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# A regular threefold of general type with $p_g = 0$ and $P_2 = 12$

**Ezio Stagnaro**

**Abstract.** *Following an idea of Ronconi, we construct a nonsingular, normal threefold of general type with $p_g = q_1 = q_2 = 0$ and $P_2 = 12$. Similar examples in the literature have $p_g = q_1 = q_2 = 0$ and $0 \leq P_2 \leq 8$.*

**Sunto.** *Seguendo un'idea di Ronconi, si costruisce una varietà tridimensionale, non singolare, normale e di tipo generale con $p_g = q_1 = q_2 = 0$ e $P_2 = 12$. Nella letteratura esempi connessi hanno $p_g = q_1 = q_2 = 0$ e $0 \leq P_2 \leq 8$.*

## 1. Introduction

It is well known that nonsingular surfaces $S$ of general type with the geometric genus $p_g(S) = 0$ have the irregularity $q(S) = 0$, and the bigenus satisfies $2 \leq P_2(S) \leq 10$. There are also known examples for all possible values of the bigenus in this range (cf. for instance [1]).

In the case of nonsingular threefolds $X$ of general type having the geometric genus $p_g = 0$ and the irregularities $q_1 = q_2 = 0$, the value of the minimum integer $n_0$, if any, such that the bigenus $P_2(X) \leq n_0$ is still unknown. Whether $P_2(X)$ can take any value up to $n_0$ also remains to be established.

The examples in the literature are as follows, depending on the range of $P_2$: $P_2 = 0$ [6, 11]; $P_2 = 1$ [9]; $P_2 = 2$ [12]; $P_2 = 3, 4$ [2]; $P_2 = 5$ [3, 8, 10]; $P_2 = 6$ [7]; $P_2 = 7, 8$ [8]. Other examples with $0 \leq P_2 \leq 4$ are produced as threefolds in weighted projective spaces [4, 5].

M.C. Ronconi found her best result in [8] using a remarkable trick. She considered a degree 7 hypersurface $F_7 \subset \mathbb{P}^4$, given by a linear system of monomials depending on 25 parameters, and having the 10 edges of the fundamental pentahedron as triple lines. By cancelling 6 suitable monomials in the equation of $F_7$ and replacing them with 3 new ones, she was able to replace 6 triple edges on $F_7$ with 6 double lines having an infinitely near double surface. No other triple edges could be replaced with double lines having an infinitely near double surface. A desingularization of the original $F_7$ has the birational invariants given by $p_g = 0$, the irregularities $q_1 = q_2 = 0$, and the bigenus $P_2 = 5$ (cf. [8]). A desingularization of the new $F_7$ constructed by Ronconi has the same invariants $p_g = 0, q_1 = q_2 = 0$,

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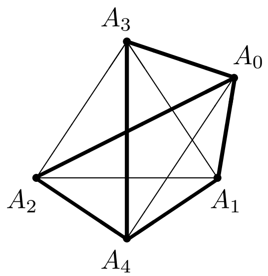
but the bigenus is increased by three, obtaining $P_2 = 8$. This is because the double lines with an infinitely near double surface give the same conditions to the canonical adjoints (which give $p_g$), and to the adjoints (giving the irregularities $q_1, q_2$), but fewer conditions to the bicanonical adjoints, which give $P_2$, because the two degree 7 hypersurfaces are normal (cf. [8, 10]).

In $\mathbb{P}^4$ with homogeneous coordinates $(X_0, X_1, X_2, X_3, X_4)$ the equations of the above 6 edges are given by $r_1$, $r_2$, $r_3$, $r_4$, $r_5$, $r_6$ of the equations

$$\begin{cases} X_0 = 0 \\ X_1 = 0 \\ X_2 = 0 \end{cases}, \begin{cases} X_1 = 0 \\ X_2 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_0 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases}, \begin{cases} X_2 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{cases}, \begin{cases} X_0 = 0 \\ X_1 = 0 \\ X_3 = 0 \end{cases}, \begin{cases} X_1 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{cases}.$$

In the two blow-ups resolving the double edges with an infinitely near double surface (which is locally given by planes), the variable $X_0$ must also produce two exceptional divisors in the case of the above first, third, and fifth lines, and the variable $X_4$ must again produce two exceptional divisors in the case of the above second, fourth, and sixth lines.

Indicating the five vertices of the fundamental pentahedron with $A_0 = (1, 0, 0, 0, 0)$, $A_1 = (0, 1, 0, 0, 0)$, $A_2 = (0, 0, 1, 0, 0)$, $A_3 = (0, 0, 0, 1, 0)$, $A_4 = (0, 0, 0, 0, 1)$, the 6 double edges $\ell_{A_iA_j}$ are shown in the following picture, where they are drawn in bold type.




That said, our aim here is: **first**, to start with a generic hypersurface $F_7 \subset \mathbb{P}^4$ of degree 7, and to impose on $F_7$ the above 6 double edges having an infinitely near double surface, and the above-mentioned properties of the variables $X_0$ and $X_4$ (Section 2); **second**, to find the conditions given by these singularities to the canonical adjoints to $F_7$ for the computation of the geometric genus $p_g$ of a desingularization of $F_7$, and the conditions given to the bicanonical adjoints to $F_7$ for the computation of the bigenus $P_2$ (Section 3); and **third**, to look for new singularities on $F_7$ to substitute the 4 triple lines considered by Ronconi, with a view to increasing the bigenus $P_2 = 8$ (Section 4).

Concerning this third step, from the normality of the last $F_7$, we obtain $P_2 = 12$ with the new singularities given by $\ell_{A_1A_2}$, $\ell_{A_1A_3}$, $\ell_{A_2A_3}$ as triple lines, according to Ronconi, but the fourth line $\ell_{A_0A_4}$ is left nonsingular and substituted by a new singularity given by an isolated ordinary 4-ple point (Sections 4,5,6).

Section 8 contains the desingularization of $F_7$, and Section 9 its property of being of general type, while we compute $p_g, P_2$, the trigenus $P_3$, in Section 10, and the irregularities $q_1 = q_2 = 0$ in Section 11.

The ground field $\mathbf{k}$, is an algebraically closed field of characteristic zero, that we can assume to be the field of complex numbers.

In all the sections, the computations are done with "muMATH".

We have to add a few words here about the desingularization of our $F_7$ because we use a method that is very long and new to us. Some different approaches to the desingularization were discussed with Alberto Calabri, and the author is grateful for his contribution.

In Ronconi's paper [8] we read: "*... apart from unimposed double lines appearing on the exceptional divisor, that we resolve after blowing up all the $\ell_i$, ...*". This means that, in solving the singularities of our $F_7$, when we start blowing up a double line with an infinitely near double surface, or a triple line, we cannot go on blowing up the unimposed double lines infinitely near the exceptional divisors. We have to go on blowing up the imposed singularities outside the exceptional divisors (cf. Remark 2, Section 8.2.2 and Remark 6, Section 8.3.1) because otherwise the singularities imposed on our $F_7$ interfere with the singularities on the exceptional divisors. With such an interference, we would obtain an infinite sequence of blow-ups moving in a circular manner.

## 2. $\mathbf{F_7} \subset \mathbb{P}^4$ with the six above singularities - our first step

The equation of a generic $F_7$ is given by a system of monomials with coefficients or parameters $a_{i_0 i_1 i_2 i_3 i_4}$

$$F_7 : \sum_{0 \leq i_0, i_1, i_2, i_3, i_4 \leq 4} a_{i_0 i_1 i_2 i_3 i_4} X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3} X_4^{i_4} = 0,\ i_0 + i_1 + i_2 + i_3 + i_4 = 7.$$

Let us impose on the generic $F_7$ the six double lines with an infinitely near double surface, which is locally given by planes. Let us start by considering the line

$$\mathbf{r_1} : \mathbf{X_0} = \mathbf{X_1} = \mathbf{X_2} = \mathbf{0}.$$

To do so, we consider the affine set $U_4$ of affine coordinates $X_0 = w, X_1 = x, X_2 = y, X_3 = z, X_4 = 1$, and we impose the double line $r_1 \cap U_4 : w = x = y = 0$ on $F_7 \cap U_4$. Let $F_7(w, x, y, z) = 0$ be the equation of $F_7 \cap U_0$.

To impose the double line $r_1 \cap U_4$, we consider the blow-up of this line, which is locally given by

$$\mathcal{B}_{w_1} : \begin{cases} w = w_1 \\ x = x_1 w_1 \\ y = y_1 w_1 \\ z = z_1 \end{cases} ; \mathcal{B}_{x_2} : \begin{cases} w = w_2 x_2 \\ x = x_2 \\ y = y_2 x_2 \\ z = z_2 \end{cases} ; \mathcal{B}_{y_3} : \begin{cases} w = w_3 y_3 \\ x = x_3 y_3 \\ y = y_3 \\ z = z_3 \end{cases} .$$

We consider $\mathcal{B}_{x_2}$. Substituting in $F_7(w,x,y,z)$, dividing by $x_2^2$, and imposing that $\frac{1}{x_2^2}F_7(w_2x_2,x_2,y_2x_2,z_2)$ be a polynomial, we obtain conditions on the coefficients $a_{i_0i_3i_2i_3i_4}$ of $F_7$.

Let us call $F_{x_2}(w_2,x_2,y_2,z_2)$ the polynomial $\frac{1}{x_2^2}F_7(w_2x_2,x_2,y_2x_2,z_2)$, so that the hypersurface $F_{x_2}(w_2,x_2,y_2,z_2)=0$ has the double line $w_2=x_2=y_2=0$.

Next, we want to impose the double plane $\pi : w_2=x_2=0$ infinitely near the double line $w_2=x_2=y_2=0$.

Locally, the blow-up of $\pi : w_2=x_2=0$ is given by

$$\mathcal{B}_{w_{21}}:\begin{cases} w_2=w_{21}\\ x_2=x_{21}w_{21}\\ y_2=y_{21}\\ z_2=z_{21}\end{cases};\quad \mathcal{B}_{x_{22}}:\begin{cases} w_2=w_{22}x_{22}\\ x_2=x_{22}\\ y_2=y_{22}\\ z_2=z_{22}\end{cases}.$$

We consider $\mathcal{B}_{x_{22}}$, and substitute in $F_{x_2}(w_2,x_2,y_2,z_2)$.

By imposing that $\frac{1}{x_{22}^2}F_{x_2}(w_{22}x_{22},x_{22},y_{22},z_{22})$ be a polynomial, we obtain conditions on the coefficients, and $r_1$ has the double plane that we wanted infinitely near.

In conclusion, the conditions obtained on the coefficients are due to the fact that we imposed the edge $r_1$ on $F_7$ as a double line with a double plane $\pi$ infinitely near.

The two blow-ups that we have considered are clearly incomplete. They will be completed in the desingularization of the last $F_7$ (Section 8).

Considering $\mathcal{B}_{x_2}$ and $\mathcal{B}_{x_{22}}$, we can nonetheless see that the variable $X_0=w$ furnishes two exceptional divisors in the two blow-ups, as required.

To obtain all the conditions on the coefficients of $F_7$ in order to obtain the six double lines having a double surface infinitely near, we produce a sketch showing only the remaining 4 blow-ups concerning $r_3$ and $r_5$:

$$r_3 : w=y=z=0,\ \mathcal{B}_{y_2}:\begin{cases} w=w_2y_2\\ x=x_2\\ y=y_2\\ z=z_2y_2\end{cases};\ \mathcal{B}_{y_{22}}:\begin{cases} w_2=w_{22}y_{22}\\ x_2=x_{22}\\ y_2=y_{22}\\ z_=z_{22}\end{cases}.$$

$$r_5 : w=x=z=0,\ \mathcal{B}_{z_3}:\begin{cases} w=w_3z_3\\ x=x_3z_3\\ y=y_3\\ z=z_3\end{cases};\ \mathcal{B}_{z_{33}}:\begin{cases} w_3=w_{33}z_{33}\\ x_3=x_{33}\\ y_3=y_{33}\\ z_3=z_{33}\end{cases}.$$

Next, we have to consider the lines $\mathbf{r_2},\mathbf{r_4}$ and $\mathbf{r_6}$, and repeat what we did for $\mathbf{r_1},\mathbf{r_3}$ and $\mathbf{r_5}$, but we omit this repetition here, which only involves changing $X_0$ to $X_4$.

Having imposed the above singularities, the equation of $F_7$ is given by

$$
\begin{aligned}
&a_{41002}X_0^4X_1X_4^2 + a_{40102}X_0^4X_2X_4^2 + a_{40012}X_0^4X_3X_4^2 + a_{32002}X_0^3X_1^2X_4^2 + a_{31111}X_0^3X_1X_2X_3X_4+\\
&a_{31102}X_0^3X_1X_2X_4^2 + a_{31012}X_0^3X_1X_3X_4^2 + a_{31003}X_0^3X_1X_4^3 + a_{30202}X_0^3X_2^2X_4^2 + a_{30112}X_0^3X_2X_3X_4^2+\\
&a_{30103}X_0^3X_2X_4^3 + a_{30022}X_0^3X_3^2X_4^2 + a_{30013}X_0^3X_3X_4^3 + a_{23002}X_0^2X_1^3X_4^2 + a_{22201}X_0^2X_1^2X_2^2X_4+\\
&a_{22111}X_0^2X_1^2X_2X_3X_4 + a_{22102}X_0^2X_1^2X_2X_4^2 + a_{22021}X_0^2X_1^2X_3^2X_4 + a_{22012}X_0^2X_1^2X_3X_4^2+\\
&a_{22003}X_0^2X_1^2X_4^3 + a_{21211}X_0^2X_1X_2^2X_3X_4 + a_{21202}X_0^2X_1X_2^2X_4^2 + a_{21121}X_0^2X_1X_2X_3^2X_4+\\
&a_{21112}X_0^2X_1X_2X_3X_4^2 + a_{21103}X_0^2X_1X_2X_4^3 + a_{21022}X_0^2X_1X_3^2X_4^2 + a_{21013}X_0^2X_1X_3X_4^3+\\
&a_{21004}X_0^2X_1X_4^4 + a_{20302}X_0^2X_2^3X_4^2 + a_{20221}X_0^2X_2^2X_3^2X_4 + a_{20212}X_0^2X_2^2X_3X_4^2 + a_{20203}X_0^2X_2^2X_4^3+\\
&a_{20122}X_0^2X_2X_3^2X_4^2 + a_{20113}X_0^2X_2X_3X_4^3 + a_{20104}X_0^2X_2X_4^4 + +a_{20032}X_0^2X_3^3X_4^2 + a_{20023}X_0^2X_3^2X_4^3+\\
&a_{20014}X_0^2X_3X_4^4+a_{13201}X_0X_1^3X_2^2X_4+a_{13111}X_0X_1^3X_2X_3X_4+a_{13021}X_0X_1^3X_3^2X_4+a_{12301}X_0X_1^2X_2^3X_4+\\
&a_{12220}X_0X_1^2X_2^2X_3^2 + a_{12211}X_0X_1^2X_2^2X_3X_4 + a_{12202}X_0X_1^2X_2^2X_4^2 + a_{12121}X_0X_1^2X_2X_3^2X_4+\\
&a_{12112}X_0X_1^2X_2X_3X_4^2 + a_{12031}X_0X_1^2X_3^3X_4 + a_{12022}X_0X_1^2X_3^2X_4^2 + a_{11311}X_0X_1X_2^3X_3X_4+\\
&a_{11221}X_0X_1X_2^2X_3^2X_4 + a_{11212}X_0X_1X_2^2X_3X_4^2 + a_{11131}X_0X_1X_2X_3^3X_4 + a_{11122}X_0X_1X_2X_3^2X_4^2+\\
&a_{11113}X_0X_1X_2X_3X_4^3+a_{10321}X_0X_2^3X_3^2X_4+a_{10231}X_0X_2^2X_3^3X_4+a_{10222}X_0X_2^2X_3^2X_4^2+a_{03310}X_1^3X_2^3X_3+\\
&a_{03220}X_1^3X_2^2X_3^2 + a_{03130}X_1^3X_2X_3^3 + a_{02320}X_1^2X_2^3X_3^2 + a_{02230}X_1^2X_2^2X_3^3 + a_{02221}X_1^2X_2^2X_3^2X_4+\\
&a_{01330}X_1X_2^3X_3^3 = 0.
\end{aligned}
$$

## 3. Pluricanonical adjoints to the last $\mathbf{F_7}$ - our second step

Throughout the present paper, the hypersurfaces are normal and the infinitely near surfaces are locally given by planes, so we can use the theory of adjoints and pluricanonical adjoints to $F_7$ that we revisited and developed in [10].

In the present section, we assume that the $F_7$ given at the end of Section 2 only has the six double lines with an infinitely near double surface as singularities, and no others.

Let

$$\mathbb{P}_{12} \xrightarrow{\pi_{12}} \cdots \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = \mathbb{P}^4$$

be a sequence of blow-ups resolving the singularities on $F_7$, i.e. by setting $\sigma = \pi_{12} \circ ... \circ \pi_2 \circ \pi_1$, if $X$ is the strict (or proper) transform of $F_7$ with respect to $\sigma$, then

$$\sigma_{|X} : X \longrightarrow F_7$$

is a desingularization of $F_7$.

If we call $V_i$ the strict (or proper) transform of $V_{i-1}$ with respect to $\pi_i$, then the desingularization of $F_7$ can be written as follows:

$$X = V_{12} \xrightarrow{\pi'_{12}} ... \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = F_7,$$

where $\pi'_i = \pi_{i|V_i} : V_i \longrightarrow V_{i-1}$.

Let us assume that $\pi_i$ is a blow-up of $\mathbb{P}_{i-1}$ along an algebraic subvariety $Y_{i-1} \subset \mathbb{P}_{i-1}$ of dimension $j_{i-1} = 1$ or $= 2$, that the variety $Y_{i-1}$ is also a subvariety of $V_{i-1}$, and that $V_{i-1} \subset \mathbb{P}_{i-1}$ passes with multiplicity 2 through $Y_{i-1}$. Here $Y_{i-1}$ is either a line or a surface.

Let us set $n_{i-1} = -4 + 1 + j_{i-1} + 2$, for $i = 1, ..., 12$ (cf. [10, p. 152]).

A hypersurface $\Phi_{2m}$ of degree $2m$ in $\mathbb{P}^4$ is an *m-canonical adjoint* to $F_7$ (with respect to the sequence of blow-ups $\pi_1, ..., \pi_{12}$) if the restriction to $X$ of the divisor

$$D_m = \pi_{12}^*\{\pi_{11}^*[\cdots \pi_1^*(\Phi_{2m}) - mn_0E_1 \cdots] - mn_{10}E_{11}\} - mn_{11}E_{12}$$

is effective, i.e. $D_{m|_X} \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of $\pi_i$ and $\pi_i^* \colon Div(\mathbb{P}_{i-1}) \longrightarrow Div(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups.

Note that, if $\Phi_{2m}$ is an $m$-canonical adjoint to $F_7$, then $D_{m|_X} \equiv mK$, where '$\equiv$' denotes linear equivalence, $K$ denotes a canonical divisor on $X$, and $D_{m|_X} \in |mK| = H^0(X, \mathcal{O}_X(mK))^*/k^*$, where $(...)^* = (...)\backslash\{0\}$.

The following picture is useful.

$$\{\text{linear system of m-canonical adjoints to } F_7\}|_{F_7} \longrightarrow |mK_X|$$

$$\Phi_{2m|_{F_7}} \longmapsto D_{m|_X}$$

Since $F_7$ is normal, the map in the above picture is an isomorphism of projective spaces (cf. [10, Section 4, Corollary 8]).

### 3.1. Canonical adjoints to $\mathbf{F_7}$ and $\mathbf{p_g}$

The canonical adjoints to $F_7$ are obtained for $m = 1$. They are hypersurfaces of degree 2: $\Phi_2$ passing with multiplicity $n_{i-1}$ on each singularity on $F_7$. The singularities on $F_7$ are either lines or infinitely near surfaces, so we have $n_{i-1} = 0$ for lines, and $n_{i-1} = 1$ for surfaces. For six lines with a surface infinitely near, we thus have $\Phi_2$ as a canonical adjoint to $F_7$ if $\Phi_2$ passes through the six lines.

The linear system of canonical adjoints to $F_7$ is given by

$$\Phi_2\colon a_{10001}X_0X_4 + a_{01100}X_1X_2 + a_{01010}X_1X_3 + a_{00110}X_2X_3 = 0$$

and $p_g = 4$.

### 3.2. Bicanonical adjoints to $\mathbf{F_7}$ and $\mathbf{P_2}$

The bicanonical adjoints to $F_7$ are obtained for $m = 2$. They are hypersurfaces of degree 4: $\Phi_4$ passing with multiplicity $2n_{i-1}$ on each singularity on $F_7$. The singularities on $F_7$ are either lines or infinitely near surfaces, so we have $2n_{i-1} = 0$ for lines, and $2n_{i-1} = 2$ for surfaces. For six lines with a surface infinitely near, we thus have $\Phi_4$ as a bicanonical adjoint to $F_7$ if $\Phi_4$ passes through the six lines and the proper transform of $\Phi_4$ passes through the six infinitely near surfaces.

For bicanonical adjoints we need the restriction to $F_7$, but two bicanonical adjoints cannot be identified using this restriction because the degree of $\Phi_4$ is $4 < 7$.

Thus, from the normality of $F_7$, the number of linearly independent bicanonical adjoints coincides with the bigenus $P_2$ (cf. [10]). These bicanonical adjoints are also called *global* bicanonical adjoints (loc. cit.).

The linear system of bicanonical adjoints to $F_7$ is given by

$\Phi_4$: $a_{30001}X_0^3X_4 + a_{21001}X_0^2X_1X_4 + a_{20101}X_0^2X_2X_4 + a_{20011}X_0^2X_3X_4 + a_{20002}X_0^2X_4^2 +$
$a_{12001}X_0X_1^2X_4 + a_{11110}X_0X_1X_2X_3 + a_{11101}X_0X_1X_2X_4 + a_{11011}X_0X_1X_3X_4 +$
$a_{11002}X_0X_1X_4^2 + a_{10201}X_0X_2^2X_4 + a_{10111}X_0X_2X_3X_4 + a_{10102}X_0X_2X_4^2 + a_{10021}X_0X_3^2X_4 +$
$a_{10012}X_0X_3X_4^2 + a_{10003}X_0X_4^3 + a_{02200}X_1^2X_2^2 + a_{02110}X_1^2X_2X_3 + a_{02020}X_1^2X_3^2 +$
$a_{01210}X_1X_2^2X_3 + a_{01120}X_1X_2X_3^2 + a_{01111}X_1X_2X_3X_4 + a_{00220}X_2^2X_3^2 = 0$

and $P_2 = 23$.

We do not calculate the 3-canonical adjoints and $P_3$ for now. We shall do so when we have all the singularities on $F_7$ (Section 9.3).

## 4. New singularities on $\mathbf{F_7}$ - our third step

If we agree with Ronconi, then we impose on the remaining four edges $r_7$, $r_8$, $r_9$, $r_{10}$

$$r_7 : \begin{cases} X_0 = 0 \\ X_1 = 0 \\ X_4 = 0 \end{cases}, \quad r_8 : \begin{cases} X_0 = 0 \\ X_2 = 0 \\ X_4 = 0 \end{cases}, \quad r_9 : \begin{cases} X_0 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{cases}, \quad r_{10} \begin{cases} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases},$$

that they be triple lines. In this case, the canonical adjoints to the new $F_7$ (cf. Section 3.1) must pass through the four lines because $n_{i-1} = -4 + 1 + 1 + 3 = 1$ It is easy to see that there are no canonical adjoints, so $p_g = 0$. The bicanonical adjoints to the new $F_7$ (cf. Section 3.2) must pass doubly through the four edges because $2n_{i-1} = 2$, so they have the equation

$a_{12001}X_0X_1^2X_4 + a_{10201}X_0X_2^2X_4 + a_{10021}X_0X_3^2X_4 + a_{11110}X_0X_1X_2X_3 + a_{11101}X_0X_1X_2X_4 +$
$a_{11011}X_0X_1X_3X_4 + a_{10111}X_0X_2X_3X_4 + a_{01111}X_1X_2X_3X_4 = 0$

and $P_2 = 8$.

**Remark 1.** There is an important fact to consider when imposing that the four edges $r_7$, $r_8$, $r_9$, $r_{10}$ be triple, as above. If we impose on the three edges $r_7$, $r_8$, $r_9$ that they be triple, then we kill six coefficients in the equation of the bicanonical adjoints, i.e. $a_{02200} = a_{02110} = a_{02020} = a_{01210} = a_{01120} = a_{00220} = 0$, whereas if we only impose on the edge $r_{10}$ that it be triple, then we kill nine coefficients, i.e. $a_{30001} = a_{21001} = a_{20101} = a_{20011} = a_{20002} = a_{11002} = a_{10102} = a_{10012} = a_{10003} = 0$.

This is due to the particular position of the first six double edges, and to the properties of the variables $X_0$ and $X_4$. □

The new singularities having six double edges with a double surface infinitely near that we want to impose on $F_7$ (see Section 2) are given by three triple edges $r_7$, $r_8$, $r_9$. This means that there is only one canonical adjoint to this $F_7$, and it is given by

$$\Phi_2 : a_{10001}X_0X_4 = 0.$$

To kill $a_{10001}$, we impose on $F_7$ an ordinary 4-ple point $P = (1,1,1,1,1)$. The canonical adjoints must pass through $P$, so $a_{10001} = 0$ and $p_g = 0$. The bicanonical adjoints must have a double point at $P$. For this to happen, five conditions must be satisfied, and the bigenus is therefore $P_2 = 23 - 6 - 5 = 12$. In other words, we have a gain of four numbers by comparison with the triple edge $r_{10}$, and again $P_2 = 8 + 4 = 12$ (see Remark 1).

We also tried to substitute the 4-ple point on $F_7$ with a triple point at $P = (1,1,1,1,1)$ having an infinitely near triple line, but this last singularity seems to disturb the previous singular edges and creates new singularities.

**Conclusion.** The best choice of singularities for $F_7$ is given by the above six double edges $r_1, r_2, r_3, r_4, r_5, r_6$ with an infinitely near double surface, the triple lines $r_7, r_8, r_9$, and the ordinary 4-ple point at $P = (1,1,1,1,1)$. New unimposed singularities appear as well (cf. Section 8, Propositions 2,3): they are double singular lines, and they are negligible singularities because they do not affect the canonical adjoints to $F_7$, so they leave the birational invariants unchanged. **In this case, we shall prove that the birational invariants of a desingularization of $\mathbf{F_7}$ are given by $\mathbf{q_1 = q_2 = p_g = 0}$ and $\mathbf{P_2 = 12}$** (cf. Sections 8,9,10,11).

## 5. Imposing the triple lines $\mathbf{r_7, r_8, r_9}$ on $\mathbf{F_7}$

This is easy to do by choosing $a_{13201} = a_{13021} = a_{12301} = a_{12031} = a_{10321} = a_{10231} = a_{03310} = a_{03220} = a_{03130} = a_{02320} = a_{02230} = a_{01330} = 0$.

## 6. Imposing a 4-ple point on $\mathbf{F_7}$ at $\mathbf{P = (1,1,1,1,1)}$

The equation of $F_7$ with a 4-ple point in $P$ is very long, so we cancel several coefficients $a_{ijklm}$ that are not essential to our construction. To be precise, we set $a_{22111} = \cdots a_{12121} = \cdots = a_{11122} = 0$, except for $a_{21211}$ because, in the calculation of the 4-ple point, it equals $\frac{1}{10}(-13a_{31111} + a_{13111} + a_{11311} + a_{11131} + 13a_{11113})$.

After cancelling these coefficients, and those in Section 5, the equation of $F_7$ given at the end of Section 2 now looks like

$F_7$: $a_{41002}X_0^4X_1X_4^2 + a_{40102}X_0^4X_2X_4^2 + a_{40012}X_0^4X_3X_4^2 + a_{32002}X_0^3X_1^2X_4^2 + a_{31111}X_0^3X_1X_2X_3X_4 +$
$a_{31102}X_0^3X_1X_2X_4^2 + a_{31012}X_0^3X_1X_3X_4^2 + a_{31003}X_0^3X_1X_4^3 + a_{30202}X_0^3X_2^2X_4^2 + a_{30112}X_0^3X_2X_3X_4^2 +$
$a_{30103}X_0^3X_2X_4^3 + a_{30022}X_0^3X_3^2X_4^2 + a_{30013}X_0^3X_3X_4^3 + a_{23002}X_0^2X_1^3X_4^2 + a_{22201}X_0^2X_1^2X_2^2X_4 +$
$a_{22102}X_0^2X_1^2X_2X_4^2 + a_{22021}X_0^2X_1^2X_3^2X_4 + a_{22012}X_0^2X_1^2X_3X_4^2 + a_{22003}X_0^2X_1^2X_4^3 +$

$a_{21211}X_0^2X_1X_2^2X_3X_4 + a_{21202}X_0^2X_1X_2^2X_4^2 + a_{21103}X_0^2X_1X_2X_4^3 + a_{21022}X_0^2X_1X_3^2X_4^2$
$a_{21013}X_0^2X_1X_3X_4^3 + a_{21004}X_0^2X_1X_4^4 + a_{20302}X_0^2X_2^3X_4^2 + a_{20221}X_0^2X_2^2X_3^2X_4 + a_{20212}X_0^2X_2^2X_3X_4^2+$
$a_{20203}X_0^2X_2^2X_4^3 + a_{20122}X_0^2X_2X_3^2X_4^2 + a_{20113}X_0^2X_2X_3X_4^3 + a_{20104}X_0^2X_2X_4^4 + a_{20032}X_0^2X_3^3X_4^2+$
$a_{20023}X_0^2X_3^2X_4^3 + a_{20014}X_0^2X_3X_4^4 + a_{13111}X_0X_1^3X_2X_3X_4 + a_{12220}X_0X_1^2X_2^2X_3^2 + a_{12202}X_0X_1^2X_2^2X_4^2+$
$a_{12022}X_0X_1^2X_3^2X_4^2 + a_{11311}X_0X_1X_2^3X_3X_4 + a_{11131}X_0X_1X_2X_3^3X_4 + a_{11113}X_0X_1X_2X_3X_4^3+$
$a_{10222}X_0X_2^2X_3^2X_4^2 + a_{02221}X_1^2X_2^2X_3^2X_4 = 0.$

We impose the 4-ple point $P = (1, 1, 1, 1, 1)$ considering the affine coordinate $X_0 = 1, X_1 = x, X_2 = y, X_3 = z, X_4 = t$, making the translation

$$\begin{aligned} &F_7(1, x-1, y-1, z-1, t-1) \\ &\quad = a_{41002}(x-1)(t-1)^2 + \cdots + a_{13111}(x-1)^3(y-1)(z-1)(t-1) + \cdots \\ &\qquad + a_{02221}(x-1)^2(y-1)^2(z-1)^2(t-1). \end{aligned}$$

and imposing that *the constant and the polynomials of degrees* 1,2,3, *in* $x, y, z, t$ *disappear.*

There are 35 long equalities, the three simplest of which are: $a_{23002} = -a_{13111}$, $a_{20032} = -a_{11131}, \cdots, a_{21211} = \frac{1}{10}(-13a_{31111} + a_{13111} + a_{11311} + a_{11131} + 13a_{11113})$.

## 7. The long final equation of $\mathbf{F_7}$ with the singularities described in the Conclusion at the end of Section 4

$\mathbf{F_7}$: $\mathbf{a_{41002}}[10X_0X_4^2(2X_0X_1X_2^2 - 4X_0^2X_1X_2 + 2X_0^3X_1 + X_2^2X_3^2 - X_1^2X_2^2 + 2X_0X_1^2X_2 - X_0^2X_1^2 -$
$2X_0X_2X_3^2 - 2X_0X_2^2X_3 + 4X_0^2X_2X_3 + X_0^2X_3^2 - 2X_0^3X_3)]+$
$\mathbf{a_{21004}}[10X_0^2X_4(-4X_1X_3X_4^2 + 2X_1X_3^2X_4 + 2X_1X_4^3 + 2X_1^2X_3X_4 - X_1^2X_3^2 - X_1^2X_4^2 + 4X_2X_3X_4^2 -$
$2X_2X_3^2X_4 - 2X_2X_4^3 - 2X_2^2X_3X_4 + X_2^2X_3^2 + X_2^2X_4^2)]+$
$\mathbf{a_{20014}}[10X_0^2X_4(4X_1X_2X_4^2 - 4X_1X_3X_4^2 - 2X_1X_2^2X_4 + 2X_1X_3^2X_4 - 2X_1^2X_2X_4 + 2X_1^2X_3X_4 + X_1^2X_2^2 -$
$X_1^2X_3^2 - 2X_2X_4^3 + 2X_3X_4^3 + X_2^2X_4^2 - X_3^2X_4^2)]+$
$\mathbf{a_{12022}}[20X_0X_4^2(2X_0X_1X_2^2 - 2X_0X_1X_3^2 - 4X_0^2X_1X_2 + 4X_0^2X_1X_3 - X_1^2X_2^2 + X_1^2X_3^2 + 2X_0X_1^2X_2 -$
$2X_0X_1^2X_3 - X_0^2X_2^2 + X_0^2X_3^2 + 2X_0^3X_2 - 2X_0^3X_3)]+$
$\mathbf{a_{31111}}(-4X_0^2X_1X_2X_4^3 - 6X_0^2X_1X_3X_4^3 - 26X_0^2X_1X_2^2X_3X_4 + 12X_0^2X_1X_2^2X_4^2 + 24X_0^2X_1X_3^2X_4^2 +$
$20X_0^3X_1X_2X_3X_4 + 12X_0^3X_1X_2X_4^2 - 36X_0^3X_1X_3X_4^2 + 4X_0^3X_1X_4^3 + 8X_0X_2^2X_3^2X_4^2 + 12X_0X_1^2X_2^2X_3^2 +$
$12X_0X_1^2X_2^2X_4^2 - 14X_0^2X_1^2X_2X_4^2 + 24X_0^2X_1^2X_3X_4^2 - 9X_0^2X_1^2X_2^2X_4 - 16X_0^2X_1^2X_3^2X_4 - 9X_0^2X_1^2X_4^3+$
$8X_0^3X_1^2X_4^2 - 8X_1^2X_2^2X_3^2X_4 - 4X_0^2X_2X_3X_4^3 - 6X_0^2X_2X_3^2X_4^2 + 12X_0^2X_2X_4^4 + 20X_0^2X_2^2X_3X_4^2-$
$9X_0^2X_2^2X_3^2X_4 - 28X_0^2X_2^2X_4^3 - 9X_0^2X_3^2X_4^3 - 4X_0^3X_2X_3X_4^2 + 4X_0^3X_2X_4^3 + 4X_0^3X_3X_4^3 + 16X_0^3X_2^2X_4^2+$
$4X_0^3X_3^2X_4^2 - 16X_0^4X_2X_4^2 + 8X_0^4X_3X_4^2)+$
$\mathbf{a_{13111}}(-12X_0^2X_1X_2X_4^3 - 18X_0^2X_1X_3X_4^3 + 2X_0^2X_1X_2^2X_3X_4 + 56X_0^2X_1X_2^2X_4^2 + 32X_0^2X_1X_3^2X_4^2 -$
$44X_0^3X_1X_2X_4^2 + 12X_0^3X_1X_3X_4^2 - 28X_0^3X_1X_4^3 + 14X_0X_2^2X_3^2X_4^2 - 4X_0X_1^2X_2^2X_3^2 - 14X_0X_1^2X_2^2X_4^2-$
$2X_0^2X_1^2X_2X_4^2 - 28X_0^2X_1^2X_3X_4^2 - 7X_0^2X_1^2X_2^2X_4 - 8X_0^2X_1^2X_3^2X_4 + 33X_0^2X_1^2X_4^3 + 34X_0^3X_1^2X_4^2-$
$4X_1^2X_2^2X_3^2X_4 + 20X_0X_1^3X_2X_3X_4 - 20X_0^2X_1^3X_4^2 + 28X_0^2X_2X_3X_4^3 - 38X_0^2X_2X_3^2X_4^2 - 4X_0^2X_2X_4^4-$
$40X_0^2X_2^2X_3X_4^2 + 13X_0^2X_2^2X_3^2X_4 - 4X_0^2X_2^2X_4^3 - 7X_0^2X_3^2X_4^3 + 28X_0^3X_2X_3X_4^2 + 12X_0^3X_2X_4^3 + 12X_0^3X_3X_4^3-$
$12X_0^3X_2^2X_4^2 + 2X_0^3X_3^2X_4^2 + 12X_0^4X_2X_4^2 - 16X_0^4X_3X_4^2)+$

$\mathbf{a_{11311}}(20X_0X_1X_2^3X_3X_4 - 12X_0^2X_1X_2X_4^3 + 22X_0^2X_1X_3X_4^3 + 2X_0^2X_1X_2^2X_3X_4 - 44X_0^2X_1X_2^2X_4^2 - 8X_0^2X_1X_3^2X_4^2 + 36X_0^3X_1X_2X_4^2 - 28X_0^3X_1X_3X_4^2 + 12X_0^3X_1X_4^3 - 6X_0X_2^2X_3^2X_4^2 - 4X_0X_1^2X_2^2X_3^2 + 6X_0X_1^2X_2^2X_4^2 + 18X_0^2X_1^2X_2X_4^2 - 8X_0^2X_1^2X_3X_4^2 - 7X_0^2X_1^2X_2^2X_4 + 12X_0^2X_1^2X_3^2X_4 - 7X_0^2X_1^2X_4^3 - 6X_0^3X_1^2X_4^2 - 4X_1^2X_2^2X_3^2X_4 - 12X_0^2X_2X_3X_4^3 + 42X_0^2X_2X_3^2X_4^2 - 4X_0^2X_2X_4^4 - 20X_0^2X_2^2X_3X_4^2 - 7X_0^2X_2^2X_3^2X_4 + 36X_0^2X_2^2X_4^3 - 20X_0^2X_2^3X_4^2 - 7X_0^2X_3^2X_4^3 - 12X_0^3X_2X_3X_4^2 - 28X_0^3X_2X_4^3 + 12X_0^3X_3X_4^3 + 48X_0^3X_2^2X_4^2 - 18X_0^3X_3^2X_4^2 - 28X_0^4X_2X_4^2 + 24X_0^4X_3X_4^2) +$
$\mathbf{a_{11131}}(20X_0X_1X_2X_3^3X_4 + 28X_0^2X_1X_2X_4^3 - 18X_0^2X_1X_3X_4^3 + 2X_0^2X_1X_2^2X_3X_4 - 24X_0^2X_1X_2^2X_4^2 - 28X_0^2X_1X_3^2X_4^2 - 4X_0^3X_1X_2X_4^2 + 12X_0^3X_1X_3X_4^2 + 12X_0^3X_1X_4^3 - 6X_0X_2^2X_3^2X_4^2 - 4X_0X_1^2X_2^2X_3^2 + 6X_0X_1^2X_2^2X_4^2 - 22X_0^2X_1^2X_2X_4^2 + 32X_0^2X_1^2X_3X_4^2 + 13X_0^2X_1^2X_2^2X_4 - 8X_0^2X_1^2X_3^2X_4 - 7X_0^2X_1^2X_4^3 - 6X_0^3X_1^2X_4^2 - 4X_1^2X_2^2X_3^2X_4 - 12X_0^2X_2X_3X_4^3 - 18X_0^2X_2X_3^2X_4^2 - 4X_0^2X_2X_4^4 + 40X_0^2X_2^2X_3X_4^2 - 7X_0^2X_2^2X_3^2X_4 - 4X_0^2X_2^2X_4^3 + 33X_0^2X_3^2X_4^3 - 20X_0^2X_3^3X_4^2 - 12X_0^3X_2X_3X_4^2 + 12X_0^3X_2X_4^3 - 28X_0^3X_3X_4^3 - 12X_0^3X_2^2X_4^2 + 42X_0^3X_3^2X_4^2 + 12X_0^4X_2X_4^2 - 16X_0^4X_3X_4^2) +$
$\mathbf{a_{11113}}(20X_0X_1X_2X_3X_4^3 - 4X_0^2X_1X_2X_4^3 - 46X_0^2X_1X_3X_4^3 - 26X_0^2X_1X_2^2X_3X_4 + 52X_0^2X_1X_2^2X_4^2 - 16X_0^2X_1X_3^2X_4^2 - 68X_0^3X_1X_2X_4^2 + 84X_0^3X_1X_3X_4^2 + 4X_0^3X_1X_4^3 - 22X_0X_2^2X_3^2X_4^2 - 8X_0X_1^2X_2^2X_3^2 - 38X_0X_1^2X_2^2X_4^2 + 26X_0^2X_1^2X_2X_4^2 - 16X_0^2X_1^2X_3X_4^2 + 21X_0^2X_1^2X_2^2X_4 + 4X_0^2X_1^2X_3^2X_4 + 21X_0^2X_1^2X_4^3 - 22X_0^3X_1^2X_4^2 + 12X_1^2X_2^2X_3^2X_4 - 4X_0^2X_2X_3X_4^3 - 6X_0^2X_2X_3^2X_4^2 - 8X_0^2X_2X_4^4 + 20X_0^2X_2^2X_3X_4^2 + 21X_0^2X_2^2X_3^2X_4 + 12X_0^2X_2^2X_4^3 + 21X_0^2X_3^2X_4^3 - 4X_0^3X_2X_3X_4^2 + 4X_0^3X_2X_4^3 + 4X_0^3X_3X_4^3 - 44X_0^3X_2^2X_4^2 - 6X_0^3X_3^2X_4^2 + 44X_0^4X_2X_4^2 - 32X_0^4X_3X_4^2) = 0.$

This $F_7$ is a linear system of hypersurfaces of degree 7. From now on, by $F_7$ we mean the generic element of the above linear system (often omitting the adjective "generic").

## 8. Desingularization of the (generic) $\mathbf{F_7}$

### 8.1. On the actual singularities on $\mathbf{F_7}$ and the normality of $\mathbf{F_7}$

We call the singularities on $F_7$ *actual singularities* to distinguish them from those infinitely near.

We have imposed the singularities described in the Conclusion at the end of Section 4 on a degree 7 hypersurface. In particular, we have imposed the actual singularities among them. In the present section, we prove the following:

**Proposition 1.** *The actual singularities on* $\mathbf{F_7}$ *are given by six double lines* $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4}, \mathbf{r_5}, \mathbf{r_6}$ *(cf. Section 2), three triple lines* $\mathbf{r_7}, \mathbf{r_8}, \mathbf{r_9}$ *(cf. Section 5) and the* 4*-ple point* $\mathbf{P} = (1,1,1,1,1)$ *(cf. Section 6), and no others. In particular,* $F_7$ *is normal.*

*Proof.* (*The proof is very long.*) Using Bertini's theorem, the actual singularities on the generic $F_7$ belong to the base points of the linear system. These base points are points that are the zeros of all the polynomials defining the hypersurfaces of the linear system.

**First step [(1)]**: Let us start by considering $a_{41002}10X_0X_4^2(X_0 - X_2)^2(X_1 - X_3)(2X_0 - X_1 - X_3) = 0$, and $X_0 = 0$ in particular. By intersecting $F_7 = 0$ and

$X_0 = 0$, we obtain

$$\left\{\begin{array}{l} F_7 = 0 \\ X_0 = 0 \end{array}\right. = \left\{\begin{array}{l} X_1^2X_2^2X_3^2X_4(-8a_{31111} - 4a_{13111} - 4a_{11311} - 4a_{11131} - \\ \quad 12a_{11113}) = 0 \\ X_0 = 0 \end{array}\right.$$

and we deduce the following 4 planes that are in the base points of the linear system:

$$\left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \end{array}\right. \cup \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \end{array}\right. \cup \left\{\begin{array}{l} X_0 = 0 \\ X_3 = 0 \end{array}\right. \cup \left\{\begin{array}{l} X_0 = 0 \\ X_4 = 0 \end{array}\right. .$$

According to Bertini's theorem, the actual singularities on $F_7$, that are contained in $X_0 = 0$, belong to the 4 above planes.

**I)** Let us consider the singularities on $\left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \end{array}\right.$.

From $(\frac{\partial F_7}{\partial X_0})_{X_0=X_1=0} = X_2^2X_3^2X_4^2(10a_{41002} + 8a_{31111} + \cdots - 22a_{11113}) = 0$, we obtain the three imposed singular lines

$$r_1 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_2 = 0 \end{array}\right. , \; r_5 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_3 = 0 \end{array}\right. , \; r_7 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_4 = 0 \end{array}\right. .$$

**II)** Let us consider the singularities on $\left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \end{array}\right.$.

From $(\frac{\partial F_7}{\partial X_0})_{X_0=X_2=0} = 20a_{12022}X_1^2X_3^2X_4^2 = 0$, we obtain the three imposed singular lines

$$r_1 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_2 = 0 \end{array}\right. , \; r_3 : \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{array}\right. , \; r_8 : \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \\ X_4 = 0 \end{array}\right. .$$

**III)** Let us consider the singularities on $\left\{\begin{array}{l} X_0 = 0 \\ X_3 = 0 \end{array}\right.$. From $(\frac{\partial F_7}{\partial X_0})_{X_0=X_3=0} = X_1^2X_2^2X_4^2(-10a_{41002} - 20a_{12022} + 12a_{31111} + \cdots - 38a_{11113}) = 0$, we obtain the three imposed singular lines

$$r_5 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_3 = 0 \end{array}\right. , \; r_3 : \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{array}\right. , \; r_8 : \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \\ X_4 = 0 \end{array}\right. .$$

**IV)** Let us consider the singularities on $\left\{\begin{array}{l} X_0 = 0 \\ X_4 = 0 \end{array}\right.$. From $(\frac{\partial F_7}{\partial X_0})_{X_0=X_4=0} =$
$X_1^2X_2^2X_3^2(12a_{31111} + \cdots - 8a_{11113}) = 0$, we obtain the three imposed triple lines

$$r_7 : \left\{\begin{array}{l} X_0 = 0 \\ X_1 = 0 \\ X_4 = 0 \end{array}\right. , \; r_8 : \left\{\begin{array}{l} X_0 = 0 \\ X_2 = 0 \\ X_4 = 0 \end{array}\right. , \; r_9 : \left\{\begin{array}{l} X_0 = 0 \\ X_3 = 0 \\ X_4 = 0 \end{array}\right. .$$

**Second step [(2)]**: Let us continue to consider $X_4 = 0$ in

$a_{41002}10X_0X_4^2(X_0-X_2)^2(X_1-X_3)(2X_0-X_1-X_3)=0$. By replacing $X_0$ with $X_4$ and omitting this duplicate, as before, we obtain the three imposed double lines

$$r_2:\begin{cases} X_1=0\\ X_2=0\\ X_4=0\end{cases},\quad r_4:\begin{cases} X_2=0\\ X_3=0\\ X_4=0\end{cases},\quad r_6:\begin{cases} X_1=0\\ X_3=0\\ X_4=0\end{cases}.$$

From now on, we consider $X_0X_4\neq 0$, because $X_0X_4=0$ has already been considered.

To argue our next point, it is convenient to consider the first four parts of the linear system defining $F_7$ in the following way:

$$(*)\quad\begin{cases} 10a_{41002}X_0X_4^2(X_0-X_2)^2(X_1-X_3)(2X_0-X_1-X_3)=0\\ 10a_{21004}X_0^2X_4(X_3-X_4)^2(X_1-X_2)(2X_4-X_1-X_2)=0\\ 10a_{20014}X_0^2X_4(X_1-X_4)^2(X_2-X_3)(-2X_4+X_2+X_3)=0\\ 20a_{12022}X_0X_4^2(X_0-X_1)^2(X_2-X_3)(2X_0-X_2-X_3)=0\end{cases}.$$

**Third step [(*)(3)]**: In the first equation of $(*)$, we choose $X_0=X_2$. In this step, substituting $X_0=X_2$, we have to find the zeros of all the hypersurfaces of the linear system defining $F_7$. Let us start with the fourth equation of $(*)$. Substituting $X_0=X_2$, we obtain

$$X_2X_4^2(X_1-X_2)^2(X_2-X_3)^2=0\,.$$

Since we have already considered $X_0=0$ and $X_4=0$, it remains for us to consider two cases:

$$\textbf{[(*)(3)(A)]}\ X_1=X_2\ ;\quad \textbf{[(*)(3)(B)]}\ X_3=X_2.$$

In case **[(*)(3)(A)]**, the second equation of $(*)$ is vanishing. So, we consider the third of $(*)$. It is vanishing in the event of three possibilities:

$$\textbf{[(A)I]}\ \ X_4=X_2,\ \ \textbf{[(A)II]}\ \ X_3=X_2,\ \ \textbf{[(A)III]}\ \ -2X_4+X_2+X_3=0.$$

In possibility **[(A)I]**, we obtain the line $X_0-X_2=X_1-X_2=X_4-X_2=0$. The points of this line are base points of $F_7$.

From $(\frac{\partial F_7}{\partial X_0})_{X_0-X_2=X_1-X_2=X_4-X_2=0}=20a_{11131}X_2^3(X_3-X_2)^3=0$ and from $X_2=X_0\neq 0$ we obtain the singular 4-ple point $P=(1,1,1,1,1)$. We note that if we consider $X_2=0$, then we obtain the singular point $(0,0,0,1,0)\in r_1\cap r_2\cap r_7\cap r_8$.

In possibility **[(A)II]**, we obtain the line $X_0-X_2=X_1-X_2=X_3-X_2=0$. Its points are not base points of $F_7$, so we intersect the line with $F_7$. Considering the particular equation of the linear system defining $F_7$ given by the coefficient $a_{11113}$, for example, we obtain the equation

$$-8a_{11113}X_2^3(X_4-X_2)^4=0$$

which again furnishes the singular point $P$.

In possibility [**(A)III**], we obtain the line $X_0 - X_2 = X_1 - X_2 = X_4 - (X_2 + X_3)/2 = 0$. Its points are not base points of $F_7$, so we intersect the line with $F_7$. Considering the particular equation of the linear system defining $F_7$ given by the coefficient $a_{31111}$, for example, we obtain the equation

$$3/8a_{31111}X_2^2(X_3 - X_2)^4(-3X_3 - X_2) = 0\,.$$

Considering $X_3 - X_2 = 0$, we obtain the singular point $P$ and from $X_3 = -X_2/3$, we have the point $(X_2, X_2, X_2, -X_2/3, X_2/3)$. Substituting again in $F_7$, we obtain

$$320/81X_2^7(-2/3a_{13111} - 2/3a_{11311} - 2/3a_{11131} + a_{11113}).$$

This last expression cannot be equal to zero, because $X_2 \neq 0$. (If $X_2 = 0$, then we obtain $(0, 0, 0, 0, 0) \notin \mathbb{P}^4$).

In conclusion, the point $(X_2, X_2, X_2, -X_2/3, X_2/3) \notin F_7$.

In case [**(*)(3)(B)**] the third equation of $(*)$ is vanishing. So we consider the second equation of $(*)$. It is vanishing in the event of three possibilities:

[**(B)I**] $X_4 = X_2$, [**(B)II**] $X_1 = X_2$, [**(B)III**] $2X_4 - X_1 - X_2 = 0$.

In possibility [**(B)I**], we obtain the line $X_0 - X_2 = X_3 - X_2 = X_4 - X_2 = 0$. The points of this line are base points of $F_7$.

Let us consider $(\frac{\partial F_7}{\partial X_0})_{X_0 - X_2 = X_3 - X_2 = X_4 - X_2 = 0} = 20a_{13111}X_2^3(X_2 - X_1)^3 = 0$, finding the singular point $P$ again.

Possibility [**(B)II**] coincides with [**(A)I**].

In possibility [**(B)III**], we obtain the line $X_0 - X_2 = X_3 - X_2 = X_4 - (X_1 + X_2)/2 = 0$. Its points are not base points of $F_7$, so we intersect the line with $F_7$. Considering the particular equation of the linear system defining $F_7$ given by the coefficient $a_{31111}$, for example, we obtain the equation

$$3/8a_{31111}X_2^2(X_1 - X_2)^4(-3X_1 - X_2) = 0.$$

Considering $X_1 - X_2 = 0$, we obtain the singular point $P$, and from $X_1 = -X_2/3$ we have the point $(X_2, -X_2/3, X_2, X_2, 2/3X_2)$. Substituting again in $F_7$, we obtain

$$320/81X_2^7(-2/3a_{13111} - 2/3a_{11311} - 2/3a_{11131} + a_{11113}).$$

This last expression cannot be equal to zero, because $X_2 \neq 0$.

In conclusion, the point $(X_2, -X_2/3, X_2, X_2, 2/3X_2) \notin F_7$.

**Fourth step [(*)(4)]**: In the first equation of $(*)$, we choose $X_1 - X_2 = 0$, considering $X_0X_4(X_0 - X_2) \neq 0$. This step is very similar to the **third step [(*)(3)]**, so we omit it here.

**Fifth step [(*)(5)]**: In the first equation of $(*)$, we choose $2X_0 - X_1 - X_2 = 0$, considering $X_0X_4(X_0 - X_2)(X_1 - X_2) \neq 0$. Again this step is very similar to the **third step [(*)(3)]**, so we omit it again.

As an example, we make the calculation in the last possibility of the **Fifth step [(*)(5)]**.

The last possibility starts with $2X_0 - X_1 - X_3 = 0$ and $2X_4 - X_1 - X_2 = 0$. Substituting in $-2X_4 + X_2 + X_3 = 0$, we obtain $X_1 = X_2$ and the line $X_0 - (X_2 + X_3)/2 = X_4 - X_2 = X_1 - X_2 = 0$. Substituting the last equalities in the fourth $(*)$, we obtain zero. We thus substitute in $F_7$. Considering the particular equation of the linear system defining $F_7$ given by the coefficient $a_{31111}$, for example, we obtain the equation

$$1/2a_{31111}X_2^2(X_2 - X_3)^4(X_2 + 2X_3) = 0.$$

Considering $X_2 = 0$, we obtain the singular point $(X_3/2, 0, 0, X_3, 0) \in r_2$, and from $X_3 = X_2$ we obtain the 4-ple point $P$.

Finally, from $X_3 = -X_2/2$, we have the point $(X_2/4, X_2, X_2, -X_2/2, X_2)$, and this point $\notin F_7$.

We have thus found the 9 singular edges, the 4-ple point $P$ and no other singularities.

This proves Proposition 1. □

### 8.2. Resolution of the singularities on $\mathbf{F_7}$ starting with the resolution of the double lines $\mathbf{r_1}, \mathbf{r_2}, \dots, \mathbf{r_6}$ on $\mathbf{F_7}$

**Proposition 2.** *We start the desingularization of* $\mathbf{F_7}$ *blowing up the double line* $\mathbf{r_1}$ *on* $\mathbf{F_7}$ *having a double surface infinitely near, given locally by planes. We also blow up the singularities infinitely near the double surface, which are given by double lines that are negligible singularities.*

*Proof.* (*The proof is again very long.*) Let us consider $r_1 : X_0 = X_1 = X_2 = 0$ and the affine set $U_4$ of affine coordinates $X_0 = w, X_1 = x, X_2 = y, X_3 = z, X_4 = 1$.

#### 8.2.1. Let us blow up the double line $r_1 \cap U_4 : w = x = y = 0$.

Locally the blow-up of this double line is given by $\mathcal{B}_{w_1}, \mathcal{B}_{x_2}, \mathcal{B}_{y_3}$, which are described in Section 2.

The strict transform $F'_{w_1}$ of $F_7$ with respect to the local blow-up $\mathcal{B}_{w_1}$ is the total transform divided by $w_1^2$ given by

- $F'_{w_1} : z_1(20a_{20014} - 10a_{20014}z_1 - 9a_{31111}z_1 - 7a_{13111}z_1 - 7a_{11311}z_1 + 33a_{11131}z_1+$

$+21a_{11113}z_1 - 20a_{11131}z_1^2) + w_1(12a_{31111}y_1 + \cdots + 20a_{21004}x_1 + \cdots) = 0.$

The base points of the (generic) $F'_{w_1}$ on the exceptional divisor $w_1 = 0$ are given by the plane $w_1 = z_1 = 0$. According to Bertini's theorem, the possible singular points on $F'_{w_1}$ that are on the exceptional divisor belong to this plane. To find the possible singular points, we use partial derivatives. Considering $(\frac{\partial F'_{w_1}}{\partial z_1})_{w_1=z_1=0} = 20a_{20014} \neq 0$, we see that there are no singular points on the plane and no singular points on the exceptional divisor.

The strict transform $F'_{x_2}$ of $F_7$ with respect to the local blow-up $\mathcal{B}_{x_2}$ is the total transform divided by $x_2^2$, given by

• $F'_{x_2} : w_2^2 z_2(20a_{20014} - 10a_{20014}z_2 - 9a_{31111}z_2 - 7a_{13111}z_2 - 7a_{11311}z_2 + 33a_{11131}z_2 + 21a_{11113}z_2 - 20a_{11131}z_2^2) + x_2(20a_{21004}w_2^2 + \cdots) + x_2^2 y_2^2 z_2^2(-4a_{13111} + \cdots) = 0.$

On the exceptional divisor $x_2 = 0$ there is the imposed double plane $w_2 = x_2 = 0$.

The strict transform $F'_{y_3}$ of $F_7$ with respect to the local blow-up $\mathcal{B}_{y_3}$ is the total transform divided by $y_3^2$, given by

• $F'_{y_3} : w_3^2 z_3(20a_{20014} - 10a_{20014}z_3 - 9a_{31111}z_3 - 7a_{13111}z_3 - 7a_{11311}z_3 + 33a_{11131}z_3 + 21a_{11113}z_3 - 20a_{11131}z_3^2) + y_3(-20a_{20014}w_3^2 + \cdots) + x_3^2 y_3^2 z_3^2(-4a_{13111} + \cdots) = 0.$

On the exceptional divisor $y_3 = 0$ there is the imposed double plane $w_3 = y_3 = 0$.

### 8.2.2. Let us blow up the double plane $w_2 = x_2 = 0$ on $F'_{x_2}$.

Locally, its blow-up is given by $\mathcal{B}_{w_{21}}, \mathcal{B}_{x_{22}}$, which are described in Section 2.

The strict transform $F''_{w_{21}}$ of $F'_{x_2}$ with respect to the local blow-up $\mathcal{B}_{w_{21}}$ is the total transform divided by $w_{21}^2$, given by

•• $F''_{w_{21}} : 20a_{20014}z_{21} + \cdots + w_{21}(20a_{21004}x_{21} + \cdots) = 0.$

As before, the base points of the (generic) $F''_{w_{21}}$ on the exceptional divisor $w_{21} = 0$ are given by the plane $w_{21} = z_{21} = 0$. Using Bertini's theorem, the possible singular points on $F''_{w_{21}}$ that are on the exceptional divisor belong to this plane. To find the possible singular points, we use partial derivatives. Considering $(\frac{\partial F''_{w_{21}}}{\partial z_{21}})_{w_{21}=z_{21}=0} = 20a_{20014} \neq 0$, we see that there are no singular points on the plane and no singular points on the exceptional divisor.

The strict transform $F''_{x_{22}}$ of $F'_{x_2}$ with respect to the local blow-up $\mathcal{B}_{x_{22}}$ is the total transform divided by $x_{22}^2$, given by

•• $F''_{x_{22}} : 20a_{20014}w_{22}^2 z_{22} + 20a_{12022}w_{22}z_{22}^2 - 8a_{31111}y_{22}^2 z_{22}^2 + \cdots + x_{22}(20a_{21004}w_{22}^2 + \cdots) = 0.$

According to Bertini's theorem, the singularities on $F''_{x_{22}}$ are given by the unimposed double line $w_{22} = x_{22} = z_{22} = 0$, which is on the exceptional divisor $x_{22} = 0$, and by the imposed double line $w_{22} = y_{22} = z_{22} = 0$, which is outside the exceptional divisor. The first double line is negligible, while the second is the image of the imposed double line $r_3 : X_0 = X_2 = X_3 = 0$ having a double plane infinitely near (cf. Introduction and Section 2).

Let us blow up the double plane $w_3 = y_3 = 0$ on $F'_{y_3}$. Locally, this blow-up is given by

$$\mathcal{B}_{w_{31}} : \begin{cases} w_3 = w_{31} \\ x_3 = x_{31} \\ y_3 = y_{31}w_{31} \\ z_3 = z_{31} \end{cases} ; \mathcal{B}_{y_{32}} : \begin{cases} w_3 = w_{32}y_{32} \\ x_3 = x_{32} \\ y_3 = y_{32} \\ z_3 = z_{32} \end{cases} .$$

The strict transform $F''_{w_{31}}$ of $F'_{y_3}$ with respect to the local blow-up $\mathcal{B}_{w_{31}}$ is the total transform divided by $w_{31}^2$ given by

•• $F''_{w_{31}} : 20a_{20014}z_{31} + \cdots + w_{31}(20a_{21004}y_{31} + \cdots) = 0.$

As before, the base points of $F''_{w_{31}}$ on the exceptional divisor $w_{31} = 0$ are given by the plane $w_{31} = z_{31} = 0$. Using Bertini's theorem, and $(\frac{\partial F''_{w_{31}}}{\partial z_{31}})_{w_{31}=z_{31}=0} = 20a_{20014} \neq 0$, it follows that this threefold is nonsingular on the plane $w_{31} = z_{31} = 0$ and nonsingular on the exceptional divisor.

The strict transform $F''_{y_{32}}$ of $F'_{y_3}$ with respect to the local blow-up $\mathcal{B}_{y_{32}}$ is the total transform divided by $y_{32}^2$, given by

•• $F''_{y_{32}} : 20a_{20014}w_{32}^2z_{32} + 10a_{41002}w_{32}z_{32}^2 + 12a_{11113}x_{32}^2z_{32}^2 + \cdots$
$+y_{32}(-20a_{21004}w_{32}^2 + \cdots) = 0.$

Using Bertini's theorem, the singularities on $F''_{y_{32}}$ are given by the unimposed double line $w_{32} = y_{32} = z_{32} = 0$, which is on the exceptional divisor $y_{32} = 0$, and by the imposed double line $w_{32} = x_{32} = z_{32} = 0$, which is outside the exceptional divisor. The first double line is negligible, while the second is the image of the imposed double line $r_5 : X_0 = X_2 = X_3 = 0$ having a double plane infinitely near (cf. Introduction and Section 2).

**Remark 2.** Based on our aim (as stated at the end of the Introduction), we now have to blow up the double lines that are outside the exceptional divisors. □

From now on, reference to Bertini's theorem is taken for granted, and not mentioned again.

#### 8.2.3. We blow up the imposed double line $w_{22} = y_{22} = z_{22} = 0$ on $F''_{x_{22}}$, which is outside the exceptional divisor.

Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{221}} : \begin{cases} w_{22} = w_{221} \\ x_{22} = x_{221} \\ y_{22} = y_{221}w_{221} \\ z_{22} = z_{221}w_{221} \end{cases}; \ \mathcal{B}_{y_{222}} : \begin{cases} w_{22} = w_{222}y_{222} \\ x_{22} = x_{222} \\ y_{22} = y_{222} \\ z_{22} = z_{222}y_{222} \end{cases}; \ \mathcal{B}_{z_{223}} : \begin{cases} w_{22} = w_{223}z_{223} \\ x_{22} = x_{223} \\ y_{22} = y_{223}z_{223} \\ z_{22} = z_{223} \end{cases}.$$

The strict transform $F'''_{w_{221}}$ of $F''_{x_{22}}$ with respect to the local blow-up $\mathcal{B}_{w_{221}}$ is given by

●●● $F'''_{w_{221}} : 20a_{21004}x_{221} + \cdots + w_{221}(\cdots) = 0$. Operating with the instruments that we used before, we obtain that this threefold is also nonsingular on the plane $w_{221} = x_{221} = 0$ and nonsingular on the exceptional divisor.

●●● The strict transform $F'''_{y_{222}}$ of $F''_{x_{22}}$ with respect to $\mathcal{B}_{y_{222}}$ has the double plane $w_{222} = y_{222} = 0$.

●●● The strict transform $F'''_{z_{223}}$ of $F''_{x_{22}}$ with respect to $\mathcal{B}_{z_{223}}$ has the double plane $w_{223} = y_{223} = 0$.

We blow up the imposed double line $w_{32} = x_{32} = z_{32} = 0$ on $F''_{y_{32}}$, which is outside the exceptional divisor. Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{321}} : \begin{cases} w_{32} = w_{321} \\ x_{32} = x_{321}w_{321} \\ y_{32} = y_{321} \\ z_{32} = z_{321}w_{321} \end{cases}; \ \mathcal{B}_{x_{322}} : \begin{cases} w_{32} = w_{322}x_{322} \\ x_{32} = x_{322} \\ y_{32} = y_{322} \\ z_{32} = z_{322}x_{322} \end{cases}; \ \mathcal{B}_{z_{323}} : \begin{cases} w_{32} = w_{323}z_{323} \\ x_{32} = x_{323}z_{323} \\ y_{32} = y_{323} \\ z_{32} = z_{323} \end{cases}.$$

The strict transform $F'''_{w_{321}}$ of $F''_{y_{32}}$ with respect to the local blow-up $\mathcal{B}_{w_{321}}$ is given by

●●● $F'''_{w_{321}} : y_{321}(-20a_{21004} + \cdots - 8a_{11113}) + \cdots + w_{321}(\cdots) = 0$. Operating with the instruments that we used before, we obtain that this threefold is also nonsingular on the plane $w_{321} = y_{321} = 0$ and nonsingular on the exceptional divisor.

●●● The strict transform $F'''_{x_{322}}$ of $F''_{y_{32}}$ with respect to $\mathcal{B}_{x_{322}}$ has the double plane $w_{322} = x_{322} = 0$.

●●● The strict transform $F'''_{z_{323}}$ of $F''_{y_{32}}$ with respect to $\mathcal{B}_{z_{323}}$ has the double plane $w_{323} = z_{323} = 0$.

### 8.2.4. We blow up the double plane $w_{222} = y_{222} = 0$ on $F'''_{y_{222}}$.

Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{2221}} : \begin{cases} w_{222} = w_{2221} \\ x_{222} = x_{2221} \\ y_{222} = y_{2221}w_{2221} \\ z_{222} = z_{2221} \end{cases}; \ \mathcal{B}_{y_{2222}} : \begin{cases} w_{222} = w_{2222}y_{2222} \\ x_{222} = x_{2222} \\ y_{222} = y_{2222} \\ z_{222} = z_{2222} \end{cases}.$$

The strict transform $F'^{v}_{w_{2221}}$ of $F'''_{y_{222}}$ with respect to the local blow-up $\mathcal{B}_{w_{2221}}$ is given by

•**v** $F'^{v}_{w_{2221}} : a_{21004}(20-10x_{2221})x_{2221}+x_{2221}(\cdots)+y_{2221}z_{2221}(\cdots)+w_{2221}(\cdots)=0.$

The base points of $F'^{v}_{w_{2221}}$, on the exceptional divisor $w_{2221}=0$, are given by two lines: $w_{2221}=x_{2221}=y_{2221}=0$ and $w_{2221}=x_{2221}=z_{2221}=0$. Calculating the partial derivatives $(\frac{\partial F'^{v}_{w_{2221}}}{\partial x_{2221}})_{w_{2221}=x_{2221}=y_{2221}=0}=20a_{21004}\neq 0$, $(\frac{\partial F'^{v}_{w_{2221}}}{\partial x_{2221}})_{w_{2221}=x_{2221}=z_{2221}=0}=20a_{21004}\neq 0$, we obtain that the threefold is nonsingular on the two lines and on the exceptional divisor.

•**v** The strict transform $F'^{v}_{y_{2222}}$ of $F'''_{x_{222}}$ with respect to $\mathcal{B}_{y_{2222}}$ has the double line $w_{2222}=x_{2222}=z_{2222}=0$. This double line is outside the exceptional divisor and it is the image of the unimposed double line $w_{22}=x_{22}=z_{22}=0$ on the exceptional divisor on $F''_{x_{22}}$.

We blow up the double plane $w_{223}=z_{223}=0$ on $F'''_{z_{223}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{2231}}:\begin{cases} w_{223}=w_{2231}\\ x_{223}=x_{2231}\\ y_{223}=y_{2231}\\ z_{223}=z_{2231}w_{2231}\end{cases};\ \mathcal{B}_{z_{2232}}:\begin{cases} w_{223}=w_{2232}z_{2232}\\ x_{223}=x_{2232}\\ y_{223}=y_{2232}\\ z_{223}=z_{2232}\end{cases}.$$

The strict transform $F'^{v}_{w_{2231}}$ of $F'''_{z_{223}}$ with respect to the local blow-up $\mathcal{B}_{w_{2231}}$ is given by

•**v** $F'^{v}_{w_{2231}} : 10a_{21004}(2x_{2231}-x^2_{2231}y^2_{2231}z_{2231})+10a_{21004}(2x_{2231}-x^2_{2231})+$
$20a_{12022}(z_{2231}-x^2_{2231}y^2_{2231}z_{2231})+a_{31111}(\cdots)+\cdots a_{11113}(\cdots)+w_{2231}(\cdots)=0.$

The base points of $F'^{v}_{w_{2231}}$, on the exceptional divisor $w_{2231}=0$, are given by the line: $w_{2231}=x_{2231}=z_{2231}=0$. Calculating the partial derivative $(\frac{\partial F'^{v}_{w_{2231}}}{\partial x_{2231}})_{w_{2231}=x_{2231}=z_{2231}=0}=20a_{41002}+20a_{23004}\neq 0$, we obtain that the threefold is nonsingular on the line and on the exceptional divisor.

The strict transform $F'^{v}_{z_{2232}}$ of $F'''_{z_{223}}$ with respect to the local blow-up $\mathcal{B}_{z_{2232}}$ is given by

•**v** $F'^{v}_{z_{2232}} : -10a_{41002}w_{2232}x^2_{2232}y^2_{2232}+10a_{21004}(2w^2_{2232}x_{2232}-w^2_{2232}x_{2232})+$
$20a_{12022}(w_{2232}+w_{2232}x^2_{2232}y^2_{2232})+a_{31111}(\cdots)+\cdots+a_{11113}(\cdots)+z_{2232}(\cdots)=0.$

The base points of $F'^{v}_{z_{2232}}$ on the exceptional divisor $z_{2232}=0$ are given by the plane $w_{22321}=z_{2232}=0$. Considering $(\frac{\partial F'^{v}_{z_{2232}}}{\partial w_{2232}})_{w_{2232}=z_{2232}=0}=20a_{12022}\neq 0$, it follows that this threefold is nonsingular on the plane $w_{2232}=z_{2232}=0$ and nonsingular on the exceptional divisor.

We blow up the double plane $w_{322}=x_{322}=0$ on $F'''_{x_{322}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{3221}} : \begin{cases} w_{322} = w_{3221} \\ x_{322} = x_{3221} w_{3221} \\ y_{322} = y_{3221} \\ z_{322} = z_{3221} \end{cases} ; \ \mathcal{B}_{x_{3222}} : \begin{cases} w_{322} = w_{3222} x_{3222} \\ x_{322} = x_{3222} \\ y_{322} = y_{3222} \\ z_{322} = z_{3222} \end{cases} .$$

The strict transform $F'^{v}_{w_{3221}}$ of $F'''_{x_{322}}$ with respect to the local blow-up $\mathcal{B}_{w_{3221}}$ is given by

•**v** $F'^{v}_{w_{3221}} : y_{3221}(-20a_{21004} + \cdots - 8a_{11113}) + x_{3221}z_{3221}(\cdots) + w_{3221}(\cdots) = 0.$

The base points of $F'^{v}_{w_{3221}}$ on the exceptional divisor $w_{3221} = 0$ are given by two lines: $w_{3221} = x_{3221} = y_{3221} = 0$, $w_{3221} = y_{3221} = z_{3221} = 0$.

$(\frac{\partial F'^{v}_{w_{3221}}}{\partial y_{3221}})_{w_{3221}=x_{3221}=y_{3221}=0} = -20a_{21004} - \cdots - 8a_{11113} \neq 0,$

$(\frac{\partial F'^{v}_{w_{3221}}}{\partial y_{3221}})_{w_{3221}=y_{3221}=z_{3221}=0} = -20a_{21004} - \cdots - 8a_{11113} \neq 0.$

It follows that this threefold is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{x_{3222}}$ of $F'''_{x_{322}}$ with respect to $\mathcal{B}_{x_{3222}}$ is given by

•**v** $F'^{v}_{x_{3222}} : -10a_{41002}w_{3222}y^2_{3222} + 10a_{20014}w^2_{3222} + 12a_{11113}z^2_{3222} + \cdots + x_{3222}(\cdots) = 0.$

The threefold $F'^{v}_{x_{3222}}$ on the exceptional divisor $x_{3222} = 0$ has the double point $(0,0,0,0)$. Outside the exceptional divisor, $F'^{v}_{x_{3222}}$ has the double line $w_{3222} = y_{3222} = z_{3222} = 0$. This double line is the image of the unimposed double line $w_{32} = y_{32} = z_{32} = 0$ on the exceptional divisor on $F''_{y_{32}}$. We note that the double point $(0,0,0,0)$ is on the above double line, so we can consider $F'^{v}_{x_{3222}}$ nonsingular on the exceptional divisor.

We blow up the double plane $w_{323} = z_{323} = 0$ on $F'''_{z_{323}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{3231}} : \begin{cases} w_{323} = w_{3231} \\ x_{323} = x_{3231} \\ y_{323} = y_{2231} \\ z_{323} = z_{3231} w_{3231} \end{cases} ; \ \mathcal{B}_{z_{3232}} : \begin{cases} w_{323} = w_{3232} z_{3232} \\ x_{323} = x_{3232} \\ y_{323} = y_{3232} \\ z_{323} = z_{3232} \end{cases} .$$

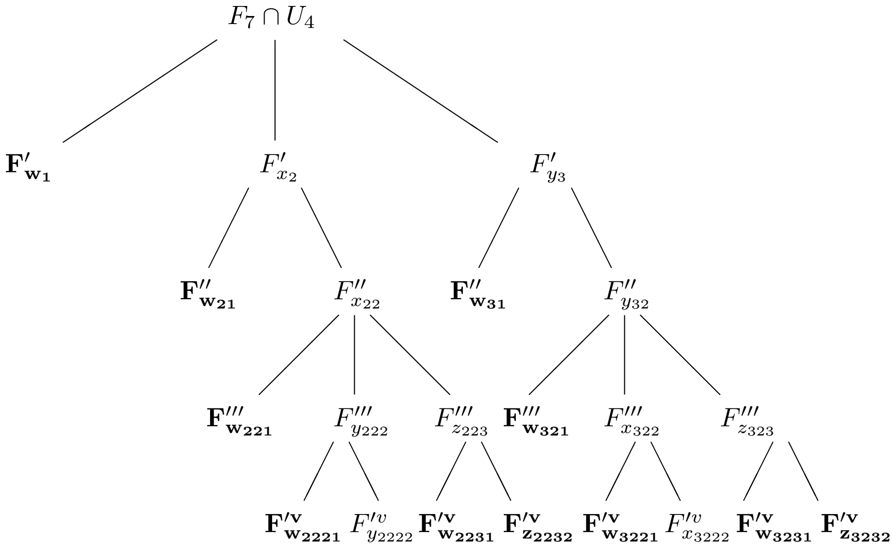
The strict transform $F'^{v}_{w_{3231}}$ of $F'''_{z_{323}}$ with respect to the local blow-up $\mathcal{B}_{w_{3231}}$ is given by

•**v** $F'^{v}_{w_{3231}} : 10a_{21004}z_{3231} - 20a_{21004}y_{3231} + \cdots + w_{3231}(\dots) = 0$, which is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{z_{3232}}$ of $F'''_{z_{323}}$ with respect to the local blow-up $\mathcal{B}_{z_{3232}}$ is given by

•**v** $F'^{v}_{z_{3232}} : w_{3232}(10a_{41002} + \cdots - 22a_{11113}) + \cdots + z_{3232}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

The tree of the blow-ups solving almost all of the singularities of $F_7 \cap U_4$ is shown below.




where the nonsingular threefolds on the exceptional divisor are drawn in bold type.

**Remark 3.** We still need to blow up two unimposed double lines: $w_{2222} = x_{2222} = z_{2222} = 0$ outside the exceptional divisor on $F'^{v}_{y_{2222}}$, and $w_{3222} = y_{3222} = z_{3222} = 0$ outside the exceptional divisor on $F'^{v}_{x_{3222}}$. As we have already said, they are images of the unimposed double line $w_{22} = x_{22} = z_{22} = 0$ on the exceptional divisor on $F''_{x_{22}}$, and of the unimposed double line $w_{32} = y_{32} = z_{32} = 0$ on the exceptional divisor on $F''_{y_{32}}$. Infinitely near each of these lines there is another double line. The four double lines terminate the desingularization of Proposition 2, showing that they are four negligible singularities infinitely near the imposed singularities. □

### 8.2.5. We blow up the unimposed double line $w_{2222} = x_{2222} = z_{2222} = 0$ outside the exceptional divisor on $F'^{v}_{y_{2222}}$.

Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_1} : \begin{cases} w_{2222} = W_1 \\ x_{2222} = X_1 W_1 \\ y_{2222} = Y_1 \\ z_{2222} = Z_1 W_1 \end{cases}; \quad \mathcal{B}_{X_2} : \begin{cases} w_{2222} = W_2 X_2 \\ x_{2222} = X_2 \\ y_{2222} = Y_2 \\ z_{2222} = Z_2 X_2 \end{cases}; \quad \mathcal{B}_{Z_3} : \begin{cases} w_{2222} = W_3 Z_3 \\ x_{2222} = X_3 Z_3 \\ y_{2222} = Y_3 \\ z_{2222} = Z_3 \end{cases}.$$

The strict transform $F^{v}_{W_1}$ of $F'^{v}_{y_{2222}}$ with respect to the local blow-up $\mathcal{B}_{W_1}$ is given by

$\mathbf{v}$ $F^{v}_{W_1} : 20a_{11113}Z_1 + \cdots + W_1(\cdots) = 0$, which is nonsingular on the exceptional divisor.

**v** The strict transform $F^v_{X_2}$ of $F'^v_{y2222}$ with respect to $\mathcal{B}_{X_2}$ has the double line $W_2 = X_2 = Z_2 = 0$.

The strict transform $F^v_{Z_3}$ of $F'^v_{y2222}$ with respect to $\mathcal{B}_{Z_3}$ is given by

**v** $F^v_{Z_3} : -8a_{31111} + \cdots + 12a_{11113} + \cdots + Z_3(\cdots) = 0$, which is nonsingular on the exceptional divisor.

We blow up the unimposed double line $w_{3222} = y_{3222} = z_{3222} = 0$ outside the exceptional divisor on $F'^v_{x3222}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_4} : \begin{cases} w_{3222} = W_4 \\ x_{3222} = X_4 \\ y_{3222} = Y_4W_4 \\ z_{3222} = Z_4W_4 \end{cases}; \ \mathcal{B}_{Y_5} : \begin{cases} w_{3222} = W_5Y_5 \\ x_{3222} = X_5 \\ y_{3222} = Y_5 \\ z_{3222} = Z_5Y_5 \end{cases}; \ \mathcal{B}_{Z_6} : \begin{cases} w_{3222} = W_6Z_6 \\ x_{3222} = X_6 \\ y_{3222} = Y_6Z_6 \\ z_{3222} = Z_6 \end{cases}.$$

The strict transform $F^v_{W_4}$ of $F'^v_{x3222}$ with respect to the local blow-up $\mathcal{B}_{W_4}$ is given by

**v** $F^v_{W_4} : 20a_{11113}Z_4 + \cdots + W_4(\cdots) = 0$, which is nonsingular on the exceptional divisor.

**v** The strict transform $F^v_{Y_5}$ of $F'^v_{x3222}$ with respect to $\mathcal{B}_{Y_5}$ has the double line $W_5 = Y_5 = Z_5 = 0$.

The strict transform $F^v_{Z_6}$ of $F'^v_{x3222}$ with respect to $\mathcal{B}_{Z_6}$ is given by

**v** $F^v_{Z_6} : -8a_{31111} + \cdots + 12a_{11113} + \cdots + Z_6(\cdots) = 0$, which is nonsingular on the exceptional divisor.

### 8.2.6. We blow up the double line $W_2 = X_2 = Z_2 = 0$ on the exceptional divisor on $F^v_{X_2}$.

Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_{21}} : \begin{cases} W_2 = W_{21} \\ X_2 = X_{21}W_{21} \\ Y_2 = Y_{21} \\ Z_2 = Z_{21}W_{21} \end{cases}; \ \mathcal{B}_{X_{22}} : \begin{cases} W_2 = W_{22}X_{22} \\ X_2 = X_{22} \\ Y_2 = Y_{22} \\ Z_2 = Z_{22}X_{22} \end{cases}; \ \mathcal{B}_{Z_{23}} : \begin{cases} W_2 = W_{23}Z_{23} \\ X_2 = X_{23}Z_{23} \\ Y_2 = Y_{23} \\ Z_2 = Z_{23} \end{cases}.$$

The strict transform ${F^v}'_{W_{21}}$ of $F^v_{X_2}$ with respect to the local blow-up $\mathcal{B}_{W_{21}}$ is given by

**v•** ${F^v}'_{W_{21}} : X_{21}(-10a_{41002} + \cdots + 20a_{11113}) + \cdots + W_{21}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

The strict transform ${F^v}'_{X_{22}}$ of $F^v_{X_2}$ with respect to $\mathcal{B}_{X_{22}}$ is given by

**v**• ${F^{v}}'_{X_{22}} : W_{22}(-10a_{41002} + \cdots - 38a_{11113}) + \cdots + X_{22}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

The strict transform ${F^{v}}'_{Z_{23}}$ of $F^{v}_{X_2}$ with respect to $\mathcal{B}_{Z_{23}}$ is given by

**v**• ${F^{v}}'_{Z_{23}} : -8a_{31111} + \cdots + 12a_{11113} + \cdots + Z_{23}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

We blow up the double line $W_5 = Y_5 = Z_5 = 0$ on the exceptional divisor on $F^{v}_{X_2}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_{51}} : \begin{cases} W_5 = W_{51} \\ X_5 = X_{51} \\ Y_5 = Y_{51}W_{51} \\ Z_5 = Z_{51}W_{51} \end{cases} ; \mathcal{B}_{Y_{52}} : \begin{cases} W_5 = W_{52}Y_{52} \\ X_5 = X_{52} \\ Y_5 = Y_{52} \\ Z_5 = Z_{52}Y_{52} \end{cases} ; \mathcal{B}_{Z_{53}} : \begin{cases} W_5 = W_{53}Z_{53} \\ X_5 = X_{53} \\ Y_5 = Y_{53}Z_{53} \\ z_5 = Z_{53} \end{cases} .$$

The strict transform ${F^{v}}'_{W_{51}}$ of $F^{v}_{Y_5}$ with respect to the local blow-up $\mathcal{B}_{W_{51}}$ is given by

**v**• ${F^{v}}'_{W_{51}} : -38a_{11113}Y_{51} + 20a_{11113}Z_{51} + \cdots + W_{51}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

The strict transform ${F^{v}}'_{Y_{52}}$ of $F^{v}_{Y_5}$ with respect to $\mathcal{B}_{Y_{52}}$ is given by

**v**• ${F^{v}}'_{Y_{52}} : W_{52}(-10a_{41002} + \cdots - 38a_{11113}) + \cdots + Y_{52}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

The strict transform ${F^{v}}'_{Z_{53}}$ of $F^{v}_{Y_5}$ with respect to $\mathcal{B}_{Z_{53}}$ is given by

**v**• ${F^{v}}'_{Z_{53}} : -8a_{31111} + \cdots + 12a_{11113} + \cdots + Z_{53}(\cdots) = 0$, which is nonsingular on the exceptional divisor.

This proves Proposition 2. □

**Remark 4.** In the proof of Proposition 2, we have solved the singularities given by the three imposed double lines $\mathbf{r_1}, \mathbf{r_3}$ and $\mathbf{r_5}$. In fact, we have already said that the double line $w_{22} = y_{22} = z_{22} = 0$ is the image of $\mathbf{r_3} : X_0 = X_2 = X_3 = 0$, and that the double line $w_{32} = x_{32} = z_{32} = 0$ is the image of $\mathbf{r_5} : X_0 = X_1 = X_3 = 0$. Since we solved the singularities given by these double lines in the proof of Proposition 2, the above statement is true. □

**Remark 5.** We have to solve the singularities given by the three imposed double lines $\mathbf{r_2}, \mathbf{r_4}$ and $\mathbf{r_6}$, but this desingularization is very similar to the one in the proof of Proposition 2. We just need to change $X_0$ to $X_4$, so we omit this duplicate here. We add that, in the proof of the following Proposition 3, we solve the singularity given by $\mathbf{r_6} : X_1 = X_3 = X_4 = 0$. □

### 8.3. Resolution of the singularities starting with the triple lines $\mathbf{r_7}$, $\mathbf{r_8}$ and $\mathbf{r_9}$ on $\mathbf{F_7}$.

**Proposition 3.** *Blowing up the triple edge* $\mathbf{r_7}$ *on* $\mathbf{F_7}$ *having negligible singularities infinitely near.*

*Proof.* (*The proof is again very long.*) Let us consider $r_7 : X_0 = X_1 = X_4 = 0$, and the affine set $U_2$ of affine coordinates $X_0 = w, X_1 = x, X_2 = 1, X_3 = z, X_4 = t$.

**8.3.1. Let us blow up the triple line $r_7 \cap U_2 : w = x = t = 0$ on $F_7 \cap U_2$.**

Locally, the blow-up of the double line $r_7 \cap U_2 : w = x = t = 0$ is given by

$$\mathcal{B}_{w_1} : \begin{cases} w = w_1 \\ x = x_1 w_1 \\ z = z_1 \\ t = t_1 w_1 \end{cases} ; \mathcal{B}_{x_2} : \begin{cases} w = w_2 x_2 \\ x = x_2 \\ z = z_2 \\ t = t_2 x_2 \end{cases} ; \mathcal{B}_{t_3} : \begin{cases} w = w_3 t_3 \\ x = x_3 t_3 \\ z = z_3 \\ t = t_3 \end{cases} .$$

The strict transform $F'_{w_1}$ of $F_7 \cap U_2$ with respect to $\mathcal{B}_{w_1}$ is obtained from the total transform by dividing by $w_1^3$

• $F'_{w_1} : 10a_{21004}z_1^2 t_1 + 20a_{11311}x_1 z_1 t_1 - 8a_{11113}x_1^2 z_1^2 + \cdots + w_1(-20a_{11311}t_1^2 + 2a_{13111}x_1 z_1 t_1 + \cdots) = 0.$

The base points of $F'_{w_1}$ that are on the exceptional divisor $w_1 = 0$ are given by the points of the simple plane $w_1 = z_1 = 0$. On this plane there is the unique singularity given by the unimposed double line $w_1 = z_1 = t_1 = 0$, because $(\frac{\partial F'_{w_1}}{\partial w_1})_{w_1 = z_1 = 0} = -20a_{11311}t_1^2 = 0$.

Outside the exceptional divisor there is the imposed double line $x_1 = z_1 = t_1 = 0$, which is the image of $\mathbf{r_6} : X_1 = X_3 = X_4 = 0$.

The strict transform $F'_{x_2}$ of $F_7$ with respect to $\mathcal{B}_{x_2}$ is obtained from the total transform by dividing by $x_2^3$

• $F'_{x_2} : 12a_{11113}z_2^2 t_2 - 8a_{11113}w_2 z_2^2 + 20a_{11311}w_2 z_2 t_2 + \cdots + x_2(2a_{11131}w_2^2 z_2 t_2 - 20a_{11311}w_2^2 t_2^2 + \cdots) = 0.$

The base points of $F'_{x_2}$ that are on the exceptional divisor $x_2 = 0$ are given by the points of the plane $x_2 = z_2 = 0$. We have $(\frac{\partial F'_{x_2}}{\partial z_2})_{x_2 = z_2 = 0} = 20a_{11311}w_2 t_2 = 0$. The plane is therefore simple and it contains the two unimposed lines $w_2 = x_2 = z_2 = 0$, $x_2 = z_2 = t_2 = 0$ and the two lines are double lines.

Outside the exceptional divisor there is the imposed triple line $w_2 = z_2 = t_2 = 0$, which is the image of $\mathbf{r_9} : X_0 = X_3 = X_4 = 0$.

The strict transform $F'_{t_3}$ of $F_7$ with respect to $\mathcal{B}_{t_3}$ is obtained from the total transform by dividing by $t_3^3$

• $F'_{t_3}: 22a_{11113}w_3z_3^2 - 4a_{11131}x_3^2z_3^2 + 20a_{11311}w_3x_3z_3 + \cdots + t_3(-20a_{41002}w_3^2z_3 - 20a_{11311}w_3^2 + \cdots) = 0$.

The base points of $F'_{t_3}$ that are on the exceptional divisor $t_3 = 0$ are given by the points of the simple plane $z_3 = t_3 = 0$. On this plane there is the unimposed double line $w_3 = z_3 = t_3 = 0$ because $(\frac{\partial F'_{t_3}}{\partial t_3})_{z_3=t_3=0} = -20a_{11311}w_3^2 = 0$.

Outside the exceptional divisor there is the imposed double line $w_3 = x_3 = z_3 = 0$, which is the image of $\mathbf{r_5}: X_0 = X_1 = X_3 = 0$.

**Remark 6.** Based on our aim (as stated at the end of the Introduction), here again, we now have to blow up the singularities that are outside the exceptional divisors. □

**8.3.2. Let us blow up the double line $x_1 = z_1 = t_1 = 0$ on $F'_{w_1}$.**

Locally, the blow-up of this double line is given by

$$\mathcal{B}_{x_{11}}: \begin{cases} w_1 = w_{11} \\ x_1 = x_{11} \\ z_1 = z_{11}x_{11} \\ t1 = t_{11}x_{11} \end{cases} ; \mathcal{B}_{z_{12}}: \begin{cases} w_1 = w_{12} \\ x_1 = x_{12}z_{12} \\ z_1 = z_{12} \\ t_1 = t_{12}z_{12} \end{cases} ; \mathcal{B}_{t_{13}}: \begin{cases} w_1 = w_{13} \\ x_1 = x_{13}t_{13} \\ z_1 = z_{13}t_{13} \\ t_1 = t_{13} \end{cases} .$$

•• The strict transform $F''_{x_{11}}$ of $F'_{w_1}$ with respect to $\mathcal{B}_{x_{11}}$ has the double plane $x_{11} = t_{11} = 0$ on the exceptional divisor.

•• The strict transform $F''_{z_{12}}$ of $F'_{w_1}$ with respect to $\mathcal{B}_{z_{12}}$ has the double plane $z_{12} = t_{12} = 0$ on the exceptional divisor.

The strict transform $F''_{t_{13}}$ of $F'_{w_1}$ with respect to $\mathcal{B}_{t_{13}}$ is given by

•• $F''_{t_{13}}: -20a_{11311}w_{13} + \cdots + t_{13}(\cdots) = 0$.

The base points of $F''_{t_{13}}$ that are on the exceptional divisor $t_{13} = 0$ are given by the points of the plane $w_{13} = t_{13} = 0$. From $(\frac{\partial F''_{t_{13}}}{\partial w_{13}})_{w_{13}=t_{13}=0} = -20a_{11311} \neq 0$, we deduce that there are no singular points on the plane or on the exceptional divisor.

Let us blow up the triple line $w_2 = z_2 = t_2 = 0$ on $F'_{x_2}$. Locally, the blow-up of this triple line is given by

$$\mathcal{B}_{w_{21}}: \begin{cases} w_2 = w_{21} \\ x_2 = x_{21} \\ z_2 = z_{21}w_{21} \\ t2 = t_{21}w_{21} \end{cases} ; \mathcal{B}_{z_{22}}: \begin{cases} w_2 = w_{22}z_{22} \\ x_2 = x_{22} \\ z_2 = z_{22} \\ t_2 = t_{22}z_{22} \end{cases} ; \mathcal{B}_{t_{23}}: \begin{cases} w_2 = w_{23}t_{23} \\ x_2 = x_{23} \\ z_2 = z_{23}t_{23} \\ t_2 = t_{23} \end{cases} .$$

•• The strict transform $F''_{w_{21}}$ of $F'_{x_2}$ with respect to $\mathcal{B}_{w_{21}}$ has the double point $(0, 0, 0, 0)$ on the exceptional divisor. This double point is on the double line

$x_{21} = z_{21} = t_{21} = 0$ outside the exceptional divisor. So, we can say that $F''_{w_{21}}$ is nonsingular on the exceptional divisr.

**Remark 7.** We note that this double line is the image of the unimposed double line $x_2 = z_2 = t_2 = 0$ on the exceptional divisor of threefold $F'_{z_2}$. □

The strict transform $F''_{z_{22}}$ of $F'_{x_2}$ with respect to $\mathcal{B}_{z_{22}}$ is given by

•• $F''_{z_{22}} : w_{22}(12a_{31111}+\cdots-8a_{11113})+t_{22}(-8a_{31111}+\cdots+12a_{11113})+z_{22}(\cdots) = 0.$

The base points of $F''_{z_{22}}$ that are on the exceptional divisor $z_{22} = 0$ are given by the points of the line $w_{22} = t_{22} = 0$. From $(\frac{\partial F''_{z_{22}}}{\partial w_{22}})_{w_{22}=z_{22}=t_{22}=0} = 12a_{31111} + \cdots - 8a_{11113} \neq 0$, we deduce that there are no singular points on the line or on the exceptional divisor.

•• The strict transform $F''_{t_{23}}$ of $F'_{x_2}$ with respect to $\mathcal{B}_{t_{23}}$ has the double point $(0,0,0,0)$ on the exceptional divisor. It is on the double line $w_{23} = x_{23} = z_{23} = 0$ outside the exceptional divisor. So, we can say that $F''_{t_{23}}$ is nonsingular on the exceptional divisor.

**Remark 8.** We note that this double line is the image of the unimposed double line $w_2 = x_2 = z_2 = 0$ on the exceptional divisor of threefold $F'_{x_2}$. □

Let us blow up the double line $w_3 = x_3 = z_3 = 0$ on $F'_{t_3}$. Locally, the blow-up of this double line is given by

$$\mathcal{B}_{w_{31}} : \begin{cases} w_3 = w_{31} \\ x_3 = x_{31}w_{31} \\ z_3 = z_{31}w_{31} \\ t3 = t_{31} \end{cases} ; \mathcal{B}_{x_{32}} : \begin{cases} w_3 = w_{32}z_{32} \\ x_3 = x_{32} \\ z_3 = z_{32}x_{32} \\ t_3 = t_{32} \end{cases} ; \mathcal{B}_{t_{33}} : \begin{cases} w_3 = w_{33}z_{33} \\ x_3 = x_{33}z_{33} \\ z_3 = z_{33} \\ t_3 = t_{33} \end{cases} .$$

The strict transform $F''_{w_{31}}$ of $F'_{t_3}$ with respect to $\mathcal{B}_{w_{31}}$ is given by

•• $F''_{w_{31}} : -20a_{11311}t_{31} + \cdots - w_{31}(\cdots) = 0$. The base points of $F''_{w_{31}}$ on the exceptional divisor are given by the simple plane $w_{31} = t_{31} = 0$. From $(\frac{\partial F''_{w_{31}}}{\partial t_{31}})_{w_{31}=t_{31}=0}$ $= -20a_{11311} \neq 0$, we deduce that there are no singular points on the plane or on the exceptional divisor.

•• The strict transform $F''_{x_{32}}$ of $F'_{t_3}$ with respect to $\mathcal{B}_{t_{32}}$ has the double plane $w_{32} = x_{32} = 0$ on the exceptional divisor.

•• The strict transform $F''_{z_{33}}$ of $F'_{t_3}$ with respect to $\mathcal{B}_{t_{33}}$ has the double plane $w_{33} = z_{33} = 0$ on the exceptional divisor.

### 8.3.3. We blow up the double plane $x_{11} = t_{11} = 0$ on $F''_{x_{11}}$.

Locally, the blow-up of this plane is given by

$$\mathcal{B}_{x_{111}}:\begin{cases} w_{11}=w_{111}\\ x_{11}=x_{111}\\ z_{11}=z_{111}\\ t_{11}=t_{111}x_{111}\end{cases} ;\ \mathcal{B}_{t_{112}}:\begin{cases} w_{11}=w_{112}\\ x_{11}=x_{112}t_{112}\\ z_{11}=z_{112}\\ t_{11}=t_{112}\end{cases}.$$

●●● The strict transform $F'''_{x_{111}}$ of $F''_{x_{11}}$ with respect to $\mathcal{B}_{x_{111}}$ has the double point $(0,0,0,0)$ on the exceptional divisor. It is on the double line $w_{111}=z_{111}=t_{111}=0$ outside the exceptional divisor.

**Remark 9.** We note that this double line is the image of the unimposed double line $w_1=z_1=t_1=0$ on the exceptional divisor of threefold $F'_{w_1}$. □

The strict transform $F'''_{t_{112}}$ of $F''_{x_{11}}$ with respect to $\mathcal{B}_{t_{112}}$ is given by

●●● $F'''_{t_{112}}: -20a_{11311}w_{112}+\cdots+x_{112}z_{112}(\cdots)+t_{112}(\cdots)=0.$

The base points of $F'''_{t_{112}}$ on the exceptional divisor are given by the two lines $w_{112}=x_{112}=t_{112}=0$, $w_{112}=z_{112}=t_{112}=0$.

From $(\frac{\partial F'''_{t_{112}}}{\partial w_{112}})_{w_{112}=x_{112}=t_{112}=0}=-20a_{11311}\neq 0$, $(\frac{\partial F'''_{t_{112}}}{\partial w_{112}})_{w_{112}=z_{112}=t_{112}=0}=-20a_{11311}\neq 0$, we deduce that $F'''_{t_{112}}$ is nonsingular on the exceptional divisor.

We blow up the double plane $z_{12}=t_{12}=0$ on $F''_{z_{12}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{z_{121}}:\begin{cases} w_{12}=w_{121}\\ x_{12}=x_{121}\\ z_{12}=z_{121}\\ t_{12}=t_{121}z_{121}\end{cases} ;\ \mathcal{B}_{t_{122}}:\begin{cases} w_{12}=w_{122}\\ x_{12}=x_{122}\\ z_{12}=z_{122}t_{122}\\ t_{12}=t_{122}\end{cases}.$$

The strict transform $F'''_{z_{121}}$ of $F''_{z_{12}}$ with respect to $\mathcal{B}_{z_{121}}$ is given by

●●● $F'''_{z_{121}}: 10a_{21004}t_{121}+x_{121}^2(\cdots)+\cdots+z_{121}(\cdots)=0.$ If we intersect

The base points of $F'''_{z_{121}}$ on the exceptional divisor are given by the line $x_{121}=z_{121}=t_{121}=0$.

From $(\frac{\partial F'''_{z_{121}}}{\partial t_{112}})_{x_{121}=z_{121}=t_{121}=0}=10a_{21004}+\cdots+21a_{11113}\neq 0$, we deduce that $F'''_{t_{112}}$ is nonsingular on the exceptional divisor.

The strict transform $F'''_{t_{122}}$ of $F''_{z_{12}}$ with respect to $\mathcal{B}_{t_{122}}$ is given by

●●● $F'''_{t_{122}}: 10a_{21004}z_{122}+w_{122}^2(\cdots)+\cdots+t_{122}(\cdots)=0.$

The base points of $F'''_{t_{122}}$ on the exceptional divisor are given by the line $w_{122}=z_{122}=t_{122}=0$.

From $(\frac{\partial F'''_{t_{122}}}{\partial t_{122}})_{w_{122}=z_{122}=t_{122}=0}=10a_{21004}+\cdots+21a_{11113}+20a_{11311}x_{122}\neq 0$, we deduce that $F'''_{t_{122}}$ is nonsingular on the exceptional divisor.

We blow up the double line $x_{21} = z_{21} = t_{21} = 0$ on $F''_{w_{21}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{x_{211}} : \begin{cases} w_{21} = w_{211} \\ x_{21} = x_{211} \\ z_{21} = z_{211}x_{211} \\ t_{21} = t_{211}x_{211} \end{cases} ; \mathcal{B}_{z_{212}} : \begin{cases} w_{21} = w_{212} \\ x_{21} = x_{212}z_{212} \\ z_{21} = z_{212} \\ t_{21} = t_{212}z_{212} \end{cases} ; \mathcal{B}_{t_{213}} : \begin{cases} w_{21} = w_{213} \\ x_{21} = x_{213}t_{213} \\ z_{21} = z_{213}t_{213} \\ t_{21} = t_{213} \end{cases} .$$

••• The strict transform $F'''_{x_{211}}$ of $F''_{w_{21}}$ with respect to $\mathcal{B}_{x_{211}}$ has the double line $x_{211} = z_{211} = t_{211} = 0$.

The strict transform $F'''_{z_{212}}$ of $F''_{w_{21}}$ with respect to $\mathcal{B}_{z_{212}}$ is given by

••• $F'''_{z_{212}} : 12a_{31111} + \cdots - 8a_{11113}) + 20a_{11211}t_{212} + \cdots + z_{212}(\cdots) = 0$.
$F'''_{z_{212}}$ is nonsingular on the exceptional divisor.

The strict transform $F'''_{t_{213}}$ of $F''_{w_{21}}$ with respect to $\mathcal{B}_{t_{213}}$ is given by

••• $F'''_{t_{213}} : 20a_{11211}z_{213} + \cdots + t_{213}(\cdots) = 0$. The base points on the exceptional divisor are given by the plane $z_{213} = t_{213} = 0$. From $(\frac{\partial F'''_{t_{213}}}{\partial z_{213}})_{z_{213}=t_{213}=0} = 20a_{11311} \neq 0$, we deduce that $F'''_{t_{213}}$ is nonsingular on the exceptional divisor.

We blow up the double line $w_{23} = x_{23} = z_{23} = 0$ on $F''_{t_{23}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{231}} : \begin{cases} w_{23} = w_{231} \\ x_{23} = x_{231}w_{231} \\ z_{23} = z_{231}w_{231} \\ t_{23} = t_{231} \end{cases} ; \mathcal{B}_{x_{232}} : \begin{cases} w_{23} = w_{232}x_{232} \\ x_{23} = x_{232} \\ z_{23} = z_{232}x_{232} \\ t_{23} = t_{232} \end{cases} ; \mathcal{B}_{z_{233}} : \begin{cases} w_{23} = w_{233}z_{233} \\ x_{23} = x_{233}z_{233} \\ z_{23} = z_{233} \\ t_{23} = t_{233} \end{cases} .$$

The strict transform $F'''_{w_{231}}$ of $F''_{t_{23}}$ with respect to $\mathcal{B}_{w_{231}}$ is given by

••• $F'''_{w_{231}} : 20a_{11311}z_{231} + \cdots + w_{231}(\cdots) = 0$. The base points on the exceptional divisor are given by the plane $w_{231} = z_{231} = 0$. From $(\frac{\partial F'''_{w_{231}}}{\partial z_{231}})_{w_{231}=z_{231}=0} = 20a_{11311} \neq 0$, we deduce that the threefold is nonsingular on the exceptional divisor.

••• The strict transform $F'''_{x_{232}}$ of $F''_{t_{23}}$ with respect to $\mathcal{B}_{x_{232}}$ has the double line $w_{232} = x_{232} = z_{232} = 0$.

The strict transform $F'''_{z_{233}}$ of $F''_{t_{23}}$ with respect to $\mathcal{B}_{z_{233}}$ is given by

••• $F'''_{z_{233}} : -8a_{31111} + \cdots + 12a_{11113} + \cdots + 20a_{11311}w_{233} + z_{233}(\cdots) = 0$.
$F'''_{z_{233}}$ is nonsingular on the exceptional divisor.

We blow up the double plane $w_{32} = x_{32} = 0$ on $F''_{x_{32}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{321}}: \begin{cases} w_{32} = w_{321} \\ x_{32} = x_{321}w_{321} \\ z_{32} = z_{321} \\ t_{32} = t_{321} \end{cases} ; \mathcal{B}_{x_{322}}: \begin{cases} w_{32} = w_{322}x_{322} \\ x_{32} = x_{322} \\ z_{32} = z_{322} \\ t_{32} = t_{322} \end{cases} .$$

The strict transform $F'''_{w_{321}}$ of $F''_{x_{32}}$ with respect to $\mathcal{B}_{w_{321}}$ is given by

••• $F'''_{w_{321}}: -20a_{11311}t_{321} + \cdots + x_{321}z_{321}(\cdots) + w_{321}(\cdots) = 0$. The base points on the exceptional divisor are given by two lines. Considering $(\frac{\partial F'''_{w_{321}}}{\partial t_{321}})_{w_{321}=x_{321}=t_{321}=0} = -20a_{11311} \neq 0$, $(\frac{\partial F'''_{w_{321}}}{\partial t_{321}})_{w_{321}=z_{321}=t_{321}=0} = -20a_{11311} \neq 0$, we deduce that $F'''_{w_{321}}$ is nonsingular on the exceptional divisor.

••• The strict transform $F'''_{x_{322}}$ of $F''_{x_{32}}$ with respect to $\mathcal{B}_{x_{322}}$ has the double line $w_{322} = z_{322} = t_{322} = 0$ outside the exceptional divisor.

**Remark 10.** We note that this double line is the image of the unimposed double line $w_3 = z_3 = t_3 = 0$ on the exceptional divisor of threefold $F'_{t_3}$. □

We blow up the double plane $w_{33} = z_{33} = 0$ on $F''_{z_{33}}$. Locally, the blow-up of this plane is given by

$$\mathcal{B}_{w_{331}}: \begin{cases} w_{33} = w_{331} \\ x_{33} = x_{331} \\ z_{33} = z_{331}w_{331} \\ t_{33} = t_{331} \end{cases} ; \mathcal{B}_{z_{332}}: \begin{cases} w_{33} = w_{332}z_{332} \\ x_{33} = x_{332} \\ z_{33} = z_{332} \\ t_{33} = t_{332} \end{cases} .$$

The strict transform $F'''_{w_{331}}$ of $F''_{z_{33}}$ with respect to $\mathcal{B}_{w_{331}}$ is given by

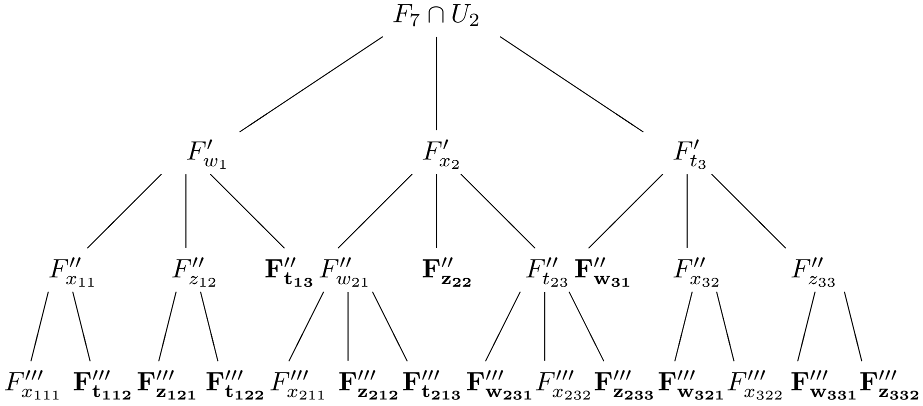
••• $F'''_{w_{331}}: 10a_{41002}z_{331} + \cdots - 20a_{11311}t_{331} + \cdots + w_{331}(\cdots) = 0$,

The base points on the exceptional divisor are given by the line $w_{331} = z_{331} = t_{331} = 0$. From $(\frac{\partial F'''_{w_{331}}}{\partial t_{331}})_{w_{331}=z_{331}=t_{331}=0} = -20a_{11311} \neq 0$, we deduce that $F'''_{w_{331}}$ is nonsingular on the exceptional divisor.

The strict transform $F'''_{z_{332}}$ of $F''_{z_{32}}$ with respect to $\mathcal{B}_{z_{332}}$ is given by

••• $F'''_{z_{332}}: 10a_{41002}w_{332} + \cdots + x_{332}^2(\cdots) + z_{332}(\cdots) = 0$. The base points on the exceptional divisor are given by the line $w_{332} = x_{332} = z_{332} = 0$. From $(\frac{\partial F'''_{z_{332}}}{\partial w_{332}})_{w_{332}=x_{332}=z_{332}=0} = 10a_{41002} + \cdots - 22a_{11113} \neq 0$, we deduce that $F'''_{z_{332}}$ is nonsingular on the exceptional divisor.

The tree of the blow-ups solving almost all of the singularities of $F_7 \cap U_2$ is shown below.




where the nonsingular threefolds on the exceptional divisor are drawn in bold type.

**8.3.4. We blow up the double line $w_{111} = z_{111} = t_{111} = 0$ on $F'''_{x_{111}}$.**

Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{1111}} : \begin{cases} w_{111} = w_{1111} \\ x_{111} = x_{1111} \\ z_{111} = z_{1111}w_{1111} \\ t_{111} = t_{1111}w_{1111} \end{cases}; \ \mathcal{B}_{z_{1112}} : \begin{cases} w_{111} = w_{1112}z_{1112} \\ x_{111} = x_{1112} \\ z_{111} = z_{1112} \\ t_{111} = t_{1112}z_{1112} \end{cases};$$

$$\mathcal{B}_{t_{1113}} : \begin{cases} w_{111} = w_{1113}t_{1113} \\ x_{111} = x_{1113} \\ z_{111} = z_{1113}t_{1112} \\ t_{111} = t_{1113} \end{cases}.$$

•**v** The strict transform $F'^{v}_{w_{1111}}$ of $F'''_{x_{111}}$ with respect to $\mathcal{B}_{w_{1111}}$ has the double line $w_{1111} = z_{1111} = t_{1111} = 0$.

The strict transform $F'^{v}_{z_{1112}}$ of $F'''_{x_{111}}$ with respect to $\mathcal{B}_{z_{1112}}$ is given by

•**v** $F'^{v}_{z_{1112}} : 12a_{31111} + \cdots - 8a_{11113} + \cdots + z_{1112}(\cdots) = 0$.
$F'^{v}_{z_{1112}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{t_{1113}}$ of $F'''_{x_{111}}$ with respect to $\mathcal{B}_{t_{1113}}$ is given by

•**v** $F'^{v}_{t_{1113}} : 20a_{11311}z_{1113} + \cdots + t_{1113}(\cdots) = 0$. The base points on the exceptional divisor are given by the plane $z_{1113} = t_{1113} = 0$. From $(\frac{\partial F'^{v}_{t_{1113}}}{\partial z_{1113}})_{z_{1113}=t_{1113}=0} = 20a_{11311} \neq 0$, we deduce that $F'^{v}_{t_{1113}}$ is nonsingular on the exceptional divisor.

We blow up the double line $x_{211} = z_{211} = t_{211} = 0$ on $F'''_{x_{211}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{x_{2111}}:\begin{cases} w_{211}=w_{2111}\\ x_{211}=x_{2111}\\ z_{211}=z_{2111}x_{2111}\\ t_{211}=t_{2111}x_{2111}\end{cases};\ \mathcal{B}_{z_{2112}}:\begin{cases} w_{211}=w_{2112}\\ x_{211}=x_{2112}z_{2112}\\ z_{211}=z_{2112}\\ t_{211}=t_{2112}z_{2112}\end{cases};$$

$$\mathcal{B}_{t_{2113}}:\begin{cases} w_{211}=w_{2113}\\ x_{211}=x_{2113}t_{2113}\\ z_{211}=z_{2113}t_{2113}\\ t_{211}=t_{2113}\end{cases}.$$

The strict transform $F'^{v}_{x_{2111}}$ of $F'''_{x_{211}}$ with respect to $\mathcal{B}_{x_{2111}}$ is given by

**•v** $F'^{v}_{x_{2111}}: 10a_{20014}t_{2111}+\cdots+z_{2111}^2(\cdots)+x_{2111})(\cdots)=0$. The base points on the exceptional divisor are given by the line $x_{2111}=z_{2111}=t_{2111}=0$. From $(\frac{\partial F'^{v}_{x_{2111}}}{\partial z_{1113}})_{x_{2111}=z_{2111}=t_{2111}=0}=10a_{20014}+\cdots+21a_{11113}\neq 0$, we deduce that $F'^{v}_{x_{2111}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{z_{2112}}$ of $F'''_{x_{211}}$ with respect to $\mathcal{B}_{z_{2112}}$ is given by

**•v** $F'^{v}_{z_{2112}}: 12a_{31111}+\cdots-8a_{11113}+10a_{20014}x_{2112}t_{2112}+\cdots+z_{2112}(\cdots)=0$.
$F'^{v}_{z_{2112}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{t_{2113}}$ of $F'''_{x_{211}}$ with respect to $\mathcal{B}_{t_{2113}}$ is given by

**•v** $F'^{v}_{t_{2113}}: x_{2113}(10a_{20014}+\cdots+21a_{11113})+20a_{1131}z_{2113}+z_{2113}^2(\cdots)+\cdots+t_{2113}(\cdots)=0$. The base points on the exceptional divisor are given by the line $x_{2113}=z_{211}=t_{2113}=0$. From $(\frac{\partial F'^{v}_{t_{2113}}}{\partial z_{2113}})_{x_{2113}=z_{2113}=t_{2113}=0}=20a_{11311}\neq 0$, we deduce that $F'^{v}_{t_{2113}}$ is nonsingular on the exceptional divisor.

We blow up the double line $w_{232}=x_{232}=z_{232}=0$ on $F'''_{x_{232}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{2321}}:\begin{cases} w_{232}=w_{2321}\\ x_{232}=x_{2321}w_{2321}\\ z_{232}=z_{2321}w_{2321}\\ t_{232}=t_{2321}\end{cases};\ \mathcal{B}_{x_{2322}}:\begin{cases} w_{232}=w_{2322}x_{2322}\\ x_{232}=x_{2322}\\ z_{232}=z_{2322}x_{2322}\\ t_{232}=t_{2322}\end{cases};$$

$$\mathcal{B}_{z_{2233}}:\begin{cases} w_{232}=w_{2323}z_{2323}\\ x_{232}=x_{2323}z_{2323}\\ z_{232}=z_{2323}\\ t_{232}=t_{2323}\end{cases}.$$

The strict transform $F'^{v}_{w_{2321}}$ of $F'''_{x_{232}}$ with respect to $\mathcal{B}_{w_{2321}}$ is given by

**•v** $F'^{v}_{w_{2321}}: x_{2321}(-10a_{41002}+\cdots-38a_{11113})+20a_{11311}z_{2321}+z_{2321}^2(\cdots)+\cdots+w_{2321}(\cdots)=0$. The base points on the exceptional divisor are given by the line $w_{2321}=x_{2321}=z_{2321}=0$. From $(\frac{\partial F'^{v}_{w_{2321}}}{\partial z_{2321}})_{w_{2321}=x_{2321}=z_{2321}=0}=20a_{11311}\neq 0$, we deduce that $F'^{v}_{w_{2321}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{x_{2322}}$ of $F'''_{x_{232}}$ with respect to $\mathcal{B}_{x_{2322}}$ is given by

**•v** $F'^{v}_{x_{2322}}: w_{2321}(-10a_{41002} + \cdots - 38a_{11113}) + 20a_{11311}w_{2322}z_{2322} + z_{2322}^2(\cdots) + \cdots + x_{2322}(\cdots) = 0$. The base points on the exceptional divisor are given by the line $w_{2322} = x_{2322} = z_{2322} = 0$. From $(\frac{\partial F'^{v}_{x_{2322}}}{\partial w_{2322}})_{w_{2322}=x_{2322}=z_{2322}=0} = -10a_{41002} + \cdots - 38a_{11113} \neq 0$, we deduce that $F'^{v}_{x_{2322}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{z_{2323}}$ of $F'''_{x_{232}}$ with respect to $\mathcal{B}_{z_{2323}}$ is given by

**•v** $F'^{v}_{z_{2323}}: -8a_{31111} + \cdots + 12a_{11113} + 20a_{11311}w_{2323} + w_{2323}x_{2323}(\cdots) + z_{2323}(\cdots) = 0$.

$F'^{v}_{z_{2323}}$ is nonsingular on the exceptional divisor.

We blow up the double line $w_{322} = z_{322} = t_{322} = 0$ on $F'''_{x_{322}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{w_{3221}}: \begin{cases} w_{322} = w_{3221} \\ x_{322} = x_{3221} \\ z_{322} = z_{3221}w_{3221} \\ t_{322} = t_{3221}w_{3221} \end{cases}; \ \mathcal{B}_{z_{3222}}: \begin{cases} w_{322} = w_{3222}z_{3222} \\ x_{322} = x_{3222} \\ z_{322} = z_{3222} \\ t_{322} = t_{3222}z_{3222} \end{cases};$$
$$\mathcal{B}_{t_{3223}}: \begin{cases} w_{322} = w_{3223}t_{3223} \\ x_{322} = x_{3223} \\ z_{322} = z_{3223}t_{3223} \\ t_{322} = t_{3223} \end{cases}.$$

The strict transform $F'^{v}_{w_{3221}}$ of $F'''_{x_{322}}$ with respect to $\mathcal{B}_{w_{3221}}$ is given by

**•v** $F'^{v}_{w_{3221}}: 20a_{11311}z_{3221} + \cdots + w_{3221}(\cdots) = 0$. The base points on the exceptional divisor are given by the plane $w_{3221} = z_{3221} = 0$. From $(\frac{\partial F'^{v}_{w_{3221}}}{\partial z_{3221}})_{w_{3221}=z_{3221}=0} = 20a_{11311} \neq 0$, we deduce that $F'^{v}_{w_{3221}}$ is nonsingular on the exceptional divisor.

The strict transform $F'^{v}_{z_{3222}}$ of $F'''_{x_{322}}$ with respect to $\mathcal{B}_{z_{3222}}$ is given by

**•v** $F'^{v}_{z_{3222}}: -8a_{31111} + \cdots + 12a_{11113} + 20a_{11311}w_{3222} + \cdots + z_{3222}(\cdots) = 0$.

$F'^{v}_{z_{3222}}$ is nonsingular on the exceptional divisor.

**•v** The strict transform $F'^{v}_{t_{3223}}$ of $F'''_{x_{322}}$ with respect to $\mathcal{B}_{t_{3223}}$ has the double line $w_{3223} = z_{3223} = t_{3223} = 0$ on the exceptional divisor.

**8.3.5. We blow up the double line $w_{1111} = z_{1111} = t_{1111} = 0$ on $F'^{v}_{w_{1111}}$.**

Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_1}: \begin{cases} w_{1111} = W_1 \\ x_{1111} = X_1 \\ z_{1111} = Z_1W_1 \\ t_{1111} = T_1W_1 \end{cases}; \ \mathcal{B}_{Z_1}: \begin{cases} w_{1111} = W_2Z_2 \\ x_{1111} = X_2 \\ z_{1111} = Z_2 \\ t_{1111} = T_2Z_2 \end{cases}; \ \mathcal{B}_{T_3}: \begin{cases} w_{1111} = W_3T_3 \\ x_{1111} = X_3 \\ z_{1111} = Z_3T_3 \\ t_{1111} = T_3 \end{cases}.$$

The strict transform $F^v_{W_1}$ of $F'^v_{w_{1111}}$ with respect to $\mathcal{B}_{W_1}$ is given by
$\mathbf{v}\ F^v_{W_1}: T_1(10a_{20014}+\cdots+21a_{11113})+20a_{11311}Z_1T_1\cdots+Z1^2(\cdots)+W_1(\cdots)=0.$
From $(\frac{\partial F'^v_{W_1}}{\partial T_1})_{W_1=Z_1=T_1=0} = 10a_{20014}+\cdots+21a_{1113} \neq 0$, we deduce that $F'^v_{W_1}$ is nonsingular on the exceptional divisor.

The strict transform $F^v_{Z_2}$ of $F'^v_{t_{3223}}$ with respect to $\mathcal{B}_{Z_2}$ is given by
$\mathbf{v}\ F^v_{Z_2}: 12a_{31111}+\cdots-8a_{11113}+20a_{11311}T_2+W_2T_2(\cdots)+\cdots Z_2(\cdots)=0.$
$F'^v_{Z_2}$ is nonsingular on the exceptional divisor.

The strict transform $F^v_{T_3}$ of $F'^v_{t_{3223}}$ with respect to $\mathcal{B}_{T_3}$ is given by
$\mathbf{v}\ F^v_{T_3}: W_3(a_{20014}+\cdots+21a_{11113})+20a_{11311}Z_3Z_2^2(\cdots)+\cdots T_3(\cdots)=0$. The base points on the exceptional divisor are given by the line $W_3=Z_3=T_3=0$. From $(\frac{\partial F'^v_{T_3}}{\partial W_3})_{W_3=Z_3=T_3=0} = 20a_{11311} \neq 0$, we deduce that $F'^v_{w_{3221}}$ is nonsingular on the exceptional divisor.

We blow up the double line $w_{3223}=z_{3223}=t_{3223}=0$ on $F'^v_{t_{3223}}$. Locally, the blow-up of this line is given by

$$\mathcal{B}_{W_4}: \begin{cases} w_{3222}=W_4 \\ x_{3223}=X_4 \\ z_{3223}=Z_4W_4 \\ t_{3223}=T_4W_4 \end{cases}; \mathcal{B}_{Z_5}: \begin{cases} w_{3223}=W_5Z_5 \\ x_{3223}=X_5 \\ z_{3223}=Z_5 \\ t_{3223}=T_5Z_5 \end{cases}; \mathcal{B}_{T_6}: \begin{cases} w_{3223}=W_6T_6 \\ x_{3223}=X_6 \\ z_{3223}=Z_6T_6 \\ t_{3223}=T_6 \end{cases}.$$

The strict transform $F^v_{W_4}$ of $F'^v_{t_{3223}}$ with respect to $\mathcal{B}_{W_4}$ is given by

$\mathbf{v}\ F^v_{W_4}: T_4(-10a_{41002}+\cdots-38a_{11113})+20a_{11311}Z_4+Z_4^2(\cdots)+W_4(\cdots)=0$. The base points on the exceptional divisor are given by the line $W_4=Z_4=T_4=0$. From $(\frac{\partial F^v_{W_4}}{\partial Z_4})_{W_4=Z_4=T_4=0} = 20a_{11311} \neq 0$, we deduce that $F^v_{w_{3221}}$ is nonsingular on the exceptional divisor.

The strict transform $F^v_{Z_5}$ of $F'^v_{t_{3223}}$ with respect to $\mathcal{B}_{Z_5}$ is given by

$\mathbf{v}\ F^v_{Z_5}: -8a_{31111}+\cdots+12a_{11113}+20a_{11311}W_5+W_5T_5(\cdots)+Z_5(\cdots)=0.$
$F^v_{Z_5}$ is nonsingular on the exceptional divisor.

The strict transform $F^v_{T_6}$ of $F'^v_{t_{3223}}$ with respect to $\mathcal{B}_{T_6}$ is given by
$\mathbf{v}\ F^v_{T_6}: W_6(-10a_{41002}+\cdots-38a_{11113})+20a_{11311}W_6Z_6+Z_6^2(\cdots)+T_6(\cdots)=0.$
The base points on the exceptional divisor are given by the line $W_6=Z_6=T_6=0$. From $(\frac{\partial F^v_{T_5}}{\partial W_6})_{W_6=Z_6=T_6=0} = -10a_{41002}+\cdots-38a_{11113} \neq 0$, we deduce that $F^v_{w_{3221}}$ is nonsingular on the exceptional divisor.

The desingularization starting with the triple line $\mathbf{r_7}: X_0=X_1=X_4=0$ is over, and Proposition 3 has been proved. □

**Remark 11.** The situation in the proof of Proposition 3 is similar to the one seen in the proof of Proposition 2. In proving Proposition 3, we solved the singularity starting with $\mathbf{r_7}: X_0=X_1=X_4=0$, but we also solved the singularity starting with the triple line $\mathbf{r_9} : X_0 = X_3 = X_4 = 0$ (see $w_2 = z_2 = t_2 = 0$ on $F'_{x_2}$), and the singularities given by the double lines $\mathbf{r_6} : X_1 = X_3 = X_4 = 0$, $\mathbf{r_5} : X_0 = X_1 = X_3 = 0$ (see $x_1 = z_1 = t_1 = 0$ on $F'_{w_1}$, $w_3 = x_3 = z_3 = 0$ on $F_{t_3}$). We note that $\mathbf{r_5}$ had already been desingularized in the proof of 2, and that $\mathbf{r_6}$ is one of three double lines that we omitted blowing up (see Remark 5, Section 8.2.6). □

**Remark 12.** The triple line $\mathbf{r_8} : X_0 = X_2 = X_4 = 0$ is missing in the proof of Proposition 3. This singularity can be solved in the same way as the singularity given by $\mathbf{r_7}$, considering $U_3 = \{X_3 \neq 0\}$, for example. As explained in Remark 5, Section 8.2.6, we omit this because it is a duplicate of Proposition 3. □

### 8.4. We complete the desingularization of $\mathbf{F_7}$ with the resolution of the singularity given by the 4-ple point $\mathbf{P = (1, 1, 1, 1, 1)}$

**Proposition 4.** *Resolution of the singularity given by the 4-ple point $P$.*

*Proof.* There are two ways to prove this resolution of $P$. The first is to consider the affine equation of $F_7$, assuming that $X_0 = 1, X_1 = x, X_2 = y, X_3 = z$, and $X_4 = t$, for example, and obtaining $F_7(x, y, z, t) = 0$. Next, if we translate $P$ at the origin $(0, 0, 0, 0)$, then the singularity is solved with only one blow-up. There are no singular points infinitely near $P$. We leave the interested reader to consider the easy blow-up involved in this first method.

The second way is to consider the tangent cone at $P$, and to make sure that it has no singular lines. It follows from this that the point $P$ has no singularities infinitely near.

Taking this second approach, let us find the tangent cone at $P$. To do this in $F_7(X_0, X_1, X_2, X_3, X_4) = 0$, we set $X_0 = 1, X_1 = x, X_2 = y, X_3 = z, X_4 = t$, obtaining $F_7(x, y, z, t) = 0$ and, with the translation $x = Xu+1, y, Yu+1, z, Zu+1, t = Tu + 1$, $u \in \mathbf{k}$, we have

$$F_7(Xu + 1, Yu + 1, Zu + 1, Tu + 1) =$$
$$(\text{tangent cone})u^4 + (\cdots)u^5 + (\cdots)u^6 + (\cdots)u^7$$

To prove that there are no singular lines on the tangent cone, we intersect the tangent cone with the hyperplane $T = 1$, obtaining a surface that we call $S_P$. The equation for $S_P$ is given by

$S_P$: $a_{41002}(-10X^2Y^2 + 10Y^2Z^2)+$
$a_{12022}(-20X^2Y^2 + 20X^2Z^2)+$
$a_{21004}(40YZ - 20YZ^2 - 20Y - 40XZ + 20XZ^2 + 20X + 20X^2Z - 10X^2Z^2 - 10X^2 - 20Y^2Z + 10Y^2Z^2 + 10Y^2)+$
$a_{20014}(40XY - 20X^2Y - 20Y - 40XZ + 20XZ^2 + 20X^2Z - 10X^2Z^2 + 20Z - 10Z^2 - 20XY^2 + 10X^2Y^2 + 10Y^2)+$
$a_{31111}(-96XYZ + 16XYZ^2 + 44XY + 16X^2YZ - 14X^2Y + 44YZ - 14YZ^2 - 12Y + 42XZ - 24X - 12X^2Z^2 + 3X^2 - 24Z + 3Z^2 + 12 - 10XY^2Z + 12XY^2 + 7X^2Y^2 + 12Y^2Z + 3Y^2Z^2 - 16Y^2)+$

$a_{13111}(32XYZ - 32XYZ^2 - 28XY + 28X^2YZ - 22X^2Y + 20X^3Y + 12YZ - 2YZ^2 + 4Y - 34XZ + 40XZ^2 + 8X - 20X^2Z - 16X^2Z^2 + 29X^2 + 20X^3Z - 20X^3 + 8Z - 11Z^2 - 4 - 30XY^2Z + 36XY^2 - 29X^2Y^2 - 4Y^2Z + 19Y^2Z^2 - 8Y^2)+$
$a_{11311}(32XYZ - 32XYZ^2 - 28XY - 32X^2YZ + 38X^2Y - 28YZ + 38YZ^2 + 4Y + 6XZ + 8X + 4X^2Z^2 - 11X^2 + 8Z - 11Z^2 - 4 + 30XY^2Z - 24XY^2 - 9X^2Y^2 - 24Y^2Z - 21Y^2Z^2 + 32Y^2 + 20XY^3 + 20Y^3Z - 20Y^3)+$
$a_{11131}(28XYZ^2 + 12XY - 32X^2YZ - 2X^2Y - 28YZ - 22YZ^2 + 20YZ^3 + 4Y + 32XYZ - 34XZ - 20XZ^2 + 20XZ^3 + 8X + 40X^2Z - 16X^2Z^2 - 11X^2 + 8Z + 29Z^2 - 20Z^3 - 4 - 30XY^2Z - 4XY^2 + 11X^2Y^2 + 36Y^2Z - 21Y^2Z^2 - 8Y^2)+$
$a_{11113}(104XYZ + 16XYZ^2 - 16XY + 16X^2YZ - 34YX^2 - 16YZ - 34YZ^2 + 8Y - 58XZ + 16X + 8X^2Z^2 + 13X^2 + 16Z + 13Z^2 - 8 - 10XY^2Z - 8XY^2 - 13X^2Y^2 - 8Y^2Z + 3Y^2Z^2 + 4Y^2) = 0.$

The singular lines on the cone coincide with the singular points on the generic $S_P$. According to Bertini's theorem, these singular points are in the base points of the linear system defining $S_P$.

We leave to the interested reader the easy proof that the generic $S_P$ is nonsingular.

This proves Proposition 4. □

**With Proposition 4, the desingularization of $\mathbf{F_7}$ is complete.**

## 9. Pluricanonical adjoints to the generic $\mathbf{F_7}$

In the present section, we have to add the triple line $r_7$, $r_8$, $r_9$, and the 4-ple point $P = (1,1,1,1,1)$ that we did not consider in Section 2. With this addition, the theory of pluricanonical adjoints to the generic $F_7$ is a little longer than the one considered in Section 3, but it is much the same. The changes only involve $n_{i-1} = -4+1+1+3 = 1$ for the triple lines and $n_{i-1} = -4+1+0+4 = 1$ for $P = (1,1,1,1,1)$ (cf. [10, p. 152]). We omit this easy addition here, leaving it to the interested reader. We return to the canonical and bicanonical adjoints to $F_7$ that we had begun to consider in Sections 2.1 and 2.2. Here we also consider the tricanonical (3-canonical) adjoints to the generic $F_7$, that we omitted in Section 3. In particular, this $F_7$ remains normal too.

### 9.1. Canonical adjoints to $\mathbf{F_7}$ and $\mathbf{p_g}$

As we already said in Section 4, there are no canonical adjoints to the generic $F_7$, so the geometric genus of a desingularization is $p_g = 0$.

### 9.2. Bicanonical adjoints to $\mathbf{F_7}$ and $\mathbf{P_2}$

For bicanonical adjoints we need the restriction to $F_7$, but with this restriction we cannot identify two bicanonical adjoints to $F_7$ because the degree of $\Phi_4$ is $4 < 7$. Thus, from the normality of $F_7$, the number of linearly independent bicanonical adjoints coincides with the bigenus $P_2$. These bicanonical adjoints are also called *global* bicanonical adjoints (cf. [10]).

We must combine the conditions given to the hypersurfaces of degree 4 given by the singularities starting with the double lines $r_1, r_2, r_3, r_4, r_5, r_6$, with those given by the triple line $r_7$, $r_8$, $r_9$, and those given by the 4-ple point $P = (1,1,1,1,1)$. To be precise, the degree 4 hypersurfaces must pass doubly on the triple lines and they must have a double point at $P$.

The linear system of bicanonical adjoints to $F_7$ is therefore given by

$$\Phi_4\colon\ a_{12001}X_0X_4(X_0-X_1)^2 + a_{11110}X_0[X_0X_4(X_0-X_1) - X_0X_4(X_2-X_4) + X_3(-X_0X_4 + X_1X_2)] + a_{11101}X_0X_4(X_0-X_1)(X_0-X_2) + a_{11011}X_0X_4(X_0-X_1)(X_0-X_3) + a_{11002}X_0X_4(X_0-X_1)(X_0-X_4) + a_{10201}X_0X_4(X_0-X_2)^2 + a_{10111}X_0X_4(X_0-X_2)(X_0-X_3) + a_{10102}X_0X_4(X_0-X_2)(X_0-X_4) + a_{10021}X_0X_4(X_0-X_3)^2 + a_{10012}X_0X_4(X_0-X_3)(X_0-X_4) + a_{10003}X_0X_4(X_0-X_4)^2 + a_{01111}X_4[X_0^2(2X_0-X_1-X_2) - X_3(X_0^2-X_1X_2)] = 0.$$

From the normality of $F_7$, the bigenus is $P_2 = 12$.

### 9.3. Tricanonical adjoints to $\mathbf{F_7}$ and the trigenus $\mathbf{P_3}$

For tricanonical (3-canonical) adjoints we need the restriction to $F_7$, but with this restriction we cannot again identify two tricanonical adjoints to $F_7$ because the degree of tricanonical adjoints is $6 < 7$. The number of linearly independent tricanonical adjoints thus coincides with the trigenus $P_3$. These tricanonical adjoints are also called *global* tricanonical adjoints (cf. [10]).

The conditions applied to the hypersurfaces of degree 6 given by the singularities starting with the double lines $r_1$, $r_2$, $r_3$, $r_4$, $r_5$, $r_6$ are as follows: the hypersurfaces of degree 6 must pass doubly on each of the six lines, and singly through the double surfaces infinitely near the six lines. The hypersurfaces must also pass triply on each triple line $r_7$, $r_8$, $r_9$, and they must have a triple point at the 4-ple point $P = (1,1,1,1,1)$.

We write only 2 short tricanonical adjoints to $F_7$

$$a_{21012}(-X_0^2X_4(X_0-X_1)(X_1-X_3)(X_2-X_4) = 0,$$
$$a_{20202}X_0^2X_4(X_1-X_2)(X_3-X_4)(X_1+X_2-2X_0) = 0\ ,$$

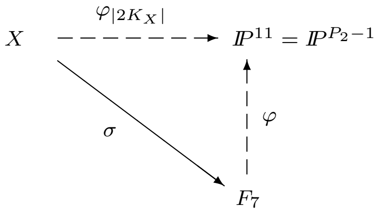
omitting the complete equation of the linear system of tricanonical adjoints because it is too long. The equation is given by 23 parameters and, from the normality of $F_7$, the trigenus is $P_3 = 23$.

## 10. The desingularization $\mathbf{X \to F_7}$ of $\mathbf{F_7}$ is of general type

Let $\sigma : X \to F_7$ be the desingularization of $F_7$. $\sigma$ is the composition of all the blow-ups resolving the singularities of $F_7$.

Let $\varphi : F_7 \dashrightarrow \mathbb{P}^{P_2-1}$ the rational transformation defined by the linear system of bicanonical adjoints to $F_7$.

There is a Zariski's open set $U \subset X$ and a Zariski's open set $U_1 \subset F_7$, that are isomorphic. By identifying $U$ and $U_1$, we find that the bicanonical transformation $\varphi_{|2K_X|} : X \dashrightarrow I\!P^{11}$ is identified with $\varphi$ as rational transformations. These results essentially follow from the commutativity of the following triangle




In particular, we establish that $\varphi_{|2K_X|}$ birational $\Longleftrightarrow$ $\varphi$ birational.

**Proposition 5.** *$\varphi$ is birational on $F_7$.*

*Proof.* In the linear system of bicanonical adjoints to $F_7$, we consider only $a_{11011}X_0X_4(X_0-X_1)(X_0-X_3)+a_{10111}X_0X_4(X_0-X_2)(X_0-X_3)+a_{10021}X_0X_4(X_0-X_3)^2+$ $+a_{10012}X_0X_4(X_0-X_3)(X_0-X_4)+a_{01111}X_4[X_0^2(2X_0-X_1-X_2)-X_3(X_0^2-X_1X_2)]=0$.

The 5 bicanonical adjoints define the rational transformation $\tau\colon I\!P^4 \dashrightarrow I\!P^4$ given by

$$\tau : \begin{cases} Y_0 = \rho\frac{1}{X_0(X_0-X_3)}[X_0^2(2X_0-X_1-X_2)-X_3(X_0^2-X_1X_2)] \\ Y_1 = \rho(X_0-X_1) \\ Y_2 = \rho(X_0-X_2) \\ Y_3 = \rho(X_0-X_3) \\ Y_4 = \rho(X_0-X_4) \end{cases},$$

where we have divided by $X_0X_4(X_0-X_3)$.

We want to find $\tau^{-1}$, so we consider $X_i = X_0 - \frac{Y_i}{\rho}$, $i = 1,2,3,4$, and we substitute them in $Y_0X_0(X_0-X_3)-\rho[X_0^2(2X_0-X_1-X_2)-X_3(X_0^2-X_1X_2)]=0$, obtaining

$$X_0 = Y_1Y_2Y_3/\rho(-Y_0Y_3+Y_1Y_2+Y_1Y_3+Y_2Y_3).$$

Substituting this $X_0$ in $X_i = X_0 - \frac{Y_i}{\rho}$, $i = 1,2,3,4$, we obtain

$$\tau^{-1} : \begin{cases} X_0 = Y_1Y_2Y_3/\rho(-Y_0Y_3+Y_1Y_2+Y_1Y_3+Y_2Y_3) \\ X_i = Y_1Y_2Y_3/\rho(-Y_0Y_3+Y_1Y_2+Y_1Y_3+Y_2Y_3) - \frac{Y_i}{\rho}, \ \ i = 1,2,3,4. \end{cases}$$

This proves that $\tau$ is birational on $I\!P^4$. Since $F_7$ is not contained in the indeterminacy locus of $\tau$, the restriction of $\tau$ to $F_7$: $\tau_{|F_7} = \varphi$ is birational too. □

**Corollary 1.** *$\varphi_{|2K_X|}$ is birational on $X$.*

□

**Corollary 2.** *$X$ is of general type.*

*Proof.* Cf. [13, Chap. II, §§ 5,6]. □

## 11. The regularity of the desingularization $\mathbf{X} \to \mathbf{F_7}$

What remains for us to prove is that $q_i(X) = \dim_{\mathbf{k}} H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$.

**Theorem.** *$X$ is totally regular, i.e. $q_i(X) = 0$ for $i = 1, 2$.*

*Proof.* We calculate $q_2(X) = \dim_{\mathbf{k}} H^2(X, \mathcal{O}_X)$ using the formula (36) in Section 4 in [10], which states that:

$$q_2(X) = p_g(X) + p_g(S) - \dim_{\mathbf{k}}(W_3),$$

where $p_g(X)$ denotes the geometric genus of $X$, and $p_g(S)$ denotes the geometric genus of a desingularization $S$ of a generic hyperplane section of $F_7$, where $W_3$ is the vector space of the degree 3 forms that define global adjoints $\Phi_3$ to $F_7$, i.e. they define hypersurfaces $\Phi_3$ of degree 3 passing through the singularities on $F_7$ with the same multiplicity as the canonical adjoints to $F_7$.

We note that $S \subset X$ is the strict transform, with respect to a desingularization $\sigma\colon X \to F_7$, of a generic hyperplane section of $F_7$ performed by a generic hyperplane $H \subset I\!P^4$. Since the hyperplane $H$ is generic, the variety $S$ can be considered nonsingular.

We remember that $q_1(X) = \dim_{\mathbf{k}} H^1(X, \mathcal{O}_X) = q_1(S) = \dim_{\mathbf{k}} H^1(S, \mathcal{O}_S)$, where $S$ is defined above (cf. [10, p. 174]).

We compute $q_1(S)$ by applying the formula (36) (loc. cit.) to $S$:

$$q_1(S) = p_g(S) + p_g(S') - \dim_{\mathbf{k}}(W_4),$$

where $W_4$ is the vector space of the degree 4 forms defining global adjoints $\Phi_4 \subset H$ to $F_7 \cap H$, and where $S' \subset S$ is the nonsingular strict transform, with respect to $\sigma$ of a generic hyperplane section of $F_7 \cap H$, performed by a generic (hyper)plane $H' \subset H$.

The singularities on $F_7 \cap H$ are given by 6 isolated double points with a double line infinitely near, called *tacnodes*, and other negligible double points (see also [8, Section 7]). In addition, the other singularities of $F_7 \cap H$ are given by triple points (at the intersections of the triple lines $r_7$, $r_8$, $r_9$ on $F_7$ with the hyperplane $H$).

**Lemma 1.** $q_1(X) = q_1(S) = 0$.

*Proof.* We have to calculate $p_g(S)$, which appears in the above formula for calculating $q_1(S)$. The geometric genus $p_g(S)$ of $S$ is given by the dimension of the vector space of the forms defining canonical adjoints to $F_7 \cap H$ in the hyperplane $H$. These canonical adjoints are hypersurfaces of degree 3 in $H$ that pass appropriately through the singularities on $F_7 \cap H$. To be precise, the degree 3 hypersurfaces must pass simply through 6 points (at the intersections of $r_i$ with $H$, $i = 1, \cdots, 6$), and they must pass simply through 3 points (at the intersections of the triple lines $r_7$, $r_8$, $r_9$ on $F_7$ with the hyperplane $H$). Note that $P = (1,1,1,1,1) \notin H$. It is not difficult to see that the 9 points give linear conditions to the degree 3 surfaces in $H$, and therefore $p_g(S) = 20 - 9 = 11$ (see also [8, Section 7]).

On $F_7 \cap H'$ there are no singularities, thus $F_7 \cap H'$ is a nonsingular curve of degree 4 and $p_g(S') = 15$.

$W_4$ are adjoints of degree 4 to $F_7 \cap H$, i.e. degree 4 surfaces passing through the 9 points described above, so $\dim_{\mathbf{k}}(W_4) = 35 - 9 = 26$.

To conclude, $q_1(S) = p_g(S) + p_g(S') - \dim_{\mathbf{k}}(W_4) = 11 + 15 - 26 = 0$.

This proves Lemma 1. $\square$

**Lemma 2.** $q_2(X) = 0$.

*Proof.* In the proof of Lemma 1, we computed $p_g(S) = 11$. In Section 9.1, we computed $p_g = p_g(X) = 0$. $W_3$ are the hypersurfaces in $\mathbb{P}^4$ passing through the 9 lines $r_1, \cdots, r_9$ and through the point $P = (1,1,1,1,1)$. We have
$W_3 : a_{20001}X_0^2X_4 + a_{11100}X_0X_1X_2 + a_{11010}X_0X_1X_3 + a_{11001}X_0X_1X_4 + a_{10110}X_0X_2X_3 + a_{10101}X_0X_2X_4 + a_{10011}X_0X_3X_4 + a_{10002}X_0X_4^2 + a_{01110}X_1X_2X_3 + a_{01101}X_1X_2X_4 + a_{01011}X_1X_3X_4 + a_{00111}X_2X_3X_4 = 0$,

with the condition

$a_{20001} + a_{11100} + a_{11010} + a_{11001} + a_{10110} + a_{10101} + a_{10011} + a_{10002} + a_{01110} + a_{01101} + a_{01011} + a_{00111} = 0$.

Therefore $\dim_{\mathbf{k}}(W_3) = 11$ and $q_2(X) = p_g(X) + p_g(S) - \dim_{\mathbf{k}}(W_3) = 0 + 11 - 11 = 0$.

This proves Lemma 2 and the Theorem. $\square$

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*Dipartimento di Tecnica e Gestione dei Sistemi Industriali.*
*Università di Padova,*
*Stradella S. Nicola, 3. 36100 Vicenza, Italy.*

stagnaro@dmsa.unipd.it

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