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Irreducibility of a specialization of the three dimensional Albeverio–Rabanovich representation of the pure braid group P_3

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Abstract. *We consider the Albeverio-Rabanovich linear representation π of the braid group B_3. After specializing the indeterminates used in defining the representation to non-zero complex numbers, we prove that its restriction to the pure braid group P_3 of dimension three is irreducible.*

1. Introduction

Let B_n be the braid group on n strands. Consider the pure braid group P_n, the kernel of the obvious surjective group homomorphism $B_n \to S_n$. Burau constructed a representations of B_n of degrees n and $n-1$, known as Burau and reduced Burau representations respectively [4]. The reduced Burau representation of B_n was proved to be irreducible [5].

Using Burau unitarizable representation, Albeverio presented a class of non trivial unitary representations for the braid groups B_3 and B_4 in the case where the dimension of the space is a multiple of 3. Researchers gave a great value for representations of the pure braid group P_n. M. Abdulrahim gave a necessary and sufficient condition for the irreducibility of the complex specialization of the reduced Gassner representation of the pure braid group P_n [1].

In our work, we mainly consider the irreducibility criteria of Albeverio–Rabanovich representation of the pure braid group P_3 with dimension three. In section 3, we write explicitly Albeverio-Rabanovich representation π of the braid group B_3 of dimension $(2n+m) \times (2n+m)$ [2]. In section 4, we let $m = n = 1$ and we write the images of the generators S and J of B_3 under a specialization of π, namely π_3. Then we deduce the images of σ_1 and σ_2, the standard generators of B_3, under π_3. After that, we consider the representation ϕ, the restriction of π_3 on the pure braid group P_3. In section 5, we prove that ϕ is an irreducible representation of P_3 of dimension three.

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2. Preliminaries

Defintion 2.1 ([3])**.** The braid group on n strings, B_n, is the abstract group with presentation

$$B_n = \left\{ \sigma_1, \dots, \sigma_{n-1}\, ; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \text{for } i = 1, 2, \dots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } |i-j| > 1 \end{array} \right\}.$$

The generators $\sigma_1, \dots, \sigma_{n-1}$ are called the standard generators of B_n.

Defintion 2.2 ([3])**.** The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \longrightarrow S_n$, defined by $\sigma_i \mapsto (i, i+1), 1 \leq i \leq n-1$. It has the following generators:

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, 1 \leq i < j \leq n$$

Defintion 2.3. A representation $\gamma\colon G \longrightarrow GL(V)$ is said to be irreducible if it has no non-trivial proper invariant subspaces.

3. Albeverio–Rabanovich representation of the braid group B_3

Consider the braid group B_3 and its standard generators σ_1 and σ_2. B_3 will be generated by J and S and has only one relation $S^2 = J^3$, where $J = \sigma_1\sigma_2$ and $S = \sigma_1 J$. Denote the representation of B_3 by π, where $\pi(S) = U$ and $\pi(J) = V$. Here U and V are $(2n+m) \times (2n+m)$ block matrices given by

$$U = 2\begin{pmatrix} A - I_n/2 & B & C \\ B^* & B^*A^{-1}B - I_n/2 & B^*A^{-1}C \\ C^* & C^*A^{-1}B & C^*A^{-1}C - I_m/2 \end{pmatrix}$$

and

$$V = \mathrm{diag}(I_n, \beta I_n, \beta^2 I_m).$$

We have $\beta = \sqrt[3]{1}$ is a primitive root of unity, $1 \leq m \leq n$, A and B are $n \times n$ matrices and C is an $n \times m$ matrix. We also have $V^3 = I_{2n+m}$. If $A = A^*$ and $BB^* + CC^* = A - A^2$, we get $U = U^*$ and $U^2 = I_{2n+m}$. For more details, see [2].

Proposition 3.1 ([2])**.** *If A and B are invertible, rank$(C) = m$, B^*B is a diagonal matrix with simple spectrum and every entry of A is non-zero then the Albeverio–Rabanovich representation is irreducible.*

4. Restriction of Albeverio–Rabanovich representation on the pure braid group P_3

Consider the braid group B_3 generated by S and V. Let π_3 be the specialization of the three dimensional Albeverio–Rabanovich representation π on B_3 by taking B and C non-zero real numbers and A is specialized to the value $\frac{1}{2}$. This implies that $B = B^*$, $C = C^*$ and $B^2 + C^2 = A - A^2$. For $A = \frac{1}{2}$, we have $B^2 = \frac{1}{4} - C^2$, where $-\frac{1}{2} < C < \frac{1}{2}$. Take $n = m = 1$ with $b = 2B$ and $c = 2C$. So b and c are non-zero real numbers with $b^2 + c^2 = 1$. Here we have $-1 < b, c < 1$. We get

$$\pi_3(S) = U = \begin{pmatrix} 0 & b & c \\ b & -c^2 & bc \\ c & bc & c^2 - 1 \end{pmatrix}$$

and

$$\pi_3(J) = V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}$$

Here, β is a primitive 3rd root of unity. That is , $\beta^3 = 1$.

Proposition 4.1. *The images of the standard generators σ_1 and σ_2 of the braid group B_3 under π_3 are given by:*

$$\pi_3(\sigma_1) = \begin{pmatrix} 0 & \beta^2 b & \beta c \\ b & -\beta^2 c^2 & \beta bc \\ c & \beta^2 bc & \beta(c^2 - 1) \end{pmatrix}$$

and

$$\pi_3(\sigma_2) = \begin{pmatrix} 0 & \beta b & \beta^2 c \\ \beta b & -\beta^2 c^2 & bc \\ \beta^2 c & bc & \beta(c^2 - 1) \end{pmatrix},$$

where b and c non are zero-real numbers with $b^2 + c^2 = 1$, $-1 < b, c < 1$, $\beta \neq 1$, and $\beta^3 = 1$.

Now, we apply Albeverio–Rabanovich representation π_3 on the pure braid group P_3. We get the following representation of dimension $d = 3$

Proposition 4.2. *Let ϕ be the restriction of Albeverio–Rabanovich representation π_3 on the pure braid group P_3. Thus ϕ is defined as follows:*

$$\phi(A_{12}) = \begin{pmatrix} I & cK & -\beta^2 bK \\ \beta cK & I + bcK & \beta^2 LK \\ -\beta bK & LK & I - \beta^2 bcK \end{pmatrix}$$

and

$$\phi(A_{23}) = \begin{pmatrix} I & \beta^2 cK & -bK \\ \beta^2 cK & I + bcK & \beta LK \\ -bK & \beta LK & I - \beta^2 bcK \end{pmatrix},$$

where $I = \beta c^2 + \beta^2 b^2$, $K = (1-\beta)bc$, $L = \beta^2 + c^2$.

Also, b *and* c *are non-zero real numbers with* $b^2 + c^2 = 1$, $-1 < b, c < 1$, $\beta \neq 1$, *and* $\beta^3 = 1$.

Since $\phi(A_{13}) = \phi(\sigma_2 \sigma_1^2 \sigma_2^{-1}) = \phi(A_{12})^{-1}$, it follows that $\phi(A_{13})$ and $\phi(A_{12})$ have the same invariant subspaces.

5. Irreducibility of Albeverio–Rabanovich representation of the pure braid group P_3 of dimension three

In this section, we prove that Albeverio–Rabanovich representation ϕ of the pure braid group P_3 of dimension three is irreducible.

Theorem 5.1. *Albeverio–Rabanovich representation* $\phi\colon P_3 \longrightarrow GL_3(\mathbb{C})$ *is irreducible.*

Proof. To get contradiction, suppose that this representation $\phi\colon P_3 \longrightarrow GL_3(C)$ is reducible. That is, there exists a proper non-zero invariant subspace T, of dimension 1 or 2. It is clear that ϕ is unitary [2]. For a unitary representation, the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. We then assume that T is one dimensional subspace generated by a vector v. We use e_1, e_2, and e_3 as the canonical basis of $\mathbb{C}^3$. It is easy to see that I, K, and L are different from zero. We consider five cases.

- *Case 1:* Let $v = e_3$, it follows that $\beta\phi(A_{12})e_3 - \phi(A_{23})e_3 \in T$. Then $LK(1-\beta)e_2 + (I - \beta^2 bcK)(\beta - 1)e_3 \in T$. Hence $e_2 \in T$. Also, we have $\phi(A_{12})e_3 - \beta\phi(A_{23})e_3 \in T$. Then

$$-\beta bK(\beta - 1)e_1 + (I - \beta^2 bcK)(1-\beta)e_3 \in T.$$

Hence $e_1 \in T$. Thus $T = \mathbb{C}^3$, a contradiction.

- *Case 2:* Let $v = e_1 + \alpha e_2$ and $\alpha \in \mathbb{C}$. It follows that $\phi(A_{12})v = a_1 v$ for some $a_1 \in \mathbb{C}^*$. Then

$$(I + \alpha cK)e_1 + (\beta cK + \alpha(I + bcK))e_2 + (-\beta bK + \alpha LK)e_3 = a_1(e_1 + \alpha e_2).$$

Hence $-\beta b + \alpha L = 0$. So $\alpha L = \beta b$. Also, there exists $a_2 \in \mathbb{C}^*$ such that $\phi(A_{23})v = a_2 v$. Then

$$(I + \alpha\beta^2 cK)e_1 + (\beta^2 cK + \alpha(I + bcK))e_2 + (-bK + \alpha\beta LK)e_3 = a_2(e_1 + \alpha e_2).$$

Hence $-b + \alpha\beta L = 0$. But, as we mention above, $\alpha L = \beta b$. Thus $b = 0$, a contradiction.

- *Case 3:* Let $v = e_1 + \alpha e_3$ and $\alpha \in \mathbb{C}$. It follows that $\phi(A_{12})v = a_1 v$ for some $a_1 \in \mathbb{C}^*$. Then $cK + \alpha\beta LK = 0$. So $\alpha\beta L = -c$. Also, there exists $a_2 \in \mathbb{C}^*$ such that $\phi(A_{23})v = a_2 v$. Then $\beta^2 c + \alpha\beta L = 0$. But $\alpha\beta L = -c$. Thus $c = 0$, a contradiction.

- *Case 4:* Let $v = e_2 + \alpha e_3$ and $\alpha \in \mathbb{C}$. It follows that $\phi(A_{12})v = a_1 v$ for some $a_1 \in \mathbb{C}^*$. Then $cK - \alpha\beta^2 bK = 0$. So $\alpha\beta^2 b = c$. Also, there exists $a_2 \in \mathbb{C}^*$ such that $\phi(A_{23})v = a_2 v$. Then $\beta^2 c - \alpha b = 0$. But $\alpha\beta^2 b = c$. Thus $c = 0$, a contradiction.

- *Case 5:* Let $v = e_1 + \alpha_1 e_2 + \alpha_2 e_3$ and $\alpha_1,\ \alpha_2 \in \mathbb{C}^*$. It follows that $\frac{1}{\beta-1}(\phi(A_{23})v - \phi(A_{12})v) = n_1 v$ for some $n_1 \in \mathbb{C}^*$. We have

$$\frac{1}{\beta-1}(\phi(A_{23}) - \phi(A_{12})) = \begin{pmatrix} 0 & -\beta^2 cK & -\beta^2 bK \\ \beta cK & 0 & -\beta LK \\ bK & LK & 0 \end{pmatrix}.$$

Thus we obtain the following equations:

$$-\alpha_1\beta^2 cK - \alpha_2\beta^2 bK = n_1 \tag{5.1}$$

$$\beta cK - \alpha_2\beta LK = n_1\alpha_1 \tag{5.2}$$

$$bK + \alpha_1 LK = n_1\alpha_2 \tag{5.3}$$

Now, using (5.1) and (5.2), we get $-\beta^2\alpha_1^2 cK - \alpha_1\alpha_2\beta^2 bK - \beta cK + \alpha_2\beta LK = 0$. Hence,

$$(-\beta\alpha_1 b + L)\alpha_2 = \alpha_1^2 c\beta + c. \tag{5.4}$$

Using (5.1) and (5.4) we get

$$(-\beta\alpha_1 b + L)n_1 = -\beta^2 cK\alpha_1(-\beta\alpha_1 b + L) - \beta^2 bcK(\alpha_1^2\beta + 1). \tag{5.5}$$

After multiplying (5.3) by $(-\alpha_1\beta bK + LK)^2$ and using (5.4) and (5.5), we get

$$L(\beta^2 b^2 + c^2)\alpha_1^3 + b(\beta^2 b^2 - 2\beta L^2 + c^2)\alpha_1^2 + L(-2\beta b^2 + \beta^2 c^2 + L^2)\alpha_1 + b(L^2 + \beta^2 c^2) = 0. \tag{5.6}$$

Similarly, there exists $n_2 \in \mathbb{C}$ such that $\dfrac{1}{1-\beta^2}(\phi(A_{23})v - \beta^2\phi(A_{12})v) = n_2 v$. We have

$$\frac{1}{1-\beta^2}(\phi(A_{23}) - \beta^2\phi(A_{12})) = \begin{pmatrix} I & 0 & \beta bK \\ -cK & I + bcK & 0 \\ 0 & -\beta^2 KL & I - \beta^2 bcK \end{pmatrix}.$$

Thus we obtain the following equations:

$$I + \beta\alpha_2 bK = n_2 \tag{5.7}$$

$$-cK + \alpha_1(I + bcK) = n_2\alpha_1 \tag{5.8}$$

$$-\beta^2 LK\alpha_1 + \alpha_2(I - \beta^2 bcK)_= n_2\alpha_2 \tag{5.9}$$

The equations (5.7) and (5.8) give

$$-\alpha_1\alpha_2\beta bK = \alpha_1 I + cK - \alpha_1(I + bcK) \tag{5.10}$$

Now, using (5.8), (5.9), and (5.10), we get

$$bL\alpha_1^3 - \beta b^2c^2\alpha_1^2 + (\beta - 1)bc^2\alpha_1 + c^2 = 0. \tag{5.11}$$

Likewise, there exists $n_3 \in \mathbb{C}$ such that $\frac{1}{1-\beta}(\phi(A_{23})v - \beta\phi(A_{12})v) = n_3 v$. Then

$$\frac{1}{1-\beta}(\phi(A_{23}) - \beta\phi(A_{12})) = \begin{pmatrix} I & -\beta cK & 0 \\ 0 & I + bcK & -LK \\ \beta^2 bK & 0 & I - \beta^2 bcK \end{pmatrix}.$$

Consider the following equations:

$$I - \beta\alpha_1 cK = n_3 \tag{5.12}$$

$$\alpha_1(I + bcK) - \alpha_2 LK = n_3\alpha_1 \tag{5.13}$$

$$\beta^2 bK + \alpha_2(I - \beta^2 bcK) = n_3\alpha_2 \tag{5.14}$$

The equations (5.12) and (5.13) give

$$\alpha_2 L = \beta c\alpha_1^2 + bc\alpha_1. \tag{5.15}$$

Using (5.12), (5.14), and (5.15) we get

$$c^2\alpha_1^3 + \beta(\beta - 1)bc^2\alpha_1^2 - b^2c^2\alpha_1 + bL = 0. \tag{5.16}$$

Now, using (5.11) and (5.16) we get

$$a_1\alpha_1^2 + b_1\alpha_1 + c_1 = 0. \tag{5.17}$$

Also, using (5.16) and (5.6) we get

$$a_2\alpha_1^2 + b_2\alpha_1 + c_2 = 0, \tag{5.18}$$

where

$$\begin{aligned} a_1 &= b^2c^2(\beta^2c^2+\beta-1),\\ b_1 &= bc^2(c^4-(1+2\beta)c^2-\beta^2),\\ c_1 &= -c^6+2(\beta+1)c^4-(3\beta+2)c^2+\beta. \end{aligned}$$

and

$$\begin{aligned} a_2 &= bc^2(-\beta c^4-2c^2+1),\\ b_2 &= Lc^2(-\beta^2c^4-\beta c^2+2\beta+1),\\ c_2 &= b(-\beta^2c^6-2\beta c^4+(2\beta-1)c^2+1). \end{aligned}$$

The equations (5.17) and (5.18) give

$$\alpha_1 = \frac{-2\beta c^2+2\beta c^4-\beta c^6+c^2-c^6-1}{bc^2+bc^4-2\beta bc^2+\beta bc^4}.$$

Substituting the obtained value of α_1 in (5.15), we get

$$\alpha_2 = \frac{4\beta c^2-\beta+2\beta c^4-12\beta c^6+12\beta c^8-4\beta c^{10}+5c^2-9c^4+3c^6+6c^8-5c^{10}+c^{12}}{17\beta c^5-3\beta c^3-16\beta c^7+\beta c^9+\beta c^{11}+5c^3+2c^5-14c^7+7c^9}.$$

Now, substituting the obtained values of α_1 and α_2 in (5.1) we obtain

$$n_1 = \frac{(1-\beta^2)(\beta(3-5c^2+4c^4+5c^6-17c^8+16c^{10}-2c^{12}-c^{14})-9c^2+12c^4-16c^6+2c^8+10c^{10}-7c^{12}+2)}{c^2(\beta c^2-2\beta+c^2+1)(3\beta-14\beta c^2+2\beta c^4+\beta c^6-7c^2+7c^4-5)}.$$

After substituting the obtained values of α_1, α_2, and n_1 in (5.2), we obtain

$$f\beta+g=0,$$

where

$$\begin{aligned} f &= 58c^4-15c^2-1-39c^6-108c^8+258c^{10}-130c^{12}-50c^{14}+29c^{16},\\ g &= 7c^2+48c^4-128c^6+115c^8+96c^{10}-204c^{12}+55c^{14}+20c^{16}-3c^{18}-3. \end{aligned}$$

Here β is a third root of unity ($\beta \neq 1$) and f, g are real-valued polynomials. Since β and 1 are linearly independent, we get $f = g = 0$. We then use the computational software MATLAB to solve $f = g = 0$.

Solving $f = 0$, we get 14 rejected solutions and two accepted solutions, which are ± 0.69 (rounded to the nearest hundredth).

Solving $g = 0$, so we get 16 rejected solutions and two accepted solutions, which are ± 0.48 (rounded to the nearest hundredth).

We observe that there is no common solution and we thus obtain a contradiction. □

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