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A note on the infrared problem in model field theories

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Dedicated to Gianfausto Dell'Antonio on the occasion of his 85th birthday

Abstract. *In this note we critically re-examine the usual procedure of quantization of classical wave equations with static sources. We point out that the origin of infrared difficulties in the so called van Hove model is related to the complex Hilbert space structure one puts on the classical phase space and the corresponding unitarization of the classical symplectic evolution. Whereas in the usual framework the condition of infrared regularity forces the total charge of the external source to vanish, in our setting the infrared regularity condition is equivalent to having a source with a finite (electrostatic) energy. A similar analysis could be applied to models of field-particle interaction such as the Nelson model.*

1 Introduction

The van Hove model is the name by which it is known the most elementary example of a quantum field beyond the free one. Basically it is given by the quantization of the d'Alembert wave equation with a constant fixed source. It was proposed by Leon van Hove [34] in the early fifties as a simplified model where some typical phenomena appearing in Quantum Field Theory, namely the infrared and ultraviolet divergencies, display themselves with manifest evidence. These phenomena were well known to the theoreticians at a heuristic level, in particular in the grand example of Quantum Electrodynamics, but the overwhelming difficulty of that theory prevented (and largely prevents nowadays) a secure comprehension of the fundamental aspects of the problems, at least from a rigorous mathematical point of view. In the van Hove model things are simpler, or at least simple enough to allow some definite statements and their rigorous proofs. Let us begin by giving a short summary of the main features of the van Hove model as it is usually described in the mathematically oriented presentations, and with an emphasis on the infrared problem. A fairly complete analysis is given in [12] and in the recent treatise [2, Chapter 13].

The classical hamiltonian of the van Hove model is given by

$$\mathcal{H}(\varphi,\pi)=\frac{1}{2}\left(\|(-\Delta)^{1/2}\varphi\|^2+\|\pi\|^2\right)+g\langle\rho,\varphi\rangle\,, \tag{1.1}$$

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where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the usual L^2 scalar product and norm respectively, and g is a coupling constant. The corresponding Hamilton equations are

$$\begin{aligned} \dot{\varphi} &= \pi \\ \dot{\pi} &= \Delta\varphi + g\rho \ , \end{aligned}$$

or equivalently

$$\ddot{\varphi} = \Delta\varphi + g\rho \ .$$

Here ρ denotes the density of the source, which in a more realistic model such as the non-relativistic QED, corresponds to the charge density, so that g is proportional to the total charge.

To the classical Hamiltonian can be applied the usual canonical quantization procedure, upgrading the classical dynamical variables φ and π to boson field operators and yielding to the hamiltonian operator

$$H(\rho) = \int_{\mathbb{R}^3} dk\, |k| a^\dagger(k) a(k) + g \int_{\mathbb{R}^3} dk\, \frac{\hat{\rho}(k)}{\sqrt{|k|}} \frac{a^\dagger(k) + a(k)}{\sqrt{2}} \tag{1.2}$$

$$\equiv d\Gamma(\sqrt{-\Delta}) + gS((-\Delta)^{-1/4}\rho) \ . \tag{1.3}$$

Here $a^\dagger$ and a are the usual bosonic creation and annihilation field operators defined on the symmetric Fock space $\mathcal{F}_s(L^2(\mathbb{R}^3))$ and satisfying the canonical commutation relations, $d\Gamma(A)$ is the second quantization of the operator A and with $S(f)$ we denote the Segal field smeared with f. By the Rellich-Kato theorem the Hamiltonian operator just described is self-adjoint on the domain of the free Hamiltonian $d\Gamma(\sqrt{-\Delta})$ if the source ρ satisfies the conditions $(-\Delta)^{-1/4}\rho \in L^2(\mathbb{R}^3)$, $(-\Delta)^{-1/2}\rho \in L^2(\mathbb{R}^3)$. These conditions, which hold true for example for every ρ in the Schwartz class, guarantee the existence of the dynamics, but rule out the case of point sources, i.e. the ultraviolet limit of the model. The point limit could be studied (see [7, 26, 27] for the case of the Pauli-Fierz model in electrodynamics) but here we are concerned with the infrared behavior of the model, so we retain these hypotheses on the density and we refer for further information to the already quoted [2, 12]. The existence of the dynamics does not prevent the model from pathological behavior in the infrared regime of low impulses, that is of the so called "soft bosons". Starting with the early papers where infrared problem was discovered (see [9, 29]), the phenomenon is defined as the divergence of the number of soft bosons, divergence disappearing when a cutoff over low k values is put on the interaction; later on, beginning with the important paper of Schroer [31], emphasis was put on the existence of the ground state for the Hamiltonian, existence which is also quite sensible to the behaviour of the form factor of the source in the region of low k; as Schroer pointed out, remotion of the infrared cutoff makes the ground state dissolving in the continuum and the particle in the ground state disappearing as a stable object, becoming an "infraparticle" (see also the seminal paper [18] and the more recent analysis in [30]). These issues are model dependent, and a definite relation between them is still lacking, at least in general. In this note, restricting ourselves to the the Van Hove model, we show how the form of the condition responsible of the infrared divergence is closely related to the choice of the single particle space used in the quantization of the classical system. In particular, we show that if the single particle space is the classical space of finite energy fields, then the standard Segal quantization procedure gives a quite natural condition for infrared regularity (absence of infrared divergence) in terms of the given source ρ: the source have to be chosen with finite electrostatic energy.

2 The van Hove model: summary of previous results.

Now we consider the van Hove model and in the following Theorem we summarize a lot of previous results from the original van Hove's paper, through the first rigorous mathematical analysis due to Cook [11] and Schroer [31] in the sixties, to the more refined present studies (see e.g. [20], [12] and [2]).

Theorem 2.1. *Assume that*

$$(-\Delta)^{-1/4}\rho \in L^2(\mathbb{R}^3)\,, \qquad (-\Delta)^{-1/2}\rho \in L^2(\mathbb{R}^3)\,. \tag{2.1}$$

1) The operator

$$H(\rho) = d\Gamma(\sqrt{-\Delta}) + gS((-\Delta)^{-1/4}\rho) \tag{2.2}$$

is self-adjoint on the domain of $d\Gamma(\sqrt{-\Delta})$.

2) $H(\rho)$ *is bounded from below and the infimum of the spectrum is given by*

$$\inf \sigma(H(\rho)) = -\frac{g^2}{2}\,\|(-\Delta)^{-1/2}\rho\|^2\,. \tag{2.3}$$

3) In the same hypothesis as above, the three following conditions are equivalent:

$$\textit{the infrared singularity condition } (-\Delta)^{-3/4}\rho \notin L^2(\mathbb{R}^3) \textit{ holds true;} \tag{2.4}$$

$$\textit{there is no ground state } \Omega(\rho) \textit{ for } H(\rho) \textit{ belonging to } \mathcal{F}_s(L^2(\mathbb{R}^3)); \tag{2.5}$$

the expectation value of the number operator $N := d\Gamma(1)$ *on the ground state diverges, i.e.,*

$$\lim_{K\downarrow 0} \langle \Omega(\rho_K), N\Omega(\rho_K)\rangle_{\mathcal{F}} = +\infty\,, \tag{2.6}$$

where ρ_K *is the infrared truncation of the source defined by* $\hat\rho_K = \chi_K\hat\rho$, χ_K *being the characteristic function of the set* $\{k \in \mathbb{R}^3 : |k| > K\}$, *and* $\Omega(\rho_K)$ *denotes the ground state of the regularized Hamiltonian* $H(\rho_K)$.

Remark 2.2. The condition $(-\Delta)^{-\frac{1}{2}}\rho \in L^2(\mathbb{R}^3)$ is not necessary to prove essential selfadjointness, but only to have boundedness from below. Essential selfadjointness can be proved to hold under the sole condition $(-\Delta)^{-\frac{1}{4}}\rho \in L^2(\mathbb{R}^3)$ on any core of $d\Gamma(\sqrt{-\Delta}+1)$ e.g. by Nelson commutator theorem. The condition $(-\Delta)^{-1/4}\rho \in L^2(\mathbb{R}^3)$ can be avoided at the price of defining $H(\rho)$ as sum of quadratic forms. These generalization however are not relevant to our main issue (see for more information and references [2, 12, 16]).

Remark 2.3. When condition (2.4) holds, the source is called *infrared singular*; in the opposite case, the source is *infrared regular*.

Remark 2.4. In the terminology of the Dereziński paper [12], when hypotheses of the theorem hold and the source is infrared regular we have a Type I infrared problem (in fact, no infrared problem) and when the infrared regularity condition fails we are in the Type II infrared problem (the Hamiltonian is bounded from below but no ground state exists). Finally, one has a Type III infrared problem if $(-\Delta)^{-\frac{1}{2}}\rho \notin L^2(\mathbb{R}^3)$.

The proof of this theorem as regard self-adjointness is an application of the Kato-Rellich theorem, because the interaction part of the Hamiltonian turns out to be, in the stated hypotheses, operator bounded w.r.t. the free part on the Fock space. The successive parts amount to the construction of a suitable canonical transformation which implements on the Fock space the "completion of the square" of the Hamiltonian operator; such canonical transformation exists under the condition $(-\Delta)^{-3/4}\rho \in L^2(\mathbb{R}^3)$. Accordingly, the ground state turns out to be a coherent state depending on the source ρ and sensible to the validity of such a condition. This goes back, in his rigorous form, to a paper of Cook [11]; further detailed informations and complete proofs can be found in [2, 12].

We give some comments on the three conditions stated in the theorem.

As already said, condition (2.4) is referred to as condition of infrared singularity and its failure infrared regularity. The two main equivalent consequences of infrared singularity are described by (2.5) and (2.6). In particular, persistence or dissolution of ground state described in (2.5) is a kind of stability, respectively instability, and it is related to the infra-particle concept recalled in the Introduction. Condition (2.6) is a version of the traditional wisdom about the infrared problem: it amounts to divergence of the number of soft bosons when infrared cutoff is removed.

The main remark we want to stress about this infrared regularity condition concerns the fact that it forces the source to have vanishing total charge. Recall that the total charge of the source ρ is given by $\hat{\rho}(0) = \int_{\mathbb{R}^3} dx\, \rho(x)$; due to the fact that $|k|^{-3/2}$ is not square integrable in any neighborhood of the origin, infrared regularity requires $\hat{\rho}(k) \to 0$ as $|k| \to 0$. Thus, to satisfy the infrared regularity condition, one has to keep vanishing the total charge of the source, and the infrared problem at this level resides precisely in this unsuspected relation between the absence or existence of the ground state, or divergence of the number of bosons, and the total charge of the source.

The same condition appears also in different models of nonrelativistic quantum field theory. In particular in the Nelson model (see [25]) the situation is quite similar to the van Hove model, which could be considered as its caricature. More precisely, it is possible to prove (see the interesting paper by M.Hirokawa [20]) that at least as far as the infrared behavior is concerned, the "singular kernel" of the massless Nelson Hamiltonian is given exactly by the van Hove contribution.

A way out of the infrared problem has been proposed by A.Arai (see [1, 28]), where it is shown that a ground state for the massless Nelson Hamiltonian exists in a non-Fock representation of the canonical commutation relations of the field. A classical interpretation of the Arai analysis is given in [10].

The situation is more structured in the Pauli-Fierz model. This model, contrarily to the ones previously discussed, is physically realistic, describing quite accurately the interaction between matter and radiation at low energy. In the Pauli-Fierz model the infrared singularity condition plays a role (see e.g. [3, 4, 13, 21]) but there exist also results of infrared regularity when spin variables are taken into account, or in presence of external potentials, such as Coulomb or more generally atomic potentials (see e.g. [5, 6, 19]). In any case, in the realistic example of quantum electrodynamics the vanishing of total charge assures, through the Gauss law, the absence of Coulomb tails, preventing a too slow decay of the field at large distance. This is a convincing interpretation of the infrared regularity assumption. In either the van Hove or the Nelson model, which are mathematical abstractions, the above interpretation is not physically tenable, but nevertheless it is usually considered as sufficiently suggestive and indicative of the general phenomenon. On the other hand, it is a quite unsatisfying fact that only globally neutral matter turns out to be free from infrared problems, at least ignoring the role of the electron spin or external potentials.

3 The Van Hove model: finite energy single particle space and infrared regularity

In the usual description outlined above, the quantization of the system proceeds from the classical (Hilbert) phase space of the couples (φ, π) such that

$$\|(-\Delta)^{1/4}\varphi\|^2 + \|(-\Delta)^{-1/4}\pi\|^2 < +\infty\,. \tag{3.1}$$

The choice of this space is due to the fact that it allows to obtain in a plain way an explicitely covariant representation of the quantum free field. A different motivation of this choice, purely non relativistic, is given in [10]. On this classical phase space, it is well defined by duality the standard symplectic form ω

$$\omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \langle \varphi_1, \pi_2 \rangle - \langle \varphi_2, \pi_1 \rangle\ ,$$

with respect to which the van Hove Hamiltonian formally generates the standard equations of the model. We take the quite spontaneous point of view to consider the equations of motion the primary object of study.

This symplectic real Hilbert space, suitably complexified through the complex structure

$$J(\varphi,\pi)=(-(-\Delta)^{-1/2}\pi,(-\Delta)^{1/2}\varphi)$$

becomes a complex Hilbert space, with inner product

$$[(\varphi_1,\pi_1),(\varphi_2,\pi_2)]:=\omega((\varphi_1,\pi_1),J(\varphi_2,\pi_2))+i\omega((\varphi_1,\pi_1),(\varphi_2,\pi_2))$$

which gives, in the usual Segal quantization scheme [33], the one particle space on which the quantum field is built up. Note that the "quantization" structure of a symplectic phase space is rigid, in the sense that given a symplectic form and a complex structure, the corresponding complex Hilbert space is fixed, and conversely, given a complex Hilbert space, there is a unique real symplectic Hilbert space whose symplectic form is the imaginary part of the complex inner product and the symmetric bilinear form corresponding to its real inner product is the real part of the complex inner product. In this representation the quantization map, i.e., the representative of the classical state in the complex Hilbert space is just the identity map. A standard procedure is then to pass to the reference complex Hilbert space of complex-valued square integrable functions. The unitary map which realizes this identification is given by

$$(\varphi,\pi)\,\mapsto\,\psi:=(-\Delta)^{1/4}\varphi+i(-\Delta)^{-1/4}\pi\;. \tag{3.2}$$

As a matter of fact, in the context of the Segal quantization the one particle structure turns out to be unique up to unitary maps once fixed the classical phase space and the symplectic flow (see [35, 24]). Different identification maps (sometimes called quantization maps) are in fact unitary equivalent. Finally, thanks to the linear character of the dynamics, the quantization map intertwines between the classical wave flow and the quantum Schrödinger flow. Alternatively, the wave equation in the classical real phase space corresponds in a unique way through the quantization map, to a Schrödinger equation in the single particle space given by

$$i\dot{\psi}=(-\Delta)^{1/2}\psi+g(-\Delta)^{-1/4}\rho\,. \tag{3.3}$$

Note the singular character of the source in the Schrödinger dynamics, as a consequence of the action of the unitary quantization map between the classical phase space and the model space $L^2(\mathbb{R}^3)$. This singularity in the Schrödinger source is the counterpart of the fact that the original dynamics takes place in a phase space which does not allow fields too much singular in the region of low momenta. For example, consider a general solution of the wave equation. Besides the free field component which simply drags along the initial data, it contains the term

$$g\left(1-\cos t(-\Delta)^{1/2}\right)(-\Delta)^{-1}\rho\,.$$

Analogously, in the first quantized, Schrödinger dynamics, the solution is given by

$$g\left(1 - e^{-it(-\Delta)^{1/2}}\right)(-\Delta)^{-1/2}(-\Delta)^{-1/4}\rho\,.$$

The dynamics on the one particle space is lifted in the standard way to the Fock space and yields the hamiltonian

$$H(\rho) = d\Gamma(\sqrt{-\Delta}) + gS((-\Delta)^{-1/4}\rho)$$

and so the second quantization simply drags along with itself the singular behavior of the Schrödinger dynamics. But this singular behavior is in the ultimate a consequence of the particular choice of the phase space for the classical dynamics, which does not contain the field attached to the source if the source itself is not infrared regular.

The situation just described is general and it is interesting to see which infrared singularity condition corresponds to different choices of single particle spaces. In particular, let us choose as a phase space where the classical dynamics takes place the space of finite energy fields. The classical energy of the field is given by the functional

$$\mathcal{E}(\varphi,\pi) = \frac{1}{2}\left(\|(-\Delta)^{\frac{1}{2}}\varphi\|^2 + \|\pi\|^2\right). \tag{3.4}$$

The idea we want to pursue is to quantize the system starting from a single particle space in which the field has finite energy.

Remark 3.1. It is true that the energy, being just a component of a four-vector, is not a covariant concept. But the model from the beginning is non relativistic, due to the presence of the fixed form factor ρ. So we do not pretend to interpret an element of the single particle space or of the corresponding Fock space as a state of a relativistic particle as a photon. Rather it could describe a kind of non-relativistic boson such as a phonon.

The set of finite energy fields is a symplectic space with respect to a symplectic form $\omega_{\mathcal{E}}$ different from the standard one, and given by

$$\omega_{\mathcal{E}}((\varphi_1,\pi_1),(\varphi_2,\pi_2)) = \langle(-\Delta)^{1/2}\varphi_1,\pi_2\rangle - \langle(-\Delta)^{1/2}\varphi_2,\pi_1\rangle\,.$$

Correspondingly there is the complex structure

$$J_{\mathcal{E}}(\varphi,\pi) = (-(-\Delta)^{-1/2}\pi, (-\Delta)^{1/2}\varphi)$$

which makes the phase space of finite energy classical fields a complex Hilbert space with inner product

$$[(\varphi_1,\pi_1),(\varphi_2,\pi_2)]_{\mathcal{E}} := \omega_{\mathcal{E}}((\varphi_1,\pi_1), J_{\mathcal{E}}(\varphi_2,\pi_2)) + i\omega_{\mathcal{E}}((\varphi_1,\pi_1),(\varphi_2,\pi_2))\,.$$

By means of the unitary map

$$(\varphi,\pi) \mapsto \psi := (-\Delta)^{1/2}\varphi + i\pi \tag{3.5}$$

the symplectic wave dynamics in the classical phase space of finite energy fields is translated into a unitary Schrödinger dynamics on the Hilbert space of complex-valued square integrable functions. The corresponding Schrödinger-like equation is

$$i\dot{\psi} = (-\Delta)^{1/2}\psi + g\rho\,, \tag{3.6}$$

which leads to the second quantization hamiltonian

$$H_{\mathcal{E}}(\rho) = d\Gamma(\sqrt{-\Delta}) + gS(\rho)\,.$$

In this case, after guaranteeing the existence of the dynamics in the space of finite energy fields, a quite simple and natural infrared regularity condition appears for the second quantized model, i.e.

$$E(\rho) := \int_{\mathbb{R}^6} dx\,dy\,\frac{\rho(x)\rho(y)}{|x-y|} \equiv \langle\rho, (-\Delta)^{-1}\rho\rangle < +\infty \tag{3.7}$$

or stated in other terms, the (electrostatic) energy of the charge is finite. More precisely, in this different framework, the analogous of Theorem 2.1 (note that the proof is the same, simply replace $(-\Delta)^{-1/4}\rho$ with ρ) is the following:

Theorem 3.2. *Assume that*

$$\rho \in L^2(\mathbb{R}^3)\,, \qquad (-\Delta)^{-1/4}\rho \in L^2(\mathbb{R}^3)\,. \tag{3.8}$$

1) The operator

$$H_{\mathcal{E}}(\rho) = d\Gamma(\sqrt{-\Delta}) + gS(\rho) \tag{3.9}$$

is self-adjoint on the domain of $d\Gamma(\sqrt{-\Delta})$.

2) $H_{\mathcal{E}}(\rho)$ *is bounded from below and the infimum of the spectrum is given by*

$$\inf\sigma(H_{\mathcal{E}}(\rho)) = -\frac{g^2}{2}\,\|(-\Delta)^{-1/4}\rho\|^2\,. \tag{3.10}$$

3) In the same hypothesis as above, the three following conditions are equivalent:

the infrared singularity condition $E(\rho) = +\infty$ *holds true;* (3.11)

there is no ground state $\Omega_{\mathcal{E}}(\rho)$ *for* $H_{\mathcal{E}}(\rho)$ *belonging to* $\mathcal{F}_s(L^2(\mathbb{R}^3))$*;* (3.12)

the expectation value of the number operator N *on the ground state diverges, i.e.,*

$$\lim_{K\downarrow 0}\langle\Omega_{\mathcal{E}}(\rho_K)\,N\Omega_{\mathcal{E}}(\rho_K)\rangle = +\infty\,, \tag{3.13}$$

where the infrared truncation of the source ρ_K *is defined as before and* $\Omega_{\mathcal{E}}(\rho_K)$ *denotes the ground state of the regularized Hamiltonian* $H_{\mathcal{E}}(\rho_K)$.

Remark 3.3. We observe that the infrared regularity condition $E(\rho) < +\infty$, by Fourier transform, Riemann-Lebesgue lemma and Hölder inequality, holds for any density in $L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Moreover $\rho \in L^2(\mathbb{R}^3)$ and $E(\rho) < +\infty$ imply $(-\Delta)^{-1/4}\rho \in L^2(\mathbb{R}^3)$. Thus the conditions of the theorem in this representation do not force any constraint on the total charge of the source, and this is the main advantage of this description.

Remark 3.4. Notice that the two different quantization procedures do not correspond to different representations of the Canonical Commutation Relations for the boson field. The classical phase space is quite different in the two constructions, and the function belonging to them have different behaviour just in the region of low impulses; nevertheless, there is a simple relation between the elements in one space and the elements in the other one: just operates with $(-\Delta)^{1/4}$ on the two components to pass from one to the other. This map is a (isometric) symplectomorphism between classical phase spaces, which thanks to the Shale theorem [32] it is implementable to a unitary map between the two Fock spaces constructed on the two single particle spaces.

Remark 3.5. In the common view, a way out of the infrared problem should consist in building a representation of the CCR inequivalent to the usual one, in which there exist a ground state for the dynamics on the Fock space (such a procedure was exploited for similar models in [17] and [8]). As already remarked in the previous section, a successful construction of this kind was given for the Nelson model by Arai [1]. This representation, which works well for the van Hove model too, is affine, and amounts to sum to the free field the "contribution of the field of the source". Correspondingly, in this representation there exists the ground state just because its classical representative has been attached to the conventional one particle space. In our description, the representation of the CCR is the usual Fock one, but the Hamiltonian $H_{\mathcal{E}}(\rho)$ differs from the standard one $H(\rho)$. However let us point out the fact that these two seemingly different Hamiltonians give rise to unitary flows which implement *the same classical dynamics*. Moreover the Hamiltonians $H(\rho)$ and $H_{\mathcal{E}}(\rho)$ turn out to be unitarily equivalent (up to a constant) if and only if the density ρ simultaneously satisfy both hypotheses of self adjointness and infrared regularity given in the Theorems 1 and 2.

Remark 3.6. Other points of view about quantization of classical field theories can be adopted, with different emphasis on different stages of the quantization process. A slightly different approach is to consider the "classical phase or state space" essentially a matter of choice of a space of test functions suitable for massless fields. Then, in order to quantize the system in a physically relevant way, the space of test functions has to be endowed with an inner product and a symplectic form. This determines if and which Fock representation could be defined, and consequently the form of the Wick quantized Hamiltonian. The choice of a test function space for the (operator valued distributions) quantized fields can be driven by the supposed properties of classical fields, and the corresponding physically relevant dynamics. The consequences of different choices can be relevant;

for example, the request of having relativistic covariant free fields yield different infrared regularity conditions on the source compared to the request of having free fields with finite energy.

We conclude with some further perspective comments.

A first open problem is the extension of the present results to more meaningful situations, for example the Nelson model. Thanks to the results shown in [20] the Van Hove Hamiltonian differs from the Nelson Hamiltonian by a term which is under control as regards its infrared behavior. So the use of finite energy spaces and corresponding unitary maps on the standard $L^2(\mathbb{R}^3)$ space, could allow to obtain existence of ground states and control of mean values of number operator under physically sensible conditions on the particle density ρ and without resorting to inequivalent representation of CCR as in the Arai paper [1].

Another interesting possible development is the relation between the classical and the quantum nonrelativistic particle-field models. In particular it seems to be relevant an understanding of the asymptotic time behavior of the classical systems of Nelson and Pauli-Fierz. There it is quite natural to consider finite energy phase space for the fields. It is interesting that in purely classical versions of the Nelson or Pauli-Fierz models, vanishing of total charge or sufficiently fast decay of the initial fields assure a satisfying time asymptotics of the solution for the coupled field-particle system (see [23] and the review [22], and for well posedness and information on classical limit [14, 15]). The expected correct asymptotics for the particle-system solution is described as an asymptotically free particle with a boosted Coulomb-like field attached to it. However in the absence of external confining fields or sufficiently fast decaying fields, this is proven only when the charge and also some higher moments of the density charge vanish ([22] and references therein). In this sense the infrared problem, as often said, has a classical origin. So to remove the problem already at this level would be a desirable and welcome completion of the view.

We plan to come back to both problems in future work.

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