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# Recent advances in symmetry of stochastic differential equations

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*To Gianfausto on his* $85^{th}$ *birthday*

**Abstract.** *We discuss some recent advances concerning the symmetry of stochastic differential equations, and on particular the interrelations between these and the integrability – complete or partial – of the equations.*

## 1 Introduction

The modern theory of Symmetry was laid down by Sophus Lie (1842-1899). The motivation behind the work of Lie was not in pure Algebra, but instead in the effort to *solve differential equations*. This was highly successful; so the question we want to answer is: can we do something similar for *stochastic* differential equations?

In this short note we first sketch how the theory of symmetry helps in determining solutions of (deterministic) differential equations, both ODEs and PDEs; we will be staying within the classical theory (Lie-point symmetries), work in coordinates, and only consider continuous symmetries [1, 4, 14, 18, 19, 22]. We will then discuss the recent extension of this theory to *stochastic (ordinary) differential equations*.

## 2 Symmetry of deterministic equations

### 2.1 The Jet space

The key idea for a proper treatment of symmetry of (deterministic) differential equations goes back to E. Cartan and Ch. Ehresmann. It consists in the introduction of the *jet bundle* (or jet space if we deal with problems in Euclidean framework) [2, 18, 19, 21].

We denote as *phase bundle* (or *phase space*) the differentiable manifold of dependent $(u^1, \ldots, u^p)$ and independent $(x^1, \ldots, x^q)$ variables; this is naturally seen as a bundle (with the manifold $B$ where the independent variables live as the basis) $(M, \pi_0, B)$.

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The Jet bundle (of order $n$) $J^n M$ is then the space of dependent $(u^1, \ldots, u^p)$ and independent $(x^1, \ldots, x^q)$ variables, together with the partial derivatives (up to order $n$) of the $u$ with respect to the $x$; this has also a natural structure of fiber bundle, $(J^n M, \pi_n, B)$.

We should however keep into account that the $u^a_J$ represents derivatives of the $u^a$ w.r.t. the $x^i$. In order to do this, the jet space should be equipped with an additional structure, the *contact structure* [2, 21].

This can be expressed by introducing the one-forms

$$\omega^a_J \ := \ \mathrm{d}u^a_J \ - \ \sum_{i=1}^q u^a_{J,i} \, \mathrm{d}x^i \ ,$$

which are called the *contact forms*, and looking at their kernel.

We also mention that moreover, each Jet $J^n M$ is also a fiber bundle over Jets of lower order; that is, we also have bundles $(J^n M, \pi_{n,k}, J^k M)$ for all $0 < k < n$. This is at the basis of the recursive construction of prolongations of vector fields (see below).

Jets are a natural generalization of the familiar geometric description of vector fields: a vector at a given point $x \in M$ can be seen as an equivalence class of curves in $M$ (mutually tangent at $x$), and a vector field as the choice of a vector at each point, and as a section of the tangent bundle $TM$. In the same way, a jet of order $k$ at a given point $x \in M$ can be seen as an equivalence class of curves in $M$ (mutually tangent of order $k$ at $x$), and a jet field as the choice of a jet at each point, and as a section of the jet bundle $J^k M$, with $J^1 M = TM$, $J^{k+1} M = T(J^k M)$. We refer e.g. to [1, 18, 19, 20, 21] for further detail on Jet bundles and their Geometry.

### 2.2 Geometry of differential equations, contact structure, prolongation

A differential equation $\Delta$ determines a manifold in $J^n M$, the *solution manifold* $S_\Delta \subset J^n M$ for $\Delta$. This is a *geometrical* object; the differential equation can be identified with it, and we can apply geometrical tools to study it.

An infinitesimal transformation of the $x$ and $u$ variables is described by a *vector field* in $M$; once this is defined the transformations of the derivatives are also implicitly defined.

The procedure of extending a vector field in $M$ to a vector field in $J^n M$ by requiring the preservation of the contact structure – thus so that derivatives transform in the natural way once the transformations of dependent and independent variables are given – is also called *prolongation* [1, 4, 14, 18, 19, 22].

If the vector field $X$ on $M$ is expressed in the local coordinates $(x, u)$ as

$$X \ = \ \xi^i(x,u) \, \frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u) \, \frac{\partial}{\partial u^a} \ ,$$

its $n$-th order prolongation $X^{(n)}$ on $J^n M$ is written – in the local coordinates $(x, u^{(n)})$ and in multi-index notation – as

$$X^{(n)} \;=\; \xi^i(x,u)\,\frac{\partial}{\partial x^i} \;+\; \psi^a_J(x,u^{(|J|)})\,\frac{\partial}{\partial u^a_J} \ ;$$

the coefficients $\psi^a_J$ are provided (recursively) by the *prolongation formula*

$$\psi^a_{J,i} \;=\; D_i \psi^a_J \;-\; u^a_{J,k}\, D_i \xi^k \ ; \qquad \psi^a_0 \;=\; \varphi^a \ .$$

## 2.3 Symmetry

A vector field $X$ defined in $M$ is then a *symmetry* of $\Delta$ if its prolongation $X^{(n)}$, satisfies

$$X^{(n)} : S_\Delta \;\to\; \mathrm{T} S_\Delta \ .$$

Note this is a (geometrical) relations among geometrical objects – a vector field and a manifold – and is hence independent of our choices of coordinates: as we expect, symmetries will still be present (or absent) if we change variables.

An equivalent characterization of symmetries is to map solutions into (generally, different) solutions. In the case a solution is mapped into itself, we speak of an *invariant solution*.

A first use of symmetry can be that of *generating new solutions from known ones*. For example, acting with (nontrivial) symmetries, the solution $u = 0$ to the heat equation get transformed into the fundamental (Gauss) solution; see e.g. Chapter 3 in [18].

As we will see, this is by far not the only way in which knowing (all or some of) the symmetries of a differential equation can help in determining (all or some of) its solutions.

In order to use the symmetries of a differential equation, we should of course first of all know what these symmetries are, i.e. determine them. Determining the symmetry of a given differential equation goes through the solution of a system of coupled *linear* PDEs, known indeed as the *determining equations*.

The procedure for solving them is in general algorithmic and can be implemented via computer algebra; the exception here is the case of (systems of) first order ODEs, i.e. Dynamical Systems.

## 2.4 Using the symmetry

The key idea is the same for ODEs and PDEs, and amounts to the use of *symmetry adapted coordinates*. But the scope of the application of symmetry methods is rather different in the two cases, and thus so is the actual meaning of “adapted”. We will only consider scalar equations for ease of discussion.

### 2.4.1 Symmetry and ODEs

If an ODE $\Delta$ of order $n$ admits a Lie-point symmetry $X$, the equation can be *reduced* to an equation of order $n-1$. The solutions to the original and to the reduced equations are in correspondence through a quadrature (which of course introduces an integration constant).

The main idea is to change variables $(x, u) \to (y, v)$, so that in the new variables the symmetry vector field $X$ reads

$$X = \partial / \partial v .$$

As $X$ *is still a symmetry*, this means that the equation will not depend on $v$, only on its derivatives.

At this point, with a new change of coordinates $w := v_y$ we reduce the equation to one of lower order.

A solution $w = h(y)$ to the reduced equation identifies solutions $v = g(y)$ to the original equation (in "intermediate" coordinates) simply by integrating,

$$v(y) = \int w(y)\, dy ;$$

a constant of integration will appear here. Finally go back to the original coordinates inverting the first change of coordinates.

Note that the reduced equation could still be too hard to solve. That is, the method can only guarantee that we are reduced to a problem of lower order, i.e. hopefully simpler than the original one.

If we are able to solve this reduced problem, then solutions to the original and the reduced problem are in (many to one) correspondence.

This approach extends, with certain algebraic conditions, to the case where multiple symmetries are present, and correspondingly multiple reductions are possible – at least if the symmetry vector field span a solvable Lie algebra [18].

### 2.4.2 Symmetry and PDEs

The approach in the case of PDEs is in a way at the opposite as the one for ODEs. If $X$ is a symmetry for $\Delta$, we change coordinates $(x, t; u) \to (y, s; v)$ so that in the new coordinates

$$X = \frac{\partial}{\partial y} .$$

Now our goal will *not* be to obtain a general reduction of the equation, but instead to obtain a (reduced) equation which determines the *invariant solutions* to the original equation.[1]

[1] The reason for this is quite clear: changing coordinates so that the vector field is written as $X = (\partial/\partial v)$, as in the ODE case, would lead to an equation not explicitly depending on $v$ (as in the ODE case), i.e. an equation in which only derivatives of $v$ appear (as in Hamilton-Jacobi). But as different partial derivatives are present, we cannot reduce the order of the equation and thus, in general, have no real advantage by such a transformation.

In the new coordinates, this is just obtained by *imposing* $v_y = 0$, i.e. $v = v(s)$. The reduced equation will have (one) less independent variables than the original one.

This reduced equation will *not* have solutions in correspondence with general solutions to the original equation: only the invariant solutions will be common to the two equations. Contrary to the ODE case, we do not need to solve any "reconstruction problem".

**Remark 2.1.** It was shown by Kumei and Bluman [15] that the (algorithmic) symmetry analysis is also able to detect if a nonlinear equation can be linearized by a change of coordinates. The reason is that the underlying linearity will show up through a Lie algebra reflecting the *superposition principle*. Similar, albeit more delicate, considerations lead to relating suitable symmetries and the presence of a *nonlinear superposition principle* [3].

**Remark 2.2.** The concept of symmetry was generalized in many ways; this extends the range of applicability of the theory. We are not discussing these, but just refer e.g. to [4, 14, 18, 22].

### 2.4.3 ODEs vs PDEs reduction

Note that in geometrical terms, the difference between the ODE and the PDE reduction approach is clearly understood in terms of the fibration $(M, \pi_0, B)$, see Section 2.1: in both cases we straighten the vector field $X$, but in the ODE case this is done so that in the new coordinates, and hence in the new fibration $(M, \widetilde{\pi}_0, \widetilde{B})$, $X$ is a vertical vector field; while in the PDE case this is so that in the new fibration $X$ has no vertical component.

Correspondingly, in the ODE case $X$ acts on (sections of $(M, \widetilde{\pi}_0, \widetilde{N})$ representing) solutions by parallel transporting them along fibers; in the new variables $(y, v)$ new solutions are obtained from known ones, acting with $X$, by the addition of a constant, which is just the integration constant arising from the quadrature linking $v(y)$ to $w(y)$.

In the PDE case instead the (sections of $(M, \widetilde{\pi}_0, \widetilde{N})$ representing) solutions are invariant when transported horizontally; this means that the corresponding sections have some flat directions, and thus depend effectively on a smaller number of variables than general solutions.

## 3 Symmetry of SDEs

We will now see how the classical symmetry theory for (deterministic) differential equations can be extended to the framework of *stochastic* differential equations.

### 3.1 Types of symmetries for SDEs

We consider an Ito SDE

$$dx^i \ = \ f^i(x,t)\, dt \ + \ \sigma^i_j(x,t)\, dw^j \tag{3.1}$$

(note by this we always mean a vector one, i.e. a system of SDEs), and a general vector field acting in the $(x,t)$ space,

$$X \ = \ \tau\, \partial_t \ + \ \xi^i\, \partial_i \ . \tag{3.2}$$

Note that we allow, in general, the coefficients $\xi^i$ of $X$ to depend on the $(x,t,w)$ variables, while it makes sense to restrict the dependence of $\tau$ to the $t$ variable alone [10].

The vector field $X$ in (3.2) is a symmetry of the Ito equation (3.1) if it satisfies the suitable determining equations; in the general case these are rather involved (see [10] for their explicit expression), and will not be reported here.

We distinguish different types of symmetries. In particular, *simple* symmetries act only on the $x$, while *general* symmetries[2] act on both the $x$ and $t$. We will also distinguish between *deterministic* symmetries, i.e. those for which – with reference to (3.2) – we have $\xi = \xi(x,t)$ and $\tau = \tau(t)$; and *random* symmetries, i.e. those with $\xi = \xi(x,t,w)$, $\tau = \tau(t)$.

We are specially interested, for reason which will be clear in the following, in *simple* (possibly random) symmetries, hence in the case $\tau = 0$ in (3.2). In this case the determining equations for (simple) symmetries of (3.1) read

$$\begin{cases} \partial_t \xi^i \ + \ f^j\, (\partial_j \xi^i) \ - \ \xi^j\, (\partial_j f^i) \ = \ -\frac{1}{2} (\triangle \xi^i) & , \\ \widehat{\partial}_k \xi^i \ + \ \sigma^j{}_k\, (\partial_j \xi^i) \ - \ \xi^j\, (\partial_j \sigma^i{}_k) \ = \ 0 & . \end{cases} \tag{3.3}$$

Here $\widehat{\partial}_i := \partial / \partial w^i$, and the symbol $\triangle$ denotes the Ito Laplacian

$$\triangle u \ := \ \sum_{j,k=1}^n \left[ \left( \sigma\, \sigma^T \right)^{jk} \frac{\partial^2 u}{\partial x^j \partial x^k} \ + \ 2\, \sigma^{ik}\, \frac{\partial^2 u}{\partial x^j \partial w^k} \ + \ \delta^{jk}\, \frac{\partial^2 u}{\partial w^j \partial w^k} \right] \ . \tag{3.4}$$

**Remark 3.1.** The case with $\xi^i = \xi^i(x,t)$ and $\tau = \tau(x,t)$ would also deserve the name of “deterministic”, but it is not acceptable in view of other considerations (roughly speaking because we want to keep $t$ as a deterministic smooth variable, while $x$ is in this context a random one, and hence we should not mix it with $t$); see the discussion in [10]. Similar considerations apply also to the case with $\xi^i(x,t;w)$, where one would be tempted to consider $\tau = \tau(x,t;w)$ rather than just $\tau = \tau(t)$.

[2] Actually, besides these, also *W-symmetries* are possible (these also act on the $w^j$), but will not be considered here. They are characterized by more general equations, reducing to (3.3) for vector fields of the form (3.2); see the discussion in [10].

### 3.2 Symmetry of SDEs and change of variables

When we look at symmetry of a SDEs *per se* a substantial problem is present.

In fact, the symmetry approach is based on passing to symmetry-adapted coordinates; vector fields transform "geometrically" (i.e. via the chain rule) under changes of coordinates, and *deterministic* differential equations are (identified with) geometrical objects, hence also transform geometrically. It is then obvious that *symmetry are preserved under changes of coordinates*, as already stressed above.

On the other hand, an Ito equation is *not* a geometrical object: in fact, it transforms under the Ito rule, not the chain rule. Thus it is *not* granted that $X$ will still be a symmetry when we change coordinates so that $X = \partial_x$! Note this is also true for *deterministic* symmetries of stochastic equations.

The easy way out of this problem would be giving up Ito equations and using *Stratonovich equations* instead. These do transform according to the chain rule, i.e. geometrically; but the relation between an Ito and the corresponding Stratonovich process is not that obvious – especially in this respect [23].

In fact, it is known that in general the two do *not* share the same symmetries [24]. But it is also known that they have the same *simple* symmetries, and this both in the deterministic [24] and in the random [7] case. This fact is specially interesting, as the Kozlov theory [11, 12, 13] relating symmetry to integrability of SDEs only makes use of simple symmetries.

We note that the determining equations for simple symmetries – which we still write in the general form (3.2) – of a Stratonovich equation

$$dx^i \ = \ b^i(x,t)\, dt \ + \ \sigma^i{}_k(x,t) \circ dw^k \tag{3.5}$$

turn out to be [10]

$$\begin{cases} \partial_t \xi^i \ + \ b^j (\partial_j \xi^i) \ - \ \xi^j (\partial_j b^i) \ = \ 0 & , \\ \widehat{\partial}_k \xi^i \ + \ \sigma^j{}_k (\partial_j \xi^i) \ - \ \xi^j (\partial_j \sigma^i{}_k) \ = \ 0 & . \end{cases} \tag{3.6}$$

(The reader is again referred to [10] for the general case.)

In particular, if we consider the Stratonovich equation associated to the Ito equation (3.1), i.e. for

$$f^i \ = \ b^i \ + \ \frac{1}{2} \left[ \frac{\partial (\sigma^T)^i{}_j}{\partial x^k} \right] \sigma^{kj} \ := \ b^i \ + \ \rho^i \ , \tag{3.7}$$

then these determining equations for simple symmetries read

$$\begin{cases} \partial_t \xi^i \ + \ f^j\, (\partial_j \xi^i) \ - \ \xi^j\, (\partial_j f^i) \ = \ \rho^j (\partial_j \xi^i) \ - \ \xi^j (\partial_j \rho^i) & , \\ \widehat{\partial}_k \xi^i \ + \ \sigma^j{}_k\, (\partial_j \xi^i) \ - \ \xi^j\, (\partial_j \sigma^i{}_k) \ = \ 0 & . \end{cases} \tag{3.8}$$

They appear to be in general *different* form the determining equations (3.3) for the Ito equation.

### 3.3 Unal type theorems

It turns out that, as can be checked by a careful explicit computation, the difference between (3.3) and (3.8) is only apparent. In fact, we have the following result, shown by Unal [24] for the deterministic case and then extended to the random one [7] (we refer to the original papers for its proof).

**Proposition 3.2.** *The simple* deterministic or random *symmetries of an Ito equation and those of the equivalent Stratonovich equation do coincide.*

In his paper, however, Unal also showed that – even in the deterministic framework – the result does *not* extend to more general symmetries; in particular, if one considers symmetries with generator of the general form (3.2), thus in general with $\tau \neq 0$, then the determining equations for the Ito and the associated Stratonovich equation are equivalent if and only if $\tau$ satisfies the additional condition

$$\sigma^k_{\ p}\,\sigma^{ip}\ \left[\partial_k\left(\partial_t\tau\ +\ f^j\,(\partial_j\tau)\ +\ \frac{1}{2}\,\sigma^m_{\ q}\,\sigma^j_{\ q}\,(\partial_m\partial_j\tau)\right)\right]\ =\ 0\ . \tag{3.9}$$

We stress that this condition is identically satisfied for $\tau = \tau(t)$, i.e. for "acceptable" cases according to the discussion in [10].

Thus we conclude that for (deterministic or random) simple symmetries, and actually also for the corresponding "acceptable" general symmetries, i.e. with $\tau = \tau(t)$, symmetries of an Ito equation and of the associated Stratonovich one do coincide. As the latter are preserved under changes of variables, it follows that the former are preserved as well. In the end [7, 8],

**Lemma 3.3.** *Simple (random or deterministic) symmetries of an Ito equation are preserved under changes of coordinates.*

**Remark 3.4.** This entails that we can hope to use (simple or acceptable general) symmetries of Stochastic Differential Equations, as the basic ingredient for applications of the theory – i.e. indeed preservation of symmetries under changes of variables – is there, albeit in a much less immediate way than for deterministic differential equations.

## 4 Kozlov theory

In the deterministic case, symmetry guarantees that an ODE can be reduced (or solved). The same holds in the SDE case, but only *simple* symmetries $X = f^i(x,t)\partial_i$ matter.[3]

We have the following theorem, which is due to R. Kozlov [11] (see also [8, 16]):

[3]This limitation may look surprising at first, but one should note that now $x$ and $t$ are intrinsically different: one is a random process (indexed by $t$), the other a smooth deterministic variable.

**Theorem 4.1.** *The scalar SDE*

$$dy = \widetilde{f}(y,t)\, dt + \widetilde{\sigma}(y,t)\, dw \tag{4.1}$$

*can be transformed by a deterministic map $y = y(x,t)$ into*

$$dx = f(t)\, dt + \sigma(t)\, dw , \tag{4.2}$$

*and hence explicitly integrated,* if and only if *it admits a simple deterministic symmetry.*

*If the generator of the latter is $X = \varphi(y,t)\partial_y$, then the change of variables $y = F(x,t)$ transforming (4.1) into (4.2) is the inverse to the map $x = \Phi(y,t)$ identified by*

$$\Phi(y,t) = \int \frac{1}{\varphi(y,t)}\, dy .$$

**Example 4.2.** The Ito equation $dy = \left[e^{-y} - (1/2)e^{-2y}\right] dt + e^{-y}\, dw$ admits the simple deterministic symmetry generated by $X = e^{-y}\partial_y$. The associated change of variable is $x = e^y$; in terms of this $X = \partial_x$, and the equation reads $dx = dt + dw$, which is readily integrated.

The same approach can be pursued to study *partial integrability*, i.e. reduction of an $n$-dimensional SDE to an SDE in dimension $n - r$ plus $r$ (stochastic) integrations.

In the deterministic case, this is possible *if and only if* there are $r$ simple symmetry generators spanning a solvable Lie algebra. In the stochastic case we obtain essentially the same result, but now it is convenient to consider separately the case of deterministic symmetries and that of random ones. Again the relevant results in this direction have been obtained by Kozlov [12, 13] (see also [8, 16]).

This has been considered in the literature only for (multiple) deterministic simple symmetries; the result below is quoted verbatim from [8], and the reader is referred there for the proof.

**Theorem 4.3.** *Suppose the system (3.1) admits an $r$-parameter solvable algebra $\mathcal{G}$ of simple deterministic symmetries, with generators*

$$\mathbf{X}_k = \sum_{i=1}^n \varphi^i_k(x,t)\, \frac{\partial}{\partial x_i} \qquad (k = 1, \ldots, r) , \tag{4.3}$$

*acting regularly with $r$-dimensional orbits.*

*Then it can be reduced to a system of $m = (n - r)$ equations,*

$$dy^i = g^i(y^1, \ldots, y^m; t)\, dt + {\sigma^i}_k(y^1, \ldots, y^m; t)\, dw^k \qquad (i, k = 1, \ldots, m) \tag{4.4}$$

*and $r$ "reconstruction equations", the solutions of which can be obtained by quadratures from the solution of the reduced $(n-r)$-order system. In particular, if $r = n$, the general solution of the system can be found by quadratures.*

We note that in Kozlov's original paper [12] (see Example 4.2 in there) the Theorem is applied to any linear two-dimensional system of SDEs

$$\begin{cases} dx_1 = (a_1 + b_{11}\, x_1 + b_{12}\, x_2)\, dt + s_{11}\, dw_1 + s_{12}\, dw_2 & , \\ dx_2 = (a_2 + b_{21}\, x_1 + b_{22}\, x_2)\, dt + s_{21}\, dw_1 + s_{22}\, , dw_2 & ; \end{cases} \tag{4.5}$$

see there for a detailed discussion and results.

We have so far only considered deterministic symmetries. In the case of random symmetries, the associated random change of variables could change the Ito equation into a random system of different nature. This problem accounts for the appearance of an extra condition, absent when one is only considering deterministic simple symmetries.

**Theorem 4.4.** *Let the Ito equation*

$$dy = F(y,t)\, dt + S(y,t)\, dw \tag{4.6}$$

*admit as Lie-point symmetry the simple random vector field*

$$X = \varphi(y,t,w)\, \partial_y \ . \tag{4.7}$$

*If there is a determination of*

$$\Phi(y,t,w) = \int \frac{1}{\varphi(y,t,w)}\, dy \tag{4.8}$$

*such that the equations*

$$\Phi_{ww} + S\, \Phi_{yw} = 0 \ ; \qquad \Phi_{tw} + F\, \Phi_{yw} + (1/2)\, (\Delta\Phi)_w = 0 \tag{4.9}$$

*are satisfied, then the equation is reduced to the explicitly integrable form*

$$dx = f(t)\, dt + \sigma(t)\, dw \tag{4.10}$$

*by passing to the variable $x = \Phi(y,t,w)$.*

**Remark 4.5.** This formulation is not fully satisfactory, in that it is based on the existence of a determination of an integral with certain properties. One would like to have a criterion based on the directly available data, i.e. the functions $F(y,t)$, $S(y,t)$ and $\varphi(y,t,w)$. This is provided by the next theorem.

**Theorem 4.6.** *Let the Ito equation (4.6) admit as Lie-point symmetry the simple random vector field (4.7); define $\gamma(y,t,w) := \partial_w(1/\varphi)$.*

*If the functions $F(y,t)$, $S(y,t)$ and $\gamma(y,t,w)$ satisfy the relation*

$$S\, \gamma_t + S_t\, \gamma = F\, \gamma_w + (1/2)\, \left[ S\, \gamma_{ww} + S^2\, \gamma_{yw} \right] \ , \tag{4.11}$$

*then the equation (4.6) can be mapped into an integrable Ito equation (4.10) by a simple random change of variables.*

**Example 4.7.** The equation $dy = y\,e^{-t}\,dt + y\,dw$ admits the simple random symmetries $X = \eta(\zeta)\partial_y$, with $\eta$ an arbitrary function of $\zeta = 2e^{-t} + t - w + \log(y)$; Equation (4.11) is satisfied. Let us choose for definiteness $\eta(\xi) = \xi$. The associated change of variable is then $x = (1/2)\log[2 + e^t(2-w) + 2e^t\log(y)] + \beta(t,w)$; the resulting equation is of Ito form for $\beta(t,w) = b(t)+cw$, with $b$ an arbitrary function. Then we get $dx = [b'(t) + (1/2)]\,dt + c\,dw$.

**Example 4.8.** The equation $dy = dt + y\,dw$ has the simple random symmetry $X = \exp[w - t/2]\partial_y$; the associated new variable is $x = \exp[t/2 - w] + \beta(t,w)$ and in this case Equation (4.11) is *not* satisfied, for any choice of $\beta$. In term of this the equation reads $dx = \exp[t/2 - w]\,dt$ ; this is *not* in Ito form, but is readily integrated.

**Remark 4.9.** The Theorems 4.4 and 4.6 identify the presence of a simple random symmetry $X = \varphi^i(x,t;w)\partial_i$ such that the compatibility condition (4.11) is satisfied as a *sufficient* condition for integrability. It is quite simple to observe this is also a *necessary* condition.

**Theorem 4.10.** *Let the Ito equation (4.6) be reducible to the integrable form (4.10) by a simple random change of variables $x = \Phi(y,t;w)$. Then necessarily (4.6) admits*

$$X = [\Phi_y(y,t,w)]^{-1}\,\partial_y := \varphi(y,t,w)\,\partial_y \tag{4.12}$$

*as a symmetry vector field, and – with $\gamma = \partial_w(1/\varphi)$ – (4.11) is satisfied.*

## 5 Discussion and conclusions

The symmetry approach is a general way to tackle Differential Equations; in the deterministic framework it proved invaluable both for the theoretical study of differential equations and for obtaining their concrete solutions. The theory is comparatively much less advanced in the case of stochastic differential equations. A first obstacle lies in that it is not at all obvious that symmetries of SDEs are preserved under changes of variables; this is the case for a special class of symmetries – the one of interest for concrete applications – as discussed in Section 3.

There is now some general agreement on what the "right" (that is, useful) definition of symmetry for SDE is; but only few applications have been considered, most of these concerning integrable or partially integrable equations.

Theorems equivalent to the standard ones for ODEs have been obtained by R. Kozlov – and recently extended – for (ordinary) SDEs, both for what concerns *solving* equations and for *reducing them*; these have been discussed in Section 4. The big difference with respect to the deterministic case is that now we cannot use *general* symmetries, but only *simple* ones.

Even beside this, there is undoubtedly ample space for considering new applications, first and foremost considering "non integrable" equations. Correspondingly, there is ample space for concrete applications, i.e. applying the approaches already existing or to be developed to new concrete stochastic systems.

We conclude by a number of observations:

$(i)$ An important topic has been completely absent from our discussion: that is, *symmetry of variational problems* (Noether theory). For this we refer e.g. to [17, 25, 26].

$(ii)$ Similarly, we have not discussed the interrelations between symmetries of an Ito equation and those of the associated diffusion (Fokker-Planck) equation; for this we refer e.g. to [6, 9].

$(iii)$ Reduction by multiple symmetries has been studied in the literature only in the case of deterministic symmetries. Albeit it appears that no obstacle is present in the case of random symmetries – except that, as for a single symmetry, the compatibility condition studied above should also be required – one would like to have precise statements in this respect.

$(iv)$ In the deterministic framework, symmetry theory flourished and expanded its role by considering generalization of the "standard" (i.e. Lie-point) symmetries in several directions [1, 14, 18, 22]. As far as we know, there is no attempt in this direction for stochastic systems yet; any work in this direction is very likely to collect success and relevant results.

$(v)$ Also, so far only *first order systems* have been considered; but Physical applications often require to consider *second order* ones (as in the familiar case of Einstein-Smoluchowsky *vs.* Ornstein-Uhlenbeck processes). This is definitely a direction requiring serious investigation, also in connection with the previous points.[4]

## Appendix. Derivation of the determining equations

In this appendix we briefly discuss how the determining equations (3.3) and (3.8) are obtained; see e.g. [6, 7, 8, 10] for further detail.

The vector field $X = \xi^i(x,t)\partial_i$ generates an infinitesimal map

$$x^i \to x^i + \varepsilon \xi^i(x,t);$$

under this the different objects appearing in (3.1) map as follows (all functions depend on $x, t$):

$$\begin{aligned} f^i &\to f^i \ + \ \varepsilon\, [\partial f^i / \partial x^j]\, \xi^j \ , \qquad {\sigma^i}_k \ \to \ {\sigma^i}_k \ + \ \varepsilon\, [\partial {\sigma^i}_k / \partial x^j]\, \xi^j \ ; \\ dx^i &\to dx^i \ + \ \varepsilon\, d\xi^i \\ &\qquad = \ dx^i \ + \ \varepsilon\, \left[ (\partial \xi^i / \partial t)\, dt \ + \ (\partial \xi^i / \partial x^j)\, dx^j \ + \ (1/2) (\Delta \xi^i)\, dt \right] \ ; \end{aligned}$$

[4] In particular, considering second (or higher) order equations opens the way to introduction of twisted prolongations and twisted symmetries [5].

note that the last term in the last line originates from Ito formula. Plugging these into (3.1), and requiring that this is actually mapped into itself – i.e. the vanishing of terms of order $\varepsilon$ – we obtain exactly the determining equations (3.3).

In the case of Stratonovich equations (3.5) we proceed in the same way, but now variables do *not* change according to the Ito rule, following instead the usual chain rule. Thus in this case

$$\begin{aligned} b^i \to b^i \ + \ \varepsilon \, [\partial b^i / \partial x^j] \, \xi^j \ , \qquad & {\sigma^i}_k \ \to \ {\sigma^i}_k \ + \ \varepsilon \, [\partial {\sigma^i}_k / \partial x^j] \, \xi^j \ ; \\ dx^i \to dx^i \ + \ \varepsilon \, d\xi^i \ = \ dx^i \ + \ \varepsilon \ & \left[ (\partial \xi^i / \partial t) \, dt \ + \ (\partial \xi^i / \partial x^j) \, dx^j \right] \ , \end{aligned}$$

with no Ito term in the $dx^i$ change. We plug these into the Stratonovich equation (3.5) and require it is mapped into itself – i.e. that terms of order $\varepsilon$ vanish – and thus obtain the determining equations (3.6).

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