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Fundamental solutions of second order elliptic linear partial differential operators with analytic coefficients and simple complex characteristics

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Abstract. *We give an explicit formula for the fundamental solutions of an elliptic linear partial differential operator of the second order with analytic coefficients and simple complex characteristics in an open set $\Omega \subset \mathbb{R}^n$. We prove that those fundamental solutions can be continued at least locally as multi-valued analytic functions $x \longrightarrow E(x, y)$ in $\mathbb{C}^n$ up to the complex bicharacteristic conoid. This extension ramifies along its singular set the bicharacteristic conoid and belongs to the Nilsson class.*

1 Introduction and main results

In this paper, we prove that every fundamental solution of an elliptic linear partial differential operator of the second order with analytic coefficients and simple complex characteristics in an open set $\Omega \subset \mathbb{R}^n$ can be continued at least locally as a multi-valued analytic function $x \longrightarrow E(x, y)$ in $\mathbb{C}^n$ up to the complex bicharacteristic conoid. This extension ramifies along its singular set the bicharacteristic conoid and has, modulo analytic functions, a very nice geometric property: it belongs to the Nilsson class. This result was announced in [13]. Later on, we proved it in [14] for elliptic operators of higher orders. The techniques of the proof are very different: explicit asymptotic expansions for the second order and integral representation for higher order operators (although the last technique applies obviously for the second order). For the second order proof, we use Hadamard's construction [6] of a fundamental solution of the wave equation, some Riemannian geometry and an analytic continuation process.

This result may be viewed as a generalization of the following result about singularities of Fuchsian type for the linear regular differential equation $(*)$, [5]. Let U be a convex open set of $\mathbb{C}$ and consider the following differential equation

$$\frac{d^2 f}{dz^2} + a_1(z)\frac{df}{dz} + a_2(z) f = 0. \qquad (*)$$

Let, for every $i \in \{1, 2\}$, $a_i(z)$ be a meromorphic function in U with simple poles $a_{i,j}$. Then a germ of a solution of $(*)$ at a pole $z_0 \in U$ is a multi-valued analytic function which belongs to the Nilsson class. This means that

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every solution of $(*)$ may be written as a finite sum of functions of the form $\varphi_{\alpha,m}(z)(z-z_o)^\alpha \log^m(z-z_0)$ where $\varphi_{\alpha,m}(z)$ is a meromorphic function in U and $\alpha \in \mathbb{C}, m \in \mathbb{N}$.

This kind of extension problems was studied for fundamental solutions of certain classes of hyperbolic differential operators in classical works of J.Leray [12]. We saw that we could generalize and adapt some of his results on hyperbolic operators to the elliptic case, starting with the Laplacian $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad (n \geqslant 2)$.

A closely related problem is the study of the ramified Cauchy problem for operators with simple characteristics. E. Leichtnam proved in [10] that Nilsson class conditions for the Cauchy data implies that the solution of the Cauchy problem belongs to the Nilsson class. The main result is

Theorem 1.1. *Let* $P(x,D) = \sum\limits_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum\limits_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x)$ *be an elliptic operator where* $a_{i,j}(x)$, $b_j(x)$, $c(x)$ *are analytic functions in* $\Omega \subset \mathbb{R}^n$, *and* $a_{i,j}(x)$ *is assumed to be symmetric without loss of generality. Denote by* $A_{i,j}(x)$ *the inverse matrix of* $[a_{i,j}(x)]$ *and* $\mathrm{d}s$ *the Riemannian metric defined by:* $\mathrm{d}s^2 = \sum\limits_{i,j=1}^{n} A_{i,j}(x)\mathrm{d}x_i\mathrm{d}x_j$. *One fixes* y, *then a local fundamental solution of* P *is given by:*

$$E(x,y) = \frac{F(x,y)}{(d(x,y)^2)^{\frac{n-2}{2}}} + G(x,y)\log d(x,y)^2 + H(x,y)$$

where F,G,H *denote three analytic functions in a neighborhood of* y *and* $d(x,y)$ *is the Riemannian distance corresponding to the Riemannian metric* $\mathrm{d}s^2$.

Now we describe briefly the contents of the following sections. In Section 2, we give the proof of this result, the technique of the proof is mainly based on asymptotic expansions. As a corollary, we obtain in section 3 the holomorphic extensions of those fundamental solutions and prove that they all belong to the Nilsson class.

In Section 4, we give a few examples upon which we started to build the theory: the Laplacian operator Δ and the generalized Helmholtz operator $\Delta + V(x)$ in $\mathbb{R}^n$. We give also an example of the simplest elliptic operator with multiple characteristics: the iterated Laplacian Δ^k $(k > 1)$ in $\mathbb{R}^n$.

In [3], L. Boutet de Monvel proved that the singular fundamental solution of an elliptic homogenous linear differential operator with constant coefficients and simple characteristics is a solution of a regular holonomic $\mathcal{D}$-module. When the coefficients are analytic and the symbol of the operator has constant coefficients, D.Meyer [15] has proved that the singular fundamental solution of such a differential operator is still a solution of a regular holonomic $\mathcal{D}$-module. In both cases, as a consequence, the fundamental solutions of these operators extend up to the bicharacteristic conoid and belong to the Nilsson class. This and our results lead to an important conjecture in $\mathcal{D}$-module theory: there exists a regular holonomic system of PDE's such that the singular fundamental solution $E(x,y)$ is a solution of this system.

Let's point out that the holomorphic extension of fundamental solutions of operators with multiple characteristics is a much harder problem, since the projection of the bicharacteristic flow is no longer proper in general. Even in the case of finite multiple characteristics, when the projection is proper, we should obtain irregular holonomic singularities. The simplest examples, like a suitable perturbation of the bi-Laplacian show it [4]. This is still an open problem.

1.1 Differential operators, Ellipticity

We use the standard multi-index notation. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω and by $P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\frac{\partial}{\partial x^\alpha}$ a linear differential operator of order m with analytic coefficients in Ω. A distribution $E(x,y) \in \mathcal{D}'(\Omega \times \Omega)$ depending on y as a parameter is called a fundamental solution of P if $PE(x,y) = \delta(x-y)$. Let $T^*\Omega$ be the cotangent bundle of Ω and let $\xi_1, \cdots, \xi_n$ be a coordinate system on its fiber at $(x_1, \cdots, x_n)$. The principal symbol of P is defined as usual: $a(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha$.

P is said to be elliptic if $a(x,\xi) \neq 0$ in $T^*\Omega \setminus 0$ where 0 denotes the zero section of $T^*\Omega$. We have the following crucial result ([9],p.61)

Proposition 1.2. *Let $P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\frac{\partial}{\partial x^\alpha}$ be a linear differential operator of order m with analytic coefficients in Ω open subset of $\mathbb{R}^n$ and fix $y \in \mathbb{R}^n$. There exists a fundamental solution $x \longrightarrow E(x,y)$ of P which is analytic in a punctured neighborhood $V(y) \setminus \{y\}$.*

Definition 1.3. Let Ω be an open subset of $\mathbb{R}^n$ and let $P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\frac{\partial}{\partial x^\alpha}$ be a linear differential operator of order m with C^∞ coefficients in Ω. P is said to be hypoelliptic if $\forall u \in \mathcal{D}'(\Omega)$, Sing supp u = Sing supp Pu.

Lars Hörmander proved that [7]

Proposition 1.4. *Let $u \in \mathcal{D}'(\Omega)$ and denote by $WF_A(u)$ its analytic wave front set. Then: $WF_A(u) \subset \Sigma \cup WF_A(Pu)$, where*

$$\Sigma = \{(x,\xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}, a(x,\xi) = 0\}.$$

It follows from this:

Corollary 1.5. *Fix $y \in \mathbb{R}^n$. Any two fundamental solutions of an elliptic differential operator differ one from another by an analytic solution $x \longrightarrow u(x,y)$ of $Pu = 0$.*

Corollary 1.6. *Fix $y \in \mathbb{R}^n$. Every fundamental solution of an elliptic differential operator P is the sum of a fundamental solution E such that Sing Supp $E = \{y\}$ (the singular solution) and an analytic solution $x \longrightarrow u(x, y)$ of $Pu = 0$.*

1.2 Characteristic variety, Bicharacteristics

The closed conic subset of $T^*\Omega$ defined by the equation $a(x, \xi) = 0$ is the characteristic variety of P. Denote by H_a the Hamiltonian field defined by $a(x, \xi)$ on $T^*\Omega$:

$$H_a = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial}{\partial \xi_i}\right).$$

The integrals curves of H_a which belong to the characteristic variety are called as usual the bicharacteristics of P. In what follows, the canonical projection of the usual bicharacteristics on Ω will also be called the bicharacteristics of P. An elliptic operator has no real bicharacteristics. However, if we consider, following Leray [11], the characteristic variety as a complex variety in $T^*\mathbb{C}^n$, one defines complex bicharacteristics associated to P: the complex bicharacteristics will be solutions included in the complex characteristic variety of the hamiltonian system ($t \in \mathbb{C}$):

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial a(x,\xi)}{\partial \xi_i}, \quad i = 1, \cdots, n \\ \dfrac{d\xi_i}{dt} = -\dfrac{\partial a(x,\xi)}{\partial x_i}, \quad i = 1, \cdots, n \end{cases} \tag{1}$$

with initial datas

$$x_i(0, \eta, y) = y_i, \qquad \xi_i(0, \eta, y) = \eta_i, \qquad i \in \{1, \cdots, n\}.$$

The bicharacteristic conoid Γ_y is defined as the union of all complex bicharacteristics with initial data $y \in \mathbb{C}^n$.

1.3 Geodesics

Definition 1.7. Let X be an analytic manifold of dimension n and $a_{i,j}(x)$ be a real positive definite $n \times n$ matrix. Denote by ds the Riemannian metric whose arclength is defined by the following quadratic form

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} a_{i,j}(x)\mathrm{d}x_i\mathrm{d}x_j.$$

The geodesics are curves $x(t)$ which minimize the arclength integral

$$\int_{t_0}^{t_1} \sqrt{\sum_{i,j=1}^{n} a_{i,j}(x)\mathrm{d}x_i\mathrm{d}x_j}$$

between the fixed points $x_0 = (x_1(t_0), \cdots, x_n(t_0))$ and $y_0 = (x_1(t_1), \cdots, x_n(t_1))$.

We recall a few facts from Riemannian geometry. Let X be a Riemannian manifold. Given a point $x \in X$ and $u \in T_x(X)$ non zero, there exists a unique geodesic, starting at x such that u is its tangent vector at x.

Definition 1.8. Let X be a Riemannian manifold. We define the exponential map as the map which associates to $(x, u) \in X \times T_x(X)$ ($T_x(X)$ being the tangent space of X at x) the value $\exp_x(u)$ at time $t = 1$ of the geodesic of initial value x and u.

Definition 1.9. Let (U, φ) be a coordinate system on the Riemannian manifold X and $\varphi(x_0) = 0$. The coordinate system (U, φ) is normal with respect to x if the inverse image under φ of straight lines through the origin in $\mathbb{R}^n$ are geodesics on X.

The exponential map can be used to define a normal coordinate system on a neighborhood of any points of X [1] (p.9).

Definition 1.10. Let X be a Riemannian manifold. We define a distance d on X such that $d^2(x, \exp_x(u)) = ||u||_x^2$, where $||\quad||_x$ is a metric on $T_x(X)$. Then

$$\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial d'}{\partial x_i} \frac{\partial d'}{\partial x_j} = 4d'.$$

(The tangent vector has indeed constant length along a geodesic [1] (p.10).)

1.4 Nilsson Class

Let X be a connected complex analytic manifold, Γ be a complex analytic hypersurface in X, $x \in X \setminus \Gamma$. The analytic germ f at x defines a multi-valued analytic function on $X \setminus \Gamma$ if it can be analytically continued along each path of $X \setminus \Gamma$ with origin x. In an equivalent way, f defines a holomorphic function $\tilde{f}$ on the universal covering of $X \setminus \Gamma$.

Definition 1.11. A multi-valued analytic function f on $X \setminus \Gamma$ has finite determination if the complex vector space generated by the local branches of f has finite dimension.

Definition 1.12. Let f be a multi-valued analytic function in $\mathbb{C}^n \setminus \Gamma$, where Γ is a complex analytic hypersurface in $\mathbb{C}^n$. We say that f belongs to the Nilsson class [16, 17] if f has finite determination and f has moderate growth along Γ: for any open set U in $\mathbb{C}^n$ such that $\Gamma \cap U = \{g(z) = 0\}$, where g is an analytic function, there exists a positive integer N such that for every semi-analytic set P, simply connected and relatively compact in $U \setminus \Gamma$, and for every local branch of f, there exists a constant C such that $\forall z \in P, |f(z)| \leq \frac{C}{|g(z)|^N}$.

We have the following result ([17],p.16):

Proposition 1.13. *Let X be a connected complex analytic manifold, Γ a divisor with normal crossings in X defined locally by $z_1 = z_2 = \cdots = z_n = 0$ where $z = (z_1, z_2, \cdots, z_n)$ is a suitable set of coordinates and let F be a multi-valued analytic function on $X \setminus \Gamma$ with finite determination. Then, F is a finite sum of functions of the form:*

$$\varphi_{\sigma,\nu}(z) z_1^{\sigma_1}, \cdots, z_n^{\sigma_n} (\log z_1)^{\nu_1}, \cdots, (\log z_n)^{\nu_n}$$

where $\varphi_{\sigma,\nu}$ are single-valued analytic functions on $X \setminus \Gamma$ and $\sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{C}^n, \nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$.
Moreover, such a multi-valued analytic function f belongs to the Nilsson class if and only if f may be written as a finite sum of functions as above:

$$\varphi_{\sigma,\nu}(z) z_1^{\sigma_1}, \cdots, z_n^{\sigma_1} (\log z_1)^{\nu_1}, \cdots, (\log z_n)^{\nu_n}$$

where this time $\varphi_{\sigma,\nu}$ are meromorphic functions on X.

Remark Using Hironaka's resolution of singularirities theorem and the fact that the Nilsson class is preserved by a biholomorphic mapping, we can always reduce the study of Nilsson class to the normal crossing case.

2 Proof of Theorem 1.1

The proof of the theorem is similar to the one given in Hadamard [6] for hyperbolic operators. First, assume that n is **odd** and that $E(x,y)d^{n-2}(x,y)$ expands in integer powers of d^2

$$E(x,y) = \frac{1}{d^{n-2}(x,y)} \sum_{k=0}^{\infty} F_k(x,y) d^{2k}(x,y).$$

Define $d' = d^2(x,y)$. Then

$$P(E(x,y)) = \sum_{k=0}^{\infty} P[F_k(x,y)(d')^{k-m}]$$

where $m = \frac{n-2}{2}$. Using the symmetry of the matrix $a_{ij}(x)$, we get

$$\begin{aligned} P[F_k(x,y)(d')^{k-m}] &= P(F_k)(d')^{k-m} \\ &+ 2(k-m)[\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial F_k}{\partial x_i} \frac{\partial d'}{\partial x_j}](d')^{k-m-1} \\ &+ (k-m)[\sum_{i=1}^{n} b_i(x) F_k \frac{\partial d'}{\partial x_i} + \sum_{i,j=1}^{n} a_{i,j}(x) F_k \frac{\partial^2 d'}{\partial x_i \partial x_j}](d')^{k-m-1} \\ &+ (k-m)(k-m-1)[\sum_{i,j=1}^{n} a_{i,j}(x) F_k \frac{\partial d'}{\partial x_i} \frac{\partial d'}{\partial x_j}](d')^{k-m-2}. \end{aligned}$$

Moving to geodesic coordinates $s = d(x,y)$, we obtain

$$\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial F_k}{\partial x_i}\frac{\partial d'}{\partial x_j} = 2s\sum_{i=1}^{n}\frac{\partial F_k}{\partial x_i}\frac{dx_i}{ds} = 2s\frac{dF_k}{ds}.$$

But we know that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial d'}{\partial x_i}\frac{\partial d'}{\partial x_j} = 4d'.$$

So it follows then from the computation above that $\forall n \in \mathbb{N}$,

$$P[F_k(d')^{k-m}] = P(F_k)(d')^{k-m} + 4(k-m)(s\frac{dF_k}{ds} + \theta(x,y) + k + m - 1)F_k)(d')^{k-m-1}$$

where

$$\theta(x,y) = \frac{1}{4}(\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 d'}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial d'}{\partial x_i}).$$

We use now the fact that $P(E(x,y)) = 0$ when $x \neq y$. Thus, we get the following crucial sequence of differential equations

$$\begin{aligned} & s\frac{dF_0(x,y)}{ds} + (\theta(x,y) - m - 1)F_0(x,y) = 0 \\ \forall k \geqslant 1 \quad & s\frac{dF_k(x,y)}{ds} + (\theta(x,y) - k - m - 1)F_k(x,y) = -\frac{1}{4(k-m)}P[F_{k-1}(x,y)] \end{aligned} \tag{2}$$

where $m = \frac{n-2}{2}$ and $\theta(x,y))$ defined just above is an analytic function of x and y.

Let us now denote by $x(\xi)$ the geodesic curve from $y = x(0)$ to $x = x(s)$ (recall that y is fixed). Now we may integrate (2) to obtain

$$F_0(x,y) = F_0(y,y)\exp\left(-\int_0^s (\theta(x(\xi),y) - m - 1)\frac{\mathrm{d}\xi}{\xi}\right)$$

The integral exists because

$$\theta(x(\xi),y) = m + 1 + O(\xi) = \frac{n}{2} + O(\xi).$$

This estimation is given by a Taylor expansion of d^2 in the neighborhood of (y,y):

$$d^2(x,y) \simeq \sum_{i,j=1}^{n} A_{i,j}(y)(x_i - y_i)(x_j - y_j).$$

From (2), we also get the equations

$$\forall k \geqslant 1, \quad F_k(x,y) = -\frac{F_0(x,y)}{4(k-m)s^k}\int_0^s \frac{P(F_{k-1}(x(\xi),y))\xi^{k-1}}{F_0(x(\xi),y)}\mathrm{d}\xi.$$

Finally, as for the Laplacian case, we need to adjust the constant in order to get a fundamental solution and we therefore put $F_0(y,y) = \frac{C_n}{2-n}$ where C_n denotes the inverse of the area of the $n-1$ dimensional geodesic unit ball (the unit ball for the geodesic distance d).

We first need to check by a technique similar to Cauchy-Kowalewska's theorem that the asymptotic expansion $\sum_{k=0}^{\infty} F_k(x,y)d^{2k}(x,y)$ is convergent for $d(x,y)$ small enough.

Instead of coordinates $x_1, x_2, \cdots, x_n$, let us work in a normal coordinate system $\eta_1, \eta_2, \cdots, \eta_n$ (as defined in [1]). A Taylor expansion of $a_{i,j}(exp_0\eta = x)$ close to $\eta = 0$ ($\eta = (\eta_1, \eta_2, \cdots, \eta_n)$) is given by

$$a_{i,j}(exp_0\eta = x) = \sum_{k_1,\cdots,k_m} \frac{1}{k_1!\cdots k_m!} \frac{\partial^{k_1+\cdots k_m}}{\partial\eta_1^{k_1}\cdots\partial\eta_n^{k_m}} a_{i,j}(0)\eta_1^{k_1}\cdots\eta_n^{k_m},$$

and by the same way as in the proof of Cauchy's Kowaleski theorem, the analytic functions $a_{i,j}(x)$, $b_j(x)$ and $c(x)$ are bounded by a number M in a neighborhood V_0 of zero such that $\forall\ (i,j) \in \{1,\cdots n\}$,

$$|a_{i,j}(x)|, |b_i(x)|, \text{ and } |c(x)| \leqslant M \sum_{n=0}^{\infty} \left(\frac{|\eta_1| + \cdots + |\eta_n|}{\rho}\right)^n = \frac{M\rho}{\rho - |\eta_1| - \cdots - |\eta_n|} \tag{3}$$

where ρ is the smallest radius of convergence of the analytic functions $a_{i,j}(x)$, $b_j(x)$ and $c(x)$. Let us prove by induction on k that

$$|F_k(\eta)| \leqslant \frac{c_k}{(1 - \frac{1}{\rho}(|\eta_1| + \cdots + |\eta_n|))^{2k}} \tag{4}$$

where c_k is a constant which depends on k. F_0 is bounded in V_1, so the last inequality is true for $k = 0$. Assume it is true at the rank k. From (3) and (4), we obtain

$$|P[F_k](\eta)| \leqslant \frac{2c_k k(2k+1)(1 + \frac{n}{\rho} + \frac{n^2}{\rho^2})M}{c_k(1 - \frac{1}{\rho}(|\eta_1| + \cdots + |\eta_n|)^{2k+3})}.$$

Integrating (2), we get a relation between F_{k+1} and F_k which allows to show that

$$|F_{k+1}(\eta)| \leqslant \frac{k(2k+1)(1 + \frac{n}{\rho} + \frac{n^2}{\rho^2})M}{2|k-m-1||\eta|^{k+1}} \int_0^{|\eta|} \frac{\xi^k \mathrm{d}\xi}{(1 - \frac{\xi}{\rho})^{2k+3}}$$

where $|\eta| = |\eta_1| + \cdots + |\eta_n|$. We then compute

$$\frac{1}{|\eta|^{k+1}} \int_0^{|\eta|} \frac{\xi^k \mathrm{d}\xi}{(1 - \frac{\xi}{\rho})^{2k+3}} \leqslant \frac{1}{(k+1)(1 - \frac{|\eta|}{\rho})^{2k+2}}.$$

Indeed,

$$\frac{\xi^k}{(1-\frac{\xi}{\rho})^{2k+3}} \leqslant (1+\frac{\xi}{\rho})\frac{\xi^k}{(1-\frac{\xi}{\rho})^{2k+3}} = \frac{1}{(k+1)}\frac{d}{d\xi}\left(\frac{\xi^{k+1}}{(1-\frac{\xi}{\rho})^{2k+2}}\right).$$

Define

$$c_{k+1} = \frac{k(2k+1)(1+\frac{n}{\rho}+\frac{n^2}{\rho^2})M}{2(k+1)|k-m-1|}c_k.$$

This allows us to prove the induction

$$|F_{k+1}(\eta)| \leqslant \frac{c_{k+1}}{(1-\frac{|\eta|}{\rho})^{2k+2}}.$$

We notice then that $\lim\limits_{k\to+\infty}\dfrac{c_{k+1}}{c_k} = M(1+\dfrac{n}{\rho}+\dfrac{n^2}{\rho^2})$. It follows from this that the asymptotic expansion $\sum\limits_{k=0}^{\infty} F_k(x,y)d^{2k}(x,y)$ is convergent for

$$d^2(x,y) \leqslant \frac{(1-\frac{|\eta|}{\rho})^2}{M(1+\frac{n}{\rho}+\frac{n^2}{\rho^2})} = \epsilon.$$

Let us now check that $E(x,y)$ is indeed a fundamental solution of P in a neighborhood $d^2(x,y) \leqslant \epsilon$ of y. So we need to show that

$$P\left[\frac{1}{d^{n-2}(x,y)}\sum_{k=0}^{\infty} F_k(x,y)d^{2k}(x,y)\right] = \delta(x-y).$$

We have $\forall\varphi \in \mathcal{D}'(\Omega)$,

$$\begin{aligned}< P(E),\varphi > &= \int_\Omega E(x,y)P\varphi(x)\mathrm{d}x \\ &= \lim_{\epsilon\to 0}\int_{d(x,y)\geqslant\epsilon} E(x,y)P\varphi(x)\mathrm{d}x\end{aligned}$$

We now use Green's second identity which remains true for all second order elliptic operator (the proof is the same)

$$\begin{aligned}\int_{d(x,y)\geqslant\epsilon} E(x,y)P\varphi(x)\mathrm{d}x = &\int_{d(x,y)\geqslant\epsilon} PE(x,y)\varphi(x)\mathrm{d}x \\ &+ \int_{d(x,y)=\epsilon}\left(E(x,y)\frac{\partial\varphi(x)}{\partial s} - \varphi(x)\frac{\partial E(x,y)}{\partial s}\right)\mathrm{d}\sigma_\epsilon\end{aligned}$$

where $\mathrm{d}\sigma_\epsilon$ is the measure on the the surface $d(x,y) = \epsilon$.

But

$$\int_{d(x,y)\geqslant\epsilon} PE(x,y)\varphi(x)\mathrm{d}x = 0$$

and

$$\int_{d(x,y)=\epsilon} E(x,y)\frac{\partial\varphi(x)}{\partial s}\mathrm{d}\sigma_\epsilon = \int_{d(x,y)=1} \frac{1}{\epsilon^{n-2}}\sum_{k=0}^{\infty} F_k(x,y)\epsilon^{2k}\frac{\partial\varphi(x)}{\partial s}\epsilon^{n-1}\mathrm{d}\sigma_1$$

so this term tends to 0 when ϵ tends to 0 because $F(x,y)$ and $\frac{\partial\varphi(x)}{\partial s}$ are bounded on the geodesic unit ball $B_1 = \{d(x,y) = 1\}$ (we assumed that $B_1 \subset U$ and moved to geodesic normal coordinates). Let's evaluate the last term

$$\int_{d(x,y)=\epsilon} \varphi(x)\frac{\partial E(x,y)}{\partial s}\mathrm{d}\sigma_\epsilon = \int_{d(x,y)=1} \varphi(x)\sum_{k=0}^{\infty}\frac{\partial(F_k(x,y)s^{2k-n+2})}{\partial s}\epsilon^{n-1}\mathrm{d}\sigma_1$$

$$= \int_{d(x,y)=1} \varphi(x)\sum_{k=0}^{\infty}\left[\frac{\partial F_k(x,y)}{\partial s}\epsilon^{2k-n+2} + F_k(x,y)(2k-n+2)\epsilon^{2k-n+1}\right]\epsilon^{n-1}\mathrm{d}\sigma_1$$

So, when $\epsilon \to 0$, this term tends to $(2-n)F_0(y,y)\varphi(y)\int_{B_1}\mathrm{d}\sigma_1 = \varphi(y)$. And this means that $PE(x,y) = \delta(x-y)$.

For n even, with the same sequence of differential equations as for n odd, we may determine only $F_0, \cdots, F_{m-1}$.

Assume then that $E(x,y)$ may be expanded in the following way:

$$E(x,y) = \sum_{k=0}^{m-1} F_k(x,y)d^{2k}(x,y) + G(x,y)\log d^2(x,y) + H(x,y)$$

With $H(x,y)$ denote an analytic function in a neighborhood of y. As for n odd, using $P(E) = 0$ for $x \neq y$, we get

$$0 = P(F_{m-1})\frac{1}{d'} - \frac{G}{d'^2}\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial d'}{\partial x_i}\frac{\partial d'}{\partial x_j}$$
$$+ 2\left(\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial G}{\partial x_i}\frac{\partial d'}{\partial x_j} + 4\theta G\right)\frac{1}{d'} + P(G)\log d' + P(H).$$

Reparametrizing with the parameter s, we obtain

$$\left[P(F_{m-1}) + 4\left((\theta(x,y)-1)G + s\frac{dG}{ds}\right)\right]\frac{1}{d'} + P(G)\log d' + P(H) = 0.$$

Since the logarithmic term must cancel out, $P(G) = 0$. $P(H)$ is analytic so the coefficient of $\frac{1}{d'}$ must vanish on the characteristic conoid $d^2(x,y) = 0$, and

$$s\frac{dG}{ds} + (\theta(x,y)-1)G = -\frac{1}{4}P[F_{m-1}(x,y)]$$

on the characteristic conoid $d^2(x,y)=0$. Let $G_0(x,y)$ be the holomorphic solution of this last differential differential equation defined in a real neighborhood of y by:

$$G_0(x,y) = -\frac{F_0(x,y)}{4s^m}\int_0^s \frac{P(F_{m-1}(x(\xi),y)\xi^{m-1}}{F_0(x(\xi),y)}\mathrm{d}\xi$$

where $x(\xi)$ is the geodesic curve from $y=x(0)$ to $x=x(s)$. So we know the value of $G_0(x,y)$ on the characteristic conoid.

We thus have a Goursat problem and this determines G in a unique way as proved in Hadamard [6, Section 64].[1] Finally, H is chosen such that it satisfies the analytic partial differential $P(H)=0$ in a neighborhood of y. The proof that $E(x,y)$ is a fundamental solution is the same that in odd dimension.

Complex coefficients case

Let us show that the theorem remains true for elliptic differential operators with complex coefficients. For the even dimension, it is not clear if our proof works by using the Goursat problem. But we may write: $G=\sum\limits_{k=0}^{\infty} G_k(x,y)d^{2k}$, take G_0 as in the proof and get the same kind of sequence of differential equations used in the proof for n odd. We obtain

$$G_{k+1}(x,y) = -\frac{F_0(x,y)}{4(k+1)s^{k+1+m}}\int_0^s \frac{P(G_k)(x(\xi),y)\xi^{k+m}}{F_0(x(\xi),y)}\mathrm{d}\xi \quad \forall k\in\mathbb{N}$$

where $x(\xi)$ is the geodesic curve from $y=x(0)$ to $x=x(s)$.
We then proceed as in the odd case to prove that this asymptotic expansion is convergent for $d^{2k}(x,y)$ small enough.

3 Holomorphic extensions

Theorem 3.1. *With the same notations, if one fixes y, then there exists a neighborhood $V(y)$ of y such that every fundamental solution of an analytic differential operator P of the second order with simple complex characteristics can be extended at least locally, in $V(y)\setminus\Gamma_y$, as a multi-valued analytic function $x\to E(x,y)$ up to $\mathbb{C}^n\setminus\Gamma_y$, where Γ_y is the bicharacteristic conoid with initial data $y\in\Omega$. All these fundamental solutions belong to the Nilsson class.*
If the dimension n is odd, we have $G=0$. There is no logarithmic term and $x\longrightarrow E(x,y)$ is ramified of order 2. If n is even, $x\longrightarrow E(x,y)$ has a polar singularity and possibly a logarithmic term.

[1] We recall that a Goursat problem is a characteristic initial value problem whose solution has a given value on the characteristic conoid.

Proof. Indeed, $x \longrightarrow d(x,y)^2$ is defined and analytic in a neighborhood of $x = y$. So $x \longrightarrow d(x,y)^2$ has a holomorphic extension in a neighborhood of $x = y$.
Now $d(x,y)^{n-2} = [d(x,y)^2]^{\frac{n-2}{2}}$, so $x \longrightarrow d(x,y)^{n-2}$ extends to a multi-valued analytic function with at most two ramifications along the isotropic cone $d^2 = 0$. Then, according to Theorem 4.1, we have proved the theorem. $\square$

Remark When the operator has constant coefficients, the extension is global. Since all we have done is purely local, we also have the following generalization:

Corollary 3.2. *The theorem remains true for such operators on an analytic manifold X.*

4 Examples

We first recall that every fundamental solution of an elliptic operator P is the sum of a fundamental solution E such that its singular support is reduced to a point $y \in \mathbb{R}^n$ (the singular solution) and an analytic solution of $Pu = 0$.

4.1 The Laplacian in $\mathbb{R}^n$

A fundamental solution of the Laplacian $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$, for $n \geqslant 3$, is:

$$E(x,y) = \frac{c_n}{(2-n)|x-y|^{n-2}}$$

where c_n denotes the area of the $(n-1)$-dimensional real unit sphere in $\mathbb{R}^n$ and $|x-y| = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$.
For $n = 2$,

$$E(x,y) = \frac{\log|x-y|}{2\pi}.$$

So $E(x,y)$ has a holomorphic extension to $\mathbb{C}^n \setminus Q$ where Q is the isotropic cone of $\mathbb{C}^n : \sum_{i=1}^{n}(z_i - z_i')^2 = 0$.

Since $|x-y|^{n-2} = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{(n-2)}{2}}$, $E(x,y)$ belongs to the Nilsson class. In particular, we stress the fact that the isotropic cone is indeed the union of all complex bicharacteristics with initial data y.
We can also describe precisely the ramification:

For $n = 2$, $E(x,y)$ has a logarithmic ramification along the isotropic cone.

For n even, $n \geqslant 4$, $E(x,y)$ is meromorphic with poles on the isotropic cone and for n odd, $E(x,y)$ has two ramifications along the isotropic cone.

4.2 The generalized Helmholtz operator in $\mathbb{R}^n$

The generalized Helmholtz operator is the operator $\Delta + V(x)$ where $V(x)$ is an analytic function. It is known that the singular fundamental solution has the form

$$E(x,y) = \frac{F(x,y)}{|x-y|^{n-2}} + G(x,y)\log|x-y|$$

where F,G are analytical functions. So it extends up to the isotropic cone of $\mathbb{C}^n$ and belongs to the Nilsson class.

4.3 Operator with multiple characteristics, the iterated laplacian in $\mathbb{R}^n$

A fundamental solution of the iterated Laplacian Δ^k $(k > 1)$ in $\mathbb{R}^n$ is

$$E_k(x,y) = \frac{(-1)^k\Gamma(\frac{n}{2}-k)}{2^{2k}\pi^{\frac{n}{2}}(k-1)!}|x-y|^{2k-n}$$

for n odd or n even and $n > 2k$.

For n even and $n \leqslant 2k$, a fundamental solution is

$$E_k(x,y) = \frac{(-1)^{\frac{n}{2}-1}}{2^{2k-1}\pi^{\frac{n}{2}}(k-1)!(k-\frac{n}{2})!}|x-y|^{2k-n}\log|x-y|.$$

We notice the fact that the fundamental solutions of the iterated Laplacian extend up to the isotropic cone and belongs to the Nilsson class. This will not be the case in general for multiple characteristics operators. For instance, the singular fundamental solution of $\Delta^2 + \dfrac{\partial^3}{\partial x_1^3}$ has an exponential singularity according to [4] (p.34), thus does not belong to the Nilsson Class.

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