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Internal observability of the wave equation in tiled domains

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Abstract. *We investigate the internal observability of the wave equation with Dirichlet boundary conditions in tilings. The paper includes a general result relating internal observability problems in general domains to their tiles, and a discussion of the case in which the domain is the* 30-60-90 *triangle.*

1. Introduction

The aim of the present paper is to investigate internal observability properties of vibrating repetitive structures. Motivated by applications of hexagonal and triangular tilings (and related subtilings) to engineering, the particular case of the half to the equilateral triangle is treated in detail.

By a repetitive structure, or tessellation, is meant a structure obtained by the assemblation of identical substructures, or tiles. For instance, two-dimensional lattices and the honeycomb lattice are examples of tessellation of $\mathbb{R}^2$; while the regular hexagon (i.e., the tile of the honeycomb lattice) and the rectangle with aspect ratio equal to $\sqrt{3}$ are bounded domains that can be both tiled with 30-60-90 triangles, see Figure 1. The interest in repetitive systems of vibrating membranes is motivated by applications in mechanical, civil and aerospace engineering [5, 24]. Modular structures have indeed the double advantage of a cost-effective manufacturing and construction (due to the repetitivity of the process) as well as a computationally cost-effective design. In particular, structural eigenproblems (e.g., vibrations and buckling) for repetitive structures in general involve a lower number of degrees of freedoms and, consequently, a less computationally demanding numerical solution [26]. Tilings involving regular triangles and hexagons (known as triangular lattice and honeycomb lattice, respectively) find countless applications in engineering, as well [27]. For instance, the use of such structures in architectural engineering is motivated by their mechanical properties, including resistance to external load and energy absorption, see for instance [6, 20] and, for a comprehensive dissertation on the topic, the book [7]. Finally, we mention that honeycomb lattice plays a crucial role in nanosciences and, in particular, in graphene technology [1].

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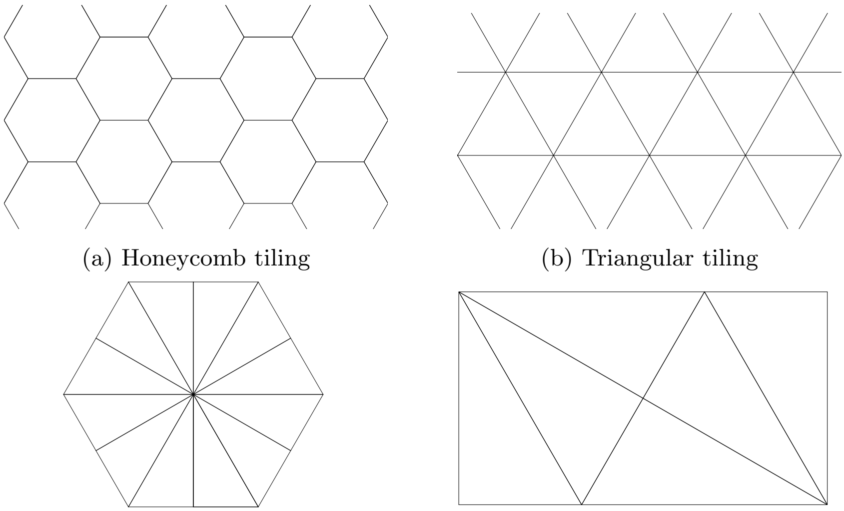

(a) Honeycomb tiling

(b) Triangular tiling

(c) Tiling of the regular hexagon

(d) Tiling of the rectangle with aspect ratio equal to $\sqrt{3}$

Figure 1: Some tilings related to the 30-60-90 triangle.

As mentioned above, we are interested in the internal observability of the wave equation, that is the problem of reconstructing initial data from the observation of the evolution of the system in a subregion of the domain. Using folding and tessellation techniques, in the spirit of [23] and [18], we provide a general class of tilings, called *admissible tilings*, for which some internal observability properties of tiled domains extend to their tiles and – under some symmetry assumptions on initial data – vice versa. In particular, we show how to bridge the well-established theory concerning rectangular domains [8, 9, 15, 16, 17, 22] to the case of a 30-60-90 triangular domain. In the remaining part of this Introduction we discuss in detail this case, while postponing the more technical, general result to Section 2.

1.1. A case study: observability in a triangular domain

We consider the problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \mathcal{T} \\ u = 0 & \text{in } \mathbb{R} \times \partial\mathcal{T} \\ u(t,0) = u_0, \ u_t(t,0) = u_1 & \text{in } \mathcal{T} \end{cases} \tag{1.1}$$

where $\mathcal{T}$ is the open triangle with vertices $(0,0), (1/\sqrt{3},0)$ and $(0,1)$. Also consider ther rectangle $\mathcal{R} := (0,\sqrt{3}) \times (0,1)$ and remark that there exists 6 rigid

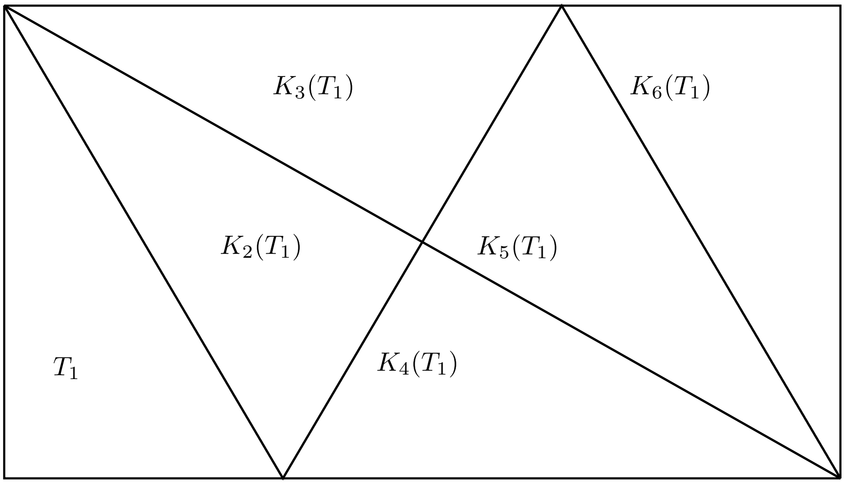

Figure 2: The tiling of $\mathcal{R}$ with $\mathcal{T}$. Note that K_1 is the identity map, hence $K_1(\mathcal{T}) = \mathcal{T}$.

transformations $K_1, \dots, K_6$ satisfying the relation

$$cl(\mathcal{R}) = \bigcup_{h=1}^{6} K_h(cl(\mathcal{T})),$$

where $cl(\Omega)$ represent the closure of a set Ω – see Figure 2. We then say that $\mathcal{T}$ *tiles*[1] $\mathcal{R}$.

As it is well known, a complete orthonormal base for $L^2(\mathcal{R})$ is given by the eigenfunctions of $-\Delta$ in $H_0^1(\mathcal{R})$

$$\overline{e}_k := \sin(\pi k_1 x_1/\sqrt{3})\sin(\pi k_2 x_2), \quad \text{where } k = (k_1, k_2),\ k_1, k_2 \in \mathbb{N}$$

and the associated eigenvalues are $\gamma_k = \dfrac{k_1^2}{3} + k_2^2$. In [23], a folding technique (that we recall in detail in Section 3) is used to derive from $\{\overline{e}_k\}$ an orthogonal base $\{e_k\}$ of $L^2(\mathcal{T})$ formed by the eigenfunctions of $-\Delta$ in $H_0^1(\mathcal{T})$. The explicit knowledge of a eigenspace for $H_0^1(\mathcal{T})$ allows us to set the problem (1.1) in the framework of Fourier analysis – see [14, 10, 8, 9, 4, 2, 3]. Our goal is to exploit the deep relation between the eigenfunctions for $H_0^1(\mathcal{R})$ and those of $H_0^1(\mathcal{T})$ in order to extend known observability results for $\mathcal{R}$ to $\mathcal{T}$.

In particular, we are interested in the *internal observability* of (1.1), i.e., in the validity of the estimates

$$\|u_0\|^2_{L^2(\mathcal{T})} + \|u_1\|^2_{H^{-1}(\mathcal{T})} \asymp \int_0^T \int_{\mathcal{T}_0} |u(t,x)|^2 dx$$

[1] For a precise definition of tilings see Definition 2.1 below, while the explicit definition of the K_h's is given in Section 3.

where $\mathcal{T}_0$ is a subset of $\mathcal{T}$ and T is sufficiently large. Here and in the sequel $A \asymp B$ means $c_1 A \le B \le c_2 A$ with some constants c_1 and c_2 which are independent from A and B. When we need to stress the dependence of these estimates on the couple of constants $c = (c_1, c_2)$, we write $A \asymp_c B$. Also by writing $A \le_c B$ we mean the inequality $cA \le B$ while the expression $A \ge_c B$ denotes $cA \ge B$.

We have

Theorem 1.1. *Let $\overline{u}$ be the solution of*

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0 & \text{in } \mathbb{R} \times \mathcal{R} \\ \overline{u} = 0 & \text{in } \mathbb{R} \times \partial\mathcal{R} \\ \overline{u}(t,0) = \overline{u}_0, \ u_t(t,0) = \overline{u}_1 & \text{in } \mathcal{R}, \end{cases} \tag{1.2}$$

let $\mathcal{R}_0$ be a subset of $\mathcal{R}$ and assume that there exists a constant $T_0 \ge 0$ such that if $T > T_0$ then there exists a couple of constants $c = (c_1, c_2)$ such that $\overline{u}$ satisfies

$$\|\overline{u}_0\|^2_{L^2(\mathcal{R})} + \|\overline{u}_1\|^2_{H^{-1}(\mathcal{R})} \asymp_c \int_0^T \int_{\mathcal{R}_0} |\overline{u}(t,x)|^2 dx dt \tag{1.3}$$

for all $(\overline{u}_0, \overline{u}_1) \in L^2(\mathcal{R}) \times H^{-1}(\mathcal{R})$. Moreover let

$$\mathcal{T}_0 := \bigcup_{h=1}^{6} K_h^{-1}(\mathcal{R}_0) \cap \mathcal{T}.$$

Then for each $T > T_0$ and $(u_0, u_1) \in L^2(\mathcal{T}) \times H^{-1}(\mathcal{T})$, the solution u of (1.1) satisfies

$$\|u_0\|^2_{L^2(\mathcal{T})} + \|u_1\|^2_{H^{-1}(\mathcal{T})} \asymp_c \int_0^T \int_{\mathcal{T}_0} |u(t,x)|^2 dx dt \,. \tag{1.4}$$

The result also holds by replacing every occurrence of $\asymp_c$ with $\le_c$ or $\ge_c$.

We point out that the time of observability T_0 stated in Theorem 1.1, as well as the couple c of constants in the estimates (1.3) and (1.4), are the same for both the domains $\mathcal{R}$ and $\mathcal{T}$. Also note that in Section 3 we prove a slightly stronger version of Theorem 1.1, that is Theorem 3.5: its precise statement requires some technicalities that we chose to avoid here, however we may anticipate to the reader that the assumption on initial data $(\overline{u}_0, \overline{u}_1) \in L^2(\mathcal{R}) \times H^{-1}(\mathcal{R})$ can be weakened by replacing $L^2(\mathcal{R}) \times H^{-1}(\mathcal{R})$ with an appropriate subspace.

1.2. Organization of the paper.

In Section 2 we consider a generic domain Ω tiling a larger domain Ω': we establish a result, Theorem 2.10, relating the observability properties of wave equation on Ω' and on its tile Ω. Section 3 is devoted to the proof of Theorem 1.1.

2. An observability result on tilings

The goal of this section is to state an equivalence between an observability problem on a domain Ω and an observability problem on a larger domain Ω', under the assumption that Ω tiles Ω'. We begin with some definitions.

Definition 2.1 (Tiling)**.** Let Ω and Ω' be two open bounded subsets of $\mathbb{R}^n$. We say that Ω *tiles* Ω' if there exists a set $\{K_h\}_{h=1}^N$ rigid transformations of $\mathbb{R}^n$ such that

$$cl(\Omega') = \bigcup_{h=1}^{N} K_h(cl(\Omega))$$

and such that $K_h(\Omega) \cap K_j(\Omega) = \emptyset$ for all $h \neq j$.

Definition 2.2 (Foldings and prolongations)**.** Let $(\Omega, \{K_h\}_{h=1}^N)$ be a tiling of Ω' and $\delta = (\delta_1, \dots, \delta_N) \in \{-1, 1\}^N$. The *prolongation with coefficients* δ of a function $u\colon \Omega \to \mathbb{R}$ to Ω' is the function $\mathcal{P}_\delta u\colon \Omega' \to \mathbb{R}$

$$\mathcal{P}_\delta u(K_h x) = \delta_h u(x) \qquad \forall h = 1, \dots, N.$$

The *folding with coefficients* δ of a function $\overline{u}\colon \Omega' \to \mathbb{R}$ is the function $\mathcal{F}_\delta \overline{u}\colon \Omega \to \mathbb{R}$ defined by

$$\mathcal{F}_\delta \overline{u}(x) = \frac{1}{N^2} \sum_{h=1}^{N} \delta_h \overline{u}(K_h x) \qquad \forall h = 1, \dots, N.$$

When the particular choice of δ is not relevant we omit it in the under scripts and we simply write $\mathcal{P}$ and $\mathcal{F}$.

Definition 2.3 (Admissible tiling)**.** A tiling $(\Omega, \{K_h\}_{h=1}^N)$ of Ω' is *admissible* if there exists $\delta \in \{-1, 1\}^N$ such that

$$\mathcal{F}_\delta \varphi \in H_0^1(\Omega) \quad \forall \varphi \in H_0^1(\Omega'). \tag{2.1}$$

Example 2.4. We show in Lemma 3.1 below that the tiling of $\mathcal{R}$ with $\mathcal{T}$ depicted in Figure 2 is admissible, in particular (2.1) holds with $\delta = (1, -1, 1, 1, -1, 1)$.

On the other hand the tiling of $\mathcal{R}' := (0, 1/\sqrt{3}) \times (0, 1)$ given by the transformations $K'_1 := id$ and

$$K'_2 : (x_1, x_2) \mapsto -(x_1, x_2) + (1/\sqrt{3}, 1),$$

see Figure 3, is not admissible. Let indeed $v_1 := (1/\sqrt{3}, 0)$, $v_2 := (0, 1)$ and $x_\lambda := \lambda v_1 + (1 - \lambda) v_2$ with $\lambda \in (0, 1)$. Then $x_\lambda \in \partial \mathcal{T}$ and

$$K_2(x_\lambda) = x_{1-\lambda}\,.$$

Therefore it suffices to choose $\varphi \in H_0^1(\mathcal{R})$ such that $\varphi(x_\lambda) \neq \pm\varphi(x_{1-\lambda})$ to obtain

$$\mathcal{F}_\delta \varphi(x_\lambda) = \delta_1 \varphi(x_\lambda) + \delta_2 \varphi(x_{1-\lambda}) \neq 0$$

for all $\delta_1, \delta_2 \in \{-1, 1\}$. Consequently $\mathcal{F}_\delta \varphi \notin H_0^1(\mathcal{T})$ for all $\delta \in \{-1, 1\}^2$.

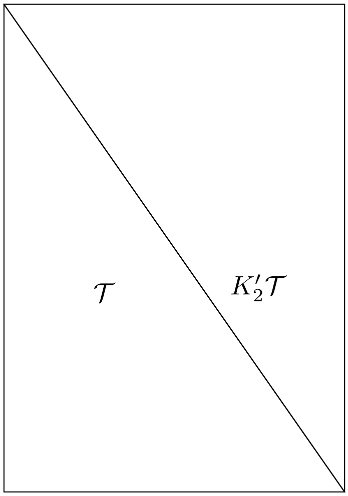

Figure 3: A non-admissible tiling of $\mathcal{R}' = (0, 1/\sqrt{3}) \times (0, 1)$ with $\mathcal{T}$.

Remark 2.5. We borrowed the notion of prolongation and folding from [23]: while our definition of $\mathcal{P}_\delta$ is exactly as it is given in [23], we introduced a normalizing term $1/N^2$ in the definition of $\mathcal{F}_\delta$ in order to enlighten the notations. Note that the following equality holds:

$$\mathcal{F}_\delta(\mathcal{P}_\delta u) = \frac{1}{N} u \tag{2.2}$$

for all $u\colon \Omega \to \mathbb{R}$.

Also remark that we shall need to prolong and fold also functions $u\colon \mathbb{R}\times\Omega \to \mathbb{R}$ and $\bar{u}\colon \mathbb{R} \times \Omega' \to \mathbb{R}$, in this case the definition of $\mathcal{P}$ and $\mathcal{F}$ naturally extends by applying the transformations K_h's to the spatial variables x. For instance if $u\colon \mathbb{R} \times \Omega \to \mathbb{R}$ then its prolongation to $\mathbb{R} \times \Omega'$ reads

$$\mathcal{P}_\delta u(t, K_h x) = \delta_h u(t, x).$$

We want to establish a relation between solutions of a wave equation with Dirichlet boundary conditions and their prolongation. To this end we introduce the notations

$$\mathcal{P}_\delta L^2(\Omega) := \{\mathcal{P}_\delta u \mid u \in L^2(\Omega)\},$$
$$\mathcal{P}_\delta H_0^1(\Omega) := \{\mathcal{P}_\delta u \mid u \in H_0^1(\Omega)\}$$

and

$$\mathcal{P}_\delta H^{-1}(\Omega) := \{\mathcal{P}_\delta u \mid u \in H^{-1}(\Omega)\}.$$

Note that $\mathcal{P}_\delta L^2(\Omega) \subset L^2(\Omega')$, $\mathcal{P}_\delta H_0^1(\Omega) \subset H_0^1(\Omega')$ and $\mathcal{P}_\delta H^{-1}(\Omega) \subset H^{-1}(\Omega)$.

All results below hold under the following assumptions on the domains Ω, Ω' and on a base $\{e_k\}$ for $L^2(\Omega)$:

Assumption 1. $(\Omega, \{K_h\}_{h=1}^N)$ is an admissible tiling of Ω'.

Assumption 2. $\{e_k\}$ is a base of eigenvectors of $-\Delta$ in $H_0^1(\Omega)$, it is defined on $\Omega \cup \Omega'$ and there exists $\delta \in \{-1,1\}^N$ such that

$$\mathcal{P}_\delta(e_k|_\Omega) = e_k|_{\Omega'}$$

for each $k \in \mathbb{N}$.

Remark 2.6 (Some remarks on Assumption 2)**.** We note that Assumption 2 can be equivalently stated as

$$e_k(K_h x) = \delta_h e_k(x) \qquad \text{for all } x \in \Omega,\ \ h = 1, \dots, N,\ \ k \in \mathbb{N}\,. \tag{2.3}$$

Indeed, by definition of prolongation and noting $\delta_h^2 \equiv 1$, we have

$$e_k(K_h x) = \delta_h^2 e_k(K_h x) = \delta_h \mathcal{P}_\delta e_k(x) = \delta_h e_k(x).$$

for every $x \in \Omega$, $h = 1, \dots, N$ and $k \in \mathbb{N}$.

Also remark that, in view of (2.2), Assumption 2 also implies

$$\mathcal{F}_\delta e_k = \frac{1}{N} e_k\,. \tag{2.4}$$

Example 2.7. Let $\Omega = (0,\pi)^2$ and $\Omega' = (0,2\pi)^2$. Consider the transformations of $\mathbb{R}^2$

$$\begin{array}{ll} K_1 := \mathrm{id}, & K_2 : (x_1,x_2) \mapsto (-x_1 + 2\pi, x_2), \\ K_3 : (x_1,x_2) \mapsto (x_1, -x_2 + 2\pi), & K_4 : (x_1,x_2) \mapsto -(x_1,x_2) + (2\pi, 2\pi) \end{array}.$$

Then $\{\Omega, \{K_h\}_{h=1}^4\}$ is a tiling for Ω'. In particular, Assumption 1 is satisfied: indeed setting $\delta = (1,-1,-1,1)$ we have for each $\varphi \in H_0^1(\Omega')$

$$\mathcal{F}_\delta \varphi(x) = 0 \quad \forall x \in \partial\Omega\,.$$

Also note that the functions

$$e_k(x) := \sin(k_1 x_1)\sin(k_2 x_2) \quad k = (k_1,k_2) \in \mathbb{N}^2$$

satisfy Assumption 2, indeed they are a base for $L^2(\Omega)$ composed by eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ and

$$e_k(K_h(x)) := \delta_h e_k(x)$$

for all $x \in \mathbb{R}^2$, $h = 1, \dots, 4$ and $k \in \mathbb{N}^2$. The space $\mathcal{P}_\delta L^2(\Omega)$ in this case coincides with the space of so-called $(2,2)$-*cyclic functions*, i.e., functions in $L^2(\Omega')$ which are odd with respect to both axes $x_1 = \pi$ and $x_2 = \pi$. We refer to [16] for some results on observability of wave equation with (p,q)-cyclic initial data.

Our starting point is to show that, under Assumption 1 and Assumption 2, the base of eigenfunctions $\{e_k\}$ is also a base of eigenfunctions also for an appropriate subspace of $L^2(\Omega')$, and to compute the associated coefficients.

Lemma 2.8. *Let Ω, Ω' and $\{e_k\}$ satisfy Assumption 1 and Assumption 2.*

Then $\{e_k\} \subset H_0^1(\Omega')$ and it is also a complete base for $\mathcal{P}_\delta L^2(\Omega)$ formed by eigenfunctions of $-\Delta$ in $\mathcal{P}_\delta H_0^1(\Omega')$.

In particular, for every $k \in \mathbb{N}$, if u_k is the coefficient of $u \in L^2(\Omega)$ (with respect to e_k) then $N u_k$ is the coefficient of $\mathcal{P}_\delta u$.

Proof. The proof is organized two steps.
Claim 1: $\{e_k\}$ is a set of eigenfunctions of $-\Delta$ in $H_0^1(\Omega')$. Extending a result given in [23], we need to show that, under Assumption 1 and Assumption 2, if $e_k \in H_0^1(\Omega)$ is a solution of the boundary value problem

$$\int_\Omega \nabla e_k \nabla \varphi dx = \int_\Omega \gamma_k e_k \varphi dx \quad \forall \varphi \in H_0^1(\Omega)$$

for some $\gamma_k \in \mathbb{R}$, then e_k is also solution of the boundary value problem on Ω'

$$\int_{\Omega'} \nabla e_k \nabla \varphi dx = \int_{\Omega'} \gamma_k e_k \varphi dx \quad \forall \varphi \in H_0^1(\Omega').$$

Now, recall from Assumption 1 that if $\varphi \in H_1^0(\Omega')$ then $\mathcal{F}_\delta \varphi \in H_1^0(\Omega)$. Then it follows again from Assumption 1 and from Assumption 2 (in particular by recalling that K_h's are isometries and (2.3)) that for all $\varphi \in H_0^1(\Omega')$

$$\begin{aligned}
\int_{\Omega'} \nabla e_k(x) \nabla \varphi(x) dx &= \int_{\bigcup_{h=1}^N K_h(\Omega)} \nabla e_k(x) \nabla \varphi(x) dx \\
&= \sum_{h=1}^N \int_\Omega \nabla e_k(K_h x) \nabla \varphi(K_h x) dx = \int_\Omega \nabla e_k(x) \sum_{h=1}^N \delta_h \nabla \varphi(K_h x) dx \\
&= \int_\Omega \nabla e_k(x) \nabla \mathcal{F}_\delta \varphi(x) dx = \int_\Omega \gamma_k e_k(x) \mathcal{F}_\delta \varphi(x) dx \\
&= \int_{\Omega'} \gamma_k e_k(x) \varphi(x) dx.
\end{aligned}$$

and this completes the proof of Claim 1.

Claim 2: completeness of $\{e_k\}$ and computation of coefficients. By Assumption 1 and Assumption 2 and by recalling $\delta_h^2 = 1$ for each $h = 1, \dots, N$, we have

$$\begin{aligned}\int_{\Omega'} \mathcal{P}_\delta u(x) e_k(x) dx &= \int_{\Omega'} \mathcal{P}_\delta u(x) \mathcal{P}_\delta e_k(x) dx \\ &= \sum_{h=1}^{N} \int_{K_h(\Omega)} \mathcal{P}_\delta u(x) \mathcal{P} e_k(x) dx \\ &= \sum_{h=1}^{N} \int_{K_h(\Omega)} \delta_h^2 u(K_h(x)) e_k(K_h(x)) dx \\ &= \sum_{h=1}^{N} \int_{\Omega} u(x) e_k(x) dx = N \int_{\Omega} u(x) e_k(x) dx,\end{aligned}$$

where the second to last equality holds because K_h's are isometries. Then we may deduce two facts: first if $\{u_k\}$ are the coefficients of $u \in L^2(\Omega)$ then $\{N u_k\}$ are coefficients of $\mathcal{P}_\delta u$. Secondly, $\{e_k\}$ is a complete base for $\mathcal{P}_\delta L^2(\Omega)$, indeed if the coefficients of $\mathcal{P}_\delta u$ are identically null, then also the coefficients of u are identically null: since $\{e_k\}$ is complete for Ω then $u \equiv 0$ and, consequently, $\mathcal{P}_\delta u \equiv 0$, as well. □

Next result establishes a relation between solutions of wave equations on tiles and their prolongations.

Lemma 2.9. *Let Ω, Ω' and $\{e_k\}$ satisfy Assumption 1 and Assumption 2. Let u be the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \Omega \\ u = 0 & \text{in } \mathbb{R} \times \partial\Omega \\ u(t,0) = u_0, \;\; u_t(t,0) = u_1 & \text{in } \Omega \end{cases} \tag{2.5}$$

Then u is well defined in $\Omega \cup \Omega'$ and $\overline{u} = N u|_{\Omega'}$ is the solution of

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0 & \text{in } \mathbb{R} \times \Omega' \\ \overline{u} = 0 & \text{in } \mathbb{R} \times \partial\Omega' \\ \overline{u}(t,0) = \mathcal{P}_\delta u_0, \;\; \overline{u}(t,0) = \mathcal{P}_\delta u_1 & \text{in } \Omega' \end{cases} \tag{2.6}$$

Conversely, if $\bar{u}$ is the solution of (2.6) then $\mathcal{F}_\delta \bar{u}$ is the solution of (2.5) and for every $h = 1, \dots, N$

$$\mathcal{F}_\delta \bar{u}(t,x) = \frac{\delta_h}{N} \bar{u}(t, K_h x) \quad \text{for each } x \in \Omega. \tag{2.7}$$

Proof. Let $\{\gamma_k\}$ be the sequence of eigenvalues associated to the base $\{e_k\}$ and set $\omega_k = \sqrt{\gamma_k}$, for every $k \in \mathbb{N}$. Expanding $u(t,x)$ with respect to e_k we obtain

$$u(t,x) = \sum_{k=1}^{\infty} (a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}) e_k(x)$$

with a_k and b_k depending only the coefficients c_k and d_k of u_0 and u_1 with respect to $\{e_k\}$. In particular $a_k + b_k = c_k$ and $a_k - b_k = -id_k/\omega_k$. We then have that the natural domain of u coincides with the one of $\{e_k\}$'s, hence it is included in $\Omega\cup\Omega'$. By Lemma 2.8 the coefficients of $\mathcal{P}_\delta u_0$ and $\mathcal{P}_\delta u_1$ are Nc_k and Nd_k, respectively. Then it is immediate to verify that

$$Nu(t,x) = \sum_{k=1}^{\infty}(Na_k e^{i\omega_k t} + Nb_k e^{-i\omega_k t})e_k(x)$$

is the solution of (2.6).

Now, let

$$\bar{u}(t,x) = \sum_{k=1}^{\infty}(\bar{a}_k e^{i\omega_k t} + \bar{b}_k e^{-i\omega_k t})e_k(x)$$

be the solution of (2.6), and note that, by the reasoning above, setting $a_k := \frac{1}{N}\bar{a}_k$ and $b_k := \frac{1}{N}\bar{b}_k$ we have that

$$u(t,x) := \sum_{k=1}^{\infty}(a_k e^{i\omega_k t} + b_k e^{-i\omega_k t})e_k(x) = \frac{1}{N}\bar{u}(t,x)$$

is the solution of (2.5). Hence to prove that $u(t,x) = \mathcal{F}_\delta \bar{u}(t,x)$ it it suffices to note that by Assumption 1 (see in particular (2.4))

$$\begin{aligned}\mathcal{F}_\delta \bar{u}(t,x) &= \sum_{k=1}^{\infty}(\bar{a}_k e^{i\omega_k t} + \bar{b}_k e^{-i\omega_k t})\mathcal{F}_\delta e_k(x) \\ &= \frac{1}{N}\sum_{k=1}^{\infty}(\bar{a}_k e^{i\omega_k t} + \bar{b}_k e^{-i\omega_k t})e_k(x) = \frac{1}{N}\bar{u}(t,x).\end{aligned}$$

Finally, we show (2.7): for each $h = 1,\dots,N$ we have

$$\begin{aligned}\bar{u}(t,x) = \delta_h^2\bar{u}(t,x) &= \delta_h \sum_{k=1}^{\infty}(\bar{a}_k e^{i\omega_k t} + \bar{b}_k e^{-i\omega_k t})\delta_h e_k(x) \\ &= \sum_{k=1}^{\infty}(\bar{a}_k e^{i\omega_k t} + \bar{b}_k e^{-i\omega_k t})e_k(K_h x) = \delta_h \bar{u}(t, K_h x)\end{aligned}$$

and this concludes the proof. □

We are now in position to state the main result of this section, that bridges observability of tiles with their prolongations.

Theorem 2.10. *Let Ω, Ω' and $\{e_k\}$ satisfy Assumption 1 and Assumption 2. Let u be the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}\times\Omega \\ u = 0 & \text{in } \mathbb{R}\times\partial\Omega \\ u(t,0) = u_0, \;\; u_t(t,0) = u_1 & \text{in } \Omega \end{cases} \tag{2.8}$$

with $u_0, u_1 \in L^2(\Omega) \times H^{-1}(\Omega)$ *and let* $\overline{u}$ *be the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times \Omega' \\ u = 0 & \text{in } \mathbb{R} \times \partial\Omega' \\ u(t,0) = \mathcal{P}_\delta u_0, \;\; u_t(t,0) = \mathcal{P}_\delta u_1 & \text{in } \Omega'. \end{cases} \tag{2.9}$$

Also let $\Omega'_0 \subset \Omega'$ *and define*

$$\Omega_0 := \bigcup_{h=1}^{N} K_h^{-1}(\Omega'_0) \cap \Omega.$$

Then for every $T > 0$ *and for every couple* $c = (c_1, c_2)$ *of positive constants, the inequalities*

$$\|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^{-1}(\Omega)} \asymp_c \int_0^T \int_{\Omega_0} |u(t,x)|^2 dx dt. \tag{2.10}$$

hold if and only if

$$\|\mathcal{P}_\delta u_0\|^2_{L^2(\Omega')} + \|\mathcal{P}_\delta u_1\|^2_{H^{-1}(\Omega')} \asymp_c \int_0^T \int_{\Omega'_0} |u(t,x)|^2 dx dt. \tag{2.11}$$

Proof. By Lemma 2.9, u and $\overline{u}$ satisfy

$$u(t,x) = \frac{\delta_h}{N} \overline{u}(t, K_h x) \quad \text{for all } h = 1, \dots, N\,.$$

Since Ω tiles Ω', then setting $\Omega_h := K_h^{-1}(\Omega') \cap \Omega$ we have $\Omega_0 = \bigcup_{h=1}^N \Omega_h$ and $\Omega'_0 = \bigcup_{h=1}^N K_h(\Omega_h)$, and that these unions are disjoint. Hence, also recalling $|\delta_h| \equiv 1$ and that K_h's are isometries, we have

$$\begin{aligned} \int_I \int_{\Omega'_0} |\overline{u}(t,x)|^2 dx &= \sum_{h=1}^N \int_I \int_{K_h(\Omega_h)} |\overline{u}(t,x)|^2 dx \\ &= \sum_{h=1}^N \int_I \int_{\Omega_h} |\overline{u}(t, K_h(x))|^2 dx \\ &= N^2 \sum_{h=1}^N \int_I \int_{\Omega_h} \left| \frac{\delta_h}{N} \overline{u}(t, K_h(x)) \right|^2 dx \\ &= N^2 \sum_{h=1}^N \int_I \int_{\Omega_h} |u(t,x)|^2 dx \\ &= N^2 \int_I \int_{\Omega_0} |u(t,x)|^2 dx \end{aligned}$$

Finally, by Lemma 2.8

$$\|\mathcal{P}_\delta u_0\|^2_{L^2(\Omega')} = N^2 \|u_0\|^2_{L^2(\Omega)} \quad \text{and} \quad \|\mathcal{P}_\delta u_1\|^2_{H^{-1}(\Omega')} = N^2 \|u_1\|^2_{H^{-1}(\Omega)}$$

and this implies the equivalence between (2.10) and (2.11). □

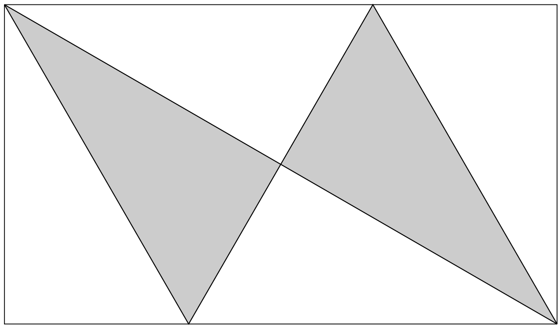

Figure 4: The tiling of $\mathcal{R}$ with $\mathcal{T}$, the grey areas correspond to negative δ_h's.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the application of Theorem 2.10 to the particular case

$$\Omega = \mathcal{T} \quad \text{and} \quad \Omega' = \mathcal{R}\,.$$

We then need to admissibly tile $\mathcal{R}$ with $\mathcal{T}$ and a base $\{e_k\}$ formed by the eigenfunctions of $-\Delta$ in $H_0^1(\mathcal{T})$ satisfying Assumption 2. Such ingredients are provided in [23]: in order to introduce them we need some notations. We consider the Pauli matrix

$$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the rotation matrix

$$R_\alpha := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

where $\alpha := \pi/3$. Now let $v_1 := (0, 1/\sqrt{3})$ and $v_2 := (0, 1)$ be two of the three vertices of $\mathcal{T}$ and define the transformations from $\mathbb{R}^2$ onto itself

$$\begin{aligned} &K_1 := id; && K_4 : x \mapsto -R_\alpha(x - v_2) + 3v_1 \\ &K_2 : x \mapsto -R_\alpha\sigma_z(x - v_2) + v_2; && K_5 : x \mapsto -R_\alpha(x - v_2) + 3v_1 + v_2 \\ &K_3 : x \mapsto R_\alpha(x - v_2) + v_2; && K_6 : x \mapsto -x + 3v_1 + v_2 \end{aligned} \tag{3.1}$$

and note $(\mathcal{T}, \{K_h\}_{h=1}^6)$ is a tiling for $\mathcal{R}$. Indeed

$$cl(\mathcal{R}) = \bigcup_{h=1}^{6} K_h cl(\mathcal{T}), \tag{3.2}$$

and the sets $K_h\mathcal{T}$, for $h = 1, \ldots, 6$, do not overlap – see Figure 4 and [23].

We set

$$\delta := (1, -1, 1, 1, -1, 1).$$

and, in next result, we prove that $\mathcal{T}$ admissibly tiles $\mathcal{R}$.

Lemma 3.1. *$(\mathcal{T}, \{K_h\}_{h=1}^{6})$ is an admissible tiling of $\mathcal{R}$.*

Proof. We want to show that if $\varphi \in H_0^1(\mathcal{R})$ then $\mathcal{F}_\delta\varphi \in H_0^1(\mathcal{T})$. To this end let $v_0 := (0,0)$, $v_1 := (1/\sqrt{3}, 0)$ and $v_2 := (0,1)$ be the vertices of $\mathcal{T}$ and define

$$x_{ij}^{\lambda} := \lambda v_i + (1-\lambda)v_j.$$

so that $\partial\mathcal{T} = \{x_{ij}^{\lambda} \mid \lambda \in [0,1], 0 \le i < j \le 2\}$. By a direct computation, for all $\lambda \in [0,1]$

$$K_1(x_{01}^{\lambda}), K_6(x_{01}^{\lambda}) \in \partial\mathcal{R},$$
$$K_2(x_{01}^{\lambda}) = K_4(x_{01}^{\lambda}),$$

and

$$K_3(x_{02}^{\lambda}) = K_5(x_{02}^{\lambda}).$$

Since $\varphi \in H_0^1(\mathcal{R})$ then $\mathcal{F}_\delta\varphi(x_{01}^{\lambda}) = 0$. Similarly, for all $\lambda \in [0,1]$

$$K_1(x_{02}^{\lambda}), K_6(x_{02}^{\lambda}) \in \partial\mathcal{R},$$
$$K_2(x_{02}^{\lambda}) = K_3(x_{02}^{\lambda}),$$

and

$$K_4(x_{02}^{\lambda}) = K_5(x_{02}^{\lambda})$$

therefore $\mathcal{F}_\delta\varphi(x_{02}^{\lambda}) = 0$ for all $\lambda \in [0,1]$. Finally for all $\lambda \in [0,1]$

$$K_3(x_{12}^{\lambda}), K_4(x_{12}^{\lambda}) \in \partial\mathcal{R},$$
$$K_1(x_{12}^{\lambda}) = K_2(x_{12}^{\lambda}),$$

and

$$K_5(x_{12}^{\lambda}) = K_6(x_{12}^{\lambda})$$

therefore we get also in this case $\mathcal{F}_\delta\varphi(x_{12}^{\lambda}) = 0$ for all $\lambda \in [0,1]$ and we may conclude that $\mathcal{F}_\delta\varphi \in H_0^1(\mathcal{T})$. □

Remark 3.2. Lemma 3.1 was remarked in [23, p.312], but to the best of our knowledge, this is the first time an explicit proof is provided.

Now, consider the eigenfunctions of $-\Delta$ in $H_0^1(\mathcal{R})$:

$$\overline{e}_k(x_1, x_2) := \sin(\pi k_1 \frac{x_1}{\sqrt{3}}) \sin(\pi k_2 x_2), \quad k = (k_1, k_2) \in \mathbb{N}^2 .$$

We finally define for every $k \in \mathbb{N}^2$

$$e_k(x) := N^2 \mathcal{F}_\delta \overline{e}_k = \sum_{h=1}^{6} \delta_h \overline{e}_k(K_h x). \tag{3.3}$$

Next result, proved in [23], states that Assumption 2 is satisfied by $\{e_k\}$.

Lemma 3.3. *The set of functions $\{e_k\}$ defined in (3.3) is a complete orthogonal base for $\mathcal{T}$ formed by the eigenfunction of $-\Delta$ in $H_0^1(\mathcal{T})$. Furthermore $\mathcal{P}_\delta e_k(x) = e_k(x)$.*

Remark 3.4. For each $k \in \mathbb{N}^2$, the eigenfunctions e_k and $\bar{e}_k$ share the same eigenvalue $\gamma_k = \pi^2 \left(\frac{k_1^2}{3} + k_2^2 \right)$, see [23].

Next gives access to classical results on observability of rectangular membranes for the study of triangular domains.

Theorem 3.5. *Let u be the solution of (1.1) with $u_0, u_1 \in L^2(\Omega) \times H^{-1}(\Omega)$ and let $\overline{u}$ be the solution of*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R} \times \mathcal{R} \\ u = 0 & \text{in } \mathbb{R} \times \partial\mathcal{R} \\ u(t,0) = \mathcal{P}_\delta u_0, \;\; u_t(t,0) = \mathcal{P}_\delta u_1 & \text{in } \mathcal{R}. \end{cases}$$

Also let $\mathcal{R}_0 \subset \mathcal{R}$ and define

$$\mathcal{T}_0 := \bigcup_{h=1}^{N} K_h^{-1}(\mathcal{T}_0) \cap \Omega.$$

Then for every $T > 0$ and for every couple $c = (c_1, c_2)$ of positive constants, the inequalities

$$\|u_0\|^2_{L^2(\mathcal{T})} + \|u_1\|^2_{H^{-1}(\mathcal{T})} \asymp_c \int_0^T \int_{\mathcal{T}_0} |u(t,x)|^2 dx dt. \tag{3.4}$$

hold if and only if

$$\|\mathcal{P}_\delta u_0\|^2_{L^2(\mathcal{R})} + \|\mathcal{P}_\delta u_1\|^2_{H^{-1}(\mathcal{R})} \asymp_c \int_0^T \int_{\mathcal{R}_0} |u(t,x)|^2 dx dt. \tag{3.5}$$

Proof. Since $\mathcal{T}, \mathcal{R}$ and $\{e_k\}$ satisfy Assumption 1 and Assumption 2, then the claim follows by a direct application of Theorem 2.10 with $\Omega = \mathcal{T}$ and $\Omega' = \mathcal{R}$. □

We conclude this section by showing that Theorem 1.1 is a direct consequence of Theorem 3.5:

Proof of Theorem 1.1. By Lemma 2.8, if $(u_0, u_1) \in L^2(\mathcal{T}) \times H^{-1}(\mathcal{T})$ then

$$(\mathcal{P}_\delta u_0, \mathcal{P}_\delta u_1) \in L^2(\mathcal{R}) \times H^{-1}(\mathcal{R}).$$

The claim hence follows by Theorem 3.5. □

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References

[1] Allen, M. J., Tung, V. C., Kaner, R. B.: Honeycomb carbon: a review of graphene. Chemical reviews **110** (1), 132–145 (2009)

[2] Baiocchi, C., Komornik, V., Loreti, P.: Ingham type theorems and applications to control theory. Bol. Un. Mat. Ital. B **2** (1), 33–63 (1999)

[3] Baiocchi, C., Komornik, V., Loreti, P.: Ingham-Beurling type theorems with weakened gap conditions. Acta Math. Hungar. **97** (1-2), 55–95 (2002)

[4] Carleson, J.N.J.W.L., Malliavin, P., editors: The Collected Works of Arne Beurling, Volume 2, Birkhäuser, 1989

[5] Evensen, D. A.: Vibration analysis of multi-symmetric structures. AIAA Journal **14** (4), 446–453 (1976)

[6] Frølich, S., Weaver, J. C., Dean, M. N., Birkedal, H.: Uncovering Nature's Design Strategies through Parametric Modeling, Multi-Material 3D Printing, and Mechanical Testing. Advanced Engineering Materials **19**(6) e201600848 (2017)

[7] Gibson, L.J., Ashby, M. F.: Cellular solids: structure and properties. Cambridge university press (1999)

[8] Haraux, A.: Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. J. Math. Pures Appl. **68** , 457–465 (1989)

[9] Haraux, A.: On a completion problem in the theory of distributed control of wave equations. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987-1988), 241-271, Pitman Res. Notes Math. Ser., 220, Longman Sci. Tech., Harlow (1991)

[10] Ingham, A.E.: Some trigonometrical inequalities with applications in the theory of series. Math. Z. **41**, 367–379 (1936)

[11] Karpov, E. G., Stephen, N. G., Dorofeev, D. L.: On static analysis of finite repetitive structures by discrete Fourier transform. International Journal of Solids and Structures **39** (16), 4291–4310 (2002)

[12] Komornik, V.: Rapid boundary stabilization of the wave equation. SIAM J. Control Optim. **29**, 197–208 (1991)

[13] Komornik, V., Loreti, P.: Ingham type theorems for vector-valued functions and observability of coupled linear systems. SIAM J. Control Optim. **37** 461–485 (1998)

[14] Komornik, V., Loreti, P.: Fourier Series in Control Theory. Springer-Verlag, New York (2005)

[15] Komornik, V., Loreti, P.: *Multiple-point internal observability of membranes and plates*, Appl. Anal. **90** (10), 1545–1555 (2011)

[16] Komornik, V., Loreti, P.: Observability of rectangular membranes and plates on small sets. Evol. Equations and Control Theory **3** (2), 287–304 (2014)

[17] Komornik, V., Miara, B.: Cross-like internal observability of rectangular membranes. Evol. Equations and Control Theory **3** (1), 135–146 (2014)

[18] Kuttler, J.R., Sigillito, V. G.: Eigenvalues of the Laplacian in two dimensions. Siam Review **26** (2), 163–193 (1984)

[19] Loreti, P., Mehrenberger, M.: An Ingham type proof for a two-grid observability theorem. ESAIM Control Optim. Calc. Var. **14** (3), 604–631 (2008)
[20] Masters, I.G., Evans, K. E.: Models for the elastic deformation of honeycombs. Composite structures **35** (4), 403–422 (1996)
[21] Martinez, P.: Boundary stabilization of the wave equation in almost star-shaped domains. SIAM journal on control and optimization **37** (3), 673–694 (1999)
[22] Mehrenberger, M.: An Ingham type proof for the boundary observability of a Nd wave equation. C. R. Math. Acad. Sci. Paris **347** (1-2), 63–68 (2009)
[23] Práger, M.: Eigenvalues and eigenfunctions of the Laplace operator on an equilateral triangle. Applications of mathematics, **43** (4), 311–320 (1998)
[24] Thomas, D. L.: Dynamics of rotationally periodic structures. International Journal for Numerical Methods in Engineering **14** (1), 81–102 (1979)
[25] Sun, J-C.: On approximation of Laplacian eigenproblem over a regular hexagon with zero boundary conditions. J. Comp. Math., international edition, **22** (2) , 275–286 (2004)
[26] Wang, D., Zhou, C., Rong, J.: Free and forced vibration of repetitive structures. International Journal of Solids and Structures **40** (20) , 5477–5494 (2003)
[27] Zhang, Q. et al.: Bioinspired engineering of honeycomb structure – Using nature to inspire human innovation. Progress in Materials Science **74**, 332–400 (2015)

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