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Quantum fields and point interactions

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Abstract. *A survey of results on quantum mechanical interactions described by idealized short range potentials, for short "point interactions", is presented. Both physical and mathematical aspects are stressed. In the first part, theory and models concerning various types of two-particle systems resp. N-particle systems for $N \geq 3$ are discussed. In the second part, certain mutual relations between point interactions and certain models of quantum fields are pointed out.*

Contents

1 Point interactions

1.1 Introduction

Point interactions is generally a name for interactions between particles (composing physical systems) supported on "thin sets" (e. g. points or low-dimensional hypersurfaces). They are also called *zero-range* or *contact interactions*.

They are used to describe situations where the range of interparticle interactions in a given physical system is much shorter than other relevant length scales. They are useful abstractions having the advantage of permitting, by their simple structure, to obtain better insights on the systems that are studied, allowing also for "explicit computations". Because of these features they are useful in the

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description of many natural and mathematical phenomena, e. g. those often mentioned with the name of "singular perturbations".

Point interactions were first introduced in *solid state physics*, through the Kronig-Penney model (1931) [101], describing a quantum mechanical particle moving on the real line under the influence of δ-potentials located at the points of a one-dimensional crystal (mathematically formalized by the set $\mathbb{Z}$ of all positive and negative integer numbers, including zero). The Hamiltonian of such a system has thus heuristically the form

$$H = -\Delta + \sum_{y \in \mathbb{Z}} \lambda_y \delta_y,$$

with δ Dirac's "delta function" (more precisely: point measure) supported by the point y.[1] H is thought of as acting (as a self-adjoint operator) in the complex Hilbert space $L^2(\mathbb{R})$.

Point interactions play also an important role in *nuclear and atomic physics*. In nuclear physics, E. Wigner (1933) underlined that nuclear potential should be considered as "strong" and of "very short range". H. Bethe and R. Peierls (1935) [35], followed by E. Fermi (1936) [87] and Breit (1947) formulated two-particle resp. many-center point interactions in the Euclidean three-dimensional space $\mathbb{R}^3$ in terms of "*boundary conditions*" (one of the methods implemented mathematically in later work, see [8, 15] and also below in this paper). L. H. Thomas [140] realized that a "renormalization" of the coupling constant λ is necessary (in the model $-\Delta + \lambda\delta(\cdot)$ for a system of particles in $\mathbb{R}^3$ described by a Hamiltonian acting in the space $L^2(\mathbb{R}^3)$ of Lebesgue square-integrable functions).

This is also connected with a new type of phenomena in systems of three or more quantum particles, as we shall discuss below, in relation with the mathematics of point interactions.

Other contexts in which point interactions appeared and have been studied include the following ones:

- N-particle models in nuclear and atomic physics (low-energy phenomena, see, e. g. [89, 61]).
- Condensed matter (e. g. polarons), see, e. g. [15, 96, 95, 106].
- Acoustics (see, e. g. [42]).
- Electromagnetism (e. g. in the study of resonances, see, e. g. references in [12, 14, 13, 40, 43, 75]).
- Quantum graphs (e. g. quantum particles moving on graphs, taken as idealized models for thin tube structures: [16, 59, 33, 34, 78, 117, 130]).

[1] This model has been well studied and used later on, see, e. g. the books [105, 92]. It has been extended to the case of higher-dimensional lattices $\mathbb{Z}^s$, $s = 2, 3$, see, e. g. the books [8] and [15]. Also in these cases H has heuristically the same form, with $\mathbb{Z}$ replaced by $\mathbb{Z}^s$, $s = 2, 3$.

- Quantum fields (see below).
- The study of the Casimir effect: [41, 86]
- Physics of cold atomic gases (where point interactions are in some sense "experimentally" implemented): [27, 28].

1.2 The mathematics of point interactions

First let us discuss the two-particle problem. How can one realize the heuristic Hamiltonian "$-\Delta + \lambda\delta(\cdot)$" for a quantum particle moving in $\mathbb{R}^s$ ($s \in \mathbb{N}$) as an operator in the complex Hilbert space $L^2(\mathbb{R}^s)$?

We shall only provide some elements, for more details see [8, 15], and also among more recent work [12, 13, 14].

After the physical considerations by L. H. Thomas (1935) [140] for $s = 3$ this became an intriguing problem, the first mathematical solution was obtained only over 25 years later through the work by F. A. Berezin and L. D. Faddeev in 1961 [31]. They used von Neumann's theory of self-adjoint extensions of densely defined symmetric lower-bounded operators to define the above heuristic Hamiltonian, looked first upon as an operator acting on the space $C_{0,0}^{\infty}$ of functions of compact support outside the origin. It turns out that there is a one-parameter family of self-adjoint extensions $-\Delta_\alpha$ characterized by a real parameter α (related to the parameter λ entering the heuristic expression for the Hamiltonian). The extensions can be characterized by exploiting the rotation symmetry of the heuristic expression, using a partial wave expansion and reducing then the considerations to the equation in the radial variable r (the interaction term $\lambda\delta(\cdot)$ being zero outside $r = 0$!). The extensions are then seen to be determined by the following boundary condition at zero for the functions ϕ in their domain:

$$\frac{\partial \phi}{\partial r}(0) = 4\pi\alpha\phi(0).$$

The parameter α takes all real values, for the limit $\alpha \to +\infty$ it characterizes the extension given by the Dirichlet boundary condition $\phi(0) = 0$. Below we shall exhibit the relation between the heuristic λ and the parameter α. The value $\alpha = 0$ corresponds to the Neumann boundary condition at the origin.

For $s = 3$, functions in the domain of $-\Delta_\alpha$ can be seen to behave as

$$\frac{q}{|x|} + \alpha q + o(1)$$

for $|x| \to 0$, with q a complex parameter.[2] Another way to characterize the family $-\Delta_\alpha$ of all point interactions with interaction at the origin is given by Krein's formula for its resolvent $(-\Delta_\alpha - k^2)^{-1}$, with (distributional) kernel:

$$(-\Delta_\alpha - k^2)^{-1}(x, y) = G_k(x - y) + \overline{G_k}(x)\Gamma_\alpha(k)G_k(y),$$

[2] For $s = 1$, the behavior for $|x| \to 0$ is regular. For $s = 2$, a logarithmic divergence arises, see, e. g. [8, 15].

with $x, y \in \mathbb{R}^3$, $\operatorname{Im} k > 0$, where $G_k(x-y)$ is the (translation invariant) kernel of the free resolvent $(-\Delta - k^2)^{-1}$, i. e. for $x \neq 0$,

$$G_k(x) = \frac{\mathrm{e}^{\,\mathrm{i}k|x|}}{4\pi|x|} \qquad \text{and} \qquad \Gamma_\alpha(k) := \left(\alpha - \frac{\mathrm{i}k}{4\pi}\right)^{-1}.$$

One shows, using this formula, that the spectrum of $-\Delta_\alpha$ as a self-adjoint operator in $L^2(\mathbb{R}^3)$ contains the positive half-line $[0, +\infty)$, for all $\alpha \in \mathbb{R}$. For $\alpha \geq 0$ this is the whole spectrum. For $\alpha < 0$, the spectrum contains an additional point (isolated bound state at $-(4\pi\alpha)^2$).

The relation between the heuristic parameter λ in "$-\Delta + \lambda\delta(\cdot)$" and the parameter α in its mathematical realization $-\Delta_\alpha$ is given by

$$\text{“} -\lambda^{-1} = G_0(0) + \alpha \text{”}.$$

Since $G_0(0)$ is $+\infty$ and α is real, λ is a "negative infinitesimal". $-\frac{1}{4\pi\alpha}$ is called the scattering length associated with the point interaction. One expresses this by saying that "the coupling constant λ has to be renormalized" and the "effective coupling constant" is α.

Remark 1.1. The way of speaking as λ being a negative infinitesimal can be made rigorous in an approach based on non-standard analysis, see [7, 9, 121, 125]

Yet another interpretation of $-\Delta_\alpha$ is via the theory of Dirichlet forms, see [10]: in fact, for $\alpha < 0$, $-\Delta_\alpha + (4\pi\alpha)^2$ is unitary equivalent to the self-adjoint positive operator uniquely associated with the classical Dirichlet form

$$\mathcal{E}_\alpha(u, v) = \int \nabla u \nabla v \phi_\alpha^2 \mathrm{d}x,$$

in $L^2(\mathbb{R}^3, \phi_\alpha^2 \mathrm{d}x)$, with

$$\phi_\alpha(x) = (-\alpha)^{1/2}\ \frac{\mathrm{e}^{4\pi\alpha|x|}}{|x|}, \qquad x \neq 0.$$

ϕ_α is the ground state eigenfunction to $-\Delta_\alpha$ to the eigenvalue $-(4\pi\alpha)^2$. For $\alpha \geq 0$, the unitary equivalence holds for $-\Delta_\alpha$ itself. Similar results can be worked out (obviously with different ϕ_α) also for the cases $s = 1, 2$.

For approximations of $-\Delta_\alpha$ by Hamiltonians with more regular potentials, see [8, 9, 15, 38]. For further references on point interactions see also, e. g., [56, 128].

1.3 Multicenter point interactions

There is an extensive literature on the study of multicenter point interactions Hamiltonians, i. e. Hamiltonians given heuristically by

$$H = -\Delta + \sum \lambda_y \delta(\bullet - y)$$

in $L^2(\mathbb{R}^s)$, where the sum is over the points $y \in Y$, and Y is a finite or infinite subset of $\mathbb{R}^s$.

Practically all results we have discussed for the case where Y consists of a single point (e. g. the origin) have been extended to these more general cases, when Y is a finite set, and partly when Y is an infinite set, see [8, 15].

In the case where Y is finite, also some detailed results have been obtained that depend essentially on the geometry of the set Y. E. g. in recent work on the distribution of eigenvalues and resonances, very interesting connections with the theory of exponential polynomials and the distributions of their zeros has been discovered, see [12, 14, 13, 75, 77]. For approximation of multicenter point interactions by regular potentials, see recent work in [53, 54, 57].

For the case where Y is infinite, some additional assumptions are needed: they include the cases where Y are regular lattices, see [8, 15, 6]. For results in the situation where Y is a discrete set with possible accumulation points, see e. g. [8, 15, 39, 106]. The case where Y is a submanifold of $\mathbb{R}^s$ is discussed e. g. in [8, 15, 29, 77]. Also for the case where the set Y is a random one some spectral results have been obtained, see e. g. references in [8, 15].

The theory of Schrödinger operators for particles moving on graphs has many aspects in common with the theory for Schrödinger operators with multicenter point interactions, see e. g. [3, 5, 16, 19, 33, 34, 76, 78, 79, 77]. For point interactions on hypersurfaces see e. g. [8, 15, 29, 30, 43, 58].

2 Point interactions for multi-particle systems

2.1 Heuristics and physical effects

Heuristically, multi-particle systems with point interactions are given by Hamiltonians of the form

$$H = \text{``} - \sum_{i=1}^{N} \Delta_{y_i} + \sum_{\substack{i,j=1 \\ i<j}}^{N} \lambda_{ij} \delta(y_i - y_j)\text{''}, \tag{2.1}$$

where λ_{ij} are real numbers. The space is $L^2(\mathbb{R}^{sN})$.

In the case $s = 1$ these Hamiltonians are easily defined as lower bounded self-adjoint operators, no renormalization is needed. They constitute interesting models for multi-particle systems on the real line, and are known also in connection with denominations like Bethe Ansatz and Yang-Baxter equation, see e. g. the books [92] and [15].

We shall discuss here the cases $s = 2$ and $s = 3$. On the basis of what we have already discussed for the special case $N = 2$ (which corresponds—after separation of the center of mass motion, which is possible due to translation invariance—to the point interactions Hamiltonians discussed above in Section 1.2) we expect to have to deal first with renormalizations in order to define H mathematically.

The discussion of this started in the years 1961–62 with a paper by Minlos and Faddeev [114, 113]. A striking phenomenon concerning these systems has been discovered early by a physicist, L. E. Thomas, already in 1935 [140].

The *Thomas effect* states the following: the quantum Hamiltonian for three bosons (three indistinguishable quantum particles of integer spin) or three distinguishable particles, interacting via local two-body *zero-range interactions* can be observed to be unbounded from below.

There is another important effect here, that was observed by another physicist, V. N. Efimov (1970) [69]. The *Efimov effect* states that if all two-body sub-systems of a three-body quantum system with regular (perhaps short-range) interactions do not have negative energy bound states, and at least two of them have a zero energy resonance (i. e. there is a solution for $E = 0$ of the Schrödinger equation which is positive, but not in $L^2(\mathbb{R}^s)$), then the total Hamiltonian shows the presence of an infinite number of negative eigenvalues (bound states) accumulating at zero.

A heuristic observation, discussed e. g. in [11], shows that a suitable scaling transformation connects the spectrum of the Hamiltonian for the Efimov effect (with possibly short-range but regular two-particle potentials) to the Hamiltonian of the Thomas effect (with zero-range two-particle potentials).

Before mathematically discussing these effects, we have to mention how a mathematical definition of H as given heuristically by equation (2.1) can be achieved.

2.2 The mathematical definition of N-particle Hamiltonians with two-particle zero-range interactions

Minlos and Faddeev [114, 113] gave the first mathematical definition of such Hamiltonians, using von Neumann's theory of self-adjoint extensions of lower bounded symmetric operators.[3] They considered the symmetric operator

$$\mathring{H} := -\sum_{i=1}^{N} \Delta_{x_i} \text{ as acting on } C_0^\infty\left(\mathbb{R}^N \setminus \bigcup_{i<j} \sigma_{ij}\right),$$

where

$$\sigma_{ij} = \langle x = (x_1, \ldots, x_{sN}) \in \mathbb{R}^{sN} \mid x_i = x_j \rangle$$

(σ_{ij} is thus the hyperplane where the particles with coordinates x_i resp. x_j come together.)

$\mathring{H}$ is looked upon as acting on functions in $L^2(\mathbb{R}^{sN})$ that are symmetric under exchange of labelling of coordinates (boson systems).

Applying von Neumann's theory Minlos and Faddeev discussed first abstractly the self-adjoint extensions of $\mathring{H}$ for the case $N = 3, s = 3$, then specialize to those

[3] It is interesting to notice that, parallel to the work of von Neumann, a theory of self-adjoint extensions for (not necessarily lower bounded) self-adjoint operators has been worked out precisely in the mathematics department in Leningrad by M. G. Krein and coworkers, see, e. g. [109, 80, 129].

discussed in terms of a symmetric operator introduced by Skornyakov and Ter-Martirosian as the simplest generalization to the case $N = 3, s = 3$, of a two-body point interaction potential (that we discussed above in Section 1.2).

They found that these extensions of $\mathring{H}$ (after splitting off the center of mass coordinates) which we call STM-Hamiltonians, have a sequence of eigenvalues going down to $-\infty$. They also observe that there might exist other extensions given by "two-body zero-range potentials" which are bounded from below, but do not manage to exhibit examples of these.

The case of three particles in $\mathbb{R}^3$ with different masses has been discussed later by Melnikov and Minlos [108, 107], who prove that also these STM-Hamiltonians present infinitely many eigenvalues going to $-\infty$.

It has been observed by Albeverio, Høegh-Krohn and Streit [10] that a lower bounded self-adjoint extension of $\mathring{H}$ can be obtained by using the theory of Dirichlet-forms. This Hamiltonian is not of the STM-form, since it contains an effective interaction at points where more than two particles coordinates coincide.

In recent years there were attempts devoted to associate to heuristic Hamiltonians of the form (2.1) lower bounded self-adjoint Hamiltonians with only two particle zero range interactions. See e. g. [53, 55, 54, 90, 139, 26].

2.3 Absence of the Thomas effect for systems involving fermions

In the case of N-particle Hamiltonians with two-body point interactions of the heuristic form (2.1) but with Fermi statistics (i. e. acting on spaces of functions which satisfy Fermi statistics) many results on the construction of self-adjoint lower bounded realizations of H heuristically given by (2.1) (hence of the absence of Thomas effect in this case) have been obtained.

Minlos and his coworkers have worked since 1989 especially on the case $N = 3, s = 3$, with two fermions and a particle of another type, see, e. g. references in [26].

In a paper by the "Italian school" the absence of the Thomas effect and the construction of a lower bounded self-adjoint Hamiltonian heuristically given by (2.1) have been obtained for $s = 3$ in the case where all the particles are fermions, but for one of them, see [47, 88, 112, 111, 109, 110, 115, 116] and [26].

More recent work with additional resp. complementary results is in [27, 28, 115, 96, 95].

There are interesting connections of this work with the study of "unitary gases", see, e. g. [45].

Whereas all this work was essentially concerned with the case $s = 3$, the case $s = 2$ (less singular but with its own peculiarities due to the logarithmic type of renormalization required) has been rigorously treated in work by Dell'Antonio, Figari and Teta [60] and by Dimock and Rajeev [65]. These authors constructed self-adjoint extensions H_α (parametrized by a real parameter α, the extensions concerning the symmetric operator acting as $-\Delta$ outside the coincidence planes).

These extensions are lower bounded and given by STM-type boundary conditions on the coincidence hyperplanes.

2.4 The Efimov effect

As we already mentioned, the Efimov effect was discovered (mainly by numerical methods) by V. N. Efimov in 1970. In a talk given almost 40 years later (at the "Symposium on Efimov Physics" in Osnabrück, 2009) Efimov described the development concerning this effect in these decades as going "from questionable to exotic to a hot topic".

In a similar mood, Ferlaino and Grimm have stated "After forty years, Efimov's scenario is now well established as the paradigm of few-body physics and, in a much broader sense, it stands for a new research field with many intriguing opportunities".

In fact, in the last few years, the presence of an Efimov effect was experimentally verified in trapped ultra cold gases and it is believed to serve as a starting point for a better description of low energy nuclear phenomena.

Let us start with a few remarks on the physics of the Efimov effect, see also, e. g. [1, 3, 5, 20, 21, 22, 17, 18, 19, 36, 37, 44, 45, 46, 47, 48, 49, 94, 100, 102, 118, 131, 143, 144].

By numerically investigating a model of three bosons with two-particle sectors without bound states and with the two-particle potentials such that the scattering length is roughly infinite (a situation called nowadays unitary limit) Efimov could see the presence of an infinity of eigenvalues accumulating at a continuum threshold (in fact the point spectrum was seen to have a discrete scaling property).

Two years later Efimov argued (again on the basis of numerical considerations) that a system of two fermions and a different particle shows this effect when the mass m of the latter particle exceeds $(13.607)^{-1}$ times the common mass of the fermions [70]. It was argued that the presence of the Efimov effect is due to an effective long-range interaction created by the influence of the third particle on a given two-body subsystem, due to the presence of a zero energy resonance in this subsystem, see [11, 25]

The challenge for mathematical work was now to rigorously prove the presence of this effect. This was done in the case of three indistinguishable particles with different masses by Yafaev [145] by using a symmetrized form of the three-body Faddeev equations.

The case of two distinguishable particles of the same mass M and a different particle of mass $m \neq M$ was investigated by Ovchinnikov and Sigal in [127] using variational techniques based on a Born Oppenheimer approximation. The two particle potentials are assumed to be rotationally invariant and such that the subsystems made of particles with different masses have a zero energy resonance. In this case they proved results corresponding to those of Yafaev and they also studied the number of eigenvalues in the limit of a very large scattering length.

The case of three indistinguishable particles in $\mathbb{R}^3$ with different masses was taken up again by Sobolev [134] with negative two-body interaction potentials with no bound states, and such that at least two of the two-particle subsystems possess a zero energy resonance. He proved that the discrete part of the spectrum of the three-particle system consists of a number $N(z)$ of eigenvalues (i. e. bound states) of energy less than z, and

$$\lim_{z\to 0} \frac{N(z)}{|\log|z||} = U_0\,,$$

where $U_0 > 0$ is a function depending only on the ratios of the masses of the particles and is independent of the specific form of the two-body potential.

This universality phenomenon was conjectured in [11]. U_0 was estimated by Sobolev and in the case of identical masses an explicit expression was exhibited (this was also conjectured in [11]). These results were extended by Tamura [137]. In [11] a conjecture on the asymptotic behavior of the ratio of successive eigenvalues was also formulated. No proof of this was published (to the best of our knowledge).

2.5 Results on the absence of the Efimov effect

It is also important to know when effectively the phenomenon of the presence of infinitely many bound states in multi-particle systems with short-range interactions is absent.

In the case of three distinguishable particles with different masses Yafaev [145] proved that the point spectrum is finite if there are no two-body bound states, but only one two-body subsystem with a zero energy resonance.

In the case of three particles in space dimensions $s = 1, 2$, Vugalter and Zhislin [141] proved in a quite general setting the absence of the Efimov effect (see also [138]).

The absence of the Efimov effect for three distinguishable particles in a uniform magnetic field was proven in [142].

Many further results have been obtained on the absence of the Efimov effect, see the work by Basti and Teta [27, 28], where in addition to new results a nice survey on further results is presented.

3 Quantum fields and their relations with point interactions

3.1 What are quantum fields?

Quantum field theory is a natural extension of non-relativistic quantum mechanics to the case of fields, e. g. electromagnetic fields. In fact, the study of the classical electromagnetic field played a basic role even at the origin of quantum mechanics, e. g. in establishing Planck's law.

The quantized version of the electromagnetic field, or other relativistic fields, can be looked upon as representing a synthesis of quantum theory and relativity theory. The case where interactions are present (e. g. of the electromagnetic field with matter as described in relativity theory) presents difficulties due to the joint constraints created by the requirements of relativistic invariance and quantum theory. To shortly explain this let us concentrate on the prototypical case of self-interacting relativistic scalar fields φ_{cl} as classically described by a non-linear Klein-Gordon equation:

$$\Box\varphi_{\text{cl}} = -V'(\varphi_{\text{cl}}), \tag{3.1}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta, \quad x \in \mathbb{R}^s,\ t \in \mathbb{R}$$

is the d'Alembert operator (s the space dimension). The classical field φ_{cl} is a real-valued function of time t and space x. V is a potential function of the form

$$V(y) = m_0^2 y^2 + \tilde{V}(y),$$

$m_0 \geq 0$ being a mass, $\tilde{V}$ a real-valued function on $\mathbb{R}$, $y \in \mathbb{R}$.

In the Lagrangian approach by Feynman the quantization is heuristically given by a functional path integral

$$I(f) = \int_\Gamma e^{iS(\gamma)} f(\gamma)\mathrm{d}\gamma,$$

with f a real-valued functional defined on Γ, Γ being the space of all maps γ from $\mathbb{R} \times \mathbb{R}^s$ to $\mathbb{R}$. The relation to the above non-linear Klein-Gordon equation is embodied in the choice of S as the classical action functional

$$S(\gamma) = \frac{1}{2}\int \gamma(t,x)(\Box - m_0^2)\gamma(t,x)\mathrm{d}t\mathrm{d}x - \int \tilde{V}(\gamma(t,x))\mathrm{d}t\mathrm{d}x,$$

such that the field obtained as solution of the variational Euler equation derived from S is precisely the field φ_{cl} satisfying (3.1).

By taking $f(\gamma)$ of the form

$$f(\gamma) = \prod_{i=1}^n \int \gamma(t_i,x_i)\chi_i(t_i,x_i)\mathrm{d}t_i\mathrm{d}x_i,$$

for $\chi_i \in \mathcal{S}(\mathbb{R}^{s+1})$, one obtains from $I(f)$ the Wightman functions $W(\chi_1,\ldots,\chi_n)$ expressing the vacuum expectation values of the product of quantized relativistic fields φ_{Q} associated with φ_{cl}, as operator valued distribution with test functions χ_i (for basic concepts of relativistic quantum field theory like Wightman functions see, e. g., [51, 52, 99, 136]).

Let us observe that, for $\tilde{V} \equiv 0$, $I(f)$ is a Gaussian functional, determined already by its values for f of the above form with $n = 2$. In this case the Wightman function $W(\chi_1,\chi_2)$ (determining all other ones) is

$$\int \chi_1(t_1,x_1)(\Box - m_0^2)^{-1}\chi_2(t_2,x_2)\mathrm{d}t_1\mathrm{d}x_1\mathrm{d}t_2\mathrm{d}x_2,$$

where $(\Box - m_0^2)^{-1}$ is Feynman's propagator.

In the case $\tilde{V} \not\equiv 0$ one performs heuristically an expansion of $I(f)$ in powers of λ_0, λ_0 being a factor put in front of $\tilde{V}$ (i.e., one expands e^{iS} in powers of λ_0, and finally exchanges the summation with the integral).

Taking for $\tilde{V}$ a non-linear function, e. g. $\tilde{V}$ equal a fourth order monomial, one sees, however, (already for space dimension $s = 1$ and in particular for the physical dimension $s = 3$) that the second-order term (i. e. the coefficient of λ_0^2) "diverges" (let us recall that a model where $\tilde{V}$ is chosen as a monomial of order four is called a φ_{s+1}^4-model).

To give a term-wise meaning to the terms on the expansion one needs modifications, usually achieved by renormalization. As a result of such a "perturbative" approach one gets, by hard work, a formal power series, with well defined summation terms (see, e. g. [71, 97]) The next problem is presented by the attribution of a sum to this formal power series. This has been achieved for special $\tilde{V}$ and only for small dimensions $s = 1, 2$ as part of the results of a constructive quantum field theoretical approach, developed since the late 50's for $s = 1, 2$ under special assumptions on $\tilde{V}$ in work first by Friedrichs, Hepp, Segal, Dell'Antonio and others, and then by Glimm, Jaffe, Guerra, Høegh-Krohn, Nelson, Simon and many others, see, e. g. [9, 23, 64, 93, 133].

The above expansion has been justified for these cases as an asymptotic (non-analytic) series, and a summation formula has been shown to hold. However, for the physical case $s = 3$ no constructive work is available, only a term by term well defined (renormalized) divergent perturbation series (at least for the above quartic $\tilde{V}$). At this point a further insight might be helpful, and in any case it was present at an early stage of developments of renormalization theory itself, namely the relation with point interactions.

3.2 Relation with point interactions

Already in 1954, T. D. Lee discussed a simplified model without vacuum polarization, but still needing renormalization. This work inspired Berezin's basic work on point interactions, it was later extended to the case of non-relativistic particles resp. relativistic kinematics interacting with a relativistic field [2, 68, 91, 119, 120] (see also, e. g. [103, 123, 124, 126]).

Further Lee-type models have been discussed later, e. g. in [66, 72, 73, 74] on manifolds.

Already in these models a relation with point interaction models arises, at various stages (see e. g. [9, 15, 97, 135]).

Let us explain heuristically the idea of the relations of point interactions with quantum fields in the non-relativistic limit. For this it is useful to reintroduce the velocity of light in all quantities entering both the classical field equation and the functional $I(f)$ we associated to it.

For the φ_2^4-model, the heuristic Hamiltonian with interaction restricted to a bounded subset Λ of $\mathbb{R}$ has the form

$$H_\Lambda = H_0 + \frac{\lambda_0}{4}\int_\Lambda :\varphi_{\rm Q}^4:(0,x)\mathrm{d}x\,,$$

where $:\varphi_{\rm Q}^4:$ is the Wick-ordered forth power of the operator in Fock space associated with the classical field $\varphi_{\rm cl}(0,x)$ (at time 0 and space point x) and H_0 is the interaction free Hamiltonian (i. e. the Hamiltonian we would have if $\lambda_0 = 0$).

Heuristically the interaction term containing λ_0 is a sum of monomials in creation-annihilation operators a^* resp. a. It is not difficult to show heuristically that the only terms which survive the limit when one sends to $+\infty$ the velocity of light c are those of the form

$$\frac{3\lambda_0}{8m_0^2}\int_{\Lambda\times\Lambda} a^*(x)a^*(y)\delta(x-y)a(x)a(y)\mathrm{d}x\mathrm{d}y$$

These leave the N-particle sectors in the overall Fock space invariant. In the N-particle sector one has then N non-relativistic particles with interactions

$$\text{“}\,\frac{3\lambda_0}{8m_0^2}\sum_{\substack{i,j=1\\ i<j}}^{N}\delta(x_i-x_j)\,\text{”},$$

in the heuristic limit $\Lambda \to \mathbb{R}$. This discussion has been made rigorous in work by Dimock [63].

The non-relativistic limit of the φ_2^4 is thus described by a collection of Hamiltonians for N particles with point interactions in one space dimension (these Hamiltonians are well defined, with no Thomas effect appearing as we mentioned above). For the φ_3^4-model arguments were given in [32, 50, 98] indicating that the absence of the Thomas effect in the N-particle models for $s = 2$ and identical particles discussed above (with the coupling constant λ_0 replaced by a renormalized one) can be related to the necessity of inserting renormalization counterterms permitting the rigorous construction of a quantized φ_3^4-model.

The latter authors extended the discussion also to the case of the φ_4^4-model in a more conjectural way, namely the presence of the Thomas effect for $s = 3$ was put in the relation with the difficulties in proving the existence of a non-trivial renormalized quantum relativistic φ_4^4-model).

In any case it seems that a deepening of the study of point interactions and their renormalization might also give some indications about how to proceed to the construction of models of quantum fields in situations like the case $s = 3$ where this has not yet been achieved. Suggestions in this direction had been presented by E. Nelson in connection with polymer measures over $\mathbb{R}^4$ and the Symanzik relations between Poissonian random fields and the φ^4 interaction, see [9, 67, 104, 122].

Let us finally mention that L. D. Faddeev [80, 81, 82, 83, 84, 85, 62], following the work by Jackiw (in [24]), Rajeev [132] and others, sketched a new possible way to construct renormalized Yang-Mills fields over $\mathbb{R}^4$ by exploiting the dimensionless coupling constant, by a mechanism similar to the one associated with the scale invariance of the point interaction for $s = 2$.

Summarizing: point interactions have a great intrinsic interest and constitute an ideal laboratory for a better understanding of singular phenomena, like those in quantum field theory and related areas.

There are still many open problems, which should be clarified, in particular those concerning N-particle systems, quantum fields and their universality phenomena.

Dedication

Gianfausto Dell'Antonio is one of the founders and promoters of the areas concerned by this paper. It is a great pleasure to dedicate this work to him on the occasion of his 85th anniversary. He has always been for both of us a steady source of inspiration, a teacher and a very close friend.

We congratulate him wholeheartedly on this occasion, we express him all our gratitude and wish him warmly many more years in good health and success in all his undertakings.

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