_2^2X_3X_4+\\
& a_{20202}X_0^2X_2^2X_4^2 + a_{12120}X_0X_1^2X_2X_3^2 + a_{12111}X_0X_1^2X_2X_3X_4+\\
& a_{12021}X_0X_1^2X_3^2X_4 + a_{11211}X_0X_1X_2^2X_3X_4 + a_{11202}X_0X_1X_2^2X_4^2+\\
& a_{11121}X_0X_1X_2X_3^2X_4 + a_{11112}X_0X_1X_2X_3X_4^2 + a_{11022}X_0X_1X_3^2X_4^2+\\
& a_{10212}X_0X_2^2X_3X_4^2 + a_{10122}X_0X_2X_3^2X_4^2 + a_{02121}X_1^2X_2X_3^2X_4+
\end{aligned}
$$

$a_{02022}X_1^2X_3^2X_4^2 + a_{01212}X_1X_2^2X_3X_4^2 + a_{01122}X_1X_2X_3^2X_4^2 +$
$a_{00222}X_2^2X_3^2X_4^2 = 0, \quad a_{ijklm} \in \mathbf{k}.$

Our threefold V_1 is the *generic* element of the above linear system of threefolds. The singularities of V_1 are given by five triple points having an infinitely near triple curve, locally isomorphic to a straight line, and other negligible singularities, i.e. singularities that do not affect the birational invariants. In particular, V_1 is normal.

If $\varphi : X \to V_1$ denotes a resolution of the singularities of V_1, then the birational invariants of X are given by $q_1 = q_2 = 0$, i.e. the two irregularities of X vanish; the first three plurigenera are given by: $p_g = 0$ and $P_2 = P_3 = 5$.

More precisely, the linear system of bicanonical adjoints to V_1 is given by:

$$\Phi_2 : a_{11000}X_0X_1 + a_{10100}X_0X_2 + a_{01010}X_1X_3 + a_{00101}X_2X_4 + a_{00011}X_3X_4 = 0,$$

This linear system defines the rational transformation $\tau_1 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ with its inverse.

$$\tau_1 : \begin{cases} Y_0 = \rho X_0X_1 \\ Y_1 = \rho X_0X_2 \\ Y_2 = \rho X_1X_3 \\ Y_3 = \rho X_2X_4 \\ Y_4 = \rho X_3X_4 \end{cases} , \rho \in \mathbf{k}, \rho \neq 0; \ \tau_1^{-1} : \begin{cases} X_0 = \rho^* Y_0Y_1Y_4 \\ X_1 = \rho^* Y_0Y_2Y_3 \\ X_2 = \rho^* Y_1Y_2Y_3 \\ X_3 = \rho^* Y_1Y_2Y_4 \\ X_4 = \rho^* Y_0Y_3Y_4 \end{cases} , \rho^* \in \mathbf{k}, \rho^* \neq 0.$$

In particular, we make the restriction $\tau_{1|V_1}$ of τ_1 to V_1 birational. There is a Zariski's open set $U \subset X$ and a Zariski's open set $U_1 \subset V_1$, that are isomorphic. By identifying U and U_1, we enable $\tau_{1|V_1}$ and the bicanonical transformation $\varphi_{|2K_X|} : X \dashrightarrow \mathbb{P}^4$ to be identified (as rational transformations). These results essentially follow from the commutativity of the following triangle

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{|2K_X|}"] \arrow[dr, "\varphi"'] & \mathbb{P}^4 = \mathbb{P}^{P_2-1} \\ & V_1 \arrow[u, dashed, "\tau_{1|V_1}"'] \end{tikzcd}$$

All the above facts are contained in [11], to which the interested reader may refer.

Using the terminology introduced in the present paper, identifying $\varphi_{|2K_X|}$ with $\tau_{1|V_1}$, we can say that the image of V_1 under τ_1 is a **bicanonical threefold in $\mathbb{P}^4$**. τ_1 is birational on $\mathbb{P}^4$, so it is easy to find the equation of $\tau_1(V_1)$ by substituting $X_0 = \rho^* Y_0Y_1Y_4$; $X_1 = \rho^* Y_0Y_2Y_3$; $X_2 = \rho^* Y_1Y_2Y_3$; $X_3 = \rho^* Y_1Y_2Y_4$; $X_4 = \rho^* Y_0Y_3Y_4$ in the equation of V_1.

We obtain

$\tau_1(V_1)$:
$$\begin{aligned}
&a_{33000}Y_0^4Y_1Y_2Y_3Y_4 + a_{32100}Y_0^3Y_1^2Y_2Y_3Y_4 + a_{31200}Y_0^2Y_1^3Y_2Y_3Y_4 + a_{30300}Y_0Y_1^4Y_2Y_3Y_4 +\\
&a_{23010}Y_0^3Y_1Y_2^2Y_3Y_4 + a_{13020}Y_0^2Y_1Y_2^3Y_3Y_4 + a_{03030}Y_0Y_1Y_2^4Y_3Y_4 + a_{20301}Y_0Y_1^3Y_2Y_3^2Y_4 +\\
&a_{10302}Y_0Y_1^2Y_2Y_3^3Y_4 + a_{00303}Y_0Y_1Y_2Y_3^4Y_4 + a_{02031}Y_0Y_1Y_2^3Y_3Y_4^2 + a_{01032}Y_0Y_1Y_2^2Y_3Y_4^3 +\\
&a_{00033}Y_0Y_1Y_2Y_3Y_4^4 + a_{00213}Y_0Y_1Y_2Y_3^3Y_4^2 + a_{00123}Y_0Y_1Y_2Y_3^2Y_4^3 + a_{22200}Y_0^2Y_1^2Y_2^2Y_3^2 +\\
&a_{22110}Y_0^2Y_1^2Y_2^2Y_3Y_4 + a_{22101}Y_0^3Y_1Y_2Y_3^2Y_4 + a_{22020}Y_0^2Y_1^2Y_2^2Y_4^2 + a_{22011}Y_0^3Y_1Y_2Y_3Y_4^2 +\\
&a_{21210}Y_0Y_1^3Y_2^2Y_3Y_4 + a_{21201}Y_0^2Y_1^2Y_2Y_3^2Y_4 + a_{21111}Y_0^2Y_1^2Y_2Y_3Y_4^2 + a_{20211}Y_0Y_1^3Y_2Y_3Y_4^2 +\\
&a_{20202}Y_0^2Y_1^2Y_3^2Y_4^2 + a_{12120}Y_0Y_1^2Y_2^3Y_3Y_4 + a_{12111}Y_0^2Y_1Y_2^2Y_3^2Y_4 + a_{12021}Y_0^2Y_1Y_2^2Y_3Y_4^2 +\\
&a_{11211}Y_0Y_1^2Y_2^2Y_3^2Y_4 + a_{11202}Y_0^2Y_1Y_2Y_3^3Y_4 + a_{11121}Y_0Y_1^2Y_2^2Y_3Y_4^2 + a_{11112}Y_0^2Y_1Y_2Y_3^2Y_4^2 +\\
&a_{11022}Y_0^2Y_1Y_2Y_3Y_4^3 + a_{10212}Y_0Y_1^2Y_2Y_3^2Y_4^2 + a_{10122}Y_0Y_1^2Y_2Y_3Y_4^3 + a_{02121}Y_0Y_1Y_2^3Y_3^2Y_4 +\\
&a_{02022}Y_0^2Y_2^2Y_3^2Y_4^2 + a_{01212}Y_0Y_1Y_2^2Y_3^3Y_4 + a_{01122}Y_0Y_1Y_2^2Y_3^2Y_4^2 + a_{00222}Y_1^2Y_2^2Y_3^2Y_4^2 = 0.
\end{aligned}$$

The threefold $\tau_1(V_1)$ has degree 8 and its desingularization has the same birational invariants $q_1 = q_2 = p_g = 0$, $P_2 = P_3 = 5$ as a desingularization of V_1. We note that $\tau_1(V_1)$ is not normal because it has the coordinate planes $\begin{cases} Y_i = 0 \\ Y_j = 0 \end{cases}$ of the fundamental pentahedron as singular planes of multiplicity 2. It also has the coordinate edges $\begin{cases} Y_i = 0 \\ Y_j = 0 \\ Y_k = 0 \end{cases}$ of the fundamental pentahedron as singular lines of multiplicity 3, and it has the vertices of the fundamental pentahedron as singular points of multiplicity 4.

Remark 3.1. The equation given by Godeaux can be written in another way

$$\begin{gathered} f_2(x_1x_2x_3x_4, x_2x_3x_4x_0, x_3x_4x_0x_1, x_4x_0x_1x_2, x_0x_1x_2x_3)+ \\ +x_0x_1x_2x_3x_4\varphi_3(x_0, x_1, x_2, x_3, x_4) = 0, \end{gathered}$$

where f_2 and φ_3 are algebraic forms in five variables.

Although this goes beyond the scope of our bicanonical and tricanonical threefolds, it is worth noting that, as well as being a desingularization of $\tau_1(V_1)$, a desingularization of Godeauxs threefold has $q_1 = q_2 = p_g = 0$, $P_2 = P_3 = 5$. So Godeaux was also the first to find a regular nonsingular threefold with $p_g = 0$ and $P_2 \neq 0$; cf. Introduction in [9], which contains a brief bibliography on this subject.

Since τ_1 is defined by the linear system of bicanonical adjoints to V_1, we know that the linear system of the bicanonical adjoints to $\tau_1(V_1)$ is given by a fixed part, multiplied by a moving part given by $a_{11000}Y_0 + a_{10100}Y_1 + a_{01010}Y_2 + a_{00101}Y_3 + a_{00011}Y_4 = 0$.

More precisely, the fixed part is given by $Y_0Y_1Y_2Y_3Y_4 = 0$, and the linear system of bicanonical adjoints to $\tau_1(V_1)$ is given by:

$$Y_0Y_1Y_2Y_3Y_4(a_{11000}Y_0 + a_{10100}Y_1 + a_{01010}Y_2 + a_{00101}Y_3 + a_{00011}Y_4) = 0.$$

In fact, they have degree 6 and $Y_0Y_1Y_2Y_3Y_4 = 0$ passes through the double singular coordinate planes on $\tau_1(V_1)$ with the correct multiplicity ≥ 2. These bicanonical adjoints also pass through the coordinate edges and through the vertices of the fundamental pentahedron with the correct multiplicities.

All the above facts concerning $\tau_1(V_1)$ follow from [11] and from the normality of V_1, even though $\tau_1(V_1)$ is not normal.

Note that, if F is any hypersurface in $\mathbb{P}^4$ and we do not remove any fixed components potentially appearing in $\tau_1(F)$, then we define a new rational transformation, that we denote $\tau_1^* : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$. If we call the image $\tau_1^*(F)$ a total transform, then the total transform $\tau_1^*(\Phi_2)$ is just the linear system of bicanonical adjoints to $\tau_1(V_1)$, i.e. $\tau_1^*(\Phi_2) : Y_0Y_1Y_2Y_3Y_4(a_{11000}Y_0 + a_{10100}Y_1 + a_{01010}Y_2 + a_{00101}Y_3 + a_{00011}Y_4) = 0$.

Moreover, the total transform $\tau_1^*(V_1)$ is given by $Y_0^2Y_1^2Y_2^2Y_3^2Y_4^2\tau_1(V_1)$.

$\tau_1(V_1)$ can also be found in [9], § 9, which states that, with a birational transformation, our V_1 is obtained from $\tau_1(V_1)$. If we replace X_0 with X_5 and X_1 with X_3, then the birational transformation given in [9] is τ_1^{-1}.

4 A second construction of the above bicanonical threefold in $\mathbb{P}^4$

In [9, § 3], Ronconi presented a threefold in $\mathbb{P}^4$ denoted by V', that we call V_2 here, the desingularization of which has the birational invariants $q_1 = q_2 = p_g = 0$ and $P_2 = P_3 = 5$: it is a degree seven hypersurface in $\mathbb{P}^4$ having triple lines at each of the edges of the fundamental pentahedron.

The linear system of all the hypersurfaces of this type depends on 35 parameters and the equation of V_2 is

$$
\begin{aligned}
V_2 : {} & a_{31111}X_0^3X_1X_2X_3X_4 + a_{13111}X_0X_1^3X_2X_3X_4 + a_{11311}X_0X_1X_2^3X_3X_4 + \\
& a_{11131}X_0X_1X_2X_3^3X_4 + a_{11113}X_0X_1X_2X_3X_4^3 + a_{22210}X_0^2X_1^2X_2^2X_3 + a_{22201}X_0^2X_1^2X_2^2X_4 + \\
& a_{22120}X_0^2X_1^2X_2X_3^2 + a_{22111}X_0^2X_1^2X_2X_3X_4 + a_{22102}X_0^2X_1^2X_2X_4^2 + a_{22021}X_0^2X_1^2X_3^2X_4 + \\
& a_{22012}X_0^2X_1^2X_3X_4^2 + a_{21220}X_0^2X_1X_2^2X_3^2 + a_{21211}X_0^2X_1X_2^2X_3X_4 + a_{21202}X_0^2X_1X_2^2X_4^2 + \\
& a_{21121}X_0^2X_1X_2X_3^2X_4 + a_{21112}X_0^2X_1X_2X_3X_4^2 + a_{21022}X_0^2X_1X_3^2X_4^2 + a_{20221}X_0^2X_2^2X_3^2X_4 + \\
& a_{20212}X_0^2X_2^2X_3X_4^2 + a_{20122}X_0^2X_2X_3^2X_4^2 + a_{12220}X_0X_1^2X_2^2X_3^2 + a_{12211}X_0X_1^2X_2^2X_3X_4 + \\
& a_{12202}X_0X_1^2X_2^2X_4^2 + a_{12121}X_0X_1^2X_2X_3^2X_4 + a_{12112}X_0X_1^2X_2X_3X_4^2 + a_{12022}X_0X_1^2X_3^2X_4^2 + \\
& a_{11221}X_0X_1X_2^2X_3^2X_4 + a_{11212}X_0X_1X_2^2X_3X_4^2 + a_{11122}X_0X_1X_2X_3^2X_4^2 + \\
& a_{10222}X_0X_2^2X_3^2X_4^2 + a_{02221}X_1^2X_2^2X_3^2X_4 + a_{02212}X_1^2X_2^2X_3X_4^2 + a_{02122}X_1^2X_2X_3^2X_4^2 + \\
& a_{01222}X_1X_2^2X_3^2X_4^2 = 0, \quad a_{ijklm} \in \mathbf{k}.
\end{aligned}
$$

To tell the truth, Ronconi wrote the equation with only 25 parameters, eliminating 10 monomials that were not essential for her purposes.

As shown in [9] the actual singularities on V_2 are only given by the coordinate edges $\begin{cases} X_i = 0 \\ X_j = 0 \\ X_k = 0 \end{cases}$ of the fundamental pentahedron, so V_2 is normal; there are other singularities infinitely near them on V_2, but they are negligible.

The linear system of bicanonical adjoints to V_2 is given by (loc. cit.)

$$b_0X_1X_2X_3X_4 + b_1X_0X_2X_3X_4 + b_2X_0X_1X_3X_4 + b_3X_0X_1X_2X_4 + b_4X_0X_1X_2X_3 = 0, b_i \in \mathbf{k}.$$

The rational transformation associated with the linear system of bicanonical adjoints can be identified with the bicanonical transformation $\varphi_{|2K_X|} : X \dashrightarrow \mathbb{P}^4$, where X is a desingularization of V_2. At the same time, this linear system defines the standard birational transformation $\sigma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$, that can also be expressed by

$$Y_i = \frac{\rho}{X_i}, \quad i = 0, 1, 2, 3, 4, \ \rho \in \mathbf{k}.$$

So, $\varphi_{|2K_X|}$ is birational. As before, $\varphi_{|2K_X|}(X)$ can be identified with $\sigma(V_2)$. Therefore, $\sigma(V_2)$ is a **bicanonical threefold in $\mathbb{P}^4$**.

Remark 4.1. The surprising fact is that **the linear system defining $\sigma(\mathbf{V_2})$ is contained in the linear system defining $\tau_1(\mathbf{V_1})$**. More precisely, if we kill the 5 parameters $a_{33000}, a_{30300}, a_{03030}, a_{00303}, a_{00033}$ in the equation of V_1, then we obtain a threefold V_1' and $\tau_1(V_1') = \sigma(V_2)$; in other words, **the bicanonical threefold in $\mathbb{P}^4$ defined by $\mathbf{V_1'}$ coincides with the bicanonical threefold in $\mathbb{P}^4$ defined by $\mathbf{V_2}$** (i.e. they have the same equation). In particular, V_1' and V_2 are birational to each other; and the birational transformation is $\sigma^{-1} \circ \tau_1$.

5 A first construction of a tricanonical threefold in $\mathbb{P}^4$

Let us return to the threefolds V_1 in Section 3, where we said that a desingularization of V_1 has $P_2 = P_3 = 5$. Here we consider the trigenus P_3. The linear system of tricanonical adjoints to V_1 is given by (cf. [11])

$$b_{11100}X_0X_1X_2 + b_{11010}X_0X_1X_3 + b_{10101}X_0X_2X_4 + b_{01011}X_1X_3X_4 + b_{00111}X_2X_3X_4 = 0.$$

This linear system defines the rational transformation $\tau_2 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ with its inverse.

$$\tau_2 : \begin{cases} Y_0 = \rho X_2X_3X_4 \\ Y_1 = \rho X_1X_3X_4 \\ Y_2 = \rho X_0X_2X_4 \\ Y_3 = \rho X_0X_1X_3 \\ Y_4 = \rho X_0X_1X_2 \end{cases}, \rho \in \mathbf{k}, \rho \neq 0; \ \tau_2^{-1} : \begin{cases} X_0 = \rho^* Y_2Y_3 \\ X_1 = \rho^* Y_1Y_4 \\ X_2 = \rho^* Y_0Y_4 \\ X_3 = \rho^* Y_0Y_3 \\ X_4 = \rho^* Y_1Y_2 \end{cases}, \rho^* \in \mathbf{k}, \rho^* \neq 0.$$

With the same arguments as we used in Section 3, we can identify $\varphi_{|3K_X|}$ with $\tau_{2|_{V_1}}$. We can say that the image of V_1 under τ_2 is a **tricanonical threefold in** $\mathbb{P}^4$ and, as in Section 3, we obtain

$$
\begin{aligned}
&\tau_2(V_1):\\
&a_{33000}Y_1^3Y_2^3Y_3^3Y_4^3 + a_{32100}Y_0Y_1^2Y_2^3Y_3^3Y_4^3 + a_{31200}Y_0^2Y_1Y_2^3Y_3^3Y_4^3 + a_{30300}Y_0^3Y_2^3Y_3^3Y_4^3 +\\
&a_{23010}Y_0Y_1^3Y_2^2Y_3^3Y_4^3 + a_{13020}Y_0^2Y_1^3Y_2Y_3^3Y_4^3 + a_{03030}Y_0^3Y_1^3Y_3^3Y_4^3 + a_{20301}Y_0^3Y_1Y_2^3Y_3^2Y_4^3 +\\
&a_{10302}Y_0^3Y_1^2Y_2^3Y_3Y_4^3 + a_{00303}Y_0^3Y_1^3Y_2^3Y_4^3 + a_{02031}Y_0^3Y_1^3Y_2Y_3^3Y_4^2 + a_{01032}Y_0^3Y_1^3Y_2^2Y_3^3Y_4 +\\
&a_{00033}Y_0^3Y_1^3Y_2^3Y_3^3 + a_{00213}Y_0^3Y_1^3Y_2^3Y_3Y_4^2 + a_{00123}Y_0^3Y_1^3Y_2^3Y_3^2Y_4 + a_{22200}Y_0^2Y_1^2Y_2^2Y_3^2Y_4^4 +\\
&a_{22110}Y_0^2Y_1^2Y_2^2Y_3^3Y_4^3 + a_{22101}Y_0Y_1^3Y_2^3Y_3^2Y_4^3 + a_{22020}Y_0^2Y_1^2Y_2^2Y_3^4Y_4^2 + a_{22011}Y_0Y_1^3Y_2^3Y_3^3Y_4^2 +\\
&a_{21210}Y_0^3Y_1Y_2^2Y_3^3Y_4^3 + a_{21201}Y_0^2Y_1^2Y_2^3Y_3^2Y_4^3 + a_{21111}Y_0^2Y_1^2Y_2^3Y_3^3Y_4^2 + a_{20211}Y_0^3Y_1Y_2^3Y_3^3Y_4^2 +\\
&a_{20202}Y_0^2Y_1^2Y_2^4Y_3^2Y_4^2 + a_{12120}Y_0^3Y_1^2Y_2Y_3^3Y_4^3 + a_{12111}Y_0^2Y_1^3Y_2^2Y_3^2Y_4^3 + a_{12021}Y_0^2Y_1^3Y_2^2Y_3^3Y_4^2 +\\
&a_{11211}Y_0^3Y_1^2Y_2^2Y_3^2Y_4^3 + a_{11202}Y_0^2Y_1^3Y_2^3Y_3Y_4^3 + a_{11121}Y_0^3Y_1^2Y_2^2Y_3^3Y_4^2 + a_{11112}Y_0^2Y_1^3Y_2^3Y_3^2Y_4^2 +\\
&a_{11022}Y_0^2Y_1^3Y_2^3Y_3^3Y_4 + a_{10212}Y_0^3Y_1^2Y_2^3Y_3^2Y_4^2 + a_{10122}Y_0^3Y_1^2Y_2^3Y_3^3Y_4 + a_{02121}Y_0^3Y_1^3Y_2Y_3^2Y_4^3 +\\
&a_{02022}Y_0^2Y_1^4Y_2^2Y_3^2Y_4^2 + a_{01212}Y_0^3Y_1^3Y_2^2Y_3Y_4^3 + a_{01122}Y_0^3Y_1^3Y_2^2Y_3^2Y_4^2 + a_{00222}Y_0^4Y_1^2Y_2^2Y_3^2Y_4^2 = 0.
\end{aligned}
$$

The threefold $\tau_2(V_1)$ has degree 12. It is not normal, because it has the coordinate planes $\begin{cases} Y_i = 0 \\ Y_j = 0 \end{cases}$ of the fundamental pentahedron as singular planes of multiplicity 3. $\tau_2(V_1)$ has the coordinate edges $\begin{cases} Y_i = 0 \\ Y_j = 0 \\ Y_k = 0 \end{cases}$ of the fundamental pentahedron as singular lines of multiplicity 6 and it has the vertices of the fundamental pentahedron as singular points of multiplicity 8.

Godeaux also considers a degree 12 equation

$$
\begin{gathered}
\varphi_3(x_1x_2x_3x_4, x_2x_3x_4x_0, x_3x_4x_0x_1, x_4x_0x_1x_2, x_0x_1x_2x_3)+\\
+(x_0x_1x_2x_3x_4)^2 f_2(x_0, x_1, x_2, x_3, x_4) = 0,
\end{gathered}
$$

but he says nothing about tricanonical threefolds; he uses this equation to obtain a degree 18 canonical surface.

Using a procedure similar to the one described in Section 3, and taking the singularities given by the edges into consideration first, the linear system of tricanonical adjoints to $\tau_2(V_1)$ is given by:

$$Y_0^4Y_1^4Y_2^4Y_3^4Y_4^4(b_{00111}Y_0 + b_{01011}Y_1 + b_{10101}Y_2 + b_{11010}Y_3 + b_{11100}Y_4) = 0.$$

In fact, they have degree 21 and $Y_0^4Y_1^4Y_2^4Y_3^4Y_4^4 = 0$ passes through the coordinate edges on $\tau_2(V_1)$ with the correct multiplicity ≥ 12 (cf. [11]). Note, however, that the tricanonical adjoints pass through the coordinate planes $\begin{cases} Y_i = 0 \\ Y_j = 0 \end{cases}$ with the correct multiplicity. More precisely, they pass through the coordinate planes with multiplicity ≥ 8, when the multiplicity required on the coordinate planes of the degree 21 hypersurfaces for them to be tricanonical adjoints to $\tau_2(V_1)$ is ≥ 6 (loc. cit.). These tricanonical adjoints pass through the vertices of the fundamental pentahedron with the correct multiplicity as well.

Remark 5.1. Starting from V_1 in $\mathbb{P}^4$, a desingularization of which has $P_2 = P_3 = 5$, we constructed two threefolds $\tau_1(V_1)$ and $\tau_2(V_1)$ in $\mathbb{P}^4$, again having a desingularization with $P_2 = P_3 = 5$. We can thus consider the tricanonical threefold in $\mathbb{P}^4$ defined by $\tau_1(V_1)$ and the bicanonical threefold in $\mathbb{P}^4$ defined by $\tau_2(V_1)$. We can see that this approach does not give us new bicanonical or tricanonical threefolds in $\mathbb{P}^4$: the linear system of the tricanonical adjoints to $\tau_1(V_1)$ defines the standard birational transformation $\sigma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$. It is easy to check that $\sigma \circ \tau_1 = \tau_2$. The tricanonical threefold in $\mathbb{P}^4$ defined by $\tau_1(V_1)$ therefore coincides with $\tau_2(V_1)$. Similarly, the linear system of the bicanonical adjoints to $\tau_2(V_1)$ defines the standard birational transformation σ. Since $\sigma = \sigma^{-1}$, the bicanonical threefold in $\mathbb{P}^4$ defined by $\tau_2(V_1)$ coincides with $\tau_1(V_1)$.

6 A normal tricanonical threefold in $\mathbb{P}^4$

Remark 6.1. The equation of $\tau_2(V_1)$ (Section 5) has degree 12, but if we consider V_1' instead of V_1, as in Remark 4.1, Section 4, i.e. if we kill a_{33000}, a_{30300}, a_{03030}, a_{00303}, a_{00033}, then we can divide the equation of $\tau_2(V_1')$ by $Y_0Y_1Y_2Y_3Y_4$ and obtain a degree 7 equation that is the equation of V_2, which is Ronconi's threefold. In fact, the following equalities hold: $\sigma \circ \tau_1 = \tau_2$ (cf. Remark 5.1) and $\sigma^{-1} = \sigma$.

According to Ronconi [9], who wrote the 5 forms defining the vector space of the tricanonical adjoints, the linear system of tricanonical adjoints to V_2 is given by

$$X_0X_1X_2X_3X_4(c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4) = 0, \quad c_i \in \mathbf{k}.$$

This proves, without any calculations, that $\mathbf{V_2}$ is itself a tricanonical threefold in $\mathbb{P}^4$ and that it is also normal (Section 4).

7 A normal bicanonical threefold in $\mathbb{P}^4$

Unlike the previous examples obtained starting from other works, here we construct a normal bicanonical threefolds starting from the beginning.

7.1 The construction of the threefold

We construct the threefold by imposing the triple line $r_1 : X_2 = X_3 = X_4 = 0$ on a generic degree 8 hypersurface

$$F_8 : \textstyle\sum_{0 \le i_0 i_1 i_2 i_3 i_4 \le 4} a_{i_0i_1i_2i_3i_4} X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3} X_4^{i_4} = 0, \; i_0 + i_1 + i_2 + i_3 + i_4 = 8,$$

in $\mathbb{P}^4$ of homogeneous coordinates $(X_0, X_1, X_2, X_3, X_4)$. To do so, we consider the affine set U_0 of affine coordinates $x = X_1/X_0, y = X_2/X_0, z = X_3/X_0, t = X_4/X_0$ and we impose the triple line $r_1 \cap U_0 : y = z = t = 0$ on $F_8 \cap U_0$. Let $F_8(x, y, z, t) = 0$ be the equation of $F_8 \cap U_0$.

To impose the triple line $r_1 \cap U_0$, we consider the blow-up of this line. Locally, the blow-up of $r_1 \cap U_0$ is given by

$$\mathcal{B}_{y_1} : \begin{cases} x = x_1 \\ y = y_1 \\ z = y_1 z_1 \\ t = y_1 t_1 \end{cases} ; \mathcal{B}_{z_2} : \begin{cases} x = x_2 \\ y = y_2 z_2 \\ z = z_2 \\ t = z_2 t_2 \end{cases} ; \mathcal{B}_{t_3} : \begin{cases} x = x_3 \\ y = y_3 t_3 \\ z = z_3 t_3 \\ t = t_3 \end{cases} .$$

It is convenient to consider $\mathcal{B}_{t_3}$. Substituting in $F_8(x, y, z, t)$, dividing by t_3^3 and imposing that $\frac{1}{t_3^3} F_8(x_3, y_3 t_3, z_3 t_3, t_3)$ be a polynomial, we obtain conditions on the coefficients $a_{i_0 i_3 i_2 i_3 i_4}$ of F_8.

Let us call $F_{t_3}(x_3, y_3, z_3, t_3)$ the polynomial $\frac{1}{t_3^3} F_8(x_3, y_3 t_3, z_3 t_3, t_3)$, so that the hypersurface $F_{t_3}(x_3, y_3, z_3, t_3) = 0$ has the triple line $y_3 = z_3 = t_3 = 0$.

Next, we want to impose the double plane $\pi : y_3 = t_3 = 0$ infinitely near on the triple line $y_3 = z_3 = t_3 = 0$.

Locally, the blow-up of $\pi : y_3 = t_3 = 0$ is given by

$$\mathcal{B}_{y_{31}} : \begin{cases} x_3 = x_{31} \\ y_3 = y_{31} \\ z_3 = z_{31} \\ t_3 = y_{31} t_{31} \end{cases} ; \quad \mathcal{B}_{t_{32}} : \begin{cases} x_3 = x_{32} \\ y_3 = y_{32} t_{32} \\ z_3 = z_{32} \\ t_3 = t_{32} \end{cases} .$$

We consider $\mathcal{B}_{t_{32}}$ and we substitute in $F_{t_3}(x_3, y_3, z_3, t_3)$. By imposing that $\frac{1}{t_{32}^2} F_{t_3}(x_{32}, y_{32} t_{32}, z_{32}, t_{32})$ be a polynomial, we obtain conditions on the coefficients $a_{i_1 i_2 i_3 i_4}$, and the hypersurface of equation $\frac{1}{t_{32}^2} F_{t_3}(x_{32}, y_{32} t_{32}, z_{32}, t_{32}) = 0$ has the double plane we wanted.

After completing these calculations, we return to the above homogeneous coordinates $(X_0, X_1, X_2, X_3, X_4)$ and impose another 4 triple lines on F_8 with the infinitely near double surfaces, using the following rotations of indices (and variables).

Rotations of indices (and variables)

$$X_0 \mapsto X_1 \mapsto X_2 \mapsto X_3 \mapsto X_4 \mapsto X_0$$

$$i_0 i_1 i_2 i_3 i_4 \mapsto i_4 i_0 i_1 i_2 i_3 \mapsto i_3 i_4 i_0 i_1 i_2 \mapsto i_2 i_3 i_4 i_0 i_1 \mapsto i_1 i_2 i_3 i_4 i_0 \mapsto i_0 i_1 i_2 i_3 i_4.$$

Above, we impose the triple line $X_2 = X_3 = X_4 = 0$ on F_8 with the double surface infinitely near, obtaining conditions on the coefficients $a_{i_0 i_1 i_2 i_3 i_4}$ of F_8. Essentially, these conditions impose which coefficients are to be killed. If the following coefficient $a_{i_0 i_1 i_2 i_3 i_4}$ must vanish, i.e. $a_{i_0 i_1 i_2 i_3 i_4} = 0$, then the following coefficients must vanish too:

$$a_{i_4 i_0 i_1 i_2 i_3} = 0, a_{i_3 i_4 i_0 i_1 i_2} = 0, a_{1_2 i_3 i_4 i_0 i_1} = 0, a_{i_1 i_2 i_3 i_4 i_0} = 0.$$

Therefore, thanks to this rotation, F_8 has the following five triple lines

$$\begin{cases} X_2 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_0 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_0 = 0 \\ X_1 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_0 = 0 \\ X_1 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases}.$$

and there is a double surface infinitely near each of them that is locally a double plane.

The final threefold has equation

$$\begin{aligned}
&F_8 : F_8(X_0, X_1, X_2, X_3, X_4) = \\
&a_{41210}X_0^4X_1X_2^2X_3 + a_{04121}X_1^4X_2X_3^2X_4 + a_{10412}X_0X_2^4X_3X_4^2 + a_{21041}X_0^2X_1X_3^4X_4+\\
&a_{12104}X_0X_1^2X_2X_4^4+\\
&a_{32210}X_0^3X_1^2X_2^2X_3 + a_{03221}X_1^3X_2^2X_3^2X_4 + a_{10322}X_0X_2^3X_3^2X_4^2 + a_{21032}X_0^2X_1X_3^3X_4^2+\\
&a_{22103}X_0^2X_1^2X_2X_4^3+\\
&a_{31211}X_0^3X_1X_2^2X_3X_4 + a_{13121}X_0X_1^3X_2X_3^2X_4 + a_{11312}X_0X_1X_2^3X_3X_4^2+\\
&a_{21131}X_0^2X_1X_2X_3^3X_4 + a_{12113}X_0X_1^2X_2X_3X_4^3+\\
&a_{31121}X_0^3X_1X_2X_3^2X_4 + a_{13112}X_0X_1^3X_2X_3X_4^2 + a_{21311}X_0^2X_1X_2^3X_3X_4+\\
&a_{12131}X_0X_1^2X_2X_3^3X_4 + a_{11213}X_0X_1X_2^2X_3X_4^3+\\
&a_{22211}X_0^2X_1^2X_2^2X_3X_4 + a_{12221}X_0X_1^2X_2^2X_3^2X_4 + a_{11222}X_0X_1X_2^2X_3^2X_4^2+\\
&a_{21122}X_0^2X_1X_2X_3^2X_4^2 + a_{22112}X_0^2X_1^2X_2X_3X_4^2 + a_{22121}X_0^2X_1^2X_2X_3^2X_4+\\
&a_{12212}X_0X_1^2X_2^2X_3X_4^2 + a_{21221}X_0^2X_1X_2^2X_3^2X_4 + a_{12122}X_0X_1^2X_2X_3^2X_4^2+\\
&a_{21212}X_0^2X_1X_2^2X_3X_4^2 = 0.
\end{aligned}$$

The above F_8 is a linear system of hypersurfaces of degree 8, which is invariant with respect to the rotation of indices (and variables). From here on, we consider the generic element of this linear system, calling it simply F_8 (omitting the term "generic"). We can apply Bertini's theorem to F_8, according to which: *the singularities of F_8 belong to the base points locus of the linear system.*

7.2 Normality of F_8

From Bertini's theorem, F_8 has no singularities in codimension 1, i.e. it is normal. In fact, the unique subvarieties of codimension 1 on F_8 belonging to the base points of F_8 are given by the 10 planes $X_i = X_j = 0$ of the fundamental pentahedron. They do not vanish the partial derivatives, i.e. they are simple planes of multiplicity 1 on F_8. For example, the plane $X_0 = X_1 = 0$ is simple because it belongs to the base point locus of F_8 and we have

$$\left(\frac{\partial F_8}{\partial X_0}\right)_{X_0=X_1=0} = a_{10412}X_2^4X_3X_4^2 + a_{10322}X_2^3X_3^2X_4^2 \neq 0.$$

To be precise, the lines $X_0 = X_1 = X_i = 0$ vanish the partial derivatives because they are singular on F_8 (cf. Section 7.1).

7.3 Desingularization of F_8

As well as the imposed singularities - five triple straight lines (cf. Section 7.2), each of them having a double surface infinitely near - there are also unimposed singularities. As usual, we call a singularity S on F_8 an *actual singularity* in order to distinguish S from those infinitely near it.

The actual unimposed singularities are given by the other five straight lines of the fundamental pentahedron, which are distinct from those given in Section 7.1.

The infinitely near unimposed singularities are given by a finite number of double curves. These double curves are locally given by straight lines. Finally, there are either double points or simple points infinitely near the double curves.

We shall see all these singularities in detail during the resolution of the singularities of F_8.

We recall that double curves and double points are *negligible singularities*, i.e. singularities that do not give the hypersurfaces conditions such that make them canonical adjoints to F_8. In other words, these singularities do not give conditions of a desingularization of F_8 to the birational invariants (cf. [11, pp. 151-152]).

The equation of F_8 is invariant with a rotation of the indices (and variables) (Section 7.1), so in the desingularization of F_8 we can limit ourselves to solving the singularities on the open set $U_0 = \{X_0 \neq 0\}$, where we place the affine coordinates $x = X_1/X_0, y = X_2/X_0, z = X_3/X_0, t = X_4/X_0$ (cf. Section 7.1). The desingularization on the other sets $U_i = \{X_i \neq 0\}$ is a consequence of the rotation of the indices.

We only consider the desingularization of F_8 locally, leaving the pasting of the local parts to the general theory of the resolution of singularities (cf. e.g. [6]).

If we consider $F_8 \cap U_0$, we see that there are two imposed actual triple lines $r_1 \cap U_0 : y = z = t = 0$, $r_2 \cap U_0 : x = y = z = 0$, and two unimposed actual double lines $s_1 \cap U_0 : x = z = t = 0$, $s_2 \cap U_0 : x = y = t = 0$. We blow up these singularities and those infinitely near, starting from $r_1 \cap U_0$.

Blow-up of the triple line $r_1 \cap U_0$

We consider the local blow-ups $\mathcal{B}_{y_1}$, $\mathcal{B}_{z_2}$ and $\mathcal{B}_{t_3}$ that we wrote in Section 7.1. If $F_8(x, y, z, t) = 0$ is the equation of $F_8 \cap U_0$, then the strict (or proper) transform F_{y_1} of $F_8 \cap U_0$ with respect to $\mathcal{B}_{y_1}$ is given by

$$F_{y_1} : \frac{1}{y_1^3} F_8(x_1, y_1, y_1 z_1, y_1 t_1) = a_{41210} x_1 z_1 + a_{22103} x_1^2 y_1 t_1^3 + a_{10412} y_1^4 z_1 t_1^2 + \cdots = 0.$$

F_{y_1} has the double line $x_1 = y_1 = z_1 = 0$ on the exceptional divisor $y_1 = 0$, and outside the exceptional divisor it has the double line $x_1 = z_1 = t_1 = 0$. This last double line is the image of s_1.

The strict (or proper) transform F_{z_2} of $F_8 \cap U_0$ with respect to $\mathcal{B}_{z_2}$ is given by

$$F_{z_2} : \frac{1}{z_2^3} F_8(x_2, y_2, y_2 z_2, z_2, z_2 t_2) = a_{41210} x_2 y_2^2 + a_{21041} x_2 z_2^2 t_2 + a_{10322} y_2^3 z_2^4 t_2^2 + \cdots = 0.$$

F_{z_2} has the double plane $y_2 = z_2 = 0$ on the exceptional divisor $z_2 = 0$, and outside the exceptional divisor it has the double line $x_2 = y_2 = t_2 = 0$. This double line is the image of s_2.

The strict transform F_{t_3} of $F_8 \cap U_0$ with respect to $\mathcal{B}_{t_3}$ is given (cf. Section 7.1) by

$$F_{t_3} : \frac{1}{t_3^3} F_8(x_3, y_3 t_3, z_3 t_3, t_3) = a_{41210} x_3 y_3^2 + a_{22103} x_3^2 y_3 t_3 + a_{21041} x_3 z_3^4 t_3^2 + a_{10412} y_3^4 z_3 t_3^4 + \cdots = 0.$$

F_{t_3} has the double plane $y_3 = t_3 = 0$ on the exceptional divisor $t_3 = 0$ (cf. Section 7.1), and outside the exceptional divisor it has the triple line $x_3 = y_3 = z_3 = 0$. This triple line is the image of r_2.

Blow-up of the double plane $y_2 = z_2 = 0$ on F_{z_2}

Locally, the blow-up of $y_2 = z_2 = 0$ is given by

$$\mathcal{B}_{y_{21}} : \begin{cases} x_2 = x_{21} \\ y_2 = y_{21} \\ z_2 = y_{21} z_{21} \\ t_2 = t_{21} \end{cases} ; \quad \mathcal{B}_{z_{22}} : \begin{cases} x_2 = x_{22} \\ y_2 = y_{22} z_{22} \\ z_2 = z_{22} \\ t_2 = t_{22} \end{cases} .$$

The strict transform $F_{y_{21}}$ of $F_{z_2} : F_{z_2}(x_2, y_2, z_2, t_2) = 0$ with respect to the local blow-up $\mathcal{B}_{y_{21}}$ is given by

$$F_{y_{21}} : \frac{1}{y_{21}^2} F_{z_2}(x_{21}, y_{21}, y_{21} z_{21}, t_{21}) = a_{41210} x_{21} + \cdots = 0.$$

$F_{y_{21}}$ is nonsingular by Bertini's theorem.

The strict transform $F_{z_{22}}$ of $F_{z_2} : F_{z_2}(x_2, y_2, z_2, t_2) = 0$ with respect to the local blow-up $\mathcal{B}_{z_{22}}$ is given by

$$F_{z_{22}} : \frac{1}{z_{22}^2} F_{z_2}(x_{22}, y_{22} z_{22}, z_{22}, t_{22}) = a_{41210} x_{22} y_{22} + a_{21041} x_{22} t_{22} + \cdots = 0.$$

$F_{z_{22}}$ is nonsingular on the exceptional divisor $z_{22} = 0$ and, outside the exceptional divisor, $F_{z_{22}}$ has the double line $x_{22} = y_{22} = t_{22} = 0$. This line is the image of s_2 and one point of this line is on the exceptional divisor, but we consider $F_{z_{22}}$ nonsingular on the exceptional divisor.

Blow-up of the double plane $y_3 = t_3 = 0$ on F_{t_3}

Locally, the blow-up of $y_3 = t_3 = 0$ is given by $\mathcal{B}_{y_{31}}$ and $\mathcal{B}_{t_{32}}$ (cf. Section 7.1).

The strict transform $F_{y_{31}}$ of $F_{t_3} : F_{t_3}(x_3, y_3, z_3, t_3) = 0$ with respect to the local blow-up $\mathcal{B}_{y_{31}}$ is given by

$$F_{y_{31}} : \frac{1}{y_{31}^2} F_{t_3}(x_{31}, y_{31}, z_{31}, y_{31} t_{31}) = a_{41210} x_{31} z_{31} + a_{22103} x_{31}^2 t_{31} + a_{10412} y_{31}^6 z_{31} t_{31}^4 + \cdots = 0.$$

There is the line $x_{31} = y_{31} = z_{31} = 0$ on the exceptional divisor $y_{31} = 0$ and, outside the exceptional divisor, there is the line $x_{31} = z_{31} = t_{31} = 0$, which is the image of s_1.

The strict transform $F_{t_{32}}$ of $F_{t_3} : F_{t_3}(x_3, y_3, z_3, t_3) = 0$ with respect to the local blow-up $\mathcal{B}_{t_{32}}$ is given by

$$F_{t_{32}} : \tfrac{1}{t_{32}^2} F_{z_2}(x_{32}, y_{32}t_{32}, z_{32}, t_{32}) = a_{41210}x_{32}y_{32}^2z_{32} + a_{22103}x_{32}^2y_{32} + a_{21041}x_{32}z_{32}^4 + a_{10412}y_{32}^4z_{32}t_{32}^6 + \cdots = 0.$$

There is the double line $x_{32} = z_{32} = t_{32} = 0$ on the exceptional divisor $t_{32} = 0$ and, outside the exceptional divisor, there is the triple line $x_{32} = y_{32} = z_{32} = 0$, which is the image of r_2.

Blow-up of the triple line $x_{32} = y_{32} = z_{32} = 0$ on $F_{t_{32}}$

Locally, the blow-up of $x_{32} = y_{32} = z_{32} = 0$ is given by

$$\mathcal{B}_{x_{321}} : \begin{cases} x_{32} = x_{321} \\ y_{32} = x_{321}y_{321} \\ z_{32} = x_{321}z_{321} \\ t_{32} = t_{321} \end{cases} ; \quad \mathcal{B}_{y_{322}} : \begin{cases} x_{32} = x_{322}y_{322} \\ y_{32} = y_{322} \\ z_{32} = y_{322}z_{322} \\ t_{32} = t_{322} \end{cases} ;$$

$$\mathcal{B}_{z_{323}} : \begin{cases} x_{32} = x_{323}z_{323} \\ y_{32} = y_{323}z_{323} \\ z_{32} = z_{323} \\ t_{32} = t_{323} \end{cases} .$$

The strict transform $F_{x_{321}}$ of $F_{t_{32}}$ with respect to $\mathcal{B}_{x_{321}}$ is given by $F_{x_{321}} : a_{22103}y_{321} + \cdots = 0$ and it is nonsingular.

The strict transform $F_{y_{322}}$ of $F_{t_{32}}$ is given by $F_{y_{322}} : a_{41210}x_{322}y_{322}z_{322} + a_{22103}x_{322}^2 + a_{10412}y_{322}^2z_{322}t_{322}^5 + \cdots = 0.$

$F_{y_{322}}$ has the double plane $x_{322} = y_{322} = 0$ on the exceptional divisor and the double line $x_{322} = z_{322} = t_{322} = 0$ outside.

The strict transform $F_{z_{323}}$ of $F_{t_{32}}$ is given by

$$F_{z_{323}} : a_{41210}x_{323}y_{323}^2z_{323} + a_{22103}x_{323}^2y_{323} + a_{21032}x_{323}z_{323} + a_{10412}y_{323}^4z_{323}^2t_{323}^6 + \cdots = 0.$$

$F_{z_{323}}$ has the double plane $x_{323} = z_{323} = 0$ on the exceptional divisor and, outside, $F_{z_{323}}$ is nonsingular.

Blow-up of the double plane $x_{322} = y_{322} = 0$ on $F_{y_{322}}$

Locally, the blow-up of $x_{322} = y_{322} = 0$ is given by

$$\mathcal{B}_{X_1}:\begin{cases} x_{322}=X_1\\ y_{322}=X_1Y_1\\ z_{322}=Z_1\\ t_{322}=T_1\end{cases};\qquad \mathcal{B}_{Y_1}:\begin{cases} x_{322}=X_2Y_2\\ y_{322}=Y_2\\ z_{322}=Z_2\\ t_{322}=T_2\end{cases}.$$

The strict transform F_{X_1} of $F_{y_{322}}$ with respect to the local blow-up $\mathcal{B}_{X_1}$ is given by

$F_{X_1}: a_{22103}+\cdots=0$ and it is nonsingular.

The strict transform F_{Y_2} of $F_{y_{322}}$ with respect to the local blow-up $\mathcal{B}_{Y_2}$ is given by

$F_{Y_2}: a_{41210}X_2Z_2+a_{22103}X_2^2+a_{21032}X_2Z_2^3+a_{10412}Z_2T_2^6+\cdots=0.$

F_{Y_2} is nonsingular on the exceptional divisor $Y_2=0$ and, outside the exceptional divisor F_{Y_2}, has the double line $X_2=Z_2=T_2=0$.

Blow-up of the double plane $x_{323}=z_{323}=0$ on $F_{z_{323}}$

Locally, the blow-up of $x_{323}=z_{323}=0$ is given by

$$\mathcal{B}_{X_3}:\begin{cases} x_{323}=X_3\\ y_{323}=X_3\\ z_{323}=X_3Z_3\\ t_{323}=T_3\end{cases};\quad \mathcal{B}_{Z_4}:\begin{cases} x_{323}=X_4Z_4\\ y_{323}=Y_4\\ z_{323}=Z_4\\ t_{323}=T_4\end{cases}.$$

The strict transform F_{X_3} with respect to the local blow-up $\mathcal{B}_{X_3}$ is given by

$F_{X_3}: a_{21032}Z_3+\cdots=0$ and it is nonsingular.

The strict transform F_{Z_4} with respect to the local blow-up $\mathcal{B}_{Z_4}$ is given by

$F_{t_{32}}: a_{21032}X_4+\cdots=0$ and it is nonsingular.

Now we have to blow up the double lines. They are negligible singularities. There is a finite number of double lines and after them there is an isolated double point or no other singularities.

All the blow-ups of these double lines are similar and the one that follows is a significant example.

Blow-up of the double line $x_{31}=y_{31}=z_{31}=0$ on $F_{y_{31}}$

Locally, the blow-up of $x_{31}=y_{31}=z_{31}=0$ is given by

$$\mathcal{B}_{x_{311}}:\begin{cases} x_{31}=x_{311}\\ y_{31}=x_{311}y_{311}\\ z_{31}=x_{311}z_{311}\\ t_{31}=t_{311}\end{cases};\quad \mathcal{B}_{y_{312}}:\begin{cases} x_{31}=x_{312}y_{312}\\ y_{31}=y_{312}\\ z_{31}=y_{312}z_{312}\\ t_{31}=t_{312}\end{cases};$$

$$\mathcal{B}_{z_{313}} : \begin{cases} x_{31} = x_{313} z_{313} \\ y_{31} = y_{313} z_{313} \\ z_{31} = z_{313} \\ t_{31} = t_{313} \end{cases} .$$

The strict transform $F_{x_{311}}$ with respect to $\mathcal{B}_{x_{311}}$ is given by

$F_{x_{311}} : a_{41210} x_{311} + \cdots = 0$ and it is nonsingular.

The strict transform $F_{z_{313}}$ with respect to $\mathcal{B}_{z_{313}}$ is given by

$F_{z_{313}} : a_{41210} z_{313} + \cdots = 0$ and it is nonsingular.

Now we need to consider the strict transform $F_{y_{312}}$ with respect to $\mathcal{B}_{y_{312}}$, that is given by

$F_{y_{312}} : a_{41210} x_{312} z_{312} + a_{22103} x_{312}^2 t_{312} + a_{10412} y_{312}^5 z_{312} t_{312}^4 + \cdots = 0.$

There is the double line $x_{312} = y_{312} = z_{312} = 0$ on the exceptional divisor $y_{312} = 0$ and, outside the exceptional divisor, there is the double line $x_{312} = z_{312} = t_{312} = 0$.

We note that the exponent of y_{31} was 6 on $F_{y_{31}}$ and became 5 for y_{312} on $F_{y_{312}}$, whereas the exponent of x_{31}, z_{31} and t_{31} remains fixed. From this remark, if we consider 5 blow-ups of $x_* = y_* = z_* = 0$, then the strict transforms with respect to x_* and to z_* are nonsingular and the strict transform with respect to y_* is given by
$F_{y_*} : a_{41210} x_* z_* + a_{22103} x_*^2 t_* + a_{10412} z_* t_*^4 + \cdots = 0.$

On F_{y_*} there is the double line $x_* = z_* = t_* = 0$. With one blow-up the exponent of t_* decreases by 1, and with 4 blow-ups of $x_* = z_* = t_* = 0$ the 3 strict transforms are nonsingular.

At this point, we consider the desingularization of $F_8 \cap U_0$ complete and, by rotating the indices, the desingularization of F_8 is completed too.

Canonical and bicanonical adjoints to $F_8 \longleftrightarrow p_g$ and P_2 of a desingularization $X \to F_8$ of F_8

Proposition 7.1. *The geometric genus of a desingularization of F_8 is $p_g = 0$.*

Proof. The linear system of canonical adjoints to F_8 is given by the cubic forms F_3 passing through the triple line and through the infinitely near double surfaces. The computations here are similar to those of Section 7.1 for the construction of F_8. In fact, we can write F_3 in non-homogeneous coordinates (x, y, z, t) and in detail

$F_3 : a_{30000} + a_{21000} x + a_{20100} y + a_{2001} z + a_{20001} t + a_{12000} x^2 + a_{11100} xy + a_{11010} xz + a_{11001} xt + a_{10200} y^2 + a_{10110} yz + a_{10101} yt + a_{10020} z^2 + a_{10011} zt + a_{10002} z^2 + a_{03000} x^3 + a_{02100} x^2 y + \cdots = 0.$

We make the line $r_1 \cap U_0 : y = z = t = 0$ be simple on $F_3 \cap U_0$, considering the partial blow-up (cf. Section 7.1)

$$\mathcal{B}_{t_3} : \begin{cases} x = x_3 \\ y = y_3 t_3 \\ z = z_3 t_3 \\ t = t_3 \end{cases} .$$

Substituting in the equation $F_3(x, y, z, t) = 0$, we obtain $F_3(x_3, y_3t_3, z_3t_3, t_3) = 0$. Dividing by t_3 and imposing that $\frac{1}{t_3} F_3(x_3, y_3t_3, z_3t_3, t_3)$ be a polynomial, we obtain the following conditions on the coefficients $a_{i_0i_1i_2i_3i_4}$ of F_3.

$$a_{30000} = a_{21000} = a_{12000} = a_{03000} = 0.$$

Let us call $F_{t_3}(x_3, y_3, z_3, t_3)$ the polynomial $\frac{1}{t_3} F_3(x_3, y_3t_3, z_3t_3, t_3)$, so that the hypersurface $F_{t_3}(x_3, y_3, z_3, t_3) = 0$ passes through the line $y_3 = z_3 = t_3 = 0$ with multiplicity 1.

Next, we want to impose the simple plane $\pi : y_3 = t_3 = 0$ infinitely near on the simple line $y_3 = z_3 = t_3 = 0$. We consider (cf. Section 7.1)

$$\mathcal{B}_{t_{32}} : \begin{cases} x_3 = x_{32} \\ y_3 = y_{32} t_{32} \\ z_3 = z_{32} \\ t_3 = t_{32} \end{cases}$$

and we substitute in $F_{t_3}(x_3, y_3, z_3, t_3) = 0$. Imposing that $\frac{1}{t_{32}} F_{t_3}(x_{32}, y_{32}t_{32}, z_{32}, t_{32})$ be a polynomial, we have the following conditions on the coefficients $a_{i_1i_2i_3i_4}$

$$a_{20010} = a_{20001} = a_{11010} = a_{11001} = a_{02010} = a_{02001} = 0.$$

The hypersurface $\frac{1}{t_{32}} F_{t_1}(x_{32}, y_{32}t_{32}, z_{32}, t_{32}) = 0$ has the plane of multiplicity 1 that we wanted.

Next, using the rotation of the indices (cf. Section 7.1)

$$i_0i_1i_2i_3i_4 \mapsto i_4i_0i_1i_2i_3 \mapsto i_3i_4i_0i_1i_2 \mapsto i_2i_3i_4i_0i_1 \mapsto i_1i_2i_3i_4i_0 \mapsto i_0i_1i_2i_3i_4,$$

we make all the coefficients of F_3 equate to zero, i.e. there are no canonical adjoints to F_8.

From [11], we therefore have $p_g = 0$.

We remember that we can apply the results of [11] because F_8 is normal and the singular curves and surfaces are locally given by straight lines and planes. □

Proposition 7.2. *The bigenus of a desingularization of* F_8 *is* $P_2 = 5$.

Proof. If we repeat for degree 6 forms what we did for canonical adjoints, but divide by t_3^2, and then by t_{32}^2, we obtain the linear system of bicanonical adjoints to F_8 given by

$$\Phi_6 : X_0X_1X_2X_3X_4(a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4) = 0,\ a_i \in \mathbf{k}.$$

From [11], we obtain $P_2 = 5$. □

Corollary 7.3. *F_8 is a normal bicanonical threefold in $\mathbb{P}^4$.*

Proof. If $X \to F_8$ is a desingularization of F_8, then the linear system of bicanonical adjoints defines the birational transformation $\tau_{|F_8} : F_8 \dashrightarrow \mathbb{P}^4$ that can be identify with the bicanonical transformation $\varphi_{|2K_X|} : X \dashrightarrow \mathbb{P}^4$ (cf. Section 3). This proves that F_8 is a bicanonical threefold in $\mathbb{P}^4$. From Section 7.2, we know that F_8 is normal. □

7.4 The regularity of a desingularization $X \to F_8$

There remains for us to prove that $q_i(X) = \dim_{\mathbf{k}} H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$.

Theorem 7.4. *X is totally regular, i.e. $q_i(X) = 0$ for $i = 1, 2$.*

Proof. We calculate $q_2(X) = \dim_{\mathbf{k}} H^2(X, \mathcal{O}_X)$ using the formula (36), Section 4 in [11], which states that:

$$q_2(X) = p_g(X) + p_g(S) - \dim_{\mathbf{k}}(W_4),$$

where $p_g(X)$ denotes the geometric genus of X, and $p_g(S)$ denotes the geometric genus of a desingularization S of a generic hyperplane section of F_8, where W_4 is the vector space of the degree 4 forms defining global adjoints Φ_4 to F_8, i.e. defining hypersurfaces Φ_4 of degree 4 passing through the triple lines and through the infinitely near double surfaces, with the same multiplicity as the canonical adjoints to F_8.

We note that $S \subset X$ is the strict transform, with respect to a desingularization $\sigma : X \to F_8$, of a generic hyperplane section of F_8 performed by a generic hyperplane $H \subset \mathbb{P}^d$. Since the hyperplane H is generic, the variety S can be considered nonsingular.

We remember that $q_1(X) = \dim_{\mathbf{k}} H^1(X, \mathcal{O}_X) = q_1(S) = \dim_{\mathbf{k}} H^1(S, \mathcal{O}_S)$, where S is defined above (cf. [11], page 174).

We compute $q_1(S)$ by applying the formula (36) (loc. cit.) to S:

$$q_1(S) = p_g(S) + p_g(S') - \dim_{\mathbf{k}}(W_5),$$

where W_5 is the vector space of the degree 5 forms defining global adjoints $\Phi_5 \subset H$ to $F_8 \cap H$, and where $S' \subset S$ is the nonsingular strict transform, with respect to σ of a generic hyperplane section of $F_8 \cap H$, performed by a generic hyperplane $H' \subset H$.

The singularities of $F_8 \cap H$ are given by isolated triple points that have an infinitely near double line and negligible double points. The triple points are given by the intersection of the actual triple lines on F_8 with the hyperplane H. To see that $F_8 \cap H$ has these singularities, it is convenient to assume, with a linear change of coordinates, that H is a coordinate hyperplane, e.g. $H = \{X_0 = 0\}$. $F_8 \cap H$ thus has homogeneous coordinates (X_1, X_2, X_3, X_4), and the lines on F_8 given by $X_{i_1} = X_{i_2} = X_{i_3} = 0$, $i_j > 0$ become points, while the planes $X_{i_1} = X_{i_2} = 0$, $i_j > 0$ become lines.

Lemma 7.5. *$q_1(X) = q_1(S) = 0$.*

Proof. We have to calculate $p_g(S)$, which appears in the above formula for calculating $q_1(S)$. The geometric genus $p_g(S)$ of S is given by the dimension of the vector space of the forms defining canonical adjoints to $F_8 \cap H$ in the hyperplane H. These canonical adjoints are hypersurfaces of degree 4 in H passing through the triple points and through the infinitely near double lines.

To ascertain the canonical adjoints to $F_8 \cap H$ of degree 4 in H, it is coveninient (as before) to consider a linear change of coordinates that change H into a coordinate hyperplane, e.g. $H = \{X_0 = 0\}$, so that $F_8 \cap H$ has homogeneous coordinates (X_1, X_2, X_3, X_4).

Let us consider the triple line $X_1 = X_2 = X_3 = 0$ on F_8. The intersection of this line with $X_0 = 0$ is given by the point $X_0 = X_1 = X_2 = X_3 = X_4 = 0$.

We can assume that in $X_0 = 0$ the forms of degree 4 are given by

$$\{F_4(X_0, X_1, X_2, X_3, X_4)\} \cap \{X_0 = 0\} = F_4(0, X_1, X_2, X_3, X_4),$$

where $F_4(X_0, X_1, X_2, X_3, X_4) = 0$ is a threefold in $\mathbb{P}^4$. This enables us to apply the rotation of the indices to the coefficients of $F_4(0, X_1, X_2, X_3, X_4)$, which are given by a_{0,i_1,i_2,i_3,i_4}. In detail, the equation is given by
$F_4 \cap \{X_0 = 0\} : a_{04000}X_1^4 + a_{03100}X_1^3X_2 + a_{03010}X_1^3X_3 + \cdots + a_{01003}X_1X_4^3 + a_{00103}X_2X_4^3 + a_{00013}X_3X_4^3 + a_{00004}X_4^4 = 0$.

Let us consider the affine coordinates $x = X_1/X_4, y = X_2/X_4, z = X_3/X_4$. The affine equation of $F_4 \cap \{X_0 = 0\}$ is given by
$F_4 \cap \{X_0 = 0\}$: $a_{04000}x^4 + a_{03100}x^3y + a_{03010}x^3z + \cdots + a_{01003}x + a_{00103}y + a_{00013}z + a_{00004} = 0$.

As in the case of the canonical adjoint to F_8, here we blow up the origin $x = y = z = 0$ and the infinitely near double line with

$$\mathcal{B}_{t_1} : \begin{cases} x = x_1 \\ y = x_1y_1 \\ z = x_1z_1 \end{cases}$$

and

$$\mathcal{B}_{t_{11}} : \begin{cases} x_1 = x_{11} \\ y_1 = x_{11}y_{11} \\ z_1 = z_{11} \end{cases}.$$

By imposing that $\frac{1}{x_1}(F_4 \cap \{X_0 = 0\})(x_1, x_1y_1, x_1z_1)$ be a polynomial, we obtain $a_{00004} = 0$.

By imposing that $\frac{1}{x_{11}}(F_4 \cap \{X_0 = 0\})(x_{11}, x_{11}y_{11}, z_{11})$ be a polynomial, we obtain $a_{01003} = a_{00013} = 0$. In conclusion, we kill 3 coefficients of F_4.

Applying the rotation of the indices, we want to prove that 15 coefficients of F_4 remain killed. In fact, if we consider $a_{01003} = a_{00013} = a_{00004} = 0$ and apply the rotation of the indices, it is easy to see that we obtain 15 distinct coefficients to be killed.

After killing the 15 coefficients, $35 - 15 = 20$ distint coefficients remain in F_4, i.e. $p_g(S) = 20$.

We remember that, here again, we can conditions on the forms of degree 4, either by starting to pass through the actual triple lines and the infinitely near double surfaces (cf. Section 7.1), or by following the desingularization of F_8 (cf. Section 7.3). The end result remains the same.

In the formula $q_1(X) = q_1(S) = p_g(S) + p_g(S') - \dim_{\mathbf{k}}(W_5)$, we still need to compute $\dim_{\mathbf{k}}(W_5)$ and $p_g(S')$. Using the same procedure as we used to compute $p_g(S)$, for W_5 we obtain $\dim_{\mathbf{k}}(W_5) = 56 - 15 = 41$. In other words, for a degree 5 form in W_5, we have
$F_5 \cap \{X_0 = 0\} : a_{05000}x^5 + a_{04100}x^4y + a_{04010}x^4z + \cdots + a_{01004}x + a_{00104}y + a_{00014}z + a_{00005} = 0$
and similarly we obtain $a_{01004} = a_{00014} = a_{00005} = 0$.

Finally, the intersection of F_8 with H' is a nonsingular plane curve of degree 8, threfore $p_g(S') = 21$ and

$$q_1(X) = q_1(S) = p_g(S) + p_g(S') - \dim_{\mathbf{k}}(W_5) = q_1(X) = 20 + 21 - 41 = 0.$$

□

Lemma 7.6. *$q_2(X) = 0$.*

Proof. In the proof of Lemma 7.5, we computed $p_g(S) = 20$. In Section 7.3, we computed $p_g = p_g(X) = 0$.

In the proof of Proposition 7.1, we showed that a cubic form passes through a triple line on F_8, and through the infinitely near double surface if 10 coefficients equate to zero. When we repeat the same procedure for the form F_4 of degree 4, we find that 13 coefficients equate to zero. Let us find these 13 coefficients.

The equation of the degree 4 hypersurface in affine coordinates (x, y, z, t) is

$F_4 : a_{40000} + a_{31000}x + a_{30100}y + a_{30010}z + a_{30001}t + a_{22000}x^2 + a_{21100}xy + a_{21010}xz + a_{21001}xt + a_{20200}y^2 + a_{20110}yz + a_{20101}yt + a_{20020}z^2 + a_{20011}zt + a_{20002}z^2 + a_{13000}x^3 + a_{12100}x^2y + \cdots = 0.$

Proceeding as in the proof of Proposition 7.1, here we find that the 13 cofficients to be killed are given by

$a_{40000} = a_{31000} = a_{30010} = a_{30001} = a_{22000} = a_{21010} = a_{13001} = a_{21001} = a_{12010} = a_{12001} = a_{04000} = a_{03010} = a_{03001} = 0.$

Now, we apply the rotation of the indices. Unlike the case of the proof of Lemma 7.5, here we find that 15 coefficients are repeated. We note that this happened in the case of the cubic forms too (cf. the proof that $p_g = 0$ in Proposition 7.1.

The remaining distinct coefficients in F_4 are $70 - 13 \cdot 5 + 15 = 20$. We have thus proved that $\dim_{\mathbf{k}}(W_4) = 20$ and

$$q_2(X) = p_g(X) + p_g(S) - \dim_{\mathbf{k}}(W_4) = 0 + 20 - 20 = 0.$$

This proves Lemma 7.6 and the Theorem 7.4. $\square$

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