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Sobolev solutions of parabolic equation in a complete Riemannian manifold

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Abstract. *We study Sobolev estimates for the solutions of parabolic equations acting on a vector bundle, in a complete Riemannian manifold M. The idea is to introduce geometric weights on M. We get global Sobolev estimates with these weights. As applications, we find and improve "classical results", i.e. results without weights. As an example we get Sobolev estimates for the solutions of the heat equation on p-forms when the manifold has "weak bounded geometry" of order* 1.

Contents

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1. Introduction

The study of L^r estimates for the solutions of parabolic equations in a complete Riemannian manifold started long time ago. For the case of the heat equation, a basic work was done by R.S. Strichartz [17]. In particular he proved that the heat kernel is a contraction on the space of functions in $L^r(M)$ for $1 \leq r \leq \infty$.

Let (M, g) be a complete Riemannian manifold and let $G := (H, \pi, M)$ be a complex $\mathcal{C}^m$ vector bundle over M of rank N with fiber H. Let A be an elliptic operator of order m acting on sections of G to themselves. Our aim here is to get Sobolev estimates on the solutions of the parabolic equation $Du := \partial_t u - Au = \omega$, where u, ω are sections of G over M.

Opposite to the usual way to do, see for instance the book by Grigor'yan [10] and the references therein or the paper by [15], we do not use estimates on the kernel associated to the semi group of the differential operator on the manifold.

We shall follow another natural path to proceed: first we use known result in $\mathbb{R}^n$ to get precise local estimates on M, then we globalise them. The advantage of this way is that, for instance when dealing with the heat equation, we need no assumptions on the heat kernel.

To present the ideas in a simple way, we first restrict ourselves to the basic case of the heat equation $Du := \partial_t u + \Delta u = \omega$, where $\Delta := dd^* + d^*d$ is the Hodge laplacian and u, ω belong to the vector bundle of differential p-forms.

We introduce (m, ϵ)-admissible balls $B_{m,\epsilon}(x)$ in (M, g). These balls are the ones defined in the work of Hebey and Herzlich [13] but without asking for the harmonicity of the local coordinates. Then we use a Theorem by Haller-Dintelmann, Heck and Hieber [11, Corollary 3.2, p. 5] done in $\mathbb{R}^n$, to get precise local results on these (m, ϵ)-admissible balls.

For x in M, the radius $R_{m,\epsilon}(x)$ of the admissible ball $B_{m,\epsilon}(x)$ tells us how far from the Euclidean geometry of $\mathbb{R}^n$ the manifold (M, g) is near the point x, and so it not surprising that our geometric weights are functions of these radius. Finally we use an adapted Vitali covering to globalise the local results we got.

Let $W_p^{k,r}(M, w)$ be the space of p-forms on M belonging in the Sobolev space $W^{k,r}(M, w)$ with the weight w. The same way $L_p^r(M, w)$ is the space of p-forms on M belonging in the Lebesgue space $L^r(M, w)$ with the weight w. This gives us the following theorem, written here in the case of the heat equation:

Theorem 1.1. *Let M be a connected complete n-dimensional $\mathcal{C}^2$ Riemannian manifold without boundary. Let $Du := \partial_t u + \Delta u$ be the heat operator acting on the bundle $\Lambda^p(M)$ of p-forms on M. Let:*

$$R(x) = R_{2,\epsilon}(x),\ w_1(x) := R(x)^{r\delta},\ w_2(x) := R(x)^{r\gamma},\ w_3(x) := R(x)^{r\beta},$$

where β, γ, δ are explicit constants. Then, for any $\alpha > 0,\ r \geq 2$, we have:

$$\forall \omega \in L^r([0, T+\alpha], L_p^r(M, w_3)) \cap L^r([0, T+\alpha], L_p^2(M)),$$
$$\exists u \in L^r([0, T], W_p^{2,r}(M, w_2)) :: Du = \omega,$$

with

$$\|\partial_t u\|_{L^r([0,T],L^r_p(M,w_1))} + \|u\|_{L^r([0,T],W^{2,r}_p(M,w_2))} \\ \leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_p(M,w_3))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_p(M))}.$$

In the case of functions instead of p-forms we have the same estimates but with $R(x) = R_{1,\epsilon}(x)$ *and the weights:*

$$w_1(x) := R(x)^{r\delta'},\ w_2(x) := R(x)^{r\gamma'},\ w_3(x) := R(x)^{r\beta'}.$$

Because our admissible radius $R_{m,\epsilon}(x)$ is smaller than one, to forget the weights, i.e. to get "classical estimates", it suffices to have $\forall x \in M,\ R_{m,\epsilon}(x) \geq \delta > 0$. In order to get this, we shall use a nice theorem by Hebey and Herzlich [13, Corollary, p. 7] which warranties us that the radius of our admissible balls is uniformly bounded below.

We introduce a weakened notion of bounded geometry: in the classical definition we replace the curvature tensor by the Ricci one:

Definition 1.2. A Riemannian manifold M has k-order **weak bounded geometry** if:

• the injectivity radius $r_{inj}(x)$ at $x \in M$ is bounded below by some constant $\delta > 0$ for any $x \in M$;

• for $0 \leq j \leq k$, the covariant derivatives $\nabla^j R_c$ of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm.

Using this notion we get the following theorem, written here in the case of the heat equation:

Theorem 1.3. *Let M be a connected complete n-dimensional $\mathcal{C}^2$ Riemannian manifold without boundary. Let $Du := \partial_t u + \Delta u$ be the heat operator acting on the bundle $\Lambda^p(M)$ of p-forms on M. Suppose moreover that (M, g) has 1 order weak bounded geometry. Then*

$$\forall\, \omega \in L^r([0,T+\alpha], L^r_p(M)) \cap L^r([0,T+\alpha], L^2_p(M)),$$

$$\exists\, u \in L^r([0,T], W^{2,r}_p(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r_p(M))} + \|u\|_{L^r([0,T],W^{2,r}_p(M))} \\ \leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_p(M))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_p(M))}.$$

In the case of functions instead of p-forms we have the same estimates just supposing that (M, g) has 0 order weak bounded geometry.

Our method extends to the study of general parabolic equation of order m acting on metric vector bundles. But even in the special case of the heat equation acting on p-forms, it gives some new insights. Let us compare with 3 papers using

the heat kernel method. These papers give estimates on the solutions of the heat equation $Du = \omega$ for $u(t,x)$ with $t \in [0,T]$ fixed. On the other hand the solutions I get are in $L^r([0,T],\ W_p^{m,r}(M))$.

• Comparing with the result of Strichartz [17] on functions, he has no condition at all to get $u(t,\cdot) \in L^r(M)$ for $\omega(t,\cdot) \in L^r(M) \cap L^2(M)$ for any $r \in [1,\infty]$. Here we get $u \in L^r([0,T],\ W^{m,r}(M))$ for $\omega \in L^r([0,T+\alpha],\ L^r(M)) \cap L^r([0,T+\alpha],\ L^2(M))$, at the price that (M,g) has 0 order weak bounded geometry.

Moreover, by Theorem 8.7 in [2], the Sobolev embeddings are true in that case, hence $u \in W^{2,r}(M) \Rightarrow u \in L^s(M)$ with $\frac{1}{s} = \frac{1}{r} - \frac{2}{n}$, and the result is improved also in the Lebesgue scale.

• The work by [15], also using the kernel associated to the semi group of the differential operator acting on metric vector bundles, contains a wide range of precise results, among them Sobolev estimates for the solutions of the parabolic equation. This is done under geometrical hypotheses on the manifold, essentially: bounded geometry of any order.

Here we allow the order m of the parabolic equation to be greater than 2 and we need only that (M,g) has $m-1$ order weak bounded geometry to get Sobolev estimates, but the price is that we have our solutions in $L^r([0,T],\ W_p^{m,r}(M))$, not in $W_p^{m,r}(M)$, for any $t \in [0,T]$.

• Comparing to the result in [14, Theorem 1.2], the hypotheses they have are directly on the kernel and on the manifold: the heat kernel must satisfy a Gaussian upper bound, M must satisfy a volume doubling condition, plus another condition on the negative part of the Ricci curvature. They get Lebesgue estimates on p-forms $u(t,\cdot) \in L_p^r(M)$ for $\omega(t,\cdot) \in L_p^r(M)$. Here again we need that (M,g) has 1 order weak bounded geometry to get Sobolev estimates, which are better than Lebesgue estimates, but in $L^r([0,T],\ W_p^{m,r}(M))$.

The proofs here are, of course, completely different than the proofs using kernels.

I thank the referee for his incisive question leading to Remark 5.5.

2. Notation, definitions and main results

2.1. Admissible balls

Definition 2.1. Let (M,g) be a Riemannian manifold and $x \in M$. We shall say that the geodesic ball $B(x,R)$ is (m,ϵ)-**admissible** if there is a chart $(B(x,R),\varphi)$ such that, with $\epsilon \in (0,1)$:

1. $(1-\epsilon)\delta_{ij} \le g_{ij} \le (1+\epsilon)\delta_{ij}$ in $B(x,R)$ as bilinear forms,
2. $\displaystyle\sum_{1\le|\beta|\le m} R^{|\beta|} \sup_{\ i,j=1,\dots,n,\ y\in B_x(R)} \left|\partial^\beta g_{ij}(y)\right| \le \epsilon.$

We shall denote $\mathcal{A}_m(\epsilon)$ the set of (m,ϵ)-admissible balls.

Definition 2.2. Let $x \in M$, we set $R'(x) = \sup\{R > 0 :: B(x,R) \in \mathcal{A}_m(\epsilon)\}$. We shall say that $R_\epsilon(x) := \min(1, R'(x)/2)$ is the (m,ϵ)-**admissible radius** at x.

Remark 2.3. Let $x, y \in M$. Suppose that $R'(x) > d_g(x,y)$, where $d_g(x,y)$ is the Riemannian distance between x and y. Consider the ball $B(y,\rho)$ of center y and radius $\rho := R'(x) - d_g(x,y)$. This ball is contained in $B(x, R'(x))$ hence, by definition of $R'(x)$, we have that all the points in $B(y,\rho)$ verify the conditions 1) and 2) so, by definition of $R'(y)$, we have that $R'(y) \geq R'(x) - d_g(x,y)$. If $R'(x) \leq d_g(x,y)$ this is also true because $R'(y) > 0$. Exchanging x and y we get that $|R'(y) - R'(x)| \leq d_g(x,y)$.

Hence $R'(x)$ is 1-lipschitzian so it is continuous. So the ϵ-admissible radius $R_\epsilon(x)$ is continuous.

Remark 2.4. Because an admissible ball $B(x, R_\epsilon(x))$ is geodesic, we get that the injectivity radius $r_{inj}(x)$ always verifies $r_{inj}(x) \geq R_\epsilon(x)$.

Lemma 2.5 (Slow variation of the admissible radius)**.** *Let (M,g) be a Riemannian manifold. With $R(x) = R_\epsilon(x) =$ the ϵ-admissible radius at $x \in M$, for every $y \in B(x, R(x))$ we have $R(x)/2 \leq R(y) \leq 2R(x)$.*

Proof. Let $x, y \in M$ and $d(x,y)$ the Riemannian distance on (M,g). Let $y \in B(x,R(x))$ then $d(x,y) \leq R(x)$ and suppose first that $R(x) \geq R(y)$. Then, because $R(x) = R'(x)/2$, we get $y \in B(x, R'(x)/2)$ hence we have $B(y, R'(x)/2) \subset B(x, R'(x))$. But by the definition of $R'(x)$, the ball $B(x, R'(x))$ is admissible and this implies that the ball $B(y, R'(x)/2)$ is also admissible for exactly the same constants and the same chart; this implies that $R'(y) \geq R'(x)/2$ hence $R(y) \geq R(x)/2$, so $R(x) \geq R(y) \geq R(x)/2$. If $R(x) \leq R(y)$ then

$$\begin{aligned} d(x,y) \leq R(x) &\Rightarrow d(x,y) \leq R(y) \Rightarrow x \in B(y, R'(y)/2) \\ &\Rightarrow B(x, R'(y)/2) \subset B(y, R'(y))\,. \end{aligned}$$

Hence the same way as above we get $R(y) \geq R(x) \geq R(y)/2 \Rightarrow R(y) \leq 2R(x)$. So in any case we proved that

$$\forall\, y \in B(x, R(x)),\ R(x)/2 \leq R(y) \leq 2R(x).$$

□

2.2. Vector bundle

Let (M,g) be a complete Riemannian manifold and let $G := (H, \pi, M)$ be a complex $\mathcal{C}^m$ vector bundle over M of rank N with fiber H. Suppose moreover that G has a smooth scalar product $(\ ,\ )$ and a *metric* connection $\nabla^G : \mathcal{C}^\infty(M,G) \to \mathcal{C}^\infty(M, G \otimes T^*M)$, i.e. verifying $d(u,v) = (\nabla^G u, v) + (u, \nabla^G v)$, where d is the exterior derivative on M acting on the scalar product (u,v). See [18, Section 13].

Lemma 2.6. *The ϵ-admissible balls $B(x, R_\epsilon(x))$ trivialise the bundle G.*

Proof. Because if $B(x,R)$ is a ϵ-admissible ball, we have by Remark 2.4 that $R \leq r_{inj}(x)$. Then, one can choose a local frame field for G on $B(x,R)$ by radial parallel translation, as done in [18, Section 13, p.86-87], see also [15, p. 4, eq. (1.3)]. This means that the ϵ-admissible ball also trivialises the bundle G. □

If $\partial_j := \partial/\partial x_j$ in a coordinate system on, say $B(x_0, R)$, and with a local frame $\{e_\alpha\}_{\alpha=1,\dots,N}$, we have, for a smooth sections of G, $u = u^\alpha e_\alpha$ with the Einstein summation convention. We set:

$$\nabla_{\partial_j} u = (\partial_j u^\alpha \ + u^\beta \Gamma^{G,\alpha}_{\beta j}) e_\alpha,$$

the Christoffel coefficients $\Gamma^{G,\alpha}_{\beta j}$ being defined by $\nabla_{\partial_j} e_\beta = \Gamma^{G,\alpha}_{\beta j} e_\alpha$. We shall make the following hypothesis on the connection on G, for $B(x_0, R) \in \mathcal{A}_m(\epsilon)$:

(CMT) $\quad \forall\, x \in B(x_0, R),\ \forall\, k \leq \ m,$

$$\left|\partial^{k-1} \Gamma^{G,\alpha}_{\beta j}(x)\right| \leq C(n, G, \epsilon) \sum_{|\beta| \leq k} \sup_{\ i,j=1,\dots,n,} \left|\partial^\beta g_{ij}(x)\right|,$$

the constant C depending only on n, ϵ and G but not on $B(x_0, R) \in \mathcal{A}_m(\epsilon)$.

This hypothesis is natural:

Lemma 2.7. *The hypothesis (CMT) is true for the Levi-Civita connection on M.*

Proof. Let Γ^k_{lj} be the Christoffel coefficients of the Levi-Civita connection on the tangent bundle TM. We have

$$\Gamma^i_{kj} = \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{2.1}$$

Now on $B(x_0, R) \in \mathcal{A}_m(\epsilon)$, we have $(1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij}$ as bilinear forms. Hence

$$\forall\, x \in B(x_0,\ R), \qquad \left|\Gamma^i_{kj}(x)\right| \leq \frac{3}{2}(1-\epsilon)^{-1} \sum_{|\beta|=1} \sup_{\ i,j=1,\dots,n,} \left|\partial^\beta g_{ij}(x)\right|,$$

in a coordinates chart on $B(x_0, R)$. We have the same with (2.1) for the derivatives of Γ^i_{kj}. □

Remark 2.8. If the hypothesis (CMT) is true for two vector bundles on M, then an easy computation gives that it is true for the tensor product of the two bundles over M. In particular (CMT) is true for tensor bundles over M. It is also true for the sub-bundle of p-forms on M.

2.3. Sobolev spaces for sections of G with weight

We have seen that $\nabla^G \colon \mathcal{C}^\infty(M, G) \to \mathcal{C}^\infty(M, G \otimes T^*M)$. On the tensor product of two Hilbert spaces we put the canonical scalar product $(u \otimes \omega, v \otimes \mu) := (u, v)(\omega, \mu)$, with $u \otimes \omega \in G \otimes T^*M$, and completed by linearity to all elements of the tensor product. On T^*M we have the Levi-Civita connection ∇^M, which is of course a

metric one, and on G we have the metric connection ∇^G so we define a connection on the tensor product $G \otimes T^*M$:

$$\nabla^{G\otimes T^*M}(u \otimes \omega) = (\nabla^G u) \otimes \omega + u \otimes (\nabla^{T^*M}\omega)$$

by asking that this connection be a derivation. We get easily that

$$\nabla^{G\otimes T^*M} \ : \ \mathcal{C}^\infty(M, G \otimes T^*M) \to \mathcal{C}^\infty(M, G \otimes (T^*M)^{\otimes 2})$$

is still a *metric* connection , i.e.

$$d(u \otimes \omega, v \otimes \mu) = (\nabla^{G\otimes T^*M}(u \otimes \omega),\ v\ \otimes \mu) + (u \otimes \omega,\ \nabla^{G\otimes T^*M}(v \otimes \mu)).$$

We define by iteration $\nabla^j u := \nabla(\nabla^{j-1}u)$ on the section u of G and the associated pointwise scalar product $(\nabla^j u(x), \nabla^j v(x))$ which is defined on $G \otimes (T^*M)^{\otimes j}$, with again the metric connection

$$d(\nabla^j u, \nabla^j v)(x) = (\nabla^{j+1}u, \nabla^j v)(x) + (\nabla^j u, \nabla^{j+1}v)(x).$$

Let w be a weight on M, i.e. a positive measurable function on M. If $k \in \mathbb{N}$ and $r \geq 1$ are given, we denote by $\mathcal{C}_G^{k,r}(M, w)$ the space of smooth sections of G $\omega \in \mathcal{C}^\infty(M)$ such that $\left|\nabla^j \omega\right| \in L^r(M, w)$ for $j = 0, \ldots, k$ with the pointwise modulus associated to the pointwise scalar product. Hence

$$\mathcal{C}_G^{k,r}(M, w) := \left\{\omega \in \mathcal{C}_G^\infty(M),\ \forall j = 0, \ldots, k,\ \int_M \left|\nabla^j \omega\right|^r (x) w(x) dv(x) < \infty\right\},$$

with dv the volume measure on (M, g).

Now we have, see M. Cantor [3, Definition 1 & 2, p. 240] for the case without weight:

Definition 2.9. The Sobolev space $W_G^{k,r}(M, w)$ is the completion of $\mathcal{C}_G^{k,r}(M, w)$ with respect to the norm:

$$\|\omega\|_{W_G^{k,r}(M,w)} = \sum_{j=0}^{k} \left(\int_M \left|\nabla^j \omega(x)\right|^r w(x) dv(x)\right)^{1/r}.$$

The usual case is when $w \equiv 1$. Then we write simply $W_G^{k,r}(M)$.

A vector bundle G verifying the following two hypotheses will be called **adapted**:

• the vector bundle G is equipped with a *metric connection*;

• the Christoffel symbols $\Gamma_{\beta j}^{G,\alpha}$ of the connection are controlled by the metric tensor (CMT) g:

(CMT) $\quad \forall\, x \in B(x_0, R),\ \forall\, k \leq\ m,$

$$\left|\partial^{k-1}\Gamma_{\beta j}^{G,\alpha}(x)\right| \leq C(n, G, \epsilon) \sum_{|\beta| \leq k} \sup_{\ i,j=1,\ldots,n,} \left|\partial^\beta g_{ij}(x)\right|,$$

the constant C depending only on n, ϵ and G but not on the admissible ball $B(x_0, R) \in \mathcal{A}_m(\epsilon)$.

2.4. Parabolic operator

We suppose that $Du := \partial_t u - Au$ is parabolic in $\mathbb{R}^n$ in the sense of [11]:

• A is a system of differential operators of the form $A = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$, where $\partial = -i(\partial_1, \ldots, \partial_n)$ and $a_\alpha \in L^\infty(\mathbb{R}^n, \mathbb{C}^{N\times N})$.

• A is (C,θ)-elliptic; this means that there exist constants $\theta \in [0,\pi)$ and $C > 0$, such that the principal part $A_\#(x,\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ of the symbol of A satisfies the following conditions:

$$\sigma(A_\#(x,\xi)) \subset \bar{S}_\theta \text{ and } \left\|A_\#(x,\xi)^{-1}\right\| \leq M \text{ for all } \xi \in \mathbb{R}^n,\ |\xi| = 1,$$

for almost all $x \in \mathbb{R}^n$. Here S_θ denotes the sector in the complex plane defined by $S_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} :: |\arg\lambda| < \theta\}$ and the spectrum of an $N\times N$-matrix $\mathcal{M}$ is denoted by $\sigma(\mathcal{M})$.

• Because we work only with the usual Lebesgue spaces, we take for the domain of A, $\mathcal{D}(A) := W^{m,r}(\mathbb{R}^n)^N$.

We shall use the following [11, Corollary 3.2, p. 5]:

Theorem 2.10. *Let $n \geq 2$, $1 < r, s < \infty$, $\theta \in (0,\pi)$ and $C > 0$. Assume that $A := \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ is a (C,θ)-elliptic operator in $L^r_w(\mathbb{R}^n)^N$ with coefficients a_α satisfying:*

a) $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N}) \cap VMO(\mathbb{R}^n; \mathbb{C}^{N\times N})$ for $|\alpha| = m$,

b) $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N})$ for $|\alpha| < m$.

Suppose that $Du := \partial_t u - Au = \omega$, $u(x,0) \equiv 0$, and assume now that $\theta < \frac{\pi}{2}$, then there exist constants $M, \mu \geq 0$ such that, with $J := [0,\infty[$,

$$\|\partial_t u\|_{L^s(J,L^r(\mathbb{R}^n)^N)} + \|(\mu + A)u\|_{L^s(J,L^r(\mathbb{R}^n)^N)} \leq M\|\omega\|_{L^s(J,L^r(\mathbb{R}^n)^N)}.$$

Moreover the solution u is unique verifying this estimate.

2.5. Global assumptions

We shall made the following global assumption on the operator A in the Riemannian manifold M in all the sequel of this work.

Definition 2.11. We say that the operator A is (C,θ)-elliptic of order m acting on sections of G in the Riemannian manifold (M,g), if for any chart (U,φ) on (M,g) which trivializes G, i.e. G_φ, the image of G, is the trivial bundle $\varphi(U)\times\mathbb{R}^N$ in $\varphi(U)$, we have, with A_φ the image of the operator A:

• A_φ is a system of differential operators of the form $A_\varphi = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$, where $\partial = -i(\partial_1, \ldots, \partial_n)$ and $a_\alpha \in L^\infty(\varphi(U), \mathbb{C}^{N\times N})$, with:

a) $a_\alpha \in L^\infty(\varphi(U); \mathbb{C}^{N\times N}) \cap VMO(\varphi(U); \mathbb{C}^{N\times N})$ for $|\alpha| = m$,

b) $a_\alpha \in L^\infty(\varphi(U); \mathbb{C}^{N\times N})$ for $|\alpha| < m$.

• A_φ is (C,θ)-elliptic; this means that there exist constants $\theta \in [0,\pi)$ and $C > 0$, such that the principal part $A_\#(x,\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ of the symbol of A satisfies the following conditions:

$$\sigma(A_\#(x,\xi)) \subset \bar{S}_\theta \text{ and } \left\|A_\#(x,\xi)^{-1}\right\| \leq M \text{ for all } \xi \in \mathbb{R}^n,\ |\xi| = 1,$$

for almost all $x \in \varphi(U)^n$. And all the bounds being independent of the chart (U, φ).

We shall also need the following "**threshold hypothesis**".

(THL2) For any $\omega \in L^s([0,T], L^2_G(M))$ there is a $u \in L^s([0,T], L^2_G(M))$ such that $Du = \omega$ with the estimate:

$$\|u\|_{L^s([0,T],L^2_G(M))} \lesssim \|\omega\|_{L^s([0,T],L^2_G(M))}.$$

This hypothesis is natural in the sense that it is true for the heat equation.

2.6. Main results

We shall use the following notation.

Definition 2.12. For $r \geq 2,\ m \in \mathbb{N},\ m \geq 1$, let $k := \left\lceil \frac{n(r-2)}{2mr} \right\rceil$ and define:

- if $k = 0,\ \beta(r,m) := m + \dfrac{n}{2} - \dfrac{n}{r}$;
- if $k \geq 1$, $\beta = \beta(r,m) := \min\left(m + \dfrac{n}{2} - \dfrac{n}{r},\ 5m\right)$; $\gamma = \gamma(r,m) = (4k+2)m$; $\delta = \delta(r,m) = (4k+1)m$.

Define also:

- if $k = 0,\ \beta' = \beta'(r,m) := m + \dfrac{n}{2} - \dfrac{n}{r}$;
- if $k \geq 1$, $\beta' = \beta'(r,m) := \min\left(m + \dfrac{n}{2} - \dfrac{n}{r},\ 4m\right)$; $\gamma' = \gamma'(r,m) = (4m-1)k + 2m$; $\delta' = \delta'(r,m) = (4m-1)k + m$.

We are in position to state the first main result of this work.

Theorem 2.13. *Let M be a connected complete n-dimensional $\mathcal{C}^m$ Riemannian manifold without boundary. Let $G := (H, \pi, M)$ be a complex $\mathcal{C}^m$ adapted vector bundle over M. Suppose $Du := \partial_t u - Au$, where A is (C,θ)-elliptic of order m acting on sections of G with $\theta < \pi/2$ in (M,g). Moreover suppose we have (THL2). Let $r \geq 2$ and:*

$$R(x) = R_{m,\epsilon}(x), \quad w_1(x) := R(x)^{r\delta}, \quad w_2(x) := R(x)^{r\gamma}, \quad w_3(x) := R(x)^{r\beta},$$

with β, γ, δ as in Definition 2.12. Then, for any $\alpha > 0,\ r \geq 2$, we have:

$$\forall\, \omega \in L^r([0,T+\alpha], L^r_G(M, w_3)) \cap L^r([0,T+\alpha], L^2_G(M)),$$
$$\exists\, u \in L^r([0,T], W^{m,r}_G(M)) :: Du = \omega,$$

with

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} + \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))}$$
$$\leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.$$

In the case of functions instead of sections of G we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and the weights:

$$w_1(x) := R(x)^{r\delta'}, \quad w_2(x) := R(x)^{r\gamma'}, \quad w_3(x) := R(x)^{r\beta'}.$$

To find and improve "classical results", i.e. results without weights, we use a Theorem by Hebey and Herzlich [13, Corollary, p. 7] which warranties us that the radius of our "admissible balls" is uniformly bounded below.

This gives the second main result of this work.

Theorem 2.14. *Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the adapted vector bundle $G := (H,\pi,M)$ in the complete Riemannian manifold (M,g), with $\theta < \pi/2$. Consider the parabolic equation $Du = \partial_t u - Au$ also acting on sections of G. Suppose moreover that (M,g) has $(m-1)$ order weak bounded geometry and (THL2) is true. Let $r \geq 2$ then*

$$\forall\, \omega \in L^r([0,T+\alpha], L^r_G(M)) \cap L^r([0,T+\alpha], L^2_G(M)),$$
$$\exists\, u \in L^r([0,T], W^{m,r}_G(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M))} + \|u\|_{L^r([0,T],W^{m,r}_G(M))}$$
$$\leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_G(M))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.$$

In the case of functions instead of sections of G we have the same estimates just supposing that (M,g) has $(m-2)$ order weak bounded geometry.

3. Local results

3.1. Local results in $\mathbb{R}^n$

The following result follows the lines of [1, Theorem 3.5]:

Theorem 3.1. *Let A be an operator of order m on G in the complete Riemannian manifold M. Suppose that A is elliptic and with $\mathcal{C}^1(M)$ smooth coefficients. Then, for any $x \in M$ and any ball $B := B(x,R)$ such that $B(x,R)$ is a basis of a chart of M around x and trivialises the bundle G, with the ball $B^1 := B(x,R/2)$, we have:*

$$\|u\|_{W^{m,r}_G(B^1)} \leq c_1\|Au\|_{L^r_G(B)} + c_2 R^{-m}\|u\|_{L^r_G(B)}.$$

Moreover the constants are independent of the radius R of the ball B.

We shall use Theorem 3.1 in the proof of the following precise interior regularity theorem in the case of $\mathbb{R}^n$. The point here is that we need to have a clear dependence in the radius R.

Theorem 3.2. *Suppose that A is a system of differential operators (C,θ)-elliptic with $\theta < \pi/2$, operating in $\mathbb{R}^n$, and suppose u is any solution of the parabolic equation $Du = \partial_t u - Au = \omega$ in a ball $B(0,R)$ with $\omega \in L^s([0,T+\alpha], L^r(B)^N)$ and $u \in L^s([0,T+\alpha], L^r(B)^N)$.*

Consider the ball $B^1 := B(0,R/2)$. We have, with $\alpha > 0$, $T > 0$ and r,s in $(1,\infty)$:

$$\|\partial_t(u)\|_{L^s([0,T],L^r(B^1)^N)} + \|u\|_{L^s([0,T],W^{m,r}(B^1)^N)} \\ \leq c_1\|D(u)\|_{L^s([0,T+\alpha],L^r(B)^N)} + c_2R^{-m}\|u\|_{L^s([0,T+\alpha],L^r(B)^N)}.$$

the constants c_j being independent of R.

Proof. Let $\chi \in \mathcal{D}(B)$ such that $\chi(x) = 1$ for $x \in B^1$. To ease the notation, let us set $L(s,r) := L^s(J, L^r(\mathbb{R}^n)^N)$. Because A is (C,θ)-elliptic we can use the uniqueness in Theorem 2.10 to get that $v := \chi u$ is the unique solution of $D(v) = D(\chi u)$ verifying, with c_1 independent of B,

$$\|\partial_t(\chi u)\|_{L(s,r)} + \|(\mu + A)(\chi u)\|_{L(s,r)} \leq c_1\|D(\chi u)\|_{L(s,r)}.$$

Because

$$\|(\mu + A)(\chi u)\|_{L(s,r)} \geq \|A(\chi u)\|_{L(s,r)} - \mu\|(\chi u)\|_{L(s,r)}$$

we have:

$$\|\partial_t(\chi u)\|_{L(s,r)} + \|A(\chi u)\|_{L(s,r)} \leq c_1\|D(\chi u)\|_{L(s,r)} + \mu\|(\chi u)\|_{L(s,r)}. \tag{3.1}$$

We shall now use the estimates given by the ellipticity of A. For t fixed, we have, by Theorem 3.1:

$$\|\chi u\|_{W^{m,r}(\mathbb{R}^n)^N} \leq c_2\|A(\chi u)\|_{L^r(\mathbb{R}^n)^N} + c_3R^{-m}\|\chi u\|_{L^r(\mathbb{R}^n)^N}$$

where c_2, c_3 are independent of R.

So we get, integrating in t and setting $W(s,r) := L^s(J, W^{m,r}(\mathbb{R}^n)^N)$,

$$\|\chi u\|_{W(s,r)} \leq c_2\|A(\chi u)\|_{L(s,r)} + c_3R^{-m}\|\chi u\|_{L(s,r)}.$$

Hence

$$\|\partial_t(\chi u)\|_{L(s,r)} + \|\chi u\|_{W(s,r)} \leq \|\partial_t(\chi u)\|_{L(s,r)} + c_2\|A(\chi u)\|_{L(s,r)} + c_3R^{-m}\|\chi u\|_{L(s,r)}.$$

Putting this in (3.1) we get with $c_4 := \max(1, c_2)$:

$$\|\partial_t(\chi u)\|_{L(s,r)} + \|\chi u\|_{W(s,r)} \leq c_4c_1\|D(\chi u)\|_{L(s,r)} + c_4\mu\|(\chi u)\|_{L(s,r)} + c_3R^{-m}\|\chi u\|_{L(s,r)}.$$

So with new constants depending on $c_1, c_2,\ c_3$ and μ only and with $R \leq 1$, we get

$$\|\partial_t(\chi u)\|_{L(s,r)} + \|\chi u\|_{W(s,r)} \leq c_1' \|D(\chi u)\|_{L(s,r)} + c_2' R^{-m} \|\chi u\|_{L(s,r)}. \tag{3.2}$$

Now we want to control $\|D(\chi u)\|_{L(s,r)}$ by $\|D(u)\|_{L(s,r)}$. We have, because χ does not depend on t:

$$D(\chi u) = \chi \partial_t u - \chi A u + E = \chi D u + E \tag{3.3}$$

with $E := \chi A u - A(\chi u)$. The point is that E contains only derivatives of the jth component of u of order strictly less than in the jth component of u in Du. So we have, fixing t,

$$\|E\|_{L^r(\mathbb{R}^n)^N} \leq \|\partial \chi\|_\infty \|\chi u\|_{W^{m-1,r}(\mathbb{R}^n)^N} \leq R^{-1} \|\chi u\|_{W^{m-1,r}(\mathbb{R}^n)^N},$$

because $\|\partial \chi\|_\infty \leq R^{-1}$.

We can use the "Peter-Paul" inequality [9, Theorem 7.28, p. 173] (see also [19, Theorem 6.18, (g) p. 232] for the case $r = 2$).

$$\exists C > 0,\ \forall \epsilon > 0 :: \|\chi u\|_{W^{m-1,r}(\mathbb{R}^n)^N} \leq \epsilon \|\chi u\|_{W^{m,r}(\mathbb{R}^n)^N} + C \epsilon^{-m+1} \|\chi u\|_{L^r(\mathbb{R}^n)^N},$$

with C independent of R of course. We choose $\epsilon = R/2$ and we get

$$\|E\|_{L^r(\mathbb{R}^n)^N} \leq R^{-1} \|\chi u\|_{W^{m-1,r}(\mathbb{R}^n)^N} \leq \frac{1}{2} \|\chi u\|_{W^{m,r}(\mathbb{R}^n)^N} + c R^{-m+1} \|\chi u\|_{L^r(\mathbb{R}^n)^N}.$$

Integrating the s power for t in J we get

$$\|E\|_{L(s,r)} \leq R^{-1} \|\chi u\|_{L^s(J, W^{m-1,r}(\mathbb{R}^n)^N)} \leq \frac{1}{2} \|\chi u\|_{W(s,r)} + c R^{-m+1} \|\chi u\|_{L(s,r)}.$$

Hence putting it in (3.3), we get:

$$\|D(\chi u)\|_{L(s,r)} \leq \|\chi D(u)\|_{L(s,r)} + \frac{1}{2} \|\chi u\|_{W(s,r)} + c R^{-m+1} \|\chi u\|_{L(s,r)}.$$

Now using (3.2) we have, because $R \leq 1 \Rightarrow R^{-m+1} \leq R^{-m}$,

$$\|\partial_t(\chi u)\|_{L(s,r)} + \frac{1}{2} \|\chi u\|_{W(s,r)} \leq c_1 \|\chi D u\|_{L(s,r)} + c_2 R^{-m} \|\chi u\|_{L(s,r)}.$$

Because $\chi = 1$ in B^1 and $\chi \geq 0$ we get

$$\|\partial_t(u)\|_{L^s(J, L^r(B^1)^N)} + \|u\|_{L^s(J, W^{m,r}(B^1)^N)} \leq \|\partial_t(\chi u)\|_{L(s,r)} + \|\chi u\|_{W(s,r)}.$$

And, because $\chi \leq 1$ with compact support in B, we deduce

$$c_1 \|\chi D u\|_{L(s,r)} + c_2 R^{-m} \|\chi u\|_{L(s,r)} \leq c_1 \|D u\|_{L^s(J, L^r(B)^N)} + c_2 R^{-m} \|\chi u\|_{L^s(J, L^r(B)^N)}.$$

So finally:

$$\begin{aligned}&\|\partial_t(u)\|_{L^s(J,L^r(B^1)^N)} + \|u\|_{L^s(J,W^{m,r}(B^1)^N)}\\ &\quad\le c_1\|Du\|_{L^s(J,L^r(B)^N)} + c_2R^{-m}\|\chi u\|_{L^s(J,L^r(B)^N)},\end{aligned}$$

with new constants still not depending on B hence nor on R.

Up to now we have $J=[0,\infty)$; to get a finite interval we just multiply u by a function $\psi(t)$ with compact support in $[0,T+\alpha)$ such that $0\le\psi\le 1,\ \psi(t)=1$ for $t\in[0,T]$ and, using that $\partial_t(\psi u)=\psi' u+\psi\partial_t u$ implies

$$\|\partial_t(\psi u)\|_{L^s(J,L^r(B^1)^N)} \ge \|\psi\partial_t u\|_{L^s(J,L^r(B^1)^N)} - \|\psi' u\|_{L^s(J,L^r(B^1)^N)}$$

we get:

$$\begin{aligned}&\|\psi\partial_t u\|_{L^s(J,L^r(B^1)^N)} - \|\psi' u\|_{L^s(J,L^r(B^1)^N)} + \|\psi u\|_{L^s(J,W^{m,r}(B^1)^N)}\\ &\quad\le c_1\|D(\psi u)\|_{L^s(J,L^r(B)^N)} + c_2R^{-m}\|\psi u\|_{L^s(J,L^r(B)^N)}.\end{aligned}$$

But, because ψ depends only on t, $D(\psi u)=\psi' u+\psi Du$ we have

$$\|D(\psi u)\|_{L^s(J,L^r(B)^N)} = \|\psi Du\|_{L^s(J,L^r(B)^N)} + \|\psi' u\|_{L^s(J,L^r(B)^N)}.$$

So we deduce

$$\begin{aligned}&\|\psi\partial_t u\|_{L^s(J,L^r(B^1)^N)} + \|\psi u\|_{L^s(J,W^{m,r}(B^1)^N)}\\ &\quad\le c_1\|\psi Du\|_{L^s(J,L^r(B)^N)} + (1+c_1)\|\psi' u\|_{L^s(J,L^r(B)^N)} + c_2R^{-m}\|\psi u\|_{L^s(J,L^r(B)^N)}.\end{aligned}$$

Now we have that $|\psi'|\le C$ and $R\le 1$ so we end with:

$$\begin{aligned}&\|\partial_t u\|_{L^s([0,T],L^r(B^1)^N)} + \|u\|_{L^s([0,T],W^{m,r}(B^1)^N)}\\ &\quad\le c_1\|Du\|_{L^s([0,T+\alpha],L^r(B)^N)} + c_2R^{-m}\|u\|_{L^s([0,T+\alpha],L^r(B)^N)},\end{aligned}$$

the new constants now depend on α (and μ) but still not on B hence not on R.

The proof is complete. □

3.2. Sobolev comparison estimates

The following two lemmas are quite well known, hence I omit the proofs.

Lemma 3.3. *Let $B(x,R)$ be a (m,ϵ)-admissible ball in M and $\varphi\colon B(x,R)\to\mathbb{R}^n$ be the admissible chart relative to $B(x,R)$. Set $v:=\varphi^*u$, then, for $m\ge 1$:*

$$\forall\, u\in W_G^{m,r}(B(x,R)),\ \|u\|_{W_G^{m,r}(B(x,R))}\le cR^{-m}\|v\|_{W^{m,r}(\varphi(B(x,R)))},$$

and, with $B_e(0,t)$ the Euclidean ball in $\mathbb{R}^n$ centered at 0 and of radius t,

$$\|v\|_{W^{m,r}(B_e(0,(1-\epsilon)R))}\le cR^{-m}\|u\|_{W_G^{m,r}(B(x,R))}.$$

We also have, for $m = 0$:

$$\forall\, u \in L^r_G(B(x,R)),\ \|u\|_{L^r_G(B(x,R))} \le (1 + C\epsilon)\|v\|_{L^r(\varphi(B(x,R)))},$$

and

$$\|v\|_{L^r(B_e(0,(1-\epsilon)R))} \le (1 + C\epsilon)\|u\|_{L^r_G(B(x,R))}.$$

The constants c, C being independent of B.

In the case of a function u on M, we have better results. Let $B(x,R)$ be a $(m-1,\epsilon)$-admissible ball in M and $\varphi\colon B(x,R) \to \mathbb{R}^n$ be the admissible chart relative to $B(x,R)$. Set $v := u \circ \varphi$. then, for $m \ge 1$:

$$\forall u \in W^{m,r}(B(x,R)),\ \|u\|_{W^{m,r}(B(x,R))} \le cR^{1-m}\|v\|_{W^{m,r}(\varphi(B(x,R)))},$$

and

$$\|v\|_{W^{m,r}(B_e(0,(1-\epsilon)R))} \le cR^{1-m}\|u\|_{W^{m,r}(B(x,R))}.$$

We also have, for $m = 0$:

$$\forall u \in L^r(B(x,R)),\ \|u\|_{L^r_G(B(x,R))} \le (1 + C\epsilon)\|v\|_{L^r(\varphi(B(x,R)))},$$

and

$$\|v\|_{L^r(B_e(0,(1-\epsilon)R))} \le (1 + C\epsilon)\|u\|_{L^r(B(x,R))}.$$

The constants c, C being independent of B.

Lemma 3.4 (Sobolev embedding)**.** *Let $B(x,R)$ is a (m,ϵ)-admissible ball in M and $\varphi\colon B(x,R) \to \mathbb{R}^n$ be the admissible chart relative to $B(x,R)$. We have the Sobolev inequality, for $m \ge 1$:*

$$\forall u \in W^{m,\rho}_G(B(x,R)),\ \|u\|_{L^\tau_G(B(x,R/2))} \le cR^{-2m}\|u\|_{W^{m,\rho}_G(B(x,R))} \text{ with } \frac{1}{\tau} = \frac{1}{\rho} - \frac{m}{n}.$$

In the special case of functions, with $B(x,R)$ a $(m-1,\epsilon)$-admissible ball in M, we have, for $m \ge 1$:

$$\forall u \in W^{m,\rho}(B(x,R)),\ \|u\|_{L^\tau(B(x,R/2))} \le cR^{1-2m}\|u\|_{W^{m,\rho}(B(x,R))}.$$

The constant c being independent of u and of the ball $B(x,R)$.

3.3. The main local estimates

We shall use the following notation to ease the writing:

Definition 3.5. For $r, s > 1$, $\alpha > 0$ fixed and $m, k \in \mathbb{N}$, $m \ge 2$, we set:

$$L(r,k) := L^s([0, T + \alpha/2^k], L^r_G(B^k)) \text{ and } W(r,k) := L^s([0, T + \alpha/2^k], W^{m,r}_G(B^k)),$$

where $B := B(x,R)$ is a ball in the Riemann manifold (M,g) and $B^k := B(x, R/2^k)$.

The following theorem follows by standard techniques but is needed for the sequel.

Theorem 3.6. *Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the vector bundle $G := (H,\pi,M)$ in the complete Riemannian manifold (M,g), with $\theta < \pi/2$, and consider the parabolic equation $Du = \partial_t u - Au$ also acting on sections of G and verifying $Du \in L^s([0,T+\alpha], L^r_G(B))$ and $u \in L^s([0,T+\alpha], L^r_G(B))$.*

Let $B := B(x,R)$ be a (m,ϵ)-admissible ball and set $B^1 := B(x,R/2)$. Then, with r,s in $(1,\infty)$, we have:

$$\|\partial_t u\|_{L^s([0,T+\alpha/2],L^r_G(B^1))} + R^m\|u\|_{L^s([0,T+\alpha/2],W^{m,r}_G(B^1))} \\ \leq c_3\|Du\|_{L^s([0,T+\alpha],L^r_G(B))} + c_4R^{-m}\|u\|_{L^s([0,T+\alpha],L^r_G(B))}.$$

In the case of functions we get, with this time $B \in \mathcal{A}_{m-1}(\epsilon)$,

$$\|\partial_t u\|_{L^s([0,T+\alpha/2],L^r_G(B^1))} + R^{m-1}\|u\|_{L^s([0,T+\alpha/2],W^{m,r}_G(B^1))} \\ \leq c_3\|Du\|_{L^s([0,T+\alpha],L^r_G(B))} + c_4R^{-m}\|u\|_{L^s([0,T+\alpha],L^r_G(B))}.$$

The constants c_3, c_4 are independent of u and of B.

Proof. The ball B being admissible, there is a diffeomorphism $\varphi\colon B \to \mathbb{R}^n$ such that G trivialises on B. I.e. we have, for any section u over B:

$$\pi^{-1}(B) \to B\times H,\ u \to (\pi(u), \chi_\varphi(u))\,.$$

So the local representation of the section u is: $u_\varphi := \chi_\varphi \circ u \circ \varphi^{-1}$.

We shall apply Theorem 3.2 with a slight change in T and α to the images of A, G, u,

$$(*) \qquad \|\partial_t u_\varphi\|_{L(r,1)} + \|u_\varphi\|_{W(r1)} \leq c_1\|(Du)_\varphi\|_{L(r,0)} + c_2R_\varphi^{-m}\|u_\varphi\|_{L(r,0)},$$

where $A_\varphi, B_\varphi, R_\varphi, u_\varphi$ are the images by φ of A, B, R, u and the image of G is the trivial bundle $\varphi(B)\times\mathbb{R}^N$ in $\mathbb{R}^n$. The constants c_1, c_2 being independent of B_φ.

First, because of the condition $(1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij}$ in the definition of the ϵ-admissible ball, we have that $R_\varphi \simeq R$.

Now we use the Sobolev comparison estimates given by Lemma 3.3 to get:

$$\|\partial_t u\|_{L^r_G(B^1)} \leq (1+C\epsilon)\|\partial_t u_\varphi\|_{L^r(\varphi(B^1))},$$

because $(\partial_t u)_\varphi = \partial_t u_\varphi$. We also have:

$$R^m\|u\|_{W^{m,r}_G(B^1)} \leq c\|u_\varphi\|_{W^{m,r}(\varphi(B^1))}.$$

The constants c, C being independent of B. Integrating the s-power with respect to t, we get for the left hand side of $(*)$

$$\|\partial_t u\|_{L(r,1)} + R^m\|u\|_{W(r,1)} \leq C(\|\partial_t u_\varphi\|_{L(r,1)} + \|u_\varphi\|_{W(r,1)}).$$

Now, still by Lemma 3.3

$$\|(Du)_\varphi\|_{L^r(B_\varphi)^N} \leq (1+C\epsilon)\|Du\|_{L^r_G(B)}$$

and

$$\|u_\varphi\|_{L^r(B_\varphi)^N} \leq (1+C\epsilon)\|u\|_{L^r_G(B)}.$$

Again integrating the s-power with respect to t, we get for the right hand side of $(*)$

$$c_1\|(Du)_\varphi\|_{L(r,0)} + c_2 R_\varphi^{-m}\|u_\varphi\|_{L(r,0)} \leq c_3\|Du\|_{L(r,0)} + c_4 R^{-m}\|u\|_{L(r,0)}.$$

Hence replacing in $(*)$ we get, with new constants:

$$\|\partial_t u\|_{L(r,1)} + R^m\|u\|_{W(r,1)} \leq c_3\|Du\|_{L(r,0)} + c_4 R^{-m}\|u\|_{L(r,0)}.$$

The constants $c_3,\ c_4$ are still independent of u and of B but depend on T and α.

In the case of functions, using Lemma 3.3, we get with this time $B \in \mathcal{A}_{m-1}(\epsilon)$,

$$\|\partial_t u\|_{L(r,1)} + R^{m-1}\|u\|_{W(r,1)} \leq c_3\|Du\|_{L(r,0)} + c_4 R^{-m}\|u\|_{L(r,0)}.$$

The proof is complete. □

The following corollary, the LIR inequality, is at the heart of the method we use. The induction step works because of the gain in regularity we get by this corollary.

Corollary 3.7 (The LIR inequality)**.** *Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the adapted vector bundle $G := (H,\pi,M)$ in the complete Riemannian manifold (M,g), with $\theta < \pi/2$, and consider the parabolic equation $Du = \partial_t u - Au$ also acting on sections of G.*

Let $B := B(x,R)$ be a (m,ϵ)-admissible ball and set $B^k := B(x,R/2^k)$. Then, with r,s in $(1,\infty)$, and $\alpha > 0$, we have:

$$\begin{aligned} R^m\|\partial_t u\|_{L^s([0,T+\alpha/2^{k+1}],L^r_G(B^{k+1}))} + R^{2m}\|u\|_{L^s([0,T+\alpha/2^{k+1}],W^{m,r}_G(B^{k+1}))} \\ \leq c_3 R^m\|Du\|_{L^s([0,T+\alpha/2^k],L^r_G(B^k))} + c_4\|u\|_{L^s([0,T+\alpha/2^k],L^r_G(B^k))}. \end{aligned}$$

With the notation of Definition 3.5 this gives:

$$R^m\|\partial_t u\|_{L(r,k+1)} + R^{2m}\|u\|_{W(r,k+1)} \leq c_3 R^m\|Du\|_{L(r,k)} + c_4\|u\|_{L(r,k)}.$$

In the case of functions instead of sections of G, we have, with $B(x,R) \in \mathcal{A}_{m-1}(\epsilon)$,

$$\begin{aligned} R^m\|\partial_t u\|_{L^s([0,T+\alpha/2^{k+1}],L^r(B^{k+1}))} + R^{2m-1}\|u\|_{L^s([0,T+\alpha/2],W^{m,r}(B^{k+1}))} \\ \leq c_3 R^m\|Du\|_{L^s([0,T+\alpha/2^k],L^r(B^k))} + c_4\|u\|_{L^s([0,T+\alpha/2^k],L^r(B^k))}. \end{aligned}$$

The constants c_3, c_4 being independent of u and B, but depend on T and α, hence on k.

Proof. We apply Theorem 3.6 to $B^{k+1} \subset B^k$ instead of $B^1 \subset B$ and with $\alpha/2^k$ instead of α. □

3.4. The induction

Remark 3.8. The idea under this method is the following one.

If we have a $u \in L^s([0,T], L^\rho_G(B)) :: Du = \omega$ then the LIR, Corollary 3.7, gives essentially that $u \in L^s([0,T], W^{m,\rho}_G(B))$. By applying the Sobolev embedding, Lemma 3.4, we get $u \in L^s([0,T], L^\tau_G(B)) :: Du = \omega$, with $\dfrac{1}{\tau} = \dfrac{1}{\rho} - \dfrac{m}{n}$.

But if $\omega \in L^s([0,T], L^\tau_G(B))$ then a new application of the LIR gives $u \in L^s([0,T], W^{m,\tau}_G(B))$. So we have a strict increase of the regularity of u. We can repeat the process up to reach the best regularity of the data ω.

The following lemma is essentially computational.

Lemma 3.9 (Induction)**.** *Provided that:*

$$IH(k) \qquad R^{d_k}\|\partial_t u\|_{L(r_k,k)} + R^{b_k}\|u\|_{W(r_k,k)} \le c_1(k)R^{a_k}\|\omega\|_{L(r,0)} + c_2(k)\|u\|_{L(2,0)}.$$

We get

$$IH(k{+}1) \qquad \begin{aligned} &R^{d_{k+1}}\|\partial_t u\|_{L(\tau,k+2)} + R^{b_{k+1}}\|u\|_{W(\tau,k+2)} \\ &\quad \le c_1(k+1)R^{a_{k+1}}\|\omega\|_{L(r,0)} + c_2(k+1)\|u\|_{L(2,0)}. \end{aligned}$$

with $\dfrac{1}{r_{k+1}} = \dfrac{1}{r_k} - \dfrac{m}{n} = \dfrac{1}{2} - (k+1)\dfrac{m}{n}$, $\tau := \min(r_{k+1}, r)$, *and for sections of* G *with* $B \in \mathcal{A}_m(\epsilon)$,

$$d_{k+1} = 3m + b_k;\ b_{k+1} = 4m + b_k;\ a_{k+1} = \min(a_k, 3m + b_k),$$

and

$$c_1(k+1) = c_3(k) + cc_4(k)c_1(k);\ c_2(k+1) = cc_4(k)c_2(k).$$

And for functions with $B \in \mathcal{A}_{m-1}(\epsilon)$,

$$d_{k+1} = 3m - 1 + b_k; \quad b_{k+1} = 4m - 1 + b_k; \quad a_{k+1} = \min(a_k, 3m - 1 + b_k),$$

and

$$c_1(k+1) = c_3(k) + cc_4(k)c_1(k); \quad c_2(k+1) = cc_4(k)c_2(k).$$

Proof. We have, by the Sobolev embedding, Lemma 3.4, with $\tau := r_{k+1}$, $\rho := r_k$ and $\dfrac{1}{r_{k+1}} = \dfrac{1}{r_k} - \dfrac{m}{n}$,

$$\|u(t,\cdot)\|_{L^{r_{k+1}}_G(B^{k+1})} \le cR^{-2m}\|u(t,\cdot)\|_{W^{m,r_k}_G(B^k)}$$

hence, integrating,

$$\|u\|_{L^s([0,T+\alpha/2^{k+1}],L^{r_{k+1}}_p(B^{k+1}))} \le cR^{-2m}\|u\|_{L^s([0,T+\alpha/2^k],W^{m,r_k}_p(B^k))}.$$

With the notation of Definition 3.5 this gives:

$$\|u\|_{L(r_{k+1},k+1)} \leq cR^{-2m}\|u\|_{W(r_k,k)}. \tag{3.4}$$

But by $IH(k)$

$$R^{b_k}\|u\|_{W(r_k,k)} \leq c_1(k)R^{a_k}\|\omega\|_{L(r,0)} + c_2(k)\|u\|_{L(2,0)}.$$

so

$$\begin{aligned}\|u\|_{L(r_{k+1},k+1)} &\leq cR^{-2m}\|u\|_{W(r_k,k)}\\ &\leq cc_1(k)R^{-2m+a_k-b_k}\|\omega\|_{L(r,0)} + cc_2(k)R^{-2m-b_k}\|u\|_{L(2,0)}.\end{aligned}$$

Now the LIR inequality, Corollary 3.7, with $\tau = \min(r, r^{k+1})$, gives:

$$R^m\|\partial_t u\|_{L(\tau,k+2)} + R^{2m}\|u\|_{W(\tau,k+2)} \leq c_3R^m\|Du\|_{L(\tau,k+1)} + c_4\|u\|_{L(\tau,k+1)}.$$

hence

$$\begin{aligned}R^m\|\partial_t u\|_{L(\tau,k+2)} + R^{2m}\|u\|_{W(\tau,k+2)} &\leq c_3R^m\|Du\|_{L(\tau,k+1)}\\ &\quad + c_4cc_1(k)R^{-2m+a_k-b_k}\|\omega\|_{L(r,0)} + c_4cc_2(k)R^{-2m-b_k}\|u\|_{L(2,0)}\end{aligned}$$

because $\|u\|_{L(\tau,k+1)} \leq \|u\|_{L(r_{k+1},k+1)}$.

But $\tau \leq r$, $[0, T+\alpha/2^{k+1}] \subset [0, T+\alpha]$, $B^{k+2} \subset B$, so we get

$$\|Du\|_{L(\tau,k+1)} \leq \|Du\|_{L(r,0)} = \|\omega\|_{L(r,0)}.$$

Hence

$$\begin{aligned}&R^m\|\partial_t u\|_{L(\tau,k+2)} + R^{2m}\|u\|_{W(\tau,k+2)}\\ &\quad\leq (c_3R^m + c_4cc_1(k))R^{-2m+a_k-b_k}\|\omega\|_{L(r,0)} + c_4cc_2(k)R^{-2m-b_k}\|u\|_{L(2,0)}.\end{aligned}$$

So, multiplying by R^{2m+b_k}, we get:

$$\begin{aligned}&R^{3m+b_k}\|\partial_t u\|_{L(\tau,k+2)} + R^{4m+b_k}\|u\|_{W(\tau,k+2)}\\ &\quad\leq (c_3R^{3m+b_k} + c_4cc_1(k)R^{a_k})\|\omega\|_{L(r,0)} + c_4cc_2(k)\|u\|_{L(2,0)}.\end{aligned}$$

Hence with

$$d_{k+1} = 3m + b_k; \quad b_{k+1} = 4m + b_k; \quad a_{k+1} = \min(a_k, 3m + b_k),$$

and

$$c_1(k+1) = c_3(\ k) + cc_4(k)c_1(k); \quad c_2(k+1) = cc_4(k)c_2(k),$$

we get

$$IH(k{+}1) \qquad \begin{aligned}&R^{d_{k+1}}\|\partial_t u\|_{L(\tau,k+2)} + R^{b_{k+1}}\|u\|_{W(\tau,k+2)}\\ &\quad\leq c_1(k+1)R^{a_{k+1}}\|\omega\|_{L(r,0)} + c_2(k+1)\|u\|_{L(2,0)}.\end{aligned}$$

In the case of functions, applying again Corollary 3.7, with

$$d_{k+1} = 2m - 1 + b_k;\ b_{k+1} = 3m - 2 + b_k;\ a_{k+1} = \min(a_k, 2m - 1 + b_k),$$

and

$$c_1(k+1) = c_3(k) + cc_4(k)c_1(k);\ c_2(k+1) = cc_4(k)c_2(k),$$

we get

$$IH(k{+}1) \qquad \begin{aligned} &R^{d_{k+1}}\|\partial_t u\|_{L(\tau,k+2)} + R^{b_{k+1}}\|u\|_{W(\tau,k+2)} \\ &\qquad \le c_1(k+1)R^{a_{k+1}}\|\omega\|_{L(r,0)} + c_2(k+1)\|u\|_{L(2,0)}. \end{aligned}$$

The proof is complete. □

Lemma 3.10. *Let $B = B(x,R)$ be a ϵ-admissible ball in M. We have, for $\omega \in L^s([0,T+\alpha], L^r_G(B))$ with $r \ge 2$:*

$$\|\omega\|_{L^s_G([0,T+\alpha],L^2(B))} \le c(n,\epsilon)R^{\frac{n}{2}-\frac{n}{r}}\|\omega\|_{L^s_G([0,T+\alpha],L^r(B))},$$

with c depending only on n and ϵ.

Proof. Let $\omega \in L^s([0,T+\alpha], L^r_G(B))$. Because $r \ge 2$ and B is relatively compact, we have $\omega \in L^s([0,T+\alpha], L^2_G(B))$. Because $\frac{dv}{|B|}$ is a probability measure on B, where $|B|$ is the volume of the ball B, we get

$$\left(\int_B |\omega(t,y)|^2 \frac{dv(y)}{|B|}\right)^{1/2} \le \left(\int_B |\omega(t,y)|^r \frac{dv(y)}{|B|}\right)^{1/r},$$

hence

$$\|\omega(t,\cdot)\|_{L^2(B)} \le |B|^{\frac{1}{2}-\frac{1}{r}}\|\omega(t,\cdot)\|_{L^r(B)}.$$

Integrating on t, we get

$$\|\omega\|_{L^s([0,T+\alpha],L^2(B))} \le |B|^{\frac{1}{2}-\frac{1}{r}}\|\omega\|_{L^s([0,T+\alpha],L^r(B))}.$$

Now on the manifold M, for $B_x := B(x,R)$ a ϵ-admissible ball, we get

$$\forall y \in B_x,\ (1-\epsilon)^n \le |\det g(y)| \le (1+\epsilon)^n,$$

hence we have, comparing the Lebesgue measure in $\mathbb{R}^n$ with the volume measure in M,

$$\forall x \in M,\ (1-\epsilon)^{n/2}\nu_n R^n \le \mathrm{Vol}(B(x,\ R_\epsilon(x))) \le (1+\epsilon)^{n/2}\nu_n R^n,$$

so, on the manifold M, we have

$$\|\omega\|_{L^s([0,T+\alpha],L^2(B))} \le c(n,\epsilon)R^{\frac{n}{2}-\frac{n}{r}}\|\omega\|_{L^s([0,T+\alpha],L^r(B))}$$

with c depending only on n and ϵ. □

For $t \geq 0$ define $k := \lceil t \rceil \in \mathbb{N}$ the integral part by excess, i.e.: $t \leq k < t+1$. Now set $k := \left\lceil \frac{n(r-2)}{2mr} \right\rceil$ then k is the smallest integer such that, with $\frac{1}{r_k} = \frac{1}{2} - \frac{mk}{n}$, we have $r_k \geq r$.

Proposition 3.11. *Let $r \geq 2$. Let $B := B(x,R)$ be a (m,ϵ)-admissible ball and set $B^{k+1} := B(x, 2^{-k-1}R)$. Then for any $\alpha > 0$ we have the estimates, using β, γ, δ from Definition 2.12:*

$$R^{\delta}\|\partial_t u\|_{L^s([0,T],L_G^{r_k}(B^{k+1}))} + R^{\gamma}\|u\|_{L^s([0,T],W_G^{m,r_k}(B^{k+1}))}$$
$$\leq c_1(k)R^{\beta}\|\omega\|_{L^s([0,T+\alpha],L_G^{r}(B))} + c_2(k)\|u\|_{L^s([0,T+\alpha],L_G^{2}(B))},$$

and the constants $c_1(k)$, $c_2(k)$ being independent of $B \in \mathcal{A}_m(\epsilon)$.

In the case of functions instead of sections of G we have the same estimates but with $B \in \mathcal{A}_{m-1}(\epsilon)$ and using β', γ', δ' from Definition 2.12 instead of β, γ, δ.

Proof. Take $B := B(x,R)$, $B^k := B(x, 2^{-k}R)$. By the LIR inequality, Corollary 3.7, we get, with $\tau = 2$:

$$R^m\|\partial_t u\|_{L(2,1)} + R^{2m}\|u\|_{W(2,1)} \leq c_3(0)R^m\|Du\|_{L(2,0)} + c_4(0)\|u\|_{L(2,0)}.$$

Now using Lemma 3.10, we get $\|\omega\|_{L(2,0)} \leq c(n,\epsilon)R^{\frac{n}{2}-\frac{n}{r}}\|\omega\|_{L(r,0)}$. Putting it above with $Du = \omega$, we get:

$$R^m\|\partial_t u\|_{L(2,1)} + R^{2m}\|u\|_{W(2,1)} \leq c_3(0)c(n,\epsilon)R^m R^{\frac{n}{2}-\frac{n}{r}}\|\omega\|_{L(r,0)} + c_4(0)\|u\|_{L(2,0)}.$$

Hence we have the induction hypothesis at level $k = 0$,

$$IH(0) \qquad R^{d_0}\|\partial_t u\|_{L(2,1)} + R^{b_0}\|u\|_{W(2,1)} \leq c_1(0)R^{a_0}\|\omega\|_{L(r,0)} + c_2(0)\|u\|_{L(2,0)},$$

with $d_0 = m$, $b_0 = 2m$, $a_0 = m + \dfrac{n}{2} - \dfrac{n}{r}$ and $c_1(0) = c(n,\epsilon)c_3(0)$, $c_2(0) = c_4(0)$.

So applying the induction Lemma 3.9, we get

$$R^{d_1}\|\partial_t u\|_{L(\tau,2)} + R^{b_{k+1}}\|u\|_{W(\tau,2)} \leq c_1(1)R^{a_1}\|\omega\|_{L(r,0)} + c_2(1)\|u\|_{L(2,0)}.$$

with $\frac{1}{r_1} = \frac{1}{2} - \frac{m}{n}$, $\tau := \min(r_1, r)$, and

$$d_1 = 3m + b_0;\ b_1 = 4m + b_0;\ a_1 = \min(a_0, 3m + b_0),$$

hence

$$d_1 = 5m;\ b_1 = 6m;\ a_1 = \min(m + \frac{n}{2} - \frac{n}{r}, 5m),$$

and

$$c_1(k+1) = c_3(k) + cc_4(k)c_1(k);\ c_2(k+1) = cc_4(k)c_2(k).$$

By induction, we get

$$b_k = 4mk + 2m;\ d_k = 4mk + m,$$

and

$$a_0 = m + \frac{n}{2} - \frac{n}{r}, \quad \forall k \geq 1, \; a_k = \min(m + \frac{n}{2} - \frac{n}{r}, \; 5m).$$

- if $r_1 \geq r \Rightarrow \tau = r$ and we get:

$$R^{d_1} \|\partial_t u\|_{L(r,2)} + R^{b_1} \|u\|_{W(r,2)} \leq c_1(1) R^{a_1} \|\omega\|_{L(r,0)} + c_2(1) \|u\|_{L(2,0)}.$$

And we are done.

- if $\tau = r_1 < r$, by the induction Lemma 3.9, after k steps, we get with $\frac{1}{r_k} = \frac{1}{2} - \frac{mk}{n}$, $\tau := \min(r_k, r)$:

$$a_k = \min\left(m + \frac{n}{2} - \frac{n}{r}, \; 5m\right), \quad b_k = (4k+2)m; \quad d_k = (4k+1)m.$$

Then

$$IH(k) \quad R^{d_k} \|\partial_t u\|_{L(\tau,k+1)} + R^{b_k} \|u\|_{W(\tau,k+1)} \leq c_1(k) R^{a_k} \|\omega\|_{L(r,0)} + c_2(k) \|u\|_{L(2,0)}.$$

Hence if $r_k \geq r$ we are done as above, if not we repeat the process. Because $\frac{1}{r_k} = \frac{1}{2} - \frac{mk}{n}$ after a finite number $k = \left\lceil \frac{n(r-2)}{2mr}) \right\rceil$ of steps we have $r_k \geq r$ and we get, with $B^k := B(x, R/2^k)$:

$$R^{d_k} \|\partial_t u\|_{L(r_k,k+1)} + R^{b_k} \|u\|_{W(r_k,k+1)} \leq c_1(k) R^{a_k} \|\omega\|_{L(r,0)} + c_2(k) \|u\|_{L(2,0)}.$$

Replacing the values of $L(r,k)$ and $W(r,k)$:

$$\begin{aligned} R^{d_k} \|\partial_t u\|_{L^s([0,T+\alpha/2^{k+1}], L_G^{r_k}(B^{k+1}))} &+ R^{b_k} \|u\|_{L^s([0,T+\alpha/2^{k+1}], W_G^{m,r_k}(B^{k+1}))} \\ &\leq c_1(k) R^{a_k} \|\omega\|_{L^s([0,T+\alpha], L_G^r(B))} + c_2(k) \|u\|_{L^s([0,T+\alpha], L_G^2(B))}. \end{aligned}$$

With $c_j(k)$ depending on $\epsilon, n, m, \alpha, k$ and not on B. Because:

$$\|\partial_t u\|_{L^s([0,T], L_G^{r_k}(B^{k+1}))} \leq \|\partial_t u\|_{L^s([0,T+\alpha/2^{k+1}], L_G^{r_k}(B^{k+1}))}$$

and

$$\|u\|_{L^s([0,T], W_G^{m,r_k}(B^{k+1}))} \leq \|u\|_{L^s([0,T+\alpha/2^{k+1}], W_G^{m,r_k}(B^{k+1}))},$$

this proves the proposition for sections of G.

In the case of functions instead of sections of G we have the same estimates but with $B \in \mathcal{A}_{m-1}(\epsilon)$ and: $\forall k \geq 1$, $a_k = \min(m + \frac{n}{2} - \frac{n}{r}, \; 4m-1)$, $b_k = k(4m-1) + 2m$, $d_k = m + k(4m-1)$, and the constants $c_1(k)$, $c_2(k)$ being independent of $B \in \mathcal{A}_{m-1}(\epsilon)$. This justifies the notation in Definition 2.12. The proof is complete. □

4. Vitali covering

The following is a well known lemma, see for instance [8, section 1.5.1].

Lemma 4.1. *Let $\mathcal{F}$ be a collection of balls $\{B(x,r(x))\}$ in a metric space, with $\forall B(x,r(x)) \in \mathcal{F},\ 0 < r(x) \leq R$. There exists a disjoint subcollection $\mathcal{G}$ of $\mathcal{F}$ with the following properties: every ball B in $\mathcal{F}$ intersects a ball C in $\mathcal{G}$ and $B \subset 5C$.*

Fix $\epsilon > 0$ and let $\forall x \in M,\ r(x) := R_\epsilon(x)/5$, where $R_\epsilon(x)$ is the admissible radius at x, we built a Vitali covering with the collection $\mathcal{F} := \{B(x,r(x))\}_{x\in M}$. The previous lemma gives a disjoint subcollection $\mathcal{G}$ such that every ball B in $\mathcal{F}$ intersects a ball C in $\mathcal{G}$ and we have $B \subset 5C$. We set

$$\mathcal{D}(\epsilon) := \{x \in M :: B(x,r(x)) \in \mathcal{G}\} \text{ and } \mathcal{C}_\epsilon := \{B(x,5r(x)),\ x \in \mathcal{D}(\epsilon)\} :$$

we shall call $\mathcal{C}_\epsilon$ a ϵ-**admissible covering** of (M,g).

More generally let $k \in \mathbb{N}$ and consider the collection $\mathcal{F}_k(\epsilon) := \{B(x,r_k(x))\}_{x\in M}$ where, for $x \in M,\ r_k(x) := 2^{-k}R_\epsilon(x)/5\eta$, still where $R_\epsilon(x)$ is the admissible ϵ-radius at x. The integer $\eta \geq 1$ will be chosen later. The previous lemma gives a disjoint subcollection $\mathcal{G}_k(\epsilon)$ such that every ball B in $\mathcal{F}_k(\epsilon)$ intersects a ball C in $\mathcal{G}_k(\epsilon)$ and we have $B \subset 5C$. We set $\mathcal{D}_k(\epsilon) := \{x \in M :: B(x,r_k(x)) \in \mathcal{G}_k(\epsilon)\}$ and $\mathcal{C}_k(\epsilon) := \{B(x,5r_k(x)),\ x \in \mathcal{D}_k(\epsilon)\}$: we shall call $\mathcal{C}_k(\epsilon)$ a (k,ϵ)-**admissible covering** of (M,g). We have the lemma:

Lemma 4.2. *Let $B(x,5r_k(x)) \in \mathcal{C}_k(\epsilon)$ then $B^0(x,\tilde{R}(x))$ with $\tilde{R}(x) := 2^k \times 5r_k(x)$ is still a ϵ-admissible ball.*

Moreover we have that all the balls $B^j(x) := B^0(x,2^{-j}\tilde{R}(x)),\ j = 0,1,\ldots,\ k$ are also ϵ-admissible balls and $\{B^j(x),\ x \in \mathcal{D}_k(\epsilon)\}$, for $j = 0,\ldots,k$, is a covering of M.

Proof. Take $x \in \mathcal{D}_k(\epsilon)$ then we have that the geodesic ball $B^0(x,\tilde{R}(x)) = B(x,R_\epsilon/\eta)$ is ϵ-admissible and because $2^{-k}R_\epsilon/\eta < R_\epsilon$, for $\eta \geq 1$, we get that $B(x,R(x))$ is also ϵ-admissible. The same for $B^j(x) = B^0(x,2^{-j}\tilde{R}(x))$ because $2^{-j}\tilde{R}(x) < 2^{-j}R_\epsilon(x)/\eta$.

The fact that $\{B^k(x),\ x \in \mathcal{D}_k(\epsilon)\}$ is a covering of M is just the Vitali lemma and, because $j \leq k \Rightarrow B^j(x) \supset B^k(x)$, we get that $\{B^j(x),\ x \in \mathcal{D}_k(\epsilon)\}$ is also a covering of M. □

Then we have:

Proposition 4.3. *Let (M,g) be a Riemannian manifold, then the overlap of a (k,ϵ)-admissible covering $\mathcal{C}_k(\epsilon)$ is less than $T = \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}(100)^n$, i.e., $\forall x \in M$, $x \in B(y,5r_k(y))$ where $B(y,r_k(y)) \in \mathcal{G}_k(\epsilon)$ for at most T such balls.*

Moreover we have

$$\forall f \in L^1(M), \qquad \sum_{j\in\mathbb{N}} \int_{B(x_j,r_k(x_j))} |f(x)|\, dv_g(x) \leq T\|f\|_{L^1(M)}.$$

Proof. Let $B_j := B(x_j,r_k(x_j)) \in \mathcal{G}_k(\epsilon)$ and suppose that $x \in \bigcap_{j=1}^l B(x_j,5r_k(x_j))$. Then we have $\forall j = 1,\ldots,l,\ d(x,x_j) \leq 5r_k(x_j)$. Hence

$$d(x_j,x_m) \leq d(x_j,x) + d(x,x_m) \leq 5(r_k(x_j) + r_k(x_m)) \leq 2^{-k}(R_\epsilon(x_j) + R_\epsilon(x_m))/\eta.$$

Suppose that $r_k(x_j) \geq r_k(x_m)$ then $x_m \in B(x_j, 10r_k(x_j)) \subset B(x_j, R_\epsilon(x_j))$ because $10r_k(x_j) = 10\times 2^{-k}R_\epsilon(x_j)/\eta \leq R_\epsilon(x_j)$ because now on *we choose* $\eta = 10$. Then, by the slow variation of the ϵ-radius Lemma 2.5, we have $R_\epsilon(x_j) \leq 2R_\epsilon(x_m)$. If $r_k(x_j) \leq r_k(x_m)$ then, the same way, $2R_\epsilon(x_j) \geq R_\epsilon(x_m)$. Hence in any case we have $\frac{1}{2}R_\epsilon(x_m) \leq R_\epsilon(x_j) \leq 2R_\epsilon(x_m)$.

So $d(x_j, x_m) \leq 2^{-k}\times 3R_\epsilon(x_j)/10$ hence $\forall m = 1, \ldots, l$,

$$\begin{aligned} B(x_m,\ r_k(x_m)) &\subset B(x_j,\ 2^{-k}\times 3R_\epsilon(x_j)/10 + 2^{-k}R_\epsilon(x_m)/10) \\ &\subset B(x_j,\ 2^{-k}\times 5R_\epsilon(x_j)/10) \end{aligned}$$

because $R_\epsilon(x_m) \leq 2R_\epsilon(x_j)$. The balls in $\mathcal{G}_k(\epsilon)$ being disjoint, we get, setting $B_m := B(x_m,\ r_k(x_m))$,

$$\sum_{m=1}^{l} \mathrm{Vol}(B_m) \leq \mathrm{Vol}(B(x_j, 2^{-k}\times 5R_\epsilon(x_j)/\eta) = \mathrm{Vol}(B(x_j, 5r_k(x_j)).$$

The Lebesgue measure read in the chart φ and the canonical measure dv_g on $B(x, R_\epsilon(x))$ are equivalent; precisely because of condition 1) in the admissible ball definition, we get that $(1-\epsilon)^n \leq |\det g| \leq (1+\epsilon)^n$, and the measure dv_g read in the chart φ is $dv_g = \sqrt{|\det g_{ij}|}d\xi$, where $d\xi$ is the Lebesgue measure in $\mathbb{R}^n$. In particular:

$$\forall x \in M,\ \mathrm{Vol}(B(x,\ R_\epsilon(x))) \leq (1+\epsilon)^{n/2}\nu_n R^n,$$

where ν_n is the Euclidean volume of the unit ball in $\mathbb{R}^n$. Now because $R_\epsilon(x_j)$ is the admissible radius and $5r_k(x_j) = 2^{-k}\times 5R_\epsilon(x_j)/10 < R_\epsilon(x_j)$, because $\eta = 10$,

$$\mathrm{Vol}(B(x_j, 5r_k(x_j))) \leq 5^n(1+\epsilon)^{n/2}v_n r_k(x_j)^n.$$

On the other hand we have also

$$\mathrm{Vol}(B_m) \geq v_n(1-\epsilon)^{n/2}r_k(x_m)^n \geq v_n(1-\epsilon)^{n/2}2^{-n}r_k(x_j)^n,$$

hence

$$\sum_{j=1}^{l}(1-\epsilon)^{n/2}2^{-n}r(x_j)^n \leq 5^n(1+\epsilon)^{n/2}r_k(x_j)^n,$$

so finally

$$l \leq (5\times 2)^n\frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}},$$

which means that $T \leq \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}(100)^n$. Saying that any $x \in M$ belongs to at most T balls of the covering $\{B_j\}$ means that $\sum_{j\in\mathbb{N}}\mathbb{1}_{B_j}(x) \leq T$, and this implies easily that:

$$\forall f \in L^1(M), \qquad \sum_{j\in\mathbb{N}}\int_{B_j}|f(x)|\,dv_g(x) \leq T\|f\|_{L^1(M)}.$$

The proof is complete. □

Corollary 4.4. *Let (M,g) be a Riemannian manifold. Consider the (k,ϵ)-admissible covering $\mathcal{C}_k(\epsilon)$. Then the overlap of the associated covering by the balls $\{B^0(x,R_\epsilon(x)/\eta)\}_{x\in\mathcal{D}_k(\epsilon)}$ verifies $T_k \leq \dfrac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}(100)^n\times 2^{nk}$.*

Proof. We start the proof exactly the same way as above. Let $B_j := B(x_j, R_\epsilon(x_j)/\eta)$, $x_j \in \mathcal{D}_k(\epsilon)$ and suppose that $x \in \bigcap_{j=1}^{l} B(x_j, R_\epsilon(x_j)/\eta)$. Then we have, as above:

$$\forall m = 1,\dots,l,\ B(x_m, R_\epsilon(x_m)) \subset B(x_j, 3R_\epsilon(x_j)/\eta + R_\epsilon(x_m)/\eta) \subset B(x_j, 5R_\epsilon(x_j)/\eta).$$

The balls in $\mathcal{G}_k(\epsilon)$ being disjoint, we get, setting $B_m := B(x_m,\ r_k(x_m))$,

$$\sum_{m=1}^{l} \mathrm{Vol}(B_m) \leq \mathrm{Vol}(B(x_j, 5R_\epsilon(x_j)/\eta) = \mathrm{Vol}(B(x_j, 5\times 2^k r_k(x_j)).$$

Exactly as above, we get because of the factor 2^k,

$$l \leq (5\times 2\times 2^k)^n \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}},$$

hence the result. □

5. The threshold

We shall need the following "threshold hypothesis".

(THL2) For any $\omega \in L^s([0,T], L^2_G(M))$ there is a $u \in L^s([0,T], L^2_G(M))$ such that $Du = \omega$ with the estimate:

$$\|u\|_{L^s([0,T],L^2_G(M))} \lesssim \|\omega\|_{L^s([0,T],L^2_G(M))}.$$

5.1. The case of the heat equation

We have, see for instance [16]:

Theorem 5.1 (Hodge decomposition)**.** *For M a compact $\mathcal{C}^\infty$ smooth Riemannian manifold, there exists a complete orthonormal basis $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ of p-forms in $L^2(M)$ consisting of eigenforms of Δ with φ_j having eigenvalue λ_j satisfying $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$. For every j we have $varphi_j \in \mathcal{C}^\infty_p(M)$ and*

$$\Phi_p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\otimes\varphi_j(y)$$

is the heat kernel for p-forms on M.

As an easy corollary we get the threshold hypothesis (THL2) because:

Corollary 5.2. *Let M be a compact $\mathcal{C}^\infty$ smooth Riemannian manifold. Let, for $t \geq 0,\ \omega(t,x) \in L^2_p(M)$ then we have that the kernel $\Phi_p(x,y,t)$ gives a solution u of the heat equation $Du := \partial_t u - \Delta u = \omega,\ u(0,x) \equiv 0$, such that $\forall t \geq 0,\ u(t,x) \in L^2_p(M)$ and we have the estimate:*

$$\forall\, t \geq 0, \quad \|u(t,\cdot)\|^2_{L^2_p(M)} \leq t \int_0^t \|\omega(\tau,\cdot)\|^2_{L^2_p(M)} d\tau.$$

Proof. We have the solution:

$$u(t,x) = \int_0^t \int_M \Phi_p(x,y,t-\tau)\omega(\tau,y)dyd\tau.$$

Because $\omega \in L^2_p(M)$, we have

$$\omega(x,t) = \sum_{j\in\mathbb{N}} \omega_j(t)\varphi_j(x) \text{ with } \sum_{j\in\mathbb{N}} |\omega_j(t)|^2 = \|\omega(t,\cdot)\|^2_{L^2_p(M)}.$$

So we get

$$u(t,x) = \int_0^t \sum_{j\in\mathbb{N}} e^{-\lambda_j(t-\tau)}\omega_j(\tau)\varphi_j(x)d\tau,$$

hence, by Cauchy-Schwarz inequality,

$$\|u(t,\cdot)\|^2_{L^2_p(M)} = \sum_{j\in\mathbb{N}} \left| \int_0^t e^{-\lambda_j\tau}\omega_j(\tau)d\tau \right|^2 \leq \sum_{\lambda_j>0} \frac{(1-e^{-2\lambda_j t})}{2\lambda_j} \int_0^t |\omega_j(\tau)|^2 d\tau.$$

Let λ_1 be the first non zero eigenvalue for the laplacian, we get for every $j \geq 1,\ \psi(2\lambda_j t) := \frac{(1-e^{-2\lambda_j t})}{2\lambda_j} \leq t$ because $\psi(s) := 1 - e^{-s} - s \leq 0$; take the derivative $\psi'(s) := e^{-s} - 1 \leq 0$ because $s \geq 0$ and $\psi(0) = 0$ imply $\forall s \geq 0,\ \psi(s) \leq 0$. With $s = 2\lambda_j t$ we get the result. So

$$\forall t \in [0,T],\ \|u(t,\cdot)\|^2_{L^2_p(M)} \leq t \sum_{\lambda_j>0} \int_0^t |\omega_j(\tau)|^2 d\tau \leq t \int_0^t \|\omega(\tau,\cdot)\|^2_{L^2_p(M)} d\tau, \quad (5.1)$$

which ends the proof of the corollary. □

For M complete non compact, we can use a sequence N_k of increasing compact sub manifolds with smooth boundary. To each of them we associate its boundaryless double to get the estimate (5.1) which gives the result for M because the constant, t, is independent of k. So we get

Theorem 5.3. *Let M be a connected complete $\mathcal{C}^\infty$ Riemannian manifold. Let, for $t \geq 0,\ \omega(t,x) \in L^2_p(M)$. Then we have a solution u of the heat equation $Du := \partial_t u - \Delta u = \omega,\ u(x,0) \equiv 0$, such that $\forall t \geq 0,\ u(t,x) \in L^2_p(M)$ with the estimate:*

$$\forall\, t \geq 0, \quad \|u(t,\cdot)\|^2_{L^2(M)} \leq t \int_0^t \|\omega(\tau,\cdot)\|^2_{L^2(M)} d\tau.$$

Clearly this result implies the hypothesis (THL2).

5.2. A more general case

Let G be an adapted vector bundle on M. If A is an essentially positive operator on the sections of G, then we have that the semi-group e^{-tA} is a contraction on the L^2 sections of G. This means

$$\forall\, \omega(t,\cdot) \in L^2_G(M),\ v(t,x;s) := e^{-sA}\omega(t,\cdot) \in L^2_G(M)$$

with $\|v(t,\cdot;s)\|_{L^2_G(M)} \leq \|\omega(t,\cdot)\|_{L^2_G(M)}$. We shall make the extra hypothesis that the kernel $\Phi_G(t,x,y)$ of the semi-group e^{-tA} is smooth i.e. $\Phi_G(t,x,y)$ is $\mathcal{C}^\infty$ in all variables for $t>0$ and

$$v(t,x;s) = \int_M \Phi_G(s,x,y)\omega(t,y)d\sigma(y).$$

If all these assumptions are fulfilled, we shall say that (A,G) is a well adapted couple.

Theorem 5.4. *Let (A,G) be a well adapted couple. Then we have, for any $\omega(t,\cdot) \in L^2_G(M)$, that there is a global solution $u(t,\cdot) \in L^2_G(M)$ of the parabolic equation $\partial_t u - Au = \omega,\ u(x,0) \equiv 0$, with:*

$$\|u(t,\cdot)\|_{L^2} \leq \int_0^t \|\omega(s,\cdot)\|_{L^2} ds.$$

Proof. Let, for the semi-group e^{-tA}, $v(t,x;s)$ be the canonical solution of the homogeneous parabolic equation with ω in $\mathcal{C}^\infty(M)$ with compact support, starting at time s:

$$\forall\, t>s, \quad \forall\, x \in M, \quad v(t,x;s) = \int_M \Phi_G(s,x,y)\omega(t,y)d\sigma(y).$$

This means

$$\forall\, t>s, \quad \forall\, x \in M, \quad \partial_t v(t,x;s) - Av(t,x;s) = 0;$$

and

$$\forall\, x \in M, \quad v(s,x;s) = \omega(s,x).$$

Now, as for instance in [7, Theorem 2] for the Laplacian in $\mathbb{R}^n$, we set:

$$u(t,x) := \int_0^t v(t,x;s)ds.$$

The hypothesis of regularity of the semi-group we made and the fact that ω is $\mathcal{C}^\infty(M)$ with compact support, allows us to differentiate under the integral sign. We get:

$$\partial_t u = v(t,x;t) + \int_0^t \partial_t v(t,x;s)ds \ \text{ and } \ Au = \int_0^t Av(t,x;s)ds.$$

So:

$$\partial_t u - Au = v(t,x;t) + \int_0^t [\partial_t v - Av]ds = v(t,x;t) = \omega(t,x).$$

Hence $u(t,x)$ is a solution of the inhomogeneous equation $\partial_t u - Au = \omega(t,x)$.

Now suppose that $\forall t \geq 0,\ \|\omega(t,\cdot)\|_{L^2_G(M)} < +\infty$, and $\omega(t,\cdot)$ is in $\mathcal{C}^\infty(M)$ with compact support. Because the semi-group e^{-tA} is bounded on L^2, we get

$$\|v(t,\cdot;s)\|_{L^2} \leq C\|\omega(s,\cdot)\|_{L^2}$$

hence

$$\|u(t,\cdot)\|_{L^2} \leq \int_0^t \|\omega(s,\cdot)\|_{L^2} ds.$$

It remains to use the density of elements in $\mathcal{C}^\infty_G(M)$ with compact support in $L^2_G(M)$ to end the proof of the theorem. □

Clearly this result implies the hypothesis (THL2).

Remark 5.5. If $G = \Lambda^p(M)$, the bundle of p-forms on M, and $A = \Delta$ the Hodge Laplacian on M, then we know that the heat semi-group $(e^{-t\Delta})_{t\geq 0}$ on the manifold is a contraction on $L^2_G(M)$, because Δ is essentially positive on p-forms, see [17, Theorem 2.4] and [14, p. 2]. In the case of functions, i.e. $p = 0$, we know that the kernel $\Phi(t,x,y)$ exists and has the right properties, see [4, Theorem 4, p. 188]. I suspect that this is also true in the case of p-forms with $p \geq 1$, but I find no reference for it, so I give the Theorem 5.3.

6. Global results

We want to globalise Theorem 3.6 by use of our Vitali coverings.

Lemma 6.1. *We have for any section $f\colon M \to G$ and $\tau \in (1,\infty)$, with $w(x) := R_\epsilon(x)^{\gamma\tau}$ and $B(x) := B(x, R_x(x)/10)$, $B^k(x) := B(x, 2^{-k}R_\epsilon(x)/10)$, that:*

$$\forall \tau \geq 1, \quad \|f\|^\tau_{W^{l,\tau}_G(M,\ w)} \simeq \sum_{x\in\mathcal{D}_k(\epsilon)} R_\epsilon(x)^{\gamma\tau}\|f\|^\tau_{W^{l,\tau}_G(B^k(x))};$$

and:

$$\forall \tau \geq 1, \quad \|f\|^\tau_{W^{l,\tau}_G(M,w)} \simeq \sum_{x\in\mathcal{D}_k(\epsilon)} R_\epsilon(x)^{\gamma\tau}\|f\|^\tau_{W^{l,\tau}_G(B(x))}.$$

Proof. Let $x \in \mathcal{D}_k(\epsilon)$, this implies that $B^k(x) := B(x, 2^{-k}R_\epsilon(x)/10) \in \mathcal{C}_k(\epsilon)$.

• First we start with $l = 0$. We shall deal with the function $|f|$. We have, because $\mathcal{C}_k(\epsilon)$ is a covering of M and with $\forall y \in B(x),\ R(y) := R_\epsilon(y)$

$$\|f\|^\tau_{L^\tau(M,w)} := \int_M |f(x)|^\tau w(x)dv(x) \leq \sum_{x\in\mathcal{D}_k(\epsilon)} \int_{B^k(x)} |f(y)|^\tau R(y)^{\gamma\tau}dv(y).$$

We have, by Lemma 2.5, $\forall y \in B,\ R(y) \leq 2R(x)$, then

$$\sum_{x\in\mathcal{D}_k(\epsilon)} \int_{B^k(x)} |f(y)|^\tau\, R(y)^{\gamma\tau} dv(y) \leq \sum_{x\in\mathcal{D}_k(\epsilon)} 2^{\gamma\tau} R(x)^{\gamma\tau} \int_{B^k(x)} |f(y)|^\tau dv(y)$$
$$\leq 2^{\gamma\tau} \sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \|f\|^\tau_{L^\tau(B^k(x))}.$$

Hence

$$\|f\|^\tau_{L^\tau_G(M,w)} \leq 2^{\gamma\tau} \sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \|f\|^\tau_{L^\tau_G(B^k)}.$$

To get the converse inequality we still use Lemma 2.5: $\forall y \in B,\ R(x) \leq 2R(y)$ so we get:

$$\sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \int_{B^k(x)} |f(y)|^\tau dv(y) \leq 2^{\gamma\tau} \sum_{x\in\mathcal{D}_k(\epsilon)} \int_{B^k(x)} R(y)^{\gamma\tau}\, |f(y)|^\tau dv(y).$$

Now we use the fact that the overlap of $\mathcal{C}_k(\epsilon)$ is bounded by T,

$$\sum_{x\in\mathcal{D}_k(\epsilon)} \int_{B^k(x)} R(y)^{\gamma\tau}\, |f(y)|^\tau dv(y) \leq 2^{\gamma\tau} T \int_M R(y)^{\gamma\tau}\, |f(y)|^\tau dv(y) = 2^{\gamma\tau} T \|f\|^\tau_{L^\tau(M,w)}.$$

So

$$\sum_{x\in\mathcal{D}_k(\epsilon)} R^{\gamma\tau} \|f\|_{L^\tau(B^k)}{}^\tau \leq 2^{\gamma\tau} T \|f\|^\tau_{L^\tau(M,w)}.$$

We already know that $\{B :: B^k \in \mathcal{C}_k(\epsilon)\}$ is a covering of M with a bounded overlap by Corollary 4.4, so we follow exactly the same lines to prove:

$$\forall \tau \geq 1, \quad \|f\|^\tau_{L^\tau_G(M,w)} \simeq \sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \|f\|^\tau_{L^\tau_G(B(x))}.$$

- Now let $l \geq 1$. We apply the case $l = 0$ to the covariant derivatives of f.

$$\forall \tau \geq 1, \quad \left\|\nabla^l f\right\|^\tau_{L^\tau_G(M,w)} \simeq \sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \left\|\nabla^l f\right\|^\tau_{L^\tau_G(B(x))}.$$

Because $\|f\|_{W^{l,\tau}} = \|f\|_{L^\tau} + \cdots + \left\|\nabla^l f\right\|_{L^\tau}$ we get

$$\forall \tau \geq 1, \quad \left\|\nabla^l f\right\|^\tau_{W^{l,\tau}_G(M,w)} \simeq \sum_{x\in\mathcal{D}_k(\epsilon)} R(x)^{\gamma\tau} \left\|\nabla^l f\right\|^\tau_{W^{l,\tau}_G(B(x))}.$$

The proof is complete. □

Theorem 6.2. *Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the adapted vector bundle $G := (H,\pi,M)$ in the complete Riemannian manifold (M,g), with $\theta < \pi/2$, and consider the parabolic equation $Du = \partial_t u - Au$*

also acting on sections of G. Let $R(x) = R_{m,\epsilon}(x)$ be the (m,ϵ) radius at the point $x \in M$. Set $w_1(x) := R(x)^{\delta}$, $w_2(x) := R(x)^{\gamma}$, $w_3(x) := R(x)^{\beta}$, with the notation in Definition 2.12. We have:

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} + \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))} \\ \leq c_1\|Du\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2\|u\|_{L^r([0,T+\alpha],L^2_G(M))}.$$

In the case of functions instead of sections of G we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and:

$$w_1(x) := R(x)^{\delta'}, \quad w_2(x) := R(x)^{\gamma'}, \quad w_3(x) := R(x)^{\beta'}.$$

Proof. Once again we shall use the notation, for $k \geq 1$,

$$L(s,r,k) := L^s([0,T], L^r_G(B^k)); \qquad W(s,r,k) := L^s([0,T], W^{m,r}_G(B^k))$$

and for $k = 0$,

$$L(s,r,0) := L^s([0,T+\alpha], L^r_G(B)); \qquad W(s,r,0) := L^s([0,T+\alpha], W^{m,r}_G(B)).$$

With this notation, the Proposition 3.11 gives, for $r \geq 2$ and $k := \left\lceil \dfrac{n(r-2)}{2mr} \right\rceil$:

$$R^{\delta}\|\partial_t u\|_{L(s,r_k,k+1)} + R^{\gamma}\|u\|_{W(s,r_k,k+1)} \leq c_1 R^{\beta}\|\omega\|_{L(s,r,0)} + c_2\|u\|_{L(s,2,0)}.$$

Because $r_k \geq r$ we get

$$R^{\delta}\|\partial_t u\|_{L(s,r,k+1)} + R^{\gamma}\|u\|_{W(s,r,k+1)} \leq c_1 R^{\beta}\|\omega\|_{L(s,r,0)} + c_2\|u\|_{L(s,2,0)}.$$

So for $s = r$ we get

$$R^{\delta}\|\partial_t u\|_{L(r,r,k+1)} + R^{\gamma}\|u\|_{W(r,r,k+1)} \leq c_1 R^{\beta}\|\omega\|_{L(r,r,0)} + c_2\|u\|_{L(r,2,0)}. \tag{6.1}$$

Because

$$a + b \leq c + d \Rightarrow a^r + b^r \leq Ac^r + Bd^r$$

with constants A, B depending on r only, the inequality (6.1) can be read with a slight change of the constants:

$$R^{r\delta}\|\partial_t u\|^r_{L(r,r,k+1)} + R^{r\gamma}\|u\|^r_{W(r,r,k+1)} \leq c_1(k) R^{r\beta}\|\omega\|^r_{L(r,r,0)} + c_2\|u\|^r_{L(r,2,0)}.$$

By use of Lemma 6.1 with $l = 0,\ \tau = r,\ w_1(x) := R(x)^{r\delta}$,

$$\|\partial_t u\|^r_{L^r_G(M,\ w_1)} \simeq \sum_{x \in \mathcal{D}_{k+1}(\epsilon)} R(x)^{\delta r}\|\partial_t u\|^r_{L^r_G(B^{k+1}(x))};$$

hence integrating in $t \in [0,T]$ we get:

$$\|\partial_t u\|^r_{L^r([0,T],L^r_G(M,w_1))} \simeq \sum_{x \in \mathcal{D}_{k+1}(\epsilon)} R(x)^{\delta r}\|\partial_t u\|^r_{L^r([0,T],L^r_G(B^{k+1}(x))}.$$

The same way, with $l = m,\ \tau = r,\ w_2(x) := R(x)^{r\gamma}$,

$$\|u\|^r_{L^r([0,T],W^{m,r}_G(M,w_2))} \simeq \sum_{x\in\mathcal{D}_{k+1}} R(x)^{\gamma r}\|u\|^r_{L^r([0,T],W^{m,r}_G(B^{k+1}(x)))}$$

with $l = 0,\ \tau = r,\ w_3(x) := R(x)^{r\beta}$,

$$\|\omega\|^r_{L^r([0,T+\alpha],L^r_G(M,w_3))} \simeq \sum_{x\in\mathcal{D}_{k+1}} R(x)^{\beta r}\|\omega\|^r_{L^r([0,T+\alpha],L^r_G(B(x)))}$$

and with $l = 0,\ \tau = r$,

$$\|u\|^r_{L^r([0,T+\alpha],L^2_G(M))} \simeq \sum_{x\in\mathcal{D}_{k+1}} \|u\|^r_{L^r([0,T+\alpha],L^2_G(B(x)))}.$$

So, putting this in Theorem 3.6, we get, with $w_1(x) := R(x)^{r\delta}$, $w_2(x) := R(x)^{r\gamma}$, $w_3(x) := R(x)^{r\beta}$,

$$\begin{aligned}&\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} + \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))}\\ &\quad\le c_1(k)\|Du\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2(k)\|u\|_{L^r([0,T+\alpha],L^2_G(M))}.\end{aligned}$$

The results, for functions instead of sections of G, follow the same lines and we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and the weights:

$$w_1(x) := R(x)^{r\delta'},\quad w_2(x) := R(x)^{r\gamma'},\quad w_3(x) := R(x)^{r\beta'}.$$

The proof is complete. □

Remark 6.3. The weights $w_j(x)$ depend on r but also on m and n via β, γ and δ given by the Definition 2.12.

Corollary 6.4. *Let M be a connected complete n-dimensional $\mathcal{C}^m$ Riemannian manifold without boundary. Let $G := (H, \pi, M)$ be a complex $\mathcal{C}^m$ adapted vector bundle over M. Suppose $Du := \partial_t u - Au$, where A is (C,θ)-elliptic of order m acting on sections of G with $\theta < \pi/2$. Moreover suppose we have (THL2). Let $r \ge 2$ and:*

$$R(x) = R_{m,\epsilon}(x),\quad w_1(x) := R(x)^{r\delta},\quad w_2(x) := R(x)^{r\gamma},\quad w_3(x) := R(x)^{r\beta},$$

with the notation in Definition 2.12. Then, for any $\alpha > 0,\ r \ge 2$, we have:

$$\begin{aligned}&\forall\omega \in L^r([0,T+\alpha], L^r_G(M,w_3)) \cap L^r([0,T+\alpha], L^2_G(M)),\\ &\exists u \in L^r([0,T], W^{m,r}_G(M)) :: Du = \omega,\end{aligned}$$

with

$$\begin{aligned}&\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} + \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))}\\ &\quad\le c_1\|\omega\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.\end{aligned}$$

In the case of functions instead of sections of G we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and the weights:

$$w_1(x) := R(x)^{r\delta'},\ w_2(x) := R(x)^{r\gamma'},\ w_3(x) := R(x)^{r\beta'}.$$

Proof. By the threshold hypothesis (THL2), because $\omega \in L^r([0,T+\alpha], L^2_G(M))$ there is a $u \in L^r([0,T], L^2_G(M))$ such that $Du = \omega$ and:

$$\|u\|_{L^r([0,T+\alpha],L^2_G(M))} \lesssim \|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}. \tag{6.2}$$

Hence, using Theorem 6.2, we get that the *same* u verifies:

$$\begin{aligned}\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} &+ \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))}\\ &\le c_1(k)\|Du\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2(k)\|u\|_{L^r([0,T+\alpha],L^2_G(M))}.\end{aligned}$$

So replacing Du by ω and using (6.2) we get

$$\begin{aligned}\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} &+ \|u\|_{L^r([0,T],W^{m,r}_G(M,w_2))}\\ &\le c_1\|\omega\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.\end{aligned}$$

The results for functions instead of sections of G, follow the same lines and we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and the weights:

$$w_1(x) := R(x)^{r\delta'}, \quad w_2(x) := R(x)^{r\gamma'}, \quad w_3(x) := R(x)^{r\beta'}.$$

The proof is complete. □

If we are more interested in $L^r - L^s$ estimates, we can use the Sobolev embedding Theorem with weights [2], valid here, which gives:

Theorem 6.5. *Let (M,g) be a complete Riemannian manifold. Let $w(x) := R(x)^\alpha$ and $w' := R(x)^\nu$ with $\nu := s(2+\alpha/r)$. Then $W^{m,r}_G(M,w)$ is embedded in $W^{k,s}_G(M,w')$, with $\frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0$ and:*

$$\forall u \in W^{m,r}_G(M,w), \quad \|u\|_{W^{k,s}_G(M,w')} \le C\|u\|_{W^{m,r}_G(M,w)}.$$

So, with $\frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0$ and $k = 0$ we get $\frac{1}{s} = \frac{1}{r} - \frac{m}{n} > 0$ i.e. $s = \frac{nr}{n-rm}$. So $w(x) := R(x)^b \Rightarrow w' := R(x)^\nu$ with:

$$\frac{\nu}{s} = 2 + \frac{b}{r} \Rightarrow \frac{\nu}{r} - m\frac{\nu}{n} = 2 + \frac{b}{r} \Rightarrow \nu\left(\frac{1}{r} - \frac{m}{n}\right) = \frac{2r+b}{r} \Rightarrow \nu(n-m) = n(2r+b)$$

and finally $\nu = \frac{n(2r+b)}{n-m}$. So we get:

Corollary 6.6. *Let M be a complete Riemannian manifold of class $\mathcal{C}^m$ without boundary. Moreover suppose we have (THL2). Then, on an adapted vector bundle G, with $\nu = \dfrac{n(2r+\gamma)}{n-m}$ and $w_4(x) := R(x)^{r\nu}$ and also $s = \dfrac{nr}{n-rm}$:*

$$\forall \omega \in L^r([0,T+\alpha], L^r_G(M,w_3)) \cap L^r([0,T+\alpha], L^2_G(M)),$$
$$\exists u \in L^r([0,T], L^s_G(M)) :: Du = \omega,$$

such that:

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M,w_1))} + \|u\|_{L^r([0,T],L^s_G(M,w_4))}$$
$$\leq c_1 \|\omega\|_{L^r([0,T+\alpha],L^r_G(M,w_3))} + c_2 \|\omega\|_{L^r([0,T+\alpha],L^2_G(M))},$$

with $w_1(x) := R(x)^{r\delta}$, $w_3(x) := R(x)^{r\beta}$, $w_4(x) := R(x)^{r\nu}$.

In the case of functions instead of sections of G we have the same estimates but with $R(x) = R_{m-1,\epsilon}(x)$ and

$$w_1(x) := R(x)^{r\delta'}, \quad w_3(x) := R(x)^{r\beta'}, \quad w_4(x) := R(x)^{r\nu}.$$

6.1. The heat equation

We shall consider the heat equation, $Du := \partial_t u + \Delta u = \omega$, with $\Delta := dd^* + d^*d$ the Hodge laplacian. Here we change the sign to use the standard notation with Δ essentially positive.

In this section we shall only consider the vector bundle of p-forms on the Riemannian manifold M. We denote $L^r_p(M)$ the space of p-forms in $L^r(M)$. The same for $W^{k,r}_p(M)$, the Sobolev spaces of p-forms on M.

We get that Δ, the Hodge laplacian, is a (C,θ)-elliptic operator on the p-forms in a complete Riemannian manifold, for any $\theta > 0$, because its spectrum is contained in $\mathbb{R}_+$ and

$$\forall x \in M, \quad \forall \xi_x \in T^*_x(M),\ |\xi_x| = 1,\ \left\|\Delta(x,\xi_x)^{-1}\right\| \leq C.$$

By Theorem 5.3 we also have that the (THL2) hypothesis is true in this case, so we can apply Corollary 6.4 to get:

Theorem 6.7. *Let M be a connected complete n-dimensional $\mathcal{C}^2$ Riemannian manifold without boundary. Let $Du := \partial_t u + \Delta u$ be the heat operator acting on the bundle $\Lambda^p(M)$ of p-forms on M. Let:*

$$R(x) = R_{2,\epsilon}(x), \quad w_1(x) := R(x)^{r\delta},\ w_2(x) := R(x)^{r\gamma}, \quad w_3(x) := R(x)^{r\beta},$$

with the notation in Definition 2.12 with $m = 2$. Then, for any $\alpha > 0$, $r \geq 2$, we have:

$$\forall \omega \in L^r([0,T+\alpha], L^r_p(M,w_3)) \cap L^r([0,T+\alpha], L^2_p(M)),$$
$$\exists u \in L^r([0,T], W^{2,r}_p(M)) :: Du = \omega,$$

with

$$\|\partial_t u\|_{L^r([0,T],L^r_p(M,w_1))} + \|u\|_{L^r([0,T],W^{2,r}_p(M,w_2))}$$
$$\leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_p(M,w_3))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_p(M))}.$$

In the case of functions i.e. $p = 0$, *we have the same estimates but with* $R(x) = R_{1,\epsilon}(x)$ *and*

$$w_1(x) := R(x)^{r\delta'}, \quad w_2(x) := R(x)^{r\gamma'}, \quad w_3(x) := R(x)^{r\beta'}.$$

7. Classical estimates

We shall give some examples where we have classical estimates using that $\forall x \in M,\ R_\epsilon(x) \geq \delta$, via [13, Corollary, p. 7] (see also Theorem 1.3 in the book by Hebey [12]):

Corollary 7.1. *Let* (M, g) *be a complete Riemannian manifold. Let* $m \geq 1$*; if we have the injectivity radius* $r_{inj}(x) \geq i > 0$ *and* $\forall j \leq m-1,\ \left|\nabla^j Rc_{(M,g)}(x)\right| \leq c$ *for all* $x \in M$*, then there exists a constant* $\delta >, 0$*, depending only on* n, ϵ, i, m *and* c*, such that:* $\forall x \in M,\ R_{m,\epsilon}(x) \geq \delta$.

Proof. The Theorem of Hebey and Herzlich gives that, under these hypotheses, for any $\alpha \in (0, 1)$ there exists a constant $\delta > 0$, depending only on $n, \epsilon, i, m, \alpha$ and c, such that:

$$\forall x \in M, \quad r_H(1 + \epsilon, m, \alpha)(x) \geq \delta > 0.$$

So even taking our definition with a harmonic coordinates patch, we have that:

$$R_{m,\epsilon}(x) \geq r_H(1 + \epsilon, m, \alpha)(x)$$

so a fortiori when we take the sup for $R_{m,\epsilon}(x)$ on *any* smooth coordinates patch, not necessarily harmonic patch. The proof is complete. □

7.1. Bounded geometry

Definition 7.2. A Riemannian manifold M has k-order **bounded geometry** if:

• the injectivity radius $r_{inj}(x)$ at $x \in M$ is bounded below by some constant $\delta > 0$ for any $x \in M$;

• and if for $0 \leq j \leq k$, the covariant derivatives $\nabla^j R$ of the curvature tensor are bounded in $L^\infty(M)$ norm.

We shall weakened this definition to suit our purpose.

Definition 7.3. A Riemannian manifold M has k-order **weak bounded geometry** if:

• the injectivity radius $r_{inj}(x)$ at $x \in M$ is bounded below by some constant $\delta > 0$ for any $x \in M$;

• and if for $0 \leq j \leq k$, the covariant derivatives $\nabla^j Rc$ of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm.

Using this notion, we get our main Theorem 6.2 without weights:

Theorem 7.4. *Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the adapted vector bundle $G := (H,\pi,M)$ in the complete Riemannian manifold (M,g), with $\theta < \pi/2$, and consider the parabolic equation $Du = \partial_t u - Au$ also acting on sections of G. Suppose moreover that (M,g) has $(m-1)$ order weak bounded geometry and (THL2) is true. Then*

$$\forall \omega \in L^r([0,T+\alpha], L^r_G(M)) \cap L^r([0,T+\alpha], L^2_G(M)),$$
$$\exists u \in L^r([0,T], W^{m,r}_G(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M))} + \|u\|_{L^r([0,T],W^{m,r}_G(M))}$$
$$\leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_G(M))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.$$

In the case of functions instead of sections of G we have the same estimates just supposing that (M,g) has $(m-2)$ order weak bounded geometry.

7.1.1. Examples of manifolds of bounded geometry

• Euclidean space with the standard metric has bounded geometry.

• A smooth, compact Riemannian manifold M has bounded geometry as well; both the injectivity radius and the curvature including derivatives are continuous functions, so these attain their finite minima and maxima, respectively on M. If $M \in \mathcal{C}^{m+2}$, then it has bounded geometry of order m.

• Non compact, smooth Riemannian manifolds that possess a transitive group of isomorphisms (such as the hyperbolic spaces $\mathbb{H}^n$) have m-order bounded geometry since the finite injectivity radius and curvature estimates at any single point translate to a uniform estimate for all points under isomorphisms.

Of course these examples have a fortiori weak bounded geometry.

7.2. Hyperbolic manifolds

These are manifolds such that the sectional curvature K_M is constantly -1. For them we have first that the Ricci curvature is bounded.

Lemma 7.5. *Let (M,g) be a complete Riemannian manifold such that $H \leq K_M \leq K$ for constants $H, K \in \mathbb{R}$. Then we have that $\|Rc\|_\infty \leq \max(|H|, |K|)$.*

This lemma is so well known than we can omit its proof.

To get that the injectivity radius $r_{inj}(x)$ is bounded below we shall use a Theorem by Cheeger, Gromov and Taylor [5]:

Theorem 7.6. *Let (M,g) be a complete Riemannian manifold such that $K_M \leq K$ for constants $K \in \mathbb{R}$. Let $0 < r < \frac{\pi}{4\sqrt{K}}$ if $K > 0$ and $r \in (0,\infty)$ if $K \leq 0$. Then*

the injectivity radius $r_{inj}(x)$ at x satisfies

$$r_{inj}(x) \geq r \frac{\mathrm{Vol}(B_M(x,r))}{\mathrm{Vol}(B_M(x,r)) + \mathrm{Vol}(B_{T_xM}(0,2r))} ,$$

where $B_{T_xM}(0,2r))$ denotes the volume of the ball of radius $2r$ in T_xM, where both the volume and the distance function are defined using the metric $g^* := \exp_p^* g$ i.e. the pull-back of the metric g to T_xM via the exponential map.

This Theorem leads to the definition:

Definition 7.7. Let (M,g) be a Riemannian manifold. We shall say that it has the **lifted doubling property** if we have:

$$\text{(LDP)} \quad \exists \alpha, \beta > 0 :: \forall x \in M,\ \exists r \geq \beta,\ \mathrm{Vol}(B_{T_xM}(0,2r)) \leq \alpha \mathrm{Vol}(B_M(x,r)),$$

where $B_{T_xM}(0,2r))$ denotes the volume of the ball of radius $2r$ in T_xM, and both the volume and the distance function are defined on T_xM using the metric $g^* := \exp_p^* g$ i.e. the pull-back of the metric g to T_xM via the exponential map.

Hence we get:

Corollary 7.8. *Let (M,g) be a complete Riemannian manifold such that $K_M \leq K$ for a constant $K \in \mathbb{R}$. For instance an hyperbolic manifold. Suppose moreover that (M,g), has the lifted doubling property. Then $\forall x \in M,\ r_{inj}(x) \geq \dfrac{\beta}{1+\alpha}$.*

Proof. By the (LDP) we get, for a $r \geq \beta$,

$$\mathrm{Vol}(B_{T_xM}(0,2r)) \leq \alpha \mathrm{Vol}(B_M(x,r)).$$

We apply Theorem 7.6 of Cheeger, Gromov and Taylor to get

$$r_{inj}(x) \geq r \frac{\mathrm{Vol}(B_M(x,r))}{\mathrm{Vol}(B_M(x,r)) + \mathrm{Vol}(B_{T_xM}(0,2r))}.$$

So

$$\frac{\mathrm{Vol}(B_M(x,r))}{\mathrm{Vol}(B_M(x,r)) + \mathrm{Vol}(B_{T_xM}(0,2r))} \geq \frac{1}{1+\alpha}$$

hence, because $r \geq \beta$, we get the result. □

As an example of application we get

Proposition 7.9. *Let (M,g) be a complete Riemannian manifold such that $H \leq K_M \leq K$ for constants $H, K \in \mathbb{R}$, where K_M is the sectional curvature of M. Suppose moreover that (M,g) has the lifted doubling property and that, for $0 \leq j \leq k$, the covariant derivatives $\nabla^j Rc$ of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm. Then (M,g) has weak bounded geometry of order k.*

Theorem 7.10. *Let (M,g) be a complete Riemannian manifold such that $H \leq K_M \leq K$ for constants $H, K \in \mathbb{R}$, where K_M is the sectional curvature of M. Suppose moreover that (M,g) has the lifted doubling property. Suppose that A is a (C,θ)-elliptic operator of order m acting on sections of the adapted vector bundle $G := (H,\pi,M)$ in (M,g), with $\theta < \pi/2$, and consider the parabolic equation $Du = \partial_t u - Au$ also acting on sections of G. Moreover suppose we have (THL2). Provided that, for $0 \leq j \leq m-1$, the covariant derivatives $\nabla^j Rc$ of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm:*

$$\forall \omega \in L^r([0,T+\alpha], L^r_G(M)) \cap L^r([0,T+\alpha], L^2_G(M)),$$
$$\exists u \in L^r([0,T], W^{m,r}_G(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r_G(M))} + \|u\|_{L^r([0,T],W^{m,r}_G(M))}$$
$$\leq c_1 \|\omega\|_{L^r([0,T+\alpha],L^r_G(M))} + c_2 \|\omega\|_{L^r([0,T+\alpha],L^2_G(M))}.$$

In the case of functions instead of sections of G we have the same estimates, just supposing that for $0 \leq j \leq m-2$, the covariant derivatives $\nabla^j Rc$ of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm.

Proof. By the Proposition 7.9, we have that (M,g) has weak bounded geometry of order k. So we can apply Theorem 7.4. □

And in the case of the heat equation:

Corollary 7.11. *Let (M,g) be a complete Riemannian manifold such that $H \leq K_M \leq K$ for constants $H, K \in \mathbb{R}$, where K_M is the sectional curvature of M. Suppose moreover that (M,g) has the lifted doubling property. Then $\exists \delta > 0$, $\forall x \in M$, $R_{1,\epsilon}(x) \geq \delta$. This implies that we get "classical solutions" for the heat equation for functions in this case. I.e.*

$$\forall \omega \in L^r([0,T+\alpha], L^r(M)) \cap L^r([0,T+\alpha], L^2(M)),$$
$$\exists u \in L^r([0,T], W^{2,r}(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r(M))} + \|u\|_{L^r([0,T],W^{2,r}(M))}$$
$$\leq c_1 \|\omega\|_{L^r([0,T+\alpha],L^r(M))} + c_2 \|\omega\|_{L^r([0,T+\alpha],L^2(M))}.$$

To get that $\exists \delta > 0$, $\forall x \in M$, $R_{2,\epsilon}(x) \geq \delta$, we need to ask that moreover the covariant derivatives ∇Rc of the Ricci curvature tensor are bounded in $L^\infty(M)$ norm. This is the case in particular if (M,g) is hyperbolic. This implies that we get "classical solutions" for the heat equation for p-forms in this case. I.e.

$$\forall \omega \in L^r([0,T+\alpha], L^r_p(M)) \cap L^r([0,T+\alpha], L^2_p(M)),$$
$$\exists u \in L^r([0,T], W^{2,r}_p(M)) :: Du = \omega,$$

with:

$$\|\partial_t u\|_{L^r([0,T],L^r_p(M))} + \|u\|_{L^r([0,T],W^{2,r}_p(M))} \\ \leq c_1\|\omega\|_{L^r([0,T+\alpha],L^r_p(M))} + c_2\|\omega\|_{L^r([0,T+\alpha],L^2_p(M))}.$$

Proof. By Lemma 7.5 we get that $\|Rc\|_\infty < \infty$. Then we apply Corollary 7.8. For forms we have to use the extra hypothesis on the covariant derivatives. □

Remark 7.12. In the case the hyperbolic manifold (M, g) is simply connected, then by the Hadamard Theorem [6, Theorem 3.1, p. 149], we get that the injectivity radius is ∞, so we have also the classical estimates in this case.

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