Rend. Mat. Appl. (7).
Volume 41 (2020), 163 – 172

Rendiconti di Matematica
e delle sue Applicazioni

An Application of the Index Theorem for Manifolds with Fibered Boundaries

Andrés Larraín-Hubach

Abstract. *We show how the index formula for manifolds with fibered boundaries can be used to compute the index of the Dirac operator on Taub-NUT space twisted by an anti-self-dual generic instanton connection.*

1. Introduction

This note derives the L^2-index theorem, first proved in [4], for a Dirac operator twisted by an Anti-Self-Dual (ASD) generic instanton on Taub-NUT space from the index formula on manifolds with fibered boundaries. The original motivation for this index problem was to establish the completeness of the bow construction of Sergey Cherkis [3]. This construction conjectured an isometry between the moduli space of bow representations and the moduli space of generic ASD-instantons on Taub-NUT space. In [4, 5] and [6] the isometry between these moduli spaces is proved.

An important technical step in verifying this conjecture is the index theorem mentioned above. The original proof used an adaptation of the powerful machinery developed by Mark Stern [10] to obtain index formulas on open spaces. The main advantages of this technique is that the argument is self-contained and avoids the use of spectral theory by computing asymptotic information instead. It is important to mention that the original proof of this index theorem applies not only to Taub-NUT space but also to the family of multi-centered Taub-NUT spaces. The statement is

Theorem ([4, Thm 43])**.** Let $\mathcal{A}$ be a generic instanton on the Taub-NUT space, then the index of the Dirac operator coupled to $\mathcal{A}$ is given by

$$\begin{aligned}\operatorname{Ind}_{L^2}\mathcal{D}_g^+ = -\frac{1}{8\pi^2}\int_{TN} \operatorname{tr} F_{\mathcal{A}}\wedge F_{\mathcal{A}} + \frac{1}{2\pi i}\int_{S^2_\infty} \operatorname{tr}_E(\{\Lambda\} - \frac{1}{2})F^E \\ - \frac{1}{2}\operatorname{tr}_E(\{\Lambda\}^2 - \{\Lambda\}).\end{aligned} \tag{1.1}$$

where $m = \operatorname{Rank}(\mathcal{E})$ and Λ is a diagonal matrix whose constant entries are related to the eigenvalues at infinity of the holonomy of $\mathcal{A}$ along the circle fibers. $\{\Lambda\}$ is the matrix Λ with its entries replaced by their corresponding fractional parts.

2020 Mathematics Subject Classification: 58J20, 53C07.
Keywords: Index Theorem, Instanton, Fibered Boundaries.

Here we show a different derivation of the theorem above using one of the index theorems for spaces with fibered boundaries (see theorem 2.3) proved in [8]. The authors of [8] impose a spectral gap condition (2.5) on the vertical boundary family. A direct proof of a more general index formula is given in [11] under the weaker assumption that the vertical boundary family has null-spaces of fixed dimension. The theorems in [8] and [11] express the index of a Dirac type operator as a sum of two terms usually called bulk and asymptotic contribution. The bulk is the usual Atiyah-Singer integrand [1], while the asymptotic contribution is generally given in terms of η-invariants of Dirac-type operators restricted to the boundary. In our case the index theorem for exact-d-metrics of [8] and [11] applies, and the boundary contribution, given in terms of the Bismut-Cheeger $\hat{\eta}$-form [2], can be computed explicitly.

2. Index Theory on Spaces with Fibered Boundaries

We start reviewing the statement of the index theorem used in this paper.

2.1. The Bismut-Cheeger $\hat{\eta}$-form.

Let $\pi\colon M \to B$ be a locally trivial fibration such that the vertical tangent bundle TM/B is spin, with base an even dimensional manifold B, and fibers isomorphic to a closed odd-dimensional manifold Z. We assume there is a connection on the fibration that induces a splitting $TM = T_H M \oplus TM/B$ into horizontal and vertical tangent vectors, such that $\pi^* TB$ can be identified with $T_H M$. Let $g^M = \pi^* g^B \oplus g^{M/B}$ be a Riemannian submersion metric, where g^B is a metric on TB pulled back to $T_H M$, and $g^{M/B}$ denotes a metric on the vertical fibers.

Let $E \to M$ be a complex vector bundle with unitary connection ∇^E and curvature F^E. The bundle E induces an infinite rank bundle $\pi_* E \to B$ with fibers given by $\Gamma(M_x, E_x)$, where M_x, E_x denote the fibers over $x \in B$. The connection ∇^E induces a connection on $\pi_* E$ denoted by $\nabla^{\pi_* E}$. See [1, Ch. 10].

We denote by $(\mathcal{S}^{M/B}, \nabla^{M/B})$ the vertical spinor bundle together with its induced spin connection coming from the metric $g^{M/B}$. We denote by $c = c^{M/B}$ the Clifford product by elements of T^*M/B. We use $\nabla^{\mathcal{S}^{M/B}\otimes E} = \nabla^{M/B} \otimes 1 + 1 \otimes \nabla^E$ and the Clifford module structure on $\mathcal{S}^{M/B} \otimes E$ with respect to the Clifford algebra of TM/B to construct a family of vertical Dirac operators denoted by $D^{M/B} = c^{M/B} \circ \nabla^{\mathcal{S}^{M/B}\otimes E}$.

Definition 2.1. Let u be a positive parameter, the Bismut superconnection, acting on $\Gamma(M, \mathcal{S}^{M/B} \otimes E) = \Gamma(B, \pi_*(\mathcal{S}^{M/B} \otimes E))$, is defined by

$$\mathfrak{A}_u = \nabla^{\pi_*(\mathcal{S}^{M/B}\otimes E)} + \sqrt{u} D^{M/B} - \frac{c(T)}{4\sqrt{u}}, \tag{2.1}$$

where T is the torsion form of the fibration $M \to B$ [1, Prop. 10.15].

Definition 2.2. The Bismut-Cheeger Eta form of the vertical family of Dirac operators $D^{M/B}$ is defined by

$$\hat{\eta}(D^{M/B}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}^{\mathrm{ev}}\big((D^{M/B} + \frac{c(T)}{4u})e^{-\mathfrak{A}_u^2}\big)\frac{du}{2\sqrt{u}}. \tag{2.2}$$

See [2, Def. 4.93] for an explanation of the notation $\mathrm{Tr}^{\mathrm{ev}}$.

2.2. The Index Formula

Consider a Riemannian manifold $(\mathcal{M}, g^{\mathcal{M}})$ such that its boundary $M = \partial\mathcal{M}$ is the total space of a fibration $\pi\colon M \to B$ like the one in Section 2.1.

We assume that, on a tubular neighborhood of the boundary $(a,\infty)_y \times M$, the metric takes the form

$$g^{\mathcal{M}} = dy^2 + \pi^* g^B + e^{-2y} g^{M/B}. \tag{2.3}$$

In the terminology of [8], this is an exact d-metric with boundary defining function $x = e^{-y}$.

Let

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \tag{2.4}$$

be a Dirac type operator on $\mathcal{S} \otimes \mathcal{E} \to \mathcal{M}$ (here $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ denotes the spin bundle), such that its boundary family $D^{M/B}$, acting on $\mathcal{S}^{M/B} \otimes E$ with $\mathcal{E}|_M = E$, satisfies the spectral gap assumption

$$\mathrm{Spec}(D_b^{M/B}) \cap (-\delta, \delta) = \emptyset, \tag{2.5}$$

for some $\delta > 0$ and for every $b \in B$.

The following index theorem is proved in [8] and [11] using different arguments.

Theorem 2.3. *If $D^{M/B}$ satisfies assumption (2.5) then*

$$\mathrm{Ind}_{L^2}(\mathcal{D}^+) = \int_{\mathcal{M}} \hat{A}(\mathcal{M}, g^{\mathcal{M}}) \wedge \mathrm{Ch}(\mathcal{E}) - \frac{1}{2\pi i}\int_{\partial\mathcal{M}} \hat{A}(B, g^B) \wedge \hat{\eta}(D^{M/B}), \tag{2.6}$$

where $\hat{A}$ denotes the A-hat genus of the corresponding space with metric $g^{\mathcal{M}}$. The $\hat{\eta}$-form is computed with respect to the submersion metric $\pi^* g^B + g^{M/B}$.

The $2\pi i$ factor does not appear in the original formula due to the different normalizations used. See [7].

For our purposes, the manifold $\mathcal{M}$ is four-dimensional. For this reason, we only need the four-form terms in $\hat{A} \wedge \mathrm{Ch}(\mathcal{E})$ (see [1, Chp. 1]) giving

$$\int_{\mathcal{M}} \hat{A}(\mathcal{M}, g^{\mathcal{M}}) \wedge \mathrm{Ch}(\mathcal{E}) = \int_{\mathcal{M}} \left(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{tr}\,\mathcal{R}_{g^{\mathcal{M}}} \wedge \mathcal{R}_{g^{\mathcal{M}}} - \frac{1}{8\pi^2}\mathrm{tr}\,F_{\mathcal{A}} \wedge F_{\mathcal{A}}\right) \tag{2.7}$$

2.3. Circle Fibrations

We only need to compute $\hat{\eta}$ in the case where $M \to B$ is a S^1-principal bundle with a Riemannian submersion metric. Several formulas simplify in this context.

Let $\{f_\alpha\}$ (resp. $\{f^\alpha\}$) denote an orthonormal frame (coframe) on B and $\{e\}, \{e^*\}$ similarly defined on the vertical fibers M/B. We use $\{\tilde{f}_\alpha\}$ to denote the horizontal lifts to $T_H M$. Set $c(e^*) = -i$ and denote $\mathcal{S}^{M/B}$ by $\mathcal{S}^{S^1}$.

The torsion form on a circle fibration are computed explicitly in [9, Sec. 5]. The result is

$$T(\tilde{f}_\alpha, \tilde{f}_\beta) = de^*(\tilde{f}_\alpha, \tilde{f}_\beta) = R(f_\alpha, f_\beta), \tag{2.8}$$

where R is the curvature of the S^1-connection inducing the splitting $TM = T_H M \oplus TB$.

Let z denote a Grassmann variable i.e. $z^2 = 0$. A well-known trick by Bismut and Cheeger rewrites $\mathfrak{A}_u^2$ as follows

$$\begin{aligned} -u\big(\nabla_e^{\pi_*(\mathcal{S}^{S^1} \otimes E)} + \frac{R}{4u} - \frac{iz}{2\sqrt{u}}\big)^2 + \sqrt{u} F^E(f_\alpha, e) f^\alpha \wedge e^* + \frac{1}{2} F^E(f_\alpha, f_\beta) f^\alpha \wedge f^\beta \\ = \mathfrak{A}_u^2 - z(\sqrt{u} D^{M/B} + \frac{c(T)}{4\sqrt{u}}) \end{aligned} \tag{2.9}$$

This is just [2, 4.68–4.70] adapted to circle fibrations. An important simplification of the original formula is that the scalar curvature of the circle fibers vanishes.

3. Taub-NUT Space

In this section we state the necessary definitions and results from [4].

3.1. Definition and Basic Properties

The single-centered Taub-NUT space, denoted by TN henceforth, is a hyperKähler 4-manifold that, outside a compact set, is a circle fibration over $\mathbb{R}^3$. It has coordinates $\{x_1, x_2, x_3, \tau\}$, where the x_j parameterize $\mathbb{R}^3$ and $\tau = \tau + 2\pi$ parameterizes the circle fiber. We use the orientation $dx_1 \wedge dx_2 \wedge dx_3 \wedge d\tau$. Define the TN metric, denoted by g_{TN}, by[1]

$$g_{\mathrm{TN}} = V(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{V}(d\tau + \omega)^2, \tag{3.1}$$

where $V = 1 + \frac{1}{2r}$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and ω is a one-form such that $\star_3 dV = d\omega$. Here $\star_3$ denotes the Hodge start in $\mathbb{R}^3$.

Using polar coordinates about the origin, we can rewrite the $\mathbb{R}^3$-metric as $dr^2 + r^2 g_{S^2}$, where g_{S^2} is the usual round metric on the two-sphere.

[1] Usually, $V = l + \frac{1}{2r}$ for some fixed $l > 0$. We simplify the computations by setting $l = 1$.

3.2. Modifications of the Metric

In order to use the index formula of [8] and [11], we first need to modify the metric on TN using a special homotopy of metrics. The corresponding twisted Dirac operators associated to the interpolating metrics in this homotopy remain Fredholm so the index is preserved (see Lemma 3.2). The metric at the end of the homotopy is of the type defined in Section 2.2.

First, we apply a conformal transformation to g_{TN} with a conformal factor that equals $e^{2u} = V^{-1}r^{-2}$ for r large. We consider e^{2u} to be equal to the identity around the origin. The resulting metric, near infinity and using the change of variables $r = e^y$, equals

$$g' = dy^2 + g_{S^2} + \frac{1}{V^2 e^{2y}}(d\tau + \omega)^2. \tag{3.2}$$

Now we use a smooth homotopy between g' and a metric g. The homotopy replaces V by $V_t = 1 + \frac{t}{2e^y}$ for y-large and it equals the identity near the origin. The resulting metric at $t = 0$, for y large, has the form

$$g = dy^2 + \pi^* g_{S^2} + \frac{1}{e^{2y}}(d\tau + \omega)^2 \tag{3.3}$$

and it is an exact d-metric [11]. The boundary fibration in this case is the Hopf fibration $\pi\colon S^3_\infty \to S^2_\infty$.

Lemma 3.1. *Let $\mathcal{R}_g$ be the Riemannian curvature of the metric g, then*

$$\frac{1}{192\pi^2}\int_{TN} \operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g = \frac{1}{12}\,.$$

Proof. The space TN with metric g is contractible and therefore topologically trivial. This implies that its tangent bundle has a global trivialization. Fixing a trivialization, we can define the Chern-Simons form $CS(g)$ such that $dCS(g) = \frac{1}{192\pi^2}\operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g$.

Let B be a ball around the origin such that the metrics g_{TN} and g coincide in on an open neighborhood of it. Then, as in [4, Lemma 32], we get

$$\begin{aligned}
\frac{1}{192\pi^2}\int \operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g &= \frac{1}{192\pi^2}\int_B \operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g + \frac{1}{192\pi^2}\int_{\mathrm{TN}\setminus B} \operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g \\
&= \frac{1}{192\pi^2}\int_B \operatorname{tr} \mathcal{R}_{g_{\mathrm{TN}}} \wedge \mathcal{R}_{g_{\mathrm{TN}}} + \frac{1}{192\pi^2}\int_{\mathrm{TN}\setminus B} \operatorname{tr} \mathcal{R}_g \wedge \mathcal{R}_g \\
&= -\frac{1}{192\pi}\int_{S^2_0} \nabla_n\Big(\frac{|\nabla V|^2}{V^3}\Big)\mathrm{vol}_{S^2_0} + \int_{S^3_\infty} CS(g),
\end{aligned}$$

where S^2_0 is a small two-sphere around the origin with outward normal vector n. The form $CS(g)$ decays exponentially and the last boundary integral equals zero. The first summand equals $\frac{1}{12}$. □

3.3. Generic ASD Instantons on TN

Let $\mathcal{E} \to \mathrm{TN}$ be a unitary bundle of rank m. A generic Anti-Self-Dual (ASD) instanton on it, is a unitary connection $\mathcal{A}$ on $\mathcal{E}$ such that its L^2-curvature form satisfies $F_{\mathcal{A}} = -\star_{\mathrm{TN}} F_{\mathcal{A}}$. Here $\star_{\mathrm{TN}}$ is the Hodge star of the original metric g_{TN}. The genericity is a technical condition explained in [4]. In [4, thm B] it is proved that there is a frame of $\mathcal{E}$ such that an ASD generic instanton on TN has the following asymptotic form outside a compact set

$$\mathcal{A} = -i\,\mathrm{Diag}\left((\lambda_j + \frac{m_j}{2r})\frac{d\tau + \omega}{V} + \eta_j \right) + \mathcal{O}(r^{-2}), \tag{3.4}$$

where η_j is a connection one-form on a complex line bundle $W(j)$ over S^2. The λ_j are related to the asymptotic eigenvalues of the holonomy of $\mathcal{A}$ along τ-circles and they are pairwise distinct and constant. Notice that, near infinity, the bundle $\mathcal{E}$ decomposes as a direct sum of eigenline-bundles of the holonomy of $\mathcal{A}$.

We assume a stronger genericity assumption by imposing

$$e^{2\pi i \lambda_j} \neq 1, \tag{3.5}$$

for every j. This is required to guarantee the Fredholmness of the Dirac operator twisted by $\mathcal{A}$. See [4, Sec. 7].

3.4. The Twisted Dirac Operator

Let $\mathcal{D}_{\mathcal{A}}$ be the twisted Dirac operator with respect to the metric g_{TN}. We write $\mathcal{D}_{g'}$ and $\mathcal{D}_g$ to denote the corresponding twisted Dirac operators with respect to the other metrics[2]. Let $\{\mathcal{D}_t\}_{0\leq t\leq 1}$ be the family of Dirac operators associated to the metrics in the homotopy between g' and g.

All of these operators admit a decomposition according to chirality as

$$\mathcal{D}_t = \begin{pmatrix} 0 & \mathcal{D}_t^- \\ \mathcal{D}_t^+ & 0 \end{pmatrix}, \tag{3.6}$$

where $\mathcal{D}_t^{\pm} \colon \Gamma(\mathcal{S}^{\pm} \otimes \mathcal{E}) \to \Gamma(\mathcal{S}^{\mp} \otimes \mathcal{E})$. Here we denote by $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the spinor bundle of TN.

The L^2-indices of these operators turn out to be the same

Lemma 3.2.

$$\mathrm{Ind}_{L^2}\mathcal{D}_A^+ = \mathrm{Ind}_{L^2}\mathcal{D}_{g'}^+ = \mathrm{Ind}_{L^2}\mathcal{D}_g^+ \tag{3.7}$$

Proof. The first equality is proved in [4, Sec. 7]. For the second one, since the λ_j are constant and satisfy assumption (3.5), the $\mathcal{D}_t$ satisfy the hypotheses of lemma 22 in [4] for every $0 \leq t \leq 1$. This implies that $\{\mathcal{D}_t\}$ is a homotopy within the space of Fredholm operators so the index is preserved.

□

[2] To simplify notation, we set $\mathcal{D}_g = \mathcal{D}_g^{\mathcal{A}}$ etc...

From now on, we only use the operator $\mathcal{D}_g$ and its chirality components $\mathcal{D}_g^{\pm}$.

The corresponding vertical family of Dirac operators, parameterized by points $p \in S^2_\infty$, is

$$D^{M/B} = D_g^{\partial} = \oplus_{j=1}^m D^{\partial,\lambda_j} = \oplus_{j=1}^m (-i)(\partial_\tau - i\lambda_j). \tag{3.8}$$

Remember that here we use the submersion metric $g_{S^3_\infty} = g_{S^2_\infty} + (d\tau + \omega)^2$ on S^3_∞. Our boundary family of operators satisfies the following spectral gap:

Lemma 3.3. *Consider the family $\{D^{\partial}_{g,p}\}_{p\in S^2_\infty}$ defined in (3.8), then there is a $\delta > 0$ such that, for every $p \in S^2_\infty$, we have Spec$\{D^{\partial}_{g,p}\} \cap (-\delta, \delta) = \emptyset$.*

Proof. Given $p \in S^2_\infty$, the boundary operator D^{∂}_p is a multiple of a direct sum of operators of the form $-i(\partial_\tau - i\lambda_j)$. Since $e^{2\pi i\lambda_j} \neq 1$, the operator is invertible. The λ_j are constant so we can take $\delta = \frac{1}{2}\min_j|\lambda_j|$. □

4. The Index of $\mathcal{D}_g^+$

From the previous sections we see that $\mathcal{D}_g$ satisfies the assumptions of the index formula in [8] and [11]. In our case, since TN is a four-dimensional manifold, formulas (2.6) and (2.7) give

$$\mathrm{Ind}_{L^2}\mathcal{D}_g^+ = \int_{TN}\left(\frac{\mathrm{Rank}(\mathcal{E})}{192\pi^2}\mathrm{tr}\,\mathcal{R}_g\wedge\mathcal{R}_g - \frac{1}{8\pi^2}\mathrm{tr}\,F_{\mathcal{A}}\wedge F_{\mathcal{A}}\right) - \frac{1}{2\pi i}\int_{S^2_\infty}\hat{\eta}\,, \tag{4.1}$$

where $\hat{\eta} \in \Omega^{\mathrm{ev}}(S^2_\infty)$ is an even form called the Bismut-Cheeger $\hat{\eta}$-form, computed with respect to the metric $g_{S^2} + (d\tau + \omega)^2$ on the boundary fibration $S^3_\infty \to S^2_\infty$.

It remains to compute $\hat{\eta}$ in terms of the asymptotic form of the twisting connection $\mathcal{A}|_{S^3_\infty} = A$.

4.1. Explicit Computation

Again, the boundary fibration on TN equals the Hopf fibration $S^3 \to S^2$ with metric $g_{S^2} \oplus (d\tau + \omega)^2$. Notice that $d\tau + \omega$ is a connection one-form for the fibration so the curvature equals

$$R = d\omega = -\frac{1}{2}\mathrm{vol}_{S^2_\infty}\,, \tag{4.2}$$

where $\mathrm{vol}_{S^2_\infty}$ is the volume form on the two-sphere.

The corresponding bundle $\mathcal{E}|_{S^3_\infty} = E \to S^3_\infty$ is the restriction of the instanton bundle $\mathcal{E}$ to the boundary fibration. The bundle E inherits a connection of the form

$$A = -i\,\mathrm{Diag}\big(\lambda_j(d\tau + \omega) + \pi^*(\eta_j)\big)\,. \tag{4.3}$$

This splitting of A and the results of [4] imply that $E \to S^3_\infty$ can be decomposed as a direct sum

$$E = \oplus_{j=1}^m \pi^* W(j), \tag{4.4}$$

where the $W(j)$ are line bundles over the base S^2_∞ with connection form η_j. Note that we identify $e^* = (d\tau + \omega)$.

The Bismut superconnection in this case equals

$$\mathfrak{A}_u = \oplus_{j=1}^{m} \big(\nabla^{\pi_*(\mathcal{S}^{S^1} \otimes W(j))} + \sqrt{u} D^{\partial, \lambda_j}\big) - \frac{c(T)}{4\sqrt{u}} \tag{4.5}$$

We simplify (2.9) further to

$$-u\big(\nabla_e + \frac{R}{4u} - \frac{iz}{2\sqrt{u}}\big)^2 + F^E(f_1, f_2) f^1 \wedge f^2 = \mathfrak{A}_u^2 - z\big(\sqrt{u} D^{M/B} + \frac{c(T)}{4\sqrt{u}}\big), \tag{4.6}$$

where $\{f_1, f_2\}$ is an oriented orthonormal frame on $S^2_\infty = S^2$ and F^E is the curvature of A. Notice that $F^E(e, f_\alpha) = 0$ for $\alpha = 1, 2$.

The identities (2.9) and (4.6) imply the following lemma.

Lemma 4.1. *Let* $\operatorname{Tr}^z(a + zb) = \operatorname{Tr} b$ *then*

$$\operatorname{Tr}^{\mathrm{ev}} \left(D^\partial + \frac{c(T)}{4u} \right) e^{-\mathfrak{A}_u^2} = u^{-1/2} \operatorname{Tr}^z \exp \left(u(\nabla_e + \frac{R}{4u} - \frac{iz}{2\sqrt{u}})^2 - F^E \right), \tag{4.7}$$

where $F^E = F^{\mathcal{E}}(f_1, f_2) f^1 \wedge f^2$.

Proof. Exponentiate both sides of (4.6) and notice that $\operatorname{Tr}^z(e^{a+bz}) = \operatorname{Tr} e^a b = \operatorname{Tr} b e^a$. □

Replacing (4.7) in the definition of $\hat{\eta}$ gives

$$\hat{\eta}(D^\partial_g) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr}^z \exp\left(u(\nabla_e + \frac{R}{4u} - \frac{iz}{2\sqrt{u}})^2 \right) \frac{du}{2u}\, e^{-F^E} \tag{4.8}$$

Let $\Lambda = \operatorname{Diag}(\lambda_j)$. In order to compute Tr^z, we use a Fourier mode decomposition in the circle variable τ of sections of E. The action of $\nabla_e = \partial_\tau - i\Lambda$ on the k^{th}-mode equals $ik - i\Lambda$. Therefore,

$$\begin{aligned}
\hat{\eta}(D^\partial_g) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \sum_{k \in \mathbb{Z}} \operatorname{Tr}^z \left(\exp(-u(k - \Lambda + \frac{R}{4ui} - \frac{z}{2\sqrt{u}})^2) \right) \frac{du}{2u}\, e^{-F^E} \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{tr}_E \sum_{k \in \mathbb{Z}} (k - \Lambda + \frac{R}{4ui}) e^{-u(k - \Lambda + \frac{R}{4ui})^2} \frac{du}{2\sqrt{u}}\, e^{-F^E}
\end{aligned}$$

We need the Poisson summation formula [4].

$$\sum_{k \in \mathbb{Z}} (k + a) e^{-4\pi^2 s (k+a)^2} = \sum_{p \geq 1} 2p \sin(2\pi p a) (4\pi s)^{-3/2} e^{-\frac{p^2}{4s}}.$$

to obtain

$$\hat{\eta}(D_g^{\partial}) = \pi \int_0^{\infty} \mathrm{tr}_E \sum_{p\geq 1} p \sin\left(2\pi p(-\Lambda + \frac{R}{4ui})\right) e^{-\frac{\pi^2 p^2}{u}} \frac{du}{u^2}\, e^{-F^E} \tag{4.9}$$

Solving the $u-$integral and simplifying gives

$$\hat{\eta}(D_g^{\partial}) = \mathrm{tr}_E \left(\sum_{p\geq 1} \frac{-\sin 2\pi p\Lambda}{\pi p} + \frac{R}{2i} \sum_{p\geq 1} \frac{\cos 2\pi p\Lambda}{\pi^2 p^2} \right) e^{-F^E} \tag{4.10}$$

Recalling the Fourier series expansions of Bernoulli polynomials we rewrite (4.10) as

$$\hat{\eta}(D_g^{\partial}) = \mathrm{tr}_E \left((\{\Lambda\} - \frac{1}{2}) + (\{\Lambda\}^2 - \{\Lambda\} + \frac{1}{6})\frac{R}{2i} \right) e^{-F^E} \tag{4.11}$$

Finally, integrating over S^2_∞ we get

$$\frac{1}{2\pi i}\int_{S^2_\infty} \hat{\eta} = -\frac{1}{2\pi i}\int_{S^2_\infty} \mathrm{tr}_E(\{\Lambda\} - \frac{1}{2})F^E + \frac{1}{2}\mathrm{tr}_E(\{\Lambda\}^2 - \{\Lambda\} + \frac{1}{6}). \tag{4.12}$$

Notice that the last $m/12$ in (4.12) cancels the first summand in (4.1). Going back to (4.1) we deduce the result of [4] stated in the introduction

Theorem 4.2 ([4, Thm 43])**.**

$$\begin{aligned} \mathrm{Ind}_{L^2}\mathcal{D}_g^+ = &-\frac{1}{8\pi^2}\int_{TN} \mathrm{tr}\, F_{\mathcal{A}} \wedge F_{\mathcal{A}} + \frac{1}{2\pi i}\int_{S^2_\infty} \mathrm{tr}_E(\{\Lambda\} - \frac{1}{2})F^E \\ &- \frac{1}{2}\mathrm{tr}_E(\{\Lambda\}^2 - \{\Lambda\}). \end{aligned} \tag{4.13}$$

Acknowledgements. The author would like to thank Sergey Cherkis and Mark Stern for helpful conversations and constructive suggestions. Part of this work was completed with the support of a Research Council SEED Grant from the University of Dayton. The author would like to thank the anonymous referee for bringing the reference [11] to his attention and for several corrections and helpful comments.

References

[1] Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Grundlehren Text Editions **298**, Springer Verlag, Berlin (2004)

[2] Bismut, J., Cheeger, J.: Eta Invariants and Their Adiabatic Limits. J. Am. Math. Sc. **2**, No 1, 33-70 (1989)

[3] Cherkis, S.: Instantons on Gravitons. Comm. Math. Phys. **306** No 2, 449-483 (2011)

[4] Cherkis, S., Larrain-Hubach, A., Stern, M.: Instantons on multi-Taub-NUT Spaces I: Asymptotic Form and Index theorem. J. Diff. Geom. (forthcoming) arXiv:1608.00018 (2019)

[5] Cherkis, S., Larrain-Hubach, A., Stern, M.: Instantons on multi-Taub-NUT Spaces II: Bow Construction. In preparation.

[6] Cherkis, S., Larrain-Hubach, A., Stern, M.: Instantons on multi-Taub-NUT Spaces III: Down Transform, Completness and Isometry. In preparation.
[7] Dai, X.: Adiabatic Limits, Nonmultiplicativity of Signature and Leray Spectral Sequence, J. Am. Math. Sc. **4** No 2, 265-321 (1991)
[8] Leichtnam, E., Mazzeo, R., Piazza, P.: The Index of Dirac Operators on Manifolds with Fibered Boundaries. Bull. Belg. Math. Sc. **13**, 845-855 (2006)
[9] Savale, N.: Asymptotics of the Eta Invariant. Comm. Math. Phys, **332** No 2, 847-884 (2014)
[10] Stern, M.: Index Theory on Certain Complete Kähler Manifolds. J. Diff. Geom, **37** No 3, 467-503 (1993)
[11] Vaillant, B.: Index and Spectral Theory for Manifolds with Generalized Fibred Cusps. Ph.D. Thesis, arXiv:math/0102072 (2001)
[12] Zhang, W.: Circle Bundles, Adiabatic Limits of Eta-Invariants and Rokhlin Congruences. Ann. Inst. Fou, **44** No 1, 249-270 (1994)

Received: 26 December 2019/Accepted: 14 April 2020/Published online: 20 April 2020

Department of Mathematics, University of Dayton, Dayton, Ohio 45469.

alarrainhubach1@udayton.edu