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The local weak solution of the wave equation

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Abstract. *This article deals with the local existence of weak solutions for a wave equation with a nonlinear integral equation at the boundary.*

1 Introduction

In this paper, we investigate the following one-dimentional wave equation

$$u_{tt} - u_{xx} + \mu(t, \|u(t)\|^2, \|u_t(t)\|^2)u_t = f(x, t, u, \|u(t)\|^2, \|u_t(t)\|^2), \tag{1.1}$$

associated with mixed boundary conditions

$$u(1, t) = 0, \tag{1.2}$$

$$u_x(0, t) = g(t) + h(u(\xi_1, t)) + \int_0^t k(t - s, u(\xi_1, s), \ldots, u(\xi_N, s))ds, \tag{1.3}$$

and initial conditions

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ (x, t) \in (0, 1) \times (0, T), \tag{1.4}$$

where ξ_1, ξ_2, $\ldots$, ξ_N are constants such that $0 = \xi_1 < \xi_2 < \cdots < \xi_N < 1$ and f, g, h, k, μ, u_0, u_1 are given functions satisfying conditions specified later. See below for some typical works.

J. Pöschel [12] showed the existence of quasi-periodic solutions for the nonlinear wave equation

$$u_{tt} - u_{xx} = \sum_{j=1}^{n} K_j u^{2j-1}, \tag{1.5}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0,\ t \in \mathbb{R}, \tag{1.6}$$

where $K_1 < 0$, $K_2 \neq 0$, $K_3, \ldots, K_n$ are constants.

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In [6], M. Li considered the existence and uniqueness of weak solutions for the Emden-Fowler type wave equation

$$t^2 u_{tt} - u_{xx} = |u|^{p-2} u, \tag{1.7}$$

subject to zero boundary values and initial values

$$u(x,1) = u_0(x),\ u_t(x,1) = u_1(x),\ (x,t) \in (a,b) \times (1,T), \tag{1.8}$$

where $p > 2$ is a constant and u_0, u_1 are given functions.

F. Ficken and B. Fleishman [4] established the global existence and stability of solutions for the following wave equation

$$u_{tt} - u_{xx} + \mu u_t = Ku + hu^3 + l, \tag{1.9}$$

where K, h, l, μ are given constants.

M. Santos [15] considered the decay rate of the solutions of the following problem

$$u_{tt} - \rho(t) u_{xx} = 0, \tag{1.10}$$

$$u(0,t) = 0, \tag{1.11}$$

$$u(1,t) + \int_0^t k(t-s)\rho(s) u_x(1,s)ds = 0, \tag{1.12}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.13}$$

with k, ρ, u_0, u_1 are given functions. Also, we see that the boundary (1.12) is equivalent to

$$-\rho(t)u_x(1,t) = g(t) + hu(1,t) + lu_t(1,t) + \int_0^t \widehat{k}(t-s)u(1,s)ds, \tag{1.14}$$

where

$$\begin{cases} g(t) = -\frac{u_0(1)}{k(0)}\widehat{k}(t),\ h = \frac{\widehat{k}(0)}{k(0)},\ l = \frac{1}{k(0)}, \\ k(0)\widehat{k}(t) + \int_0^t k'(t-s)\widehat{k}(s)ds = k'(t). \end{cases} \tag{1.15}$$

Moreover, authors in [8] constructed the asymptotic formula for the solutions of the following wave equation

$$u_{tt} - u_{xx} + \mu u_t = Ku + f(x,t), \tag{1.16}$$

$$u(1,t) = 0, \tag{1.17}$$

$$u_x(0,t) = g(t) + hu(0,t) + lu_t(0,t) + \int_0^t k(t-s)u(0,s)ds, \tag{1.18}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.19}$$

in which K, h, l, μ are constants and f, g, k, u_0, u_1 are given functions.

Our paper can be regarded as the relative extension and improvement of the corresponding results of $[1, 3, 6-9, 11, 13, 15-22]$.

2 Preliminaries

Firstly, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the scalar product and the norm in $L^2(0,1)$.

Let $u(t)$, $u'(t)=u_t(t)=\dot{u}(t)$, $u''(t)=u_{tt}(t)=\ddot{u}(t)$, $u_x(t)$ and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$ and $\frac{\partial^2 u}{\partial x^2}(x,t)$ respectively.

Also, we define a closed subspace of the Sobolev space $H^1(0,1)$ below

$$W=\left\{v\in H^1(0,1): v(1)=0\right\}, \tag{2.1}$$

with the following scalar product and norm

$$\langle u,v\rangle_W=\langle u_x,v_x\rangle \quad \text{and} \quad \|u\|_W=\|u_x\|. \tag{2.2}$$

Then the following lemmas are well-known.

Lemma 2.1. *The embedding $W\hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])}\le\|v\|_W\le\|v\|_{H^1(0,1)}\le\sqrt{2}\|v\|_W,\ \forall v\in W. \tag{2.3}$$

Lemma 2.2. *The embedding $H^1(0,1)\hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|^2_{C^0([0,1])}\le\varepsilon\|v_x\|^2+\left(1+\frac{1}{\varepsilon}\right)\|v\|^2,\ \forall v\in H^1(0,1),\ \varepsilon>0. \tag{2.4}$$

Furthermore, we have the following results.

Lemma 2.3 (see [18]). *Let $k\in\mathbb{N}$ and $\mu_j=(2j-1)\frac{\pi}{2}$, $j=\overline{1,k}$. Then*

$$\left|\sum\nolimits_{j=1}^{k}\frac{\sin\mu_j x}{\mu_j}\right|\le 1+\frac{4}{\pi},\ \forall x\in\mathbb{R}. \tag{2.5}$$

Lemma 2.4. *Let $(a,b,c_j,k)\in\mathbb{R}^3\times\mathbb{N}$ and $\mu_j=(2j-1)\frac{\pi}{2}$, $j=\overline{1,k}$. Then*

$$\int_a^b\left(\sum\nolimits_{j=1}^{k}c_j\sin(\mu_j s)\right)^2 ds\le 2\left(|a|+|b|+2\right)\int_0^1\left(\sum\nolimits_{j=1}^{k}c_j\sin(\mu_j s)\right)^2 ds, \tag{2.6}$$

$$\int_a^b\left(\sum\nolimits_{j=1}^{k}c_j\cos(\mu_j s)\right)^2 ds\le 2\left(|a|+|b|+2\right)\int_0^1\left(\sum\nolimits_{j=1}^{k}c_j\cos(\mu_j s)\right)^2 ds. \tag{2.7}$$

The proof of this lemma is not difficult, so we omit the details.

Lemma 2.5 (Integral inequality of Gronwall type [10]). *Suppose that $\alpha\in C^1([0,T])$ and β, $v\in C^0([0,T])$, β is a nonnegative function. Moreover, if*

$$v(t)\le\alpha(t)+\int_0^t\beta(s)v(s)ds,\ \forall t\in[0,T], \tag{2.8}$$

then

$$v(t)\le\alpha(0)\exp\left(\int_0^t\beta(s)ds\right)+\int_0^t\exp\left(\int_s^t\beta(\tau)d\tau\right)\alpha'(s)ds,\ \forall t\in[0,T]. \tag{2.9}$$

3 Local weak solution

Now we make the following assumptions:

(A_1) $(u_0, u_1) \in (W \cap H^2(0,1)) \times W$,

(A_2) $\mu \in C^1(\mathbb{R}_+^3)$, $f \in C^2([0,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2)$,

(A_3) $g \in W^{2,1}(\mathbb{R}_+)$, $h \in C^2(\mathbb{R})$, $k \in C^2(\mathbb{R}_+ \times \mathbb{R}^N)$,

(A_4) f, g, h, u_0 satisfy the following compatibility conditions:

$$f(\cdot, 0, u_0, \|u_0\|^2, \|u_1\|^2) = u_{0x}(0) = g(0) + h(u_0(0)) = 0.$$

Let $T_* > 0$ be a fixed constant, for each $\rho > 0$, $T \in (0, T_*]$, we put

$$\begin{cases} c(\rho, f) = \|f\|_{C^2([0,1]\times[0,T_*]\times[-\rho,\rho]\times[0,\rho^2]^2)}, \\ c(g) = \|g\|_{W^{2,1}([0,T_*])}, \ c(\rho, h) = \|h\|_{C^2([-\rho,\rho])}, \\ c(\rho, k) = \|k\|_{C^2([0,T_*]\times[-\rho,\rho]^N)}, \ c(\rho, \mu) = \|\mu\|_{C^1([0,T_*]\times[0,\rho^2]^2)}, \end{cases}$$

and

$$\begin{cases} W(\rho, T) = \{v \in L^\infty(0,T;W) : v_t \in L^\infty(0,T;W), \\ v_{tt} \in L^\infty(0,T;L^2(0,1)), \ v(\xi_i, \cdot) \in H^2(0,T), \text{ with } \|v\|_{L^\infty(0,T;W)}, \\ \|v_t\|_{L^\infty(0,T;W)}, \ \|v_{tt}\|_{L^\infty(0,T;L^2(0,1))}, \ \|v(\xi_i, \cdot)\|_{H^2(0,T)} \le \rho, \ i = \overline{1,N}\}, \\ W_*(\rho, T) = \left\{v \in W(\rho, T) : v \in L^\infty(0,T;W \cap H^2(0,1))\right\}. \end{cases}$$

Then we have the following theorem.

Theorem 3.1. *Let (A_1)-(A_4) hold. Then there exist constants $\rho, T > 0$ and a recurrent sequence $\{v_m\} \subset W_*(\rho, T)$ $(v_0 = u_0)$ satisfying the following variational problem*

$$\begin{cases} \langle v_m''(t), \varphi\rangle + \langle v_{mx}(t), \varphi_x\rangle + w_m(t)\varphi(0) \\ \quad + \mu_m(t)\langle v_m'(t), \varphi\rangle = \langle f_m(\cdot, t), \varphi\rangle, \ \forall \varphi \in W, \\ v_m(0) = u_0, \quad v_m'(0) = u_1, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \mu_m(t) = \mu(t, \|v_{m-1}(t)\|^2, \left\|v_{m-1}'(t)\right\|^2), \\ f_m(x,t) = f(x, t, v_{m-1}(t), \|v_{m-1}(t)\|^2, \left\|v_{m-1}'(t)\right\|^2), \\ w_m(t) = g(t) + h(v_{m-1}(\xi_1, t)) + \int_0^t k(t-s, v_{m-1}(\xi_1, s), \ldots, v_{m-1}(\xi_N, s))ds. \end{cases} \tag{3.2}$$

Proof of Theorem 3.1. The proof is based on the strong induction and Galerkin method. Indeed, it is not difficult to see that Theorem 3.1 holds for $m = 1$.

Suppose that it holds for $2,\ldots,m-1$ $(m\geq 2)$. Now we show below that the theorem also holds for m.

Step 1. *Galerkin approximation.* We use an orthonormal basis of W as follows

$$\varphi_j(x)=\sqrt{2/(1+\mu_j^2)}\cos(\mu_j x),\ \ \mu_j=(2j-1)\frac{\pi}{2},\ \ j=1,2,\ldots. \tag{3.3}$$

Put $v_m^{(k)}(t)=\sum_{j=1}^{k}\omega_{mj}^{(k)}(t)\varphi_j$, where the functions $\omega_{mj}^{(k)}(t)$ satisfy the following system of differential equations

$$\begin{cases}\left\langle \ddot{v}_m^{(k)}(t),\varphi_j\right\rangle+\left\langle v_{mx}^{(k)}(t),\varphi_{jx}\right\rangle+w_m(t)\varphi_j(0)\\ +\mu_m(t)\left\langle \dot{v}_m^{(k)}(t),\varphi_j\right\rangle=\left\langle f_m(\cdot,t),\varphi_j\right\rangle,\ j=\overline{1,k},\\ v_m^{(k)}(0)=\sum\limits_{j=1}^{k}a_{mj}^{(k)}\varphi_j=u_0,\ \ \dot{v}_m^{(k)}(0)=\sum\limits_{j=1}^{k}b_{mj}^{(k)}\varphi_j=u_1.\end{cases} \tag{3.4}$$

Consequently

$$\begin{cases}\ddot{\omega}_{mj}^{(k)}(t)+\mu_j^2\omega_{mj}^{(k)}(t)=-\mu_m(t)\dot{\omega}_{mj}^{(k)}(t)+\frac{1}{\|\varphi_j\|^2}\left[\left\langle f_m(\cdot,t),\varphi_j\right\rangle-w_m(t)\varphi_j(0)\right],\\ \omega_{mj}^{(k)}(0)=a_{mj}^{(k)},\ \ \dot{\omega}_{mj}^{(k)}(0)=b_{mj}^{(k)},\ \ j=\overline{1,k}.\end{cases} \tag{3.5}$$

Put $\rho_j(t)=\sin(\mu_j t)/\mu_j$, we deduce that

$$\begin{aligned}\omega_{mj}^{(k)}(t)(t)=\,&a_{mj}^{(k)}\dot{\rho}_j(t)+b_{mj}^{(k)}\rho_j(t)-\int_0^t\rho_j(t-s)\mu_m(s)\dot{\omega}_{mj}^{(k)}(s)ds\\ &-\frac{2}{\varphi_j(0)}\int_0^t ds\int_0^s\rho_j(t-s)k(s-\tau,v_{m-1}(\xi_1,\tau),\ldots,v_{m-1}(\xi_N,\tau))d\tau\\ &+\frac{2}{\varphi_j^2(0)}\int_0^t\rho_j(t-s)\left\langle f_m(\cdot,s),\varphi_j\right\rangle ds\\ &-\frac{2}{\varphi_j(0)}\int_0^t\rho_j(t-s)[h(v_{m-1}(\xi_1,s))+g(s)]ds,\ j=\overline{1,k}.\end{aligned} \tag{3.6}$$

By the Banach fixed-point theorem, we easily deduce from the assumptions of Theorem 3.1 that the system (3.6) has a unique solution $(\omega_{m1}^{(k)}(t),\ldots,\omega_{mk}^{(k)}(t))$ on the interval $[0,T]$ (see [2]).

Step 2. *A priori estimates.* This step includes two stages below.

Stage 1. *A priori estimates 1.* On account of (3.6), thus $v_m^{(k)}(\xi_i,t)$ can be rewritten as follows

$$v_m^{(k)}(\xi_i,t)=g_{mi}^{(k)}(t)-2\int_0^t k_i^{(k)}(t-s)w_m(s)ds, \tag{3.7}$$

where

$$\begin{cases} g_{mi}^{(k)}(t) = \sum_{j=1}^{k} \varphi_j(\xi_i)[a_{mj}^{(k)}\dot{\rho}_j(t) + b_{mj}^{(k)}\rho_j(t)] \\ \qquad - \sum_{j=1}^{k} \varphi_j(\xi_i) \int_0^t \rho_j(t-s)\mu_m(s)\dot{\omega}_{mj}^{(k)}(s)ds \\ \qquad + 2\sum_{j=1}^{k} \frac{\cos(\mu_j\xi_i)}{\varphi_j(0)} \int_0^t \rho_j(t-s)\left\langle f_m(\cdot,s),\varphi_j\right\rangle ds, \\ k_i^{(k)}(t) = \sum_{j=1}^{k} \cos(\mu_j\xi_i)\rho_j(t),\ i = \overline{1,N}. \end{cases} \tag{3.8}$$

Put $S_{1m}^{(k)}(t) = \left\|\dot{v}_m^{(k)}(t)\right\|^2 + \left\|v_{mx}^{(k)}(t)\right\|^2$, we get the following lemma.

Lemma 3.2. *We have*

$$\sum_{i=1}^{N} \int_0^t \left|\dot{g}_{mi}^{(k)}(s)\right|^2 ds \le Tc_*(\rho,f) + c_*(u) + c_*(\rho,\mu)\int_0^t S_{1m}^{(k)}(s)ds, \tag{3.9}$$

where

$$\begin{cases} c_*(\rho,\mu) = 6NT_*(T_*+4)c^2(\rho,\mu), \\ c_*(u) = 12N(T_*+4)(\|u_{0x}\|^2 + \|u_1\|^2), \\ c_*(\rho,f) = 38NT_*(4\rho^2+\rho+1)^2c^2(\rho,f). \end{cases} \tag{3.10}$$

Proof of Lemma 3.2. We define

$$\dot{g}_{mi}^{(k)}(t) = a_{mi}^{(k)}(t) + b_{mi}^{(k)}(t) + c_{mi}^{(k)}(t), \tag{3.11}$$

where

$$\begin{cases} a_{mi}^{(k)}(t) = \sum_{j=1}^{k} \varphi_j(0)\cos(\mu_j\xi_i)[b_{mj}^{(k)}\cos(\mu_j t) - \mu_j a_{mj}^{(k)}\sin(\mu_j t)], \\ b_{mi}^{(k)}(t) = -\sum_{j=1}^{k} \varphi_j(0)\cos(\mu_j\xi_i)\int_0^t \cos(\mu_j(t-s))\mu_m(s)\dot{\omega}_{mj}^{(k)}(s)ds, \\ c_{mi}^{(k)}(t) = 2\sum_{j=1}^{k} \frac{\cos(\mu_j\xi_i)}{\varphi_j(0)} \int_0^t \frac{\sin(\mu_j(t-s))}{\mu_j}\left\langle f_m'(\cdot,s),\varphi_j\right\rangle ds,\ i = \overline{1,N}. \end{cases} \tag{3.12}$$

Therefore

$$\int_0^t \left|\dot{g}_{mi}^{(k)}(s)\right|^2 ds \le 3\sum_{I\in\{a,b,c\}} \int_0^t \left|I_{mi}^{(k)}(s)\right|^2 ds = \sum_{j=1}^{3} I_j(t). \tag{3.13}$$

We will estimate each term on the right-hand side of this inequality.
First integral. We remark that

$$\begin{cases} 2\cos(\mu_j\xi_i)\cos(\mu_j t) = \cos(\mu_j(t+\xi_i)) + \cos(\mu_j(t-\xi_i)), \\ 2\cos(\mu_j\xi_i)\sin(\mu_j t) = \sin(\mu_j(t+\xi_i)) + \sin(\mu_j(t-\xi_i)). \end{cases} \tag{3.14}$$

Hence it follows from (3.14) and Lemma 2.4 that

$$\begin{aligned} I_1(t) &\leq \frac{3}{2}\int_{\xi_i}^{t+\xi_i}\left(\sum_{j=1}^{k}\varphi_j(0)(b_{mj}^{(k)}\cos(\mu_j s) - \mu_j a_{mj}^{(k)}\sin(\mu_j s))\right)^2 ds \\ &+ \frac{3}{2}\int_{-\xi_i}^{t-\xi_i}\left(\sum_{j=1}^{k}\varphi_j(0)(b_{mj}^{(k)}\cos(\mu_j s) - \mu_j a_{mj}^{(k)}\sin(\mu_j s))\right)^2 ds \\ &\leq 6(T_* + 2\xi_i + 2)\|u_{0x} - u_1\|^2. \end{aligned} \tag{3.15}$$

Second integral. From $(3.12)_2$ and the assumption (A_2), which implies that

$$\begin{aligned} \left|b_{mi}^{(k)}(t)\right|^2 &\leq \frac{1}{2}Tc^2(\rho,\mu)\int_0^t\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j(t-s+\xi_i))\dot{\omega}_{mj}^{(k)}(s)\right|^2 ds \\ &+ \frac{1}{2}Tc^2(\rho,\mu)\int_0^t\left|\sum_{j=1}^{k}\varphi_j(0)\cos(\mu_j(t-s-\xi_i))\dot{\omega}_{mj}^{(k)}(s)\right|^2 ds. \end{aligned} \tag{3.16}$$

Using the Fubini theorem and Lemma 2.4, it follows from (3.16) that

$$
\begin{aligned}
I_2(t) &\le \frac{3}{2}Tc^2(\rho,\mu)\int_0^t\int_\tau^t \left|\sum_{j=1}^k \varphi_j(0)\cos(\mu_j(s-\tau+\xi_i))\dot{\omega}_{mj}^{(k)}(\tau)\right|^2 dsd\tau \\
&+ \frac{3}{2}Tc^2(\rho,\mu)\int_0^t\int_\tau^t \left|\sum_{j=1}^k \varphi_j(0)\cos(\mu_j(s-\tau-\xi_i))\dot{\omega}_{mj}^{(k)}(\tau)\right|^2 dsd\tau \\
&= \frac{3}{2}Tc^2(\rho,\mu)\int_0^t \left(\int_{\xi_i}^{t-\tau+\xi_i} \left|\sum_{j=1}^k \varphi_j(0)\cos(\mu_j z)\dot{\omega}_{mj}^{(k)}(\tau)\right|^2 dz\right) d\tau \\
&+ \frac{3}{2}Tc^2(\rho,\mu)\int_0^t \left(\int_{-\xi_i}^{t-\tau-\xi_i} \left|\sum_{j=1}^k \varphi_j(0)\cos(\mu_j z)\dot{\omega}_{mj}^{(k)}(\tau)\right|^2 dz\right) d\tau \\
&\le 6T(T+2\xi_i+2)c^2(\rho,\mu)\int_0^t \left\|\dot{v}_m^{(k)}(\tau)\right\|^2 d\tau \\
&\le 6T_*(T_*+2\xi_i+2)c^2(\rho,\mu)\int_0^t S_{1m}^{(k)}(s)ds.
\end{aligned}
\tag{3.17}
$$

Third integral. Thanks to $(3.12)_3$ and Lemma 2.3, we deduce that

$$
\begin{aligned}
\left|c_{mi}^{(k)}(t)\right| &\le \frac{1}{2}\int_0^t\int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s+\xi_i+x))}{\mu_j}\right| |f'_m(x,s)|\, dxds \\
&+ \frac{1}{2}\int_0^t\int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s+\xi_i-x))}{\mu_j}\right| |f'_m(x,s)|\, dxds \\
&+ \frac{1}{2}\int_0^t\int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s-\xi_i+x))}{\mu_j}\right| |f'_m(x,s)|\, dxds \\
&+ \frac{1}{2}\int_0^t\int_0^1 \left|\sum_{j=1}^k \frac{\sin(\mu_j(t-s-\xi_i-x))}{\mu_j}\right| |f'_m(x,s)|\, dxds \\
&\le 2\left(1+\frac{4}{\pi}\right)\int_0^t \|f'_m(\cdot,s)\|_{L^1(0,1)}ds.
\end{aligned}
\tag{3.18}
$$

In addition

$$
\begin{aligned}
|f'_m(x,s)| \le & \left| D_2 f(x,s,v_{m-1}(s), \|v_{m-1}(s)\|^2, \|v'_{m-1}(s)\|^2) \right| \\
& + \left| D_3 f(x,t,v_{m-1}(s), \|v_{m-1}(s)\|^2, \|v'_{m-1}(s)\|^2) \right| |v'_{m-1}(s)| \\
& + 2\left| D_4 f(x,t,v_{m-1}(s), \|v_{m-1}(s)\|^2, \|v'_{m-1}(s)\|^2) \right| \|v'_{m-1}(s)\| \|v_{m-1}(s)\| \\
& + 2\left| D_5 f(x,t,v_{m-1}(s), \|v_{m-1}(s)\|^2, \|v'_{m-1}(s)\|^2) \right| \|v''_{m-1}(s)\| \|v'_{m-1}(s)\| \\
& \le (4\rho^2+\rho+1)c(\rho,f).
\end{aligned}
\tag{3.19}
$$

From (3.18), (3.19), we arrive at

$$
I_3(t) = 3\int_0^t \left| c_{mi}^{(k)}(s) \right|^2 ds \le 6\left(1+\frac{4}{\pi}\right)^2 T^2 (4\rho^2+\rho+1)^2 c^2(\rho,f). \tag{3.20}
$$

From (3.13), (3.15), (3.17) and (3.20), we obtain Lemma 3.2.
This completes the proof of Lemma 3.2. □

Remark 3.3. Lemma 8 in [18] is a special case of Lemma 3.2, with $\mu = 0$, $f(x,t,u,v,w) = f_1(x,u)$, $k(t,v_1,\dots,v_N) = k_1(t)k_2(v_1,\dots,v_N)$, where f_1, k_1, k_2 are given functions.

With the help of Lemma 3.2, we obtain the following lemma.

Lemma 3.4. *We have*

$$
\sum_{i=1}^N \int_0^t \left| \dot{v}_m^{(k)}(\xi_i,s) \right|^2 ds \le T c_{**}(\rho,f,g,h,k) + c_{**}(u) + c_{**}(\rho,\mu) \int_0^t S_{1m}^{(k)}(s)ds, \tag{3.21}
$$

where

$$
\begin{cases}
c_{**}(\rho,\mu) = 12NT_*(T_*+4)c^2(\rho,\mu) \\
c_{**}(u) = 24N(T_*+4)(\|u_{0x}\|^2 + \|u_1\|^2), \\
c_{**}(\rho,f,g,h,k) = 75N(2\rho+1)^4(T_*+1)^4[c(g) + \sum_{j\in\{f,h,k\}} c(\rho,j)]^2.
\end{cases}
\tag{3.22}
$$

Proof of Lemma 3.4. By $(3.2)_3$, we obtain the estimate

$$
\begin{aligned}
|w'_m(t)| \le & |g'(t)| + |h'(v_{m-1}(\xi_1,t))| \, |v'_{m-1}(\xi_1,t)| \\
& + |k(0,v_{m-1}(\xi_1,t),\dots,v_{m-1}(\xi_N,t))| \\
& + \int_0^t |D_1 k(t-s, v_{m-1}(\xi_1,s),\dots,v_{m-1}(\xi_N,s))| \, ds \\
& \le |g'(t)| + \rho c(\rho,h) + (T_*+1)c(\rho,k).
\end{aligned}
\tag{3.23}
$$

Using (3.7), (3.23) and Lemma 2.3, which leads to

$$\begin{aligned}\left|\dot{v}_m^{(k)}(\xi_i,t)\right| &= \left|\dot{g}_{mi}^{(k)}(t) - 2k_i^{(k)}(t)v_m(0) - 2\int_0^t k_i^{(k)}(t-s)w_m'(s)ds\right| \\ &\leq \left|\dot{g}_{mi}^{(k)}(t)\right| + 2\left(1+\frac{4}{\pi}\right)\int_0^t |w_m'(s)|\,ds \\ &\leq \left|\dot{g}_{mi}^{(k)}(t)\right| + 5[c(g) + T_*\rho c(\rho,h) + T_*(T_*+1)c(\rho,k)].\end{aligned} \tag{3.24}$$

Combining (3.24) and Lemma 3.2, we get Lemma 3.4. □

Next, we replace φ_j in $(3.4)_1$ by $\dot{v}_m^{(k)}(t)$. Then integrating from 0 to t, we get after some calculations

$$\begin{aligned}S_{1m}^{(k)}(t) &\leq S_{1m}^{(k)}(0) + Tc^2(\rho,f) + [2c(\rho,\mu)+1]\int_0^t S_{1m}^{(k)}(s)ds \\ &- 2\int_0^t [g(s) + h(v_{m-1}(\xi_1,s))]\dot{v}_m^{(k)}(0,s)ds \\ &- 2\int_0^t \dot{v}_m^{(k)}(0,s)ds\int_0^s k(s-\tau, v_{m-1}(\xi_1,\tau),\ldots,v_{m-1}(\xi_N,\tau))d\tau \\ &= \|u_1\|^2 + \|u_{0x}\|^2 + Tc^2(\rho,f) \\ &+ [2c(\rho,\mu)+1]\int_0^t S_{1m}^{(k)}(s)ds + I(t) + J(t).\end{aligned} \tag{3.25}$$

Thanks to Lemma 3.4, we obtain the estimates for $I(t)$ and $J(t)$ as follows:

$$\begin{aligned}I(t) &= -2\int_0^t [g(s) + h(v_{m-1}(\xi_1,s))]\dot{v}_m^{(k)}(0,s)ds \\ &\leq \int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds + 2c^2(g) + 2Tc^2(\rho,h) \\ &\leq 2Tc_{**}(\rho,f,g,h,k) + 2c^2(g) + c_{**}(u) + c_{**}(\rho,\mu)\int_0^t S_{1m}^{(k)}(s)ds,\end{aligned} \tag{3.26}$$

$$\begin{aligned}J(t) &= -2\int_0^t \dot{v}_m^{(k)}(0,s)ds\int_0^s k(s-\tau, v_{m-1}(\xi_1,\tau),\ldots,v_{m-1}(\xi_N,\tau))d\tau \\ &\leq \int_0^t \left|\dot{v}_m^{(k)}(0,s)\right|^2 ds + T^3c^2(\rho,k) \\ &\leq T[c_{**}(\rho,f,g,h,k) + {T^*}^2c^2(\rho,k)] + c_{**}(u) + c_{**}(\rho,\mu)\int_0^t S_{1m}^{(k)}(s)ds.\end{aligned} \tag{3.27}$$

Combining (3.25)-(3.27) shows that

$$S_{1m}^{(k)}(t) \leq Td_*(\rho,f,g,h,k) + d_*(u) + d_*(\rho,\mu)\int_0^t S_{1m}^{(k)}(s)ds, \tag{3.28}$$

where

$$\begin{cases} d_*(\rho,\mu) = 24NT_*(T_*+4)c^2(\rho,\mu) + 2c(\rho,\mu) + 2, \\ d_*(u) = [24N(T_*+4)+1](\|u_{0x}\|^2 + \|u_1\|^2) + 2c^2(g), \\ d_*(\rho,f,g,h,k) = 150N(2\rho+T_*+2)^4[c(g) + \sum_{j\in\{f,h,k\}} c(\rho,j) + 1]^2. \end{cases} \tag{3.29}$$

By Lemma 2.5 and Lemma 3.4, we obtain

$$\begin{cases} S_{1m}^{(k)}(t) \le d_*(\rho,T,f,g,h,k,\mu), \\ \sum\limits_{i=1}^{N} \int_0^t \left|\dot{v}_m^{(k)}(\xi_i,s)\right|^2 ds \le c_{**}(u) + d_{**}(\rho,T,f,g,h,k,\mu), \end{cases} \tag{3.30}$$

where

$$\begin{cases} d_*(\rho,T,f,g,h,k,\mu) = [Td_*(\rho,f,g,h,k) + d_*(u)]\exp(Td_*(\rho,\mu)), \\ d_{**}(\rho,T,f,g,h,k,\mu) = Tc_{**}(\rho,f,g,h,k) + Tc_{**}(\rho,\mu)d_*(\rho,T,f,g,h,k,\mu). \end{cases} \tag{3.31}$$

Stage 2. *A priori estimates 2.* Put $S_{2m}^{(k)}(t) = \left\|\ddot{v}_m^{(k)}(t)\right\|^2 + \left\|\dot{v}_{mx}^{(k)}(t)\right\|^2$. We need the following lemma.

Lemma 3.5. *We have*

$$\sum_{i=1}^{N} \int_0^t \left|\ddot{g}_{mi}^{(k)}(s)\right|^2 ds \le Tc^*(\rho,f,g,h,k,\mu) + c^*(u) + c^*(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds, \tag{3.32}$$

where

$$\begin{cases} c^*(u) = 8N(T_*+4)\|u_{0xx} + u_{1x} - \mu_* u_1\|^2, \\ c^*(\rho,\mu) = 16NT_*(T_*+4)(4\rho^2+1)^2c^2(\rho,\mu), \\ c^*(\rho,f,g,h,k,\mu) = 200Nc^2(\rho,f)(2\rho+1)^8(T_*+T_*^{-1})^2 \\ \qquad\qquad \times [c(g) + \sum_{j\in\{f,h,k,\mu\}} c(\rho,j) + 1]^2. \end{cases} \tag{3.33}$$

Proof of Lemma 3.5. On account of (3.11), we deduce that

$$\ddot{g}_{mi}^{(k)}(t) = \alpha_{mi}^{(k)}(t) + \beta_{mi}^{(k)}(t) + \gamma_{mi}^{(k)}(t) + \delta_{mi}^{(k)}(t),\ i = \overline{1,N}, \tag{3.34}$$

where

$$\begin{cases} \alpha_{mi}^{(k)}(t) = -\sum_{j=1}^{k} \varphi_j(0)\mu_j b_{mj}^{(k)} \cos(\mu_j \xi_i) \sin(\mu_j t) \\ \qquad\qquad - \sum_{j=1}^{k} \varphi_j(0)[\mu_j^2 a_{mj}^{(k)} + \mu_m(0) b_{mj}^{(k)}] \cos(\mu_j \xi_i) \cos(\mu_j t), \\ \beta_{mi}^{(k)}(t) = -\sum_{j=1}^{k} \varphi_j(0) \cos(\mu_j \xi_i) \int_0^t \cos(\mu_j(t-s))(\mu_m' \dot{\omega}_{mj}^{(k)} + \mu_m \ddot{\omega}_{mj}^{(k)})(s) ds, \\ \gamma_{mi}^{(k)}(t) = 2 \sum_{j=1}^{k} \frac{\cos(\mu_j \xi_i)}{\varphi_j(0)} \frac{\sin(\mu_j t)}{\mu_j} \langle f_m'(\cdot, 0), \varphi_j \rangle, \\ \delta_{mi}^{(k)}(t) = 2 \sum_{j=1}^{k} \frac{\cos(\mu_j \xi_i)}{\varphi_j(0)} \int_0^t \frac{\sin(\mu_j(t-s))}{\mu_j} \langle f_m''(\cdot, s), \varphi_j \rangle \, ds. \end{cases} \tag{3.35}$$

By the inequality $(\alpha+\beta+\gamma+\delta)^2 \leq 4(\alpha^2+\beta^2+\gamma^2+\delta^2)$, $\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$, we get

$$\int_0^t \left| \ddot{g}_{mi}^{(k)}(s) \right|^2 ds \leq 4 \sum_{J \in \{\alpha,\beta,\gamma,\delta\}} \int_0^t \left| J_{mi}^{(k)}(s) \right|^2 ds = \sum_{j=1}^{4} J_j(t). \tag{3.36}$$

Put $\mu_* = \mu_m(0) = \mu(0, \|u_0\|^2, \|u_1\|^2)$. Similarly as in Lemma 3.2, we get

$$\begin{aligned} J_1(t) &\leq 2 \int_{\xi_i}^{t+\xi_i} \left(\sum_{j=1}^{k} \varphi_j(0)(\mu_j b_{mj}^{(k)} \sin(\mu_j s) + (\mu_j^2 a_{mj}^{(k)} + \mu_* b_{mj}^{(k)}) \cos(\mu_j s)) \right)^2 ds \\ &\quad + 2 \int_{-\xi_i}^{t-\xi_i} \left(\sum_{j=1}^{k} \varphi_j(0)(\mu_j b_{mj}^{(k)} \sin(\mu_j s) + (\mu_j^2 a_{mj}^{(k)} + \mu_* b_{mj}^{(k)}) \cos(\mu_j s)) \right)^2 ds \\ &\leq 8(T_* + 2\xi_N + 2) \|u_{0xx} + u_{1x} - \mu_* u_1\|^2, \end{aligned} \tag{3.37}$$

$$\begin{aligned} J_2(t) &\leq 16 T_* (T_* + 2\xi_N + 2) \|\mu_m'\|_{C^0([0,T_*])}^2 \int_0^t \left\| \dot{v}_m^{(k)}(s) \right\|^2 ds \\ &\quad + 16 T_* (T_* + 2\xi_N + 2) \|\mu_m\|_{C^0([0,T_*])}^2 \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|^2 ds, \end{aligned} \tag{3.38}$$

$$J_3(t) \leq 16T \left(1 + \frac{4}{\pi}\right)^2 \|f_m'(\cdot,0)\|_{L^1(0,1)}^2 \leq 16T \left(1 + \frac{4}{\pi}\right)^2 (4\rho^2 + \rho + 1)^2 c^2(\rho, f), \tag{3.39}$$

$$J_4(t) \leq 16 \left(1 + \frac{4}{\pi}\right)^2 \int_0^t \left(\int_0^s \|f_m''(\cdot,\tau)\|_{L^1(0,1)} d\tau \right)^2 ds. \tag{3.40}$$

Furthermore, we remark that

$$\begin{aligned}|\mu_m'(t)| &\leq \left|D_1\mu(t, \|v_{m-1}(t)\|^2, \left\|v_{m-1}'(t)\right\|^2)\right| \\ &\quad + 2\left|D_2\mu(t, \|v_{m-1}(s)\|^2, \left\|v_{m-1}'(s)\right\|^2)\right| \left\|v_{m-1}'(s)\right\| \|v_{m-1}(s)\| \\ &\quad + 2\left|D_3\mu(t, \|v_{m-1}(s)\|^2, \left\|v_{m-1}'(s)\right\|^2)\right| \left\|v_{m-1}''(s)\right\| \left\|v_{m-1}'(s)\right\| \\ &\leq (4\rho^2+1)c(\rho,\mu),\end{aligned} \tag{3.41}$$

Using Lemma 2.1, we obtain from (3.38) and (3.41) that

$$\begin{aligned}J_2(t) &\leq 16T_*(T_*+2\xi_N+2)(4\rho^2+1)^2c^2(\rho,\mu)\int_0^t\left(\left\|\ddot{v}_m^{(k)}(s)\right\|^2+\left\|\dot{v}_m^{(k)}(s)\right\|^2\right)ds \\ &\leq 16T_*(T_*+2\xi_N+2)(4\rho^2+1)^2c^2(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds.\end{aligned} \tag{3.42}$$

Similarly, we get

$$\begin{aligned}|f_m''(x,t)| &\leq \Big[\left(1+\left|v_{m-1}'(t)\right|+2\left|\left\langle v_{m-1}'(t),v_{m-1}(t)\right\rangle\right|+2\left|\left\langle v_{m-1}''(t),v_{m-1}'(t)\right\rangle\right|\right)^2 \\ &\quad + \left|v_{m-1}''(t)\right| + 2\left\|v_{m-1}'(s)\right\|^2 + 2\left\langle v_{m-1}''(t),v_{m-1}(t)\right\rangle \\ &\quad + 2\left\|v_{m-1}''(s)\right\|^2 + 2\left|\left\langle v_{m-1}'''(t),v_{m-1}'(t)\right\rangle\right|\Big]c(\rho,f) \\ &\leq c(\rho,f)\left[2\left|\left\langle v_{m-1}'''(t),v_{m-1}'(t)\right\rangle\right|+\left|v_{m-1}''(t)\right|+(4\rho^2+\rho+1)^2+6\rho^2\right].\end{aligned} \tag{3.43}$$

Also, we differentiate the equations $(3.1)_1$ (for $m-1$) with respect to t, then

$$\begin{aligned}&\left\langle v_{m-1}'''(t),\varphi\right\rangle+\left\langle\nabla v_{m-1}'(t),\varphi_x\right\rangle+w_{m-1}'(t)\varphi(0)+\mu_{m-1}'(t)\left\langle v_{m-1}'(t),\varphi\right\rangle \\ &+\mu_{m-1}(t)\left\langle v_{m-1}''(t),\varphi\right\rangle=\left\langle f_{m-1}'(\cdot,t),\varphi\right\rangle,\ \forall\varphi\in W.\end{aligned} \tag{3.44}$$

Take $\varphi = v_{m-1}'(t)$ in (3.44), then from (3.19), (3.23), (3.41) and Lemma 2.1, it is not difficult to show that

$$\begin{aligned}\left|\left\langle v_{m-1}'''(t),v_{m-1}'(t)\right\rangle\right| &\leq \left\|\nabla v_{m-1}'(t)\right\|^2+\left|w_{m-1}'(t)\right|\left|v_{m-1}'(0,t)\right| \\ &\quad + \left|\mu_{m-1}'(t)\right|\left\|v_{m-1}'(t)\right\|^2+\left|\mu_{m-1}(t)\right|\left|\left\langle v_{m-1}''(t),v_{m-1}'(t)\right\rangle\right| \\ &\quad + \left|\left\langle f_{m-1}'(\cdot,t),v_{m-1}'(t)\right\rangle\right| \\ &\leq \rho(|g'(t)|+\rho c(\rho,h)+(T_*+1)(\rho,k)) \\ &\quad + \rho(4\rho^2+\rho+1)c(\rho,f)+\rho^2(4\rho^2+2)c(\rho,\mu)+\rho^2.\end{aligned} \tag{3.45}$$

Hence it follows from (3.40), (3.43) and (3.45) that

$$\begin{aligned}J_4(t) &\leq 100TT_*^2c^2(\rho,f)[2\rho(4\rho^2+\rho+1)c(\rho,f) \\ &\quad + 2\rho(T_*^{-1}c(g)+\rho c(\rho,h)+(T_*+1)(\rho,k)) \\ &\quad + 2\rho^2(4\rho^2+2)c(\rho,\mu)+(4\rho^2+\rho+1)^2+8\rho^2+\rho]^2.\end{aligned} \tag{3.46}$$

Combining (3.36), (3.37), (3.39), (3.42) and (3.46), we get Lemma 3.5. □

Next Lemma 3.5, we have the following result.

Lemma 3.6. *We have*

$$\sum_{i=1}^{N}\int_0^t \left|\ddot{v}_m^{(k)}(\xi_i,s)\right|^2 ds \leq Tc^{**}(\rho,f,g,h,k,\mu)+c^{**}(u)+c^{**}(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds, \tag{3.47}$$

where

$$\begin{cases} c^{**}(u)=16N(T_*+4)\|u_{0xx}+u_{1x}-\mu_*u_1\|^2, \\ c^{**}(\rho,\mu)=32NT_*(T_*+4)(4\rho^2+1)^2c^2(\rho,\mu), \\ c^{**}(\rho,f,g,h,k,\mu)=50N(2\rho+1)^4(T_*+1)^4[2c(g)+\sum_{j\in\{h,k\}}c(\rho,j)]^2 \\ \quad +800Nc^2(\rho,f)(2\rho+1)^8(T_*+T_*^{-1})^2[c(g)+2\sum_{j\in\{h,k,\mu\}}c(\rho,j)+2]^2. \end{cases} \tag{3.48}$$

Proof of Lemma 3.6. Using Lemma 2.1, we get from $(3.2)_3$ that

$$\begin{aligned} |w_m''(t)| &= \Big|g''(t)+h''(v_{m-1}(\xi_1,t))\left|v_{m-1}'(\xi_1,t)\right|^2 \\ &\quad + h'(v_{m-1}(\xi_1,t))v_{m-1}''(\xi_1,t)+D_1k(0,v_{m-1}(\xi_1,t),\ldots,v_{m-1}(\xi_N,t)) \\ &\quad + \sum_{i=1}^{N}D_{i+1}k(0,v_{m-1}(\xi_1,t),\ldots,v_{m-1}(\xi_N,t))v_{m-1}'(\xi_i,t) \\ &\quad + \int_0^t D_1^2k(t-s,v_{m-1}(\xi_1,s),\ldots,v_{m-1}(\xi_N,s))ds\Big| \\ &\leq |g''(t)|+c(\rho,h)(\left|v_{m-1}''(\xi_1,t)\right|+\rho^2)+(\rho+T_*+1)c(\rho,k). \end{aligned} \tag{3.49}$$

Therefore

$$\begin{aligned} \int_0^t |w_m''(s)|\,ds &\leq \|g''\|_{L^1(0,T_*)}+T_*(\rho+T_*+1)c(\rho,k) \\ &\quad + c(\rho,h)(\sqrt{T_*}\left\|v_{m-1}''(\xi_1,\cdot)\right\|_{L^2(0,T_*)}+\rho^2T) \\ &\leq c(g)+\rho\sqrt{T_*}(\rho\sqrt{T_*}+1)c(\rho,h)+T_*(\rho+T_*+1)c(\rho,k). \end{aligned} \tag{3.50}$$

By Lemma 2.3 and the assumption (A_4), we deduce from (3.7)and (3.50) that

$$\begin{aligned} \left|\ddot{v}_m^{(k)}(\xi_i,t)\right| &= \left|\ddot{g}_{mi}^{(k)}(t)-2k_i^{(k)}(t-s)w_m'(0)-2\int_0^t k_i^{(k)}(t-s)w_m''(s)ds\right| \\ &\leq \left|\ddot{g}_{mi}^{(k)}(t)\right|+2\left(1+\frac{4}{\pi}\right)|w_m'(0)|+2\left(1+\frac{4}{\pi}\right)\int_0^t |w_m''(s)|\,ds \\ &\leq \left|\ddot{g}_{mi}^{(k)}(t)\right|+5\{2c(g)+(\rho+1)^2(T_*+1)^2[c(\rho,h)+c(\rho,k)]\}. \end{aligned} \tag{3.51}$$

Thus, using (3.51) and Lemma 3.5, we have Lemma 3.6. □

Remark 3.7. Due to the proof of Lemma 3.6, we obtain the estimates for the sequences $\{u_{tt}^m(\xi_i,\cdot)\}$, $i=\overline{0,N}$ in the proof of Theorem 5 of [18] as follows:

$$\int_0^t |u_{tt}^m(\xi_i,s)|^2 ds \le C_T,\ \forall t\in[0,T],\ i=\overline{0,N}, \tag{3.52}$$

where C_T is a positive constant independent of m.

By Lemma 3.6, we can estimate the terms $S_{2m}^{(k)}(t)$ and $\sum_{i=1}^N \int_0^t \left|\ddot{v}_m^{(k)}(\xi_i,s)\right|^2 ds$. Indeed, we differentiate the equations $(3.4)_1$ with respect to t, then

$$\begin{aligned}&\left\langle \ddot{v}_{mt}^{(k)}(t),\varphi_j\right\rangle + \left\langle \dot{v}_{mx}^{(k)}(t),\varphi_{jx}\right\rangle + w_m'(t)\varphi_j(0)\\ &+\mu_m'(t)\left\langle \dot{v}_m^{(k)}(t),\varphi_j\right\rangle + \mu_m(t)\left\langle \ddot{v}_m^{(k)}(t),\varphi_j\right\rangle = \left\langle f_m'(\cdot,t),\varphi_j\right\rangle.\end{aligned} \tag{3.53}$$

We multiply the j^{th} equation of (3.53) by $\ddot{\omega}_{mj}^{(k)}(t)$, then summing up with respect to j and integrating with respect to the time variable from 0 to t, we get

$$\begin{aligned}S_{2m}^{(k)}(t) &\le S_{2m}^{(k)}(0) + 2c(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds + 2\int_0^t |\mu_m'(s)|\left|\left\langle \ddot{v}_m^{(k)}(s),\dot{v}_m^{(k)}(s)\right\rangle\right|ds\\ &+2\int_0^t \left\langle f_m'(\cdot,s),\ddot{v}_m^{(k)}(s)\right\rangle ds - 2\int_0^t w_m'(s)\ddot{v}_m^{(k)}(0,s)ds\\ &= \|u_{1x}\|^2 + 2c(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds + \sum_{j=1}^4 K_j(t).\end{aligned} \tag{3.54}$$

Now we estimate the following terms in the right-hand side of (3.54) as follows. Estimating $K_1(t)=\left\|\ddot{v}_m^{(k)}(0)\right\|^2$. Using $(3.4)_1$ and the compatibility condition (A_4), then

$$\left\|\ddot{v}_m^{(k)}(0)\right\|^2 = \left\langle u_{0xx}-\mu_m(0)u_1,\ddot{v}_m^{(k)}(0)\right\rangle \le \|u_{0xx}-\mu_m(0)u_1\|\left\|\ddot{v}_m^{(k)}(0)\right\|. \tag{3.55}$$

Consequently

$$K_1(t)\le 2\|u_{0xx}\|^2 + 2\mu^2(0,\|u_0\|^2,\|u_1\|^2)\|u_1\|^2. \tag{3.56}$$

Estimating $K_2(t)=2\int_0^t|\mu_m'(s)|\left|\left\langle \ddot{v}_m^{(k)}(s),\dot{v}_m^{(k)}(s)\right\rangle\right|ds$. Thanks to Lemma 2.1 and (3.41), which yields that

$$K_2(t)\le 2\int_0^t |\mu_m'(s)|\left\|\ddot{v}_m^{(k)}(s)\right\|\left\|\dot{v}_{mx}^{(k)}(s)\right\|ds \le (4\rho^2+1)c(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds. \tag{3.57}$$

Estimating $K_3(t) = 2\int_0^t \left\langle f'_m(\cdot,s), \ddot{v}_m^{(k)}(s)\right\rangle ds$. We deduce from (3.19) that

$$K_3(t) \le \int_0^t S_{2m}^{(k)}(s)ds + T(4\rho^2+\rho+1)^2 c^2(\rho,f). \tag{3.58}$$

Estimating $K_4(t) = -2\int_0^t w'_m(s)\ddot{v}_m^{(k)}(0,s)ds$. From (3.23) and Lemma 3.5, then

$$\begin{aligned} K_4(t) &\le \int_0^t |w'_m(s)|^2 ds + \int_0^t \left|\ddot{v}_m^{(k)}(0,s)\right|^2 ds \\ &\le c^{**}(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds + c^{**}(u) + T\{c^{**}(\rho,f,g,h,k,\mu) \\ &+ [(T_*+T_*^{-1})c(g) + \rho c(\rho,h) + (T_*+1)c(\rho,k)]^2\}. \end{aligned} \tag{3.59}$$

Therefore, from (3.54) and (3.56)- (3.59), we arrive at

$$S_{2m}^{(k)}(t) \le Td^*(\rho,f,g,h,k,\mu) + d^*(u) + d^*(\rho,\mu)\int_0^t S_{2m}^{(k)}(s)ds, \tag{3.60}$$

where

$$\begin{cases} d^*(u) = 16N(T_*+4)\|u_{0xx}+u_{1x}-\mu_* u_1\|^2, \\ +2\|u_{0xx}\|^2 + \|u_{1x}\|^2 + 2\mu^2(0,\|u_0\|^2,\|u_1\|^2)\|u_1\|^2, \\ d^*(\rho,\mu) = 32NT_*(T_*+4)(4\rho^2+1)^2[c(\rho,\mu)+1]^2+1, \\ d^*(\rho,f,g,h,k,\mu) = 1600N[c^2(\rho,f)+1](2\rho+1)^8(T_*+1)^4 \\ \times(T_*+T_*^{-1})^2[c(g)+\sum_{j\in\{f,h,k,\mu\}} c(\rho,j)+1]^2. \end{cases} \tag{3.61}$$

By Lemma 2.5 and Lemma 3.6, we obtain from (3.60) that

$$\begin{cases} S_{2m}^{(k)}(t) \le d^*(\rho,T,f,g,h,k,\mu), \\ \sum\limits_{i=1}^N \int_0^t \left|\ddot{v}_m^{(k)}(\xi_i,s)\right|^2 ds \le c^{**}(u) + d^{**}(\rho,T,f,g,h,k,\mu). \end{cases} \tag{3.62}$$

where

$$\begin{cases} d^*(\rho,T,f,g,h,k,\mu) = [Td^*(\rho,f,g,h,k,\mu)+d^*(u)]\exp(Td^*(\rho,\mu)), \\ d^{**}(\rho,T,f,g,h,k,\mu) = Tc^{**}(\rho,f,g,h,k,\mu) + Tc^{**}(\rho,\mu)d^*(\rho,T,f,g,h,k,\mu). \end{cases} \tag{3.63}$$

Choose $\rho^2 = (d^*+d_*)(u)+1$. So, from (3.22), (3.29), (3.31), (3.48), (3.61), (3.63) and the assumption (A_2), (A_3), we have

$$\begin{cases} \lim\limits_{T\to 0^+}(d^{**}+d_{**})(\rho,T,f,g,h,k,\mu) = 0, \\ \lim\limits_{T\to 0^+}(d^*+d_*)(\rho,T,f,g,h,k,\mu) = (d^*+d_*)(u). \end{cases} \tag{3.64}$$

Consequently, there exists a positive constant $T \leq \min\{1, T_*\}$ such that

$$(d^* + d_* + d^{**} + d_{**})(\rho, T, f, g, h, k, \mu) \leq \rho^2, \forall t \in [0, T]. \tag{3.65}$$

Finally, we deduce from (3.30), (3.62) and (3.65) that

$$S_{1m}^{(k)}(t) + S_{2m}^{(k)}(t) + \sum_{i=1}^{N} \int_0^t \left(\left| \dot{v}_m^{(k)}(\xi_i, s) \right|^2 + \left| \ddot{v}_m^{(k)}(\xi_i, s) \right|^2 \right) ds \leq \rho^2, \tag{3.66}$$

$\forall t \in [0, T]$. Therefore $v_m^{(k)} \in W(\rho, T)$, $\forall k, m \in \mathbb{N}$.

Step 3. *Limiting process.* Thanks to (3.66), we can extract a subsequence of sequence $\left\{v_m^{(k)}\right\}$, still labeled by the same notation, such that

$$\begin{cases} v_m^{(k)} \to v_m & \text{weakly* in } L^\infty(0, T; H^1(0,1)), \\ \dot{v}_m^{(k)} \to v'_m & \text{weakly* in } L^\infty(0, T; H^1(0,1)), \\ \ddot{v}_m^{(k)} \to v''_m & \text{weakly* in } L^\infty(0, T; L^2(0,1)), \\ v_m^{(k)}(\xi_i, \cdot) \to v_m(\xi_i, \cdot) & \text{weakly in } H^2(0, T), \ i = \overline{1, N}, \end{cases} \tag{3.67}$$

as $k \to \infty$, and $v_m \in W(\rho, T)$.

Using the compactness of the imbedding $H^2(0, T) \hookrightarrow C^1([0, T])$ and the lemma of J.L. Lions [5], then (3.67) leads to the existence of a subsequence still denoted by $\left\{v_m^{(k)}\right\}$, such that

$$\begin{cases} v_m^{(k)} \to v_m & \text{strongly in } L^2((0,1) \times (0,T)) \text{ and a.e. in } (0,1) \times (0,T), \\ \dot{v}_m^{(k)} \to v'_m & \text{strongly in } L^2((0,1) \times (0,T)) \text{ and a.e. in } (0,1) \times (0,T), \\ v_m^{(k)}(\xi_i, \cdot) \to v_m(\xi_i, \cdot) & \text{strongly in } C^1([0,T]), \ i = \overline{1, N}. \end{cases} \tag{3.68}$$

Passing to the limit in (3.4) by $(3.67)_{1,3}$ and (3.68), we obtain that v_m satisfies the problem (3.1), (3.2) for m.

On the other hand, it is not difficut to show that

$$v_{mxx} = v''_m + \mu_m(t) v'_m - f_m(x, t) \in L^\infty(0, T; L^2(0,1)). \tag{3.69}$$

Accordingly $v_m \in W_*(\rho, T)$. Theorem 3.1 is proved. □

Remark 3.8. Theorem 3.1 gives no conclusion of the existence of a recurrent sequence $\{v_m\} \subset W_*(\rho, T)$ when the term u_t in the equation (1.1) is replaced by the nonlinear term $\varphi(u_t)$.

Applying Theorem 3.1, we obtain the existence and uniqueness theorem of weak solutions for the problem (1.1)-(1.4) below.

Theorem 3.9. *Let* (A_1)-(A_4) *hold. Then there exist positive constants* ρ *and* T *such that the problem* (1.1)-(1.4) *has a unique weak solution* $v \in W_*(\rho, T)$.

Remark 3.10. By Theorem 3.9, the problem (1.1)-(1.4) has a unique weak solution v satisfying

$$\begin{cases} v \in L^\infty(0,T;W\cap H^2(0,1))\cap C^0(0,T;W)\cap C^1(0,T;L^2(0,1)),\\ v' \in L^\infty(0,T;W),\ v''\in L^\infty(0,T;L^2(0,1)),\ v(\xi_i,\cdot)\in H^2(0,T),\ i=\overline{1,N}. \end{cases}$$

In addition, we can see that the solution $v \in H^2((0,1)\times(0,T))\cap L^\infty(0,T;H^2(0,1))$. Thus the solution v is almost classical which is rather natural since the initial data $(u_0,u_1)\notin C^2([0,1])\times C^1([0,1])$.

Proof of Theorem 3.9. We devide it into two steps.
Step 1. *Existence of weak solutions.* From the proof of Theorem 3.1, it is not difficult to show that there exists a sequence $\{v_m\}$ satisfying the problem (3.1), (3.2) and the following inequalities

$$\|v_m'(t)\|^2+\|v_{mx}(t)\|^2+\|v_m''(t)\|^2+\|v_{mx}'(t)\|^2\le\rho^2, \tag{3.70}$$

$$\int_0^t\Big(|v_m(\xi_i,s)|^2+|v_m'(\xi_i,s)|^2+|v_m''(\xi_i,s)|^2\Big)ds\le\rho^2, \tag{3.71}$$

$\forall t\in[0,T]$, $i=\overline{1,N}$, with the positive constants ρ and T satisfy (3.65).
Thus, we can extract from $\{v_m\}$ a subsequence, still denoted $\{v_m\}$, such that

$$\begin{cases} v_m\to v & \text{weakly* in } L^\infty(0,T;H^1(0,1)),\\ v_m''\to v'' & \text{weakly* in } L^\infty(0,T;L^2(0,1)),\\ v_m\to v & \text{strongly in } L^2((0,1)\times(0,T)),\\ v_m'\to v' & \text{strongly in } L^2((0,1)\times(0,T)),\\ v_m(\xi_i,\cdot)\to v(\xi_i,\cdot) & \text{strongly in } C^1([0,T]),\ i=\overline{1,N}, \end{cases} \tag{3.72}$$

and $v\in W(\rho,T)$. Also, we note that

$$\begin{cases} \Big|\mu_{m+1}(t)-\mu(t,\|v(t)\|^2,\|v'(t)\|^2)\Big|\\ \le 2\rho c(\rho,\mu)\left(\|(v_m-v)(t)\|+\|(v_m'-v')(t)\|\right),\\ \Big|f_{m+1}(x,t)-f(x,t,v,\|v(t)\|^2,\|v'(t)\|^2)\Big|\\ \le(2\rho+1)c(\rho,f)\left(|(v_m-v)(t)|+\|(v_m-v)(t)\|+\|(v_m'-v')(t)\|\right),\\ |k(t,v_m(\xi_1,t),\dots,v_m(\xi_N,t))\\ -k(t,v(\xi_1,t),\dots,v(\xi_N,t))|\le c(\rho,k)\sum_{i=1}^N|(v_m-v)(\xi_i,t)|. \end{cases} \tag{3.73}$$

Hence it follows from $(3.72)_{3-5}$ and (3.73) that

$$\begin{cases} f_m(x,t)\to f(x,t,v,\|v(t)\|^2,\|v'(t)\|^2) & \text{strongly in } L^2((0,1)\times(0,T)),\\ \mu_m(t)\to\mu(t,\|v(t)\|^2,\|v'(t)\|^2) & \text{strongly in } L^2(0,T),\\ w_m\to w & \text{strongly in } C^0([0,T]), \end{cases} \tag{3.74}$$

where

$$w(t) = g(t) + h(v(\xi_1, t)) + \int_0^t k(t-s, v(\xi_1, s), \ldots, v(\xi_N, s))ds. \tag{3.75}$$

Passing to the limit in $(3.1)_1$ by $(3.72)_{1,2,4}$ and (3.74), we see that v satisfies the following equation

$$\begin{cases} \langle v''(t), \varphi\rangle + \langle v_x(t), \varphi_x\rangle + w(t)\varphi(0) + \mu(t, \|v(t)\|^2, \|v'(t)\|^2)\langle v', \varphi\rangle \\ = \left\langle f(\cdot, t, v, \|v(t)\|^2, \|v'(t)\|^2), \varphi\right\rangle, \ \forall \varphi \in W, \\ v(x,0) = u_0(x), \quad v'(x,0) = u_1(x). \end{cases} \tag{3.76}$$

Similarly as in (3.69), we have $v_{xx} \in L^\infty(0,T; L^2(0,1))$. Thus $v \in W_*(\rho, T)$.
The existence of weak solutions is proved.
Step 2. *Uniqueness of weak solutions.* Let $v_1, v_2 \in W_*(\rho, T)$ be two weak solutions of the problem (1.1)-(1.4). Then $v = v_1 - v_2$ is a weak solution of the following problem

$$\begin{cases} \langle v''(t), \varphi\rangle + \langle v_x(t), \varphi_x\rangle + (w_1 - w_2)(t)\varphi(0) + \mu_1(t)\langle v'(s), \varphi\rangle \\ + (\mu_1 - \mu_2)(t)\langle v_2'(s), \varphi\rangle = \langle (f_1 - f_2)(\cdot, t), \varphi\rangle, \ \forall \varphi \in W, \\ v(x,0) = v'(x,0) = 0, \end{cases} \tag{3.77}$$

where

$$\begin{cases} \mu_i(t) = \mu(t, \|v_i(t)\|^2, \|v_i'(t)\|^2), \\ f_i(x,t) = f(x, t, v_i(t), \|v_i(t)\|^2, \|v_i'(t)\|^2), \\ w_i(t) = h(v_i(\xi_1, t)) + \int_0^t k(t-s, v_i(\xi_1, s), \ldots, v_i(\xi_N, s))ds, \ i = 1, 2. \end{cases} \tag{3.78}$$

Taking $\varphi = v'$ in $(3.77)_1$ and then integrating both sides of it from 0 to t, we get

$$\begin{aligned} S(t) &= -2\int_0^t \langle \mu_1(s)v'(s) + (\mu_1 - \mu_2)(s)v_2'(s), v'(s)\rangle\, ds \\ &\quad + 2\int_0^t \langle (f_1 - f_2)(\cdot, s), v'(s)\rangle\, ds - 2\int_0^t v'(0,s)\sum_{i=1}^2 (-1)^{i-1} h(v_i(\xi_1, s))ds \\ &\quad - 2\int_0^t v'(0,s)ds \int_0^s \sum_{i=1}^2 (-1)^{i-1} k(s-\tau, v_i(\xi_1, \tau), \ldots, v_i(\xi_N, \tau))d\tau \\ &= L_1(t) + L_2(t) + L_3(t) + L_4(t), \end{aligned} \tag{3.79}$$

in which $S(t) = \|v'(t)\|^2 + \|v_x(t)\|^2$.
We easily estimate the integrals on the right-hand side of (3.79) as follows.

$$
\begin{aligned}
L_1(t) &\le 2\int_0^t |\mu_1(s)|\,\|v'(s)\|^2 ds + 2\rho\int_0^t |(\mu_1-\mu_2)(s)|\,\|v'(s)\|\,ds \\
&\le 2c(\rho,\mu)\int_0^t \|v'(s)\|^2 ds + 2\rho^2 c(\rho,\mu)\int_0^t \left(\|v(s)\|^2 + 3\|v'(s)\|^2\right) ds \\
&\le 2(3\rho^2+1)c(\rho,\mu)\int_0^t S(s)ds,
\end{aligned}
\tag{3.80}
$$

$$
\begin{aligned}
L_2(t) &\le 2\int_0^t \|(f_1-f_2)(\cdot,s)\|\,\|v'(s)\|\,ds \\
&\le 2(2\rho+1)c(\rho,f)\int_0^t (2\,\|v(s)\| + \|v'(s)\|)\,\|v'(s)\|\,ds \\
&\le 4(2\rho+1)c(\rho,f)\int_0^t S(s)ds.
\end{aligned}
\tag{3.81}
$$

On the other hand, using integration by parts, we get

$$
\begin{aligned}
L_3(t) &= -v^2(0,t)\int_0^1 h'((v_2+\tau v)(0,t))d\tau \\
&\quad + \int_0^t v^2(0,s)ds \int_0^1 h''((v_2+\tau v)(0,s))(\tau v_1' + (1-\tau)v_2')(0,s)d\tau \\
&\le c(\rho,h)\left(v^2(0,t) + \rho\int_0^t S(s)ds\right),
\end{aligned}
\tag{3.82}
$$

$$
L_4(t) \le \frac{1}{2}S(t) + [2N^2Tc^2(\rho,k) + N(T+3)c(\rho,k)]\int_0^t S(s)ds. \tag{3.83}
$$

In addition, from Lemma 2.2, we get

$$
v^2(0,t) \le \varepsilon S(t) + (1+\varepsilon^{-1})T\int_0^t S(s)ds,\ \forall\varepsilon > 0. \tag{3.84}
$$

Hence it follows from (3.82) and (3.84) that

$$
L_3(t) \le \varepsilon c(\rho,h)S(t) + [(1+\varepsilon^{-1})T + \rho]c(\rho,h)\int_0^t S(s)ds. \tag{3.85}
$$

Choosing $4\varepsilon c(\rho,h) \le 1$, we get from (3.79)-(3.81), (3.83) and (3.85) that

$$
S(t) \le 4c(\rho,T,f,h,k,\mu)\int_0^t S(s)ds, \tag{3.86}
$$

where

$$
\begin{aligned}
c(\rho,T,\varepsilon,f,h,k,\mu) &= 4(2\rho+1)c(\rho,f) + [(1+\varepsilon^{-1})T+\rho]c(\rho,h) \\
&\quad + 2N^2Tc^2(\rho,k) + N(T+3)c(\rho,k) + 2(3\rho^2+1)c(\rho,\mu).
\end{aligned}
\tag{3.87}
$$

Using the Gronwall inequality, we obtain $S(t) = 0$, i.e. $v_1 = v_2$.
This completes the proof of Theorem 3.9. □

Remark 3.11. (i) In the special case of $\mu = 0$, $f(x, t, u, v, w) = f_1(x, u)$ and $k(t, v_1, \ldots, v_N) = k_1(t)k_2(v_1, \ldots, v_N)$, in which f_1, k_1, k_2 are given functions, we get some results as in [18].
(ii) If we replace the term u_t in the equation (1.1) by the nonlinear term $\varphi(u_t)$, then we have no conclusion about the existence of weak solutions of the problem (1.1)-(1.4). This is an open problem.

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