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On the some subclasses of bi-univalent functions related to the Faber polynomial expansions and the Fibonacci numbers

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Abstract. *In this investigation, by using the Tremblay fractional derivative operator, we introduce the new class* $\mathfrak{I}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$ *of bi-univalent functions based on the rule of subordination. Moreover, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient* $|a_n|$ *of the bi-univalent function class.*

1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the complex plane and $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Further, let $\mathcal{A}$ represent the class of functions analytic in $\mathbb{U}$, satisfying the condition

$$f(0) = f'(0) - 1 = 0\,.$$

Then each function f in $\mathcal{A}$ has the following Taylor series expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ represent the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{U}$. With a view to reminding the rule of subordination for analytic functions, let the functions f, g be analytic in $\mathbb{U}$. A function f is *subordinate* to g, indicated as $f \prec g$, if there exists a Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathfrak{c}_n z^n \quad (\varpi(0) = 0, \;\; |\varpi(z)| < 1)\,,$$

analytic in $\mathbb{U}$ such that

$$f(z) = g(\varpi(z)) \qquad (z \in \mathbb{U})\,.$$

For the Schwarz function $\varpi(z)$ we know that $|\mathfrak{c}_n| < 1$ (see [9]).

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According to the *Koebe-One Quarter Theorem*, every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f)\,; \quad r_0(f) \geq \frac{1}{4}\right)$, where

$$\begin{aligned} g(w) = f^{-1}(w) = w \ &- a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ &- \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief historical account and for several notable investigation of functions in the class Σ, see the pioneering work on this subject by Srivastava et al. [19] (see also [6, 7, 14, 15]). The interest on estimates for the first two coefficients $|a_2|$, $|a_3|$ of the bi-univalent functions keep on by many researchers (see, for example, [4, 12, 13, 20, 22, 23, 24]). However, in the literature, there are only a few works (by making use of the Faber polynomial expansions) determining the general coefficient bounds $|a_n|$ for bi-univalent functions ([5, 10, 11, 16, 21]). The coefficient estimate problem for each of $|a_n|$ $(\, n \in \mathbb{N}\backslash\{1,2\}\,; \quad \mathbb{N} = \{1,2,3,\ldots\})$ is still an open problem.

Now, we recall to a notion of q-operators that play a major role in Geometric Function Theory. The application of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava [17]. For the convenience, we provide some basic notation details of q-calculus which are used in this paper.

Definition 1.1. (See [18]) For a function f (analytic in a simply-connected region of $\mathbb{C}$), the fractional derivative of order ρ is stated by

$$D_z^{\rho} f(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\rho}} d\xi \quad (0 \leq \rho < 1)$$

and the fractional integral of order ρ is stated by

$$I_z^{\rho} f(z) = \frac{1}{\Gamma(\rho)} \int_0^z f(\xi)(z-\xi)^{\rho-1} d\xi \quad (\rho > 0).$$

Definition 1.2. (See [16]) The Tremblay fractional derivative operator of the function f is defined as

$$I_z^{\mu,\rho} f(z) = \frac{\Gamma(\rho)}{\Gamma(\mu)} z^{1-\rho} D_z^{\mu-\rho} z^{\mu-1} f(z) \tag{1.3}$$

$$(0 < \mu \leq 1,\, 0 < \rho \leq 1,\, \mu \geq \rho d,\, 0 < \mu - \rho < 1).$$

From (1.3), we deduce that

$$I_z^{\mu,\rho} f(z) = \frac{\mu}{\rho} z + \sum_{n=2}^{\infty} \frac{\Gamma(\rho)\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+\rho)} a_n z^n.$$

In this paper, we study the new class $\mathfrak{I}_{\Sigma,\gamma}^{\mu,\rho}(\widetilde{\mathfrak{p}})$ of bi-univalent functions established by using the Tremblay fractional derivative operator. Further, we use the Faber polynomial expansions and Fibonacci numbers to derive bounds for the general coefficient $|a_n|$ of the bi-univalent function class.

2. Preliminaries

By utilizing the Faber polynomial expansions for functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be stated by (see [2, 3])

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = {} & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$. In the following, the first three terms of K_{n-1}^{-n} are stated by

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -\left(5a_2^3 - 5a_2 a_3 + a_4\right). \end{aligned}$$

In general, the expansion of K_n^p $(p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\})$ is stated by

$$K_n^p = p a_n + \frac{p(p-1)}{2} \mathcal{G}_n^2 + \frac{p!}{(p-3)!3!} \mathcal{G}_n^3 + \cdots + \frac{p!}{(p-n)!n!} \mathcal{G}_n^n,$$

where $\mathcal{G}_n^p = \mathcal{G}_n^p(a_1, a_2, \ldots)$ and by [1],

$$\mathcal{G}_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!\,(a_1)^{\delta_1} \cdots (a_n)^{\delta_n}}{\delta_1! \cdots \delta_n!},$$

while $a_1 = 1$, the sum is taken over all nonnegative integers $\delta_1, \ldots, \delta_n$ satisfying

$$\begin{aligned} \delta_1 + \delta_2 + \cdots + \delta_n &= m, \\ \delta_1 + 2\delta_2 + \cdots + n\delta_n &= n. \end{aligned}$$

The first and the last polynomials are

$$\mathcal{G}_n^1 = a_n \qquad \mathcal{G}_n^n = a_1^n.$$

For two analytic functions $\mathfrak{u}(z)$, $\mathfrak{v}(w)$ $(\mathfrak{u}(0) = \mathfrak{v}(0) = 0,\ |\mathfrak{u}(z)| < 1,\ |\mathfrak{v}(w)| < 1)$, suppose that

$$\begin{aligned} \mathfrak{u}(z) &= \textstyle\sum_{n=1}^{\infty} t_n z^n \qquad (|z| < 1,\ z \in \mathbb{U}), \\ \mathfrak{v}(w) &= \textstyle\sum_{n=1}^{\infty} s_n w^n \quad (|w| < 1,\ w \in \mathbb{U}). \end{aligned}$$

It is well known that

$$|t_1| \le 1, \quad |t_2| \le 1 - |t_1|^2, \quad |s_1| \le 1, \quad |s_2| \le 1 - |s_1|^2. \tag{2.1}$$

Definition 2.1. A function $f \in \Sigma$ is said to be in the class

$$\mathfrak{I}_{\Sigma,\gamma}^{\mu,\rho}(\widetilde{\mathfrak{p}}) \quad (\gamma \ge 1,\ 0 < \mu \le 1,\ 0 < \rho \le 1,\ z, w \in \mathbb{U})$$

if the following subordination relationships are satisfied:

$$\left[\frac{(1-\gamma)\rho}{\mu} \frac{I_z^{\mu,\rho} f(z)}{z} + \frac{\rho\gamma}{\mu} \left(I_z^{\mu,\rho} f(z)\right)'\right] \prec \widetilde{\mathfrak{p}}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

and

$$\left[\frac{(1-\gamma)\rho}{\mu} \frac{I_z^{\mu,\rho} g(w)}{w} + \frac{\rho\gamma}{\mu} \left(I_z^{\mu,\rho} g(w)\right)'\right] \prec \widetilde{\mathfrak{p}}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2},$$

where the function g is given by (1.2) and $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Remark 2.2. The function $\widetilde{\mathfrak{p}}(z)$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$. For example, $\widetilde{\mathfrak{p}}(0) = \widetilde{\mathfrak{p}}\left(-\frac{1}{2\tau}\right)$ and $\widetilde{\mathfrak{p}}\left(e^{\pm i \arccos(1/4)}\right) = \frac{\sqrt{5}}{5}$. Also, it can be written as

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|}$$

which indicates that the number $|\tau|$ divides $[0, 1]$ such that it fulfills the golden section (see for details Dziok et al. [8]).

Additionally, Dziok et al. [8] indicate a useful connection between the function $\widetilde{\mathfrak{p}}(z)$ and the Fibonacci numbers. Let $\{\Lambda_n\}$ be the sequence of Fibonacci numbers

$$\Lambda_0 = 0, \quad \Lambda_1 = 1, \quad \Lambda_{n+2} = \Lambda_n + \Lambda_{n+1} \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}),$$

then

$$\Lambda_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \qquad \tau = \frac{1-\sqrt{5}}{2}.$$

If we set

$$\widetilde{\mathfrak{p}}(z) = 1 + \sum_{n=1}^{\infty} \widetilde{\mathfrak{p}}_n z^n = 1 + (\Lambda_0 + \Lambda_2)\tau z + (\Lambda_1 + \Lambda_3)\tau^2 z^2$$
$$+ \sum_{n=3}^{\infty} (\Lambda_{n-3} + \Lambda_{n-2} + \Lambda_{n-1} + \Lambda_n)\tau^n z^n,$$

then the coefficients $\widetilde{\mathfrak{p}}_n$ satisfy

$$\widetilde{\mathfrak{p}}_n = \begin{cases} \tau & (n=1) \\ 3\tau^2 & (n=2) \\ \tau \widetilde{\mathfrak{p}}_{n-1} + \tau^2 \widetilde{\mathfrak{p}}_{n-2} & (n = 3, 4, \ldots) \end{cases}. \tag{2.2}$$

Specializing the parameters γ, μ and ρ, we state the following definitions.

Definition 2.3. For $\mu = \rho = 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{I}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$ if it satisfies the following conditions respectively:

$$\left[(1-\gamma)\frac{f(z)}{z} + \gamma f'(z)\right] \prec \widetilde{\mathfrak{p}}(z)$$

and

$$\left[(1-\gamma)\frac{g(w)}{w} + \gamma g'(w)\right] \prec \widetilde{\mathfrak{p}}(w),$$

where $g = f^{-1}$.

Definition 2.4. For $\gamma = \mu = \rho = 1$, a function $f \in \Sigma$ is said to be in the class $\mathfrak{I}_{\Sigma}\left(\widetilde{\mathfrak{p}}\right)$ if it satisfies the following conditions respectively:

$$f'(z) \prec \widetilde{\mathfrak{p}}(z)$$

and

$$g'(w) \prec \widetilde{\mathfrak{p}}(w),$$

where $g = f^{-1}$.

3. Main Result and its consequences

Theorem 3.1. *For $\gamma \geq 1,\ 0 < \mu \leq 1$ and $0 < \rho \leq 1$, let the function f given by (1.1) be in the function class $\mathfrak{I}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m = 0\ \ (2 \leq m \leq n-1)$, then*

$$|a_n| \leq \frac{|\tau|\,\Gamma(\mu+1)\Gamma(n+\rho)}{\Gamma(\rho+1)\Gamma(n+\mu)\left[(n-1)\gamma+1\right]} \quad (n \geq 3).$$

Proof. By the definition of subordination yields

$$\left[\frac{(1-\gamma)\rho}{\mu}\frac{I_z^{\mu,\rho}f(z)}{z} + \frac{\rho\gamma}{\mu}\left(I_z^{\mu,\rho}f(z)\right)'\right] = \widetilde{\mathfrak{p}}(\mathfrak{u}(z)) \tag{3.1}$$

and

$$\left[\frac{(1-\gamma)\rho}{\mu}\frac{I_z^{\mu,\rho}g(w)}{w} + \frac{\rho\gamma}{\mu}\left(I_z^{\mu,\rho}g(w)\right)'\right] = \widetilde{\mathfrak{p}}(\mathfrak{v}(w)). \tag{3.2}$$

Using Faber polynomial expansions, we have

$$\frac{(1-\gamma)\rho}{\mu}\frac{I_z^{\mu,\rho}f(z)}{z} + \frac{\rho\gamma}{\mu}\left(I_z^{\mu,\rho}f(z)\right)' = 1 + \sum_{n=2}^{\infty}\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\left[(n-1)\gamma+1\right]a_n z^{n-1}$$

and for its inverse map $g = f^{-1}$, it is seen that

$$\begin{aligned}
&\frac{(1-\gamma)\rho}{\mu}\frac{I_z^{\mu,\rho}g(w)}{w} + \frac{\rho\gamma}{\mu}\left(I_z^{\mu,\rho}g(w)\right)' \\
&\qquad = 1 + \sum_{n=2}^{\infty}\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\left[(n-1)\gamma+1\right]\frac{1}{n}K_{n-1}^{-n}\left(a_2, a_3, \ldots\right)w^{n-1} \\
&\qquad = 1 + \sum_{n=2}^{\infty}\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)}\left[(n-1)\gamma+1\right]b_n w^{n-1}.
\end{aligned}$$

Next, the equations (3.1) and (3.2) lead to

$$\begin{aligned}
\widetilde{\mathfrak{p}}\left(\mathfrak{u}\left(z\right)\right) &= 1 + \widetilde{\mathfrak{p}}_1\mathfrak{u}(z) + \widetilde{\mathfrak{p}}_2(\mathfrak{u}(z))^2 z^2 + \cdots \\
&= 1 + \widetilde{\mathfrak{p}}_1 t_1 z + \left(\widetilde{\mathfrak{p}}_1 t_2 + \widetilde{\mathfrak{p}}_2 t_1^2\right)z^2 + \cdots \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\widetilde{\mathfrak{p}}_k \mathcal{G}_n^k\left(t_1, t_2, \ldots, t_n\right)z^n,
\end{aligned}$$

and

$$\widetilde{\mathfrak{p}}\left(\mathfrak{v}\left(w\right)\right) = 1 + \widetilde{\mathfrak{p}}_1 \mathfrak{v}(w) + \widetilde{\mathfrak{p}}_2 (\mathfrak{v}(w))^2 w^2 + \cdots$$

$$= 1 + \widetilde{\mathfrak{p}}_1 s_1 w + \left(\widetilde{\mathfrak{p}}_1 s_2 + \widetilde{\mathfrak{p}}_2 s_1^2\right) w^2 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \widetilde{\mathfrak{p}}_k \mathcal{G}_n^k \left(s_1, s_2, \ldots, s_n\right) w^n.$$

Comparing the corresponding coefficients of (3.1) and (3.2) yields

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)} \left[(n-1)\gamma + 1\right] a_n = \widetilde{\mathfrak{p}}_1 t_{n-1},$$

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)} \left[(n-1)\gamma + 1\right] b_n = \widetilde{\mathfrak{p}}_1 s_{n-1}.$$

For $a_m = 0 \ \ (2 \le m \le n-1)$, we get $b_n = -a_n$ and so

$$\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)} \left[(n-1)\gamma + 1\right] a_n = \widetilde{\mathfrak{p}}_1 t_{n-1} \tag{3.3}$$

and

$$-\frac{\Gamma(\rho+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\rho)} \left[(n-1)\gamma + 1\right] a_n = \widetilde{\mathfrak{p}}_1 s_{n-1}. \tag{3.4}$$

Now taking the absolute values of either of the above two equations and from (2.1), we obtain

$$|a_n| \le \frac{|\tau|\,\Gamma(\mu+1)\Gamma(n+\rho)}{\Gamma(\rho+1)\Gamma(n+\mu)\left[(n-1)\gamma+1\right]}.$$

□

Corollary 3.2. *For $\gamma \ge 1$, suppose that $f \in \mathfrak{I}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m = 0 \ \ (2 \le m \le n-1)$, then*

$$|a_n| \le \frac{|\tau|}{\left[(n-1)\gamma+1\right]} \quad (n \ge 3).$$

Corollary 3.3. *Suppose that $f \in \mathfrak{I}_{\Sigma}\left(\widetilde{\mathfrak{p}}\right)$. If $a_m = 0 \ \ (2 \le m \le n-1)$, then*

$$|a_n| \le \frac{|\tau|}{n} \quad (n \ge 3).$$

Theorem 3.4. *Let $f \in \mathfrak{I}_{\Sigma,\gamma}^{\mu,\rho}\left(\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \le \min\left\{ \frac{|\tau|}{\sqrt{\left|\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)} - \frac{3(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}\right| |\tau| + \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}}}, \right.$$
$$\left. |\tau| \sqrt{\frac{3(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)(2\gamma+1)}} \right\}$$

and

$$|a_3| \leq \min\left\{ \frac{3\tau^2(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)(2\gamma+1)}, \right.$$

$$\left. \frac{|\tau|}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}} \left[1 + \frac{\left[\frac{(\mu+1)(\mu+2)(2\gamma+1)|\tau|}{(\rho+1)(\rho+2)} - \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2} \right]}{\left| \frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)} - \frac{3(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2} \right| |\tau| + \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}} \right] \right\}.$$

Proof. Substituting n by 2 and 3 in (3.3) and (3.4), respectively, we find that

$$\frac{(\mu+1)}{(\rho+1)}(\gamma+1)a_2 = \widetilde{\mathfrak{p}}_1 t_1, \tag{3.5}$$

$$\frac{(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)a_3 = \widetilde{\mathfrak{p}}_1 t_2 + \widetilde{\mathfrak{p}}_2 t_1^2, \tag{3.6}$$

$$-\frac{(\mu+1)}{(\rho+1)}(\gamma+1)a_2 = \widetilde{\mathfrak{p}}_1 s_1, \tag{3.7}$$

$$\frac{(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)(2a_2^2 - a_3) = \widetilde{\mathfrak{p}}_1 s_2 + \widetilde{\mathfrak{p}}_2 s_1^2. \tag{3.8}$$

Obviously, we obtain

$$t_1 = -s_1. \tag{3.9}$$

If we add the equation (3.8) to (3.6) and use (3.9), we get

$$\frac{2(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)a_2^2 = \widetilde{\mathfrak{p}}_1 (t_2+s_2) + 2\widetilde{\mathfrak{p}}_2 t_1^2. \tag{3.10}$$

Using the value of t_1^2 from (3.5), we get

$$\left[\frac{2(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)\widetilde{\mathfrak{p}}_1^2 - \frac{2(\mu+1)^2}{(\rho+1)^2}(\gamma+1)^2\widetilde{\mathfrak{p}}_2 \right] a_2^2 = \widetilde{\mathfrak{p}}_1^3 (t_2+s_2). \tag{3.11}$$

Combining (3.11) and (2.1), we obtain

$$\begin{aligned} 2\left| \frac{(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)\widetilde{\mathfrak{p}}_1^2 - \frac{(\mu+1)^2}{(\rho+1)^2}(\gamma+1)^2\widetilde{\mathfrak{p}}_2 \right| |a_2|^2 &\leq \left|\widetilde{\mathfrak{p}}_1\right|^3 (|t_2| + |s_2|) \\ &\leq 2\left|\widetilde{\mathfrak{p}}_1\right|^3 \left(1 - |t_1|^2\right) \\ &= 2\left|\widetilde{\mathfrak{p}}_1\right|^3 - 2\left|\widetilde{\mathfrak{p}}_1\right|^3 |t_1|^2. \end{aligned}$$

It follows from (3.5) that

$$|a_2| \leq \frac{|\tau|}{\sqrt{\left| \frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)} - \frac{3(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2} \right| |\tau| + \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}}}. \tag{3.12}$$

Additionally, by (2.1) and (3.10)

$$\begin{aligned}\frac{2(\mu+1)(\mu+2)}{(\rho+1)(\rho+2)}(2\gamma+1)\left|a_2\right|^2 &\leq \left|\widetilde{\mathfrak{p}}_1\right|\left(\left|t_2\right|+\left|s_2\right|\right)+2\left|\widetilde{\mathfrak{p}}_2\right|\left|t_1\right|^2\\ &\leq 2\left|\widetilde{\mathfrak{p}}_1\right|\left(1-\left|t_1\right|^2\right)+2\left|\widetilde{\mathfrak{p}}_2\right|\left|t_1\right|^2\\ &= 2\left|\widetilde{\mathfrak{p}}_1\right|+2\left|t_1\right|^2\left(\left|\widetilde{\mathfrak{p}}_2\right|-\left|\widetilde{\mathfrak{p}}_1\right|\right).\end{aligned}$$

Since $\left|\widetilde{\mathfrak{p}}_2\right| > \left|\widetilde{\mathfrak{p}}_1\right|$, we get

$$|a_2| \leq |\tau|\sqrt{\frac{3(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)(2\gamma+1)}}.$$

Next, in order to derive the bounds on $|a_3|$, by subtracting (3.8) from (3.6), we may obtain

$$\frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}a_3 = \frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}a_2^2+\widetilde{\mathfrak{p}}_1\left(t_2-s_2\right). \tag{3.13}$$

Evidently, from (3.10), we state that

$$\begin{aligned}a_3 &= \frac{\widetilde{\mathfrak{p}}_1\left(t_2+s_2\right)+2\widetilde{\mathfrak{p}}_2t_1^2}{\frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}+\frac{\widetilde{\mathfrak{p}}_1\left(t_2-s_2\right)}{\frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}\\ &= \frac{\widetilde{\mathfrak{p}}_1t_2+\widetilde{\mathfrak{p}}_2t_1^2}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}\end{aligned}$$

and consequently

$$\begin{aligned}|a_3| &\leq \frac{\left|\widetilde{\mathfrak{p}}_1\right|\left|t_2\right|+\left|\widetilde{\mathfrak{p}}_2\right|\left|t_1\right|^2}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}\\ &\leq \frac{\left|\widetilde{\mathfrak{p}}_1\right|\left(1-\left|t_1\right|^2\right)+\left|\widetilde{\mathfrak{p}}_2\right|\left|t_1\right|^2}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}\\ &= \frac{\left|\widetilde{\mathfrak{p}}_1\right|+\left|t_1\right|^2\left(\left|\widetilde{\mathfrak{p}}_2\right|-\left|\widetilde{\mathfrak{p}}_1\right|\right)}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}.\end{aligned}$$

Since $\left|\widetilde{\mathfrak{p}}_2\right| > \left|\widetilde{\mathfrak{p}}_1\right|$, we must write

$$|a_3| \leq \frac{3\tau^2(\rho+1)(\rho+2)}{(\mu+1)(\mu+2)(2\gamma+1)}.$$

On the other hand, by (2.1) and (3.13), we have

$$\frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}|a_3| \le \frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}|a_2|^2 + |\widetilde{\mathfrak{p}}_1|(|t_2|+|s_2|)$$
$$\le \frac{2(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}|a_2|^2 + 2|\widetilde{\mathfrak{p}}_1|\left(1-|t_1|^2\right).$$

Then, with the help of (3.5), we have

$$\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}|a_3|$$
$$\le \left[\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)} - \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2|\widetilde{\mathfrak{p}}_1|}\right]|a_2|^2 + |\widetilde{\mathfrak{p}}_1|.$$

By considering (3.12), we deduce that

$$|a_3| \le \frac{|\tau|}{\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)}}\left\{1+\frac{\left[\frac{(\mu+1)(\mu+2)(2\gamma+1)|\tau|}{(\rho+1)(\rho+2)} - \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}\right]}{\left|\frac{(\mu+1)(\mu+2)(2\gamma+1)}{(\rho+1)(\rho+2)} - \frac{3(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}\right||\tau| + \frac{(\mu+1)^2(\gamma+1)^2}{(\rho+1)^2}}\right\}.$$

□

Corollary 3.5. *Let $f \in \mathfrak{I}_{\Sigma,\gamma}\left(\widetilde{\mathfrak{p}}\right)$ $(\gamma \ge 1)$. Then*

$$|a_2| \le \min\left\{\frac{|\tau|}{\sqrt{(3\gamma^2+4\gamma+2)|\tau|+(\gamma+1)^2}}, |\tau|\sqrt{\frac{3}{2\gamma+1}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{3\tau^2}{2\gamma+1}, \frac{|\tau|}{2\gamma+1}\left[1+\frac{(2\gamma+1)|\tau|-(\gamma+1)^2}{(3\gamma^2+4\gamma+2)|\tau|+(\gamma+1)^2}\right]\right\}.$$

Corollary 3.6. *Let $f \in \mathfrak{I}_{\Sigma}\left(\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \le \frac{|\tau|}{\sqrt{9|\tau|+4}}$$

and

$$|a_3| \le \frac{4|\tau|^2}{9|\tau|+4}.$$

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