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On necessary conditions for the weak lower semicontinuity of integral functionals in Musielak-Orlicz-Sobolev spaces

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Abstract. *In the present work, we prove an approximation result in Musielak-Orlicz-Sobolev spaces and we give an application of this approximation result to a necessary condition in the calculus of variations.*

1. Introduction

There are many application problems involving variational integrals of the form

$$\min J(u),$$

for open $\Omega \subset \mathbb{R}^N$, where $J(u) = \int_\Omega f(x,u,\nabla u)$ and u is a vector valued function and $f(x,u,\nabla u)$ is convex in ∇u. For example, such minimization problems are used in image denoising and edge detection, modeling the deformation of a thin plate and determining a surface of minimal area with prescribed boundary conditions. In fact, Hilbert's 19^{th} and 20^{th} problems deal with these regular problems in the calculus of variations. In 1912, Bernstein [3] used the calculus of variations method to establish existence and regularity results for the 2-dimensional real-valued Dirichlet problem. Serrin [13] applied similar methods to extend these results to n-dimensions. The major problem in the calculs of variations is to find the elements u checking in the boundary conditions required by the nature of the problem and minimizing the functional J.

It will turn out that in the L^p case the search of sufficient conditions to secure those functionals attain an extreme value has a long history (see [10]). The most important problem is to verify the weak lower semi-continuity of those functionals with respect to the space involved. This usually involves hypothesis that the integrand f is convex with respect to the gradient.

In 1992 Landes [10] studied the reverse problem at a fixed level set and have been showed that if J is weakly lower semi-continuous at one fixed (nonvoid) level set then this partical level set is an extreme value of f or the defining function f

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is convex in the gradient. The above statement for f as function of u (or of x and u) is not hard to prove (see [10]) but when $f = f(x, \nabla u)$ or $f = f(x, u, \nabla u)$ this is due to an approximation result in Sobolev-spaces.

Note that Azroul and Benkirane [2] proved the same results as Landes in the case of Orlicz-Sobolev spaces $W^1L_M(\Omega)$. Kouhaila, Azroul and Benkirane extend in [8, 9] this result in the case of Weighted Sobolev spaces $W^{1,p}(\Omega, \omega)$ and Weighted Orlicz-Sobolev spaces $W^1L_M(\Omega, \rho)$ respectively.

In the present work, our first main goal is to prove an approximation theorem in the more general setting of the Musielak-Orlicz-Sobolev spaces (for almost all $x_0 \in \Omega$, it is possible to alter any function $u \in W^1L_\varphi(\Omega)$ in such a way that u is constant in small ball with center x_0 and the altered function remain within an $\epsilon-$ neighborhood of the original function.) and the second main goal is to give an application of this approximation result to a necessary condition in the calculus of variations in the same functional framework of $W^1L_\varphi(\Omega)$. However we prove when $f = f(x, \nabla u)$ that if J is weakly lower semi-continuous at one fixed level set H_μ in the space $W^1L_\varphi(\Omega)$ then H_μ is an extreme value of J or the function f is convex with respect to the gradient.

2. Preliminaries

This section presents, some definitions and well-known about Musielak-Orlicz functions, Musielak-Orlicz-Sobolev spaces.

2.1. Musielak-Orlicz functions.

Let Ω be an open subset of $\mathbb{R}^N$, and let $\varphi\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}$ and satisfying the following conditions:
a) $\varphi(x, \cdot)$ is an N-function, i.e, continuous, convex, increasing with $\varphi(x, 0) = 0$, $\varphi(x, t) \succ 0$ for $t \succ 0$, $\varphi(x, t)/t \to 0$ as $t \to 0$ and $\frac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$.
b) $\varphi(\cdot, t)$ is a measurable function.

A function $\varphi(x, t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx. \tag{2.1}$$

2.2. Musielak-Orlicz-Sobolev space.

Let Ω be an open subset of $\mathbb{R}^N$, and let φ a Musilak-Orlicz fonction. The Musilak-Orlicz classe $K_\varphi(\Omega)$ (resp. the Musilak-Orlicz spaces $L_\varphi(\Omega)$ is the set of all real-valued measurable functions u defined in Ω and satisfying,
$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx \prec \infty$ (resp. $\varrho_{\varphi,\Omega}(\frac{u}{\lambda}) = \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \prec \infty$ for some $\lambda \succ 0$).

Let

$$\overline{\varphi}(x,s) = \sup_{t\geq 0}\left\{st - \varphi(x,t)\right\},$$

that $\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ (or conjugate to φ)) in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda \succ 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \leq 1\right\}. \tag{2.2}$$

which is called Luxemburg norm and the so called Orlicz norm by;

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}}\leq 1}\int_\Omega |u(x)v(x)|\, dx. \tag{2.3}$$

For two complementary Musielak-Orlicz functions φ and $\overline{\varphi}$ we have the Young inequality [11]:

$$s.t \leq \varphi(x,t) + \overline{\varphi}(x,s) \quad \text{for} \quad t,s\geq 0 \quad \text{and} \quad x\in\Omega.$$

We recall that the Musielac function φ is said to satisfy the Δ_2-condition (or doubling) if there exists $k > 0$ and a non-negative function C, integrable on Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + C(x) \quad \text{for all} \quad x\in\Omega \text{ and for all} \quad t\geq 0$$

For any fixed nonnegative integer m we define the closure in $L_\varphi(\Omega)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ denoted by $E_\varphi(\Omega)$ (we have usual $E_\varphi(\Omega) \subset K_\varphi(\Omega) \subset L_\varphi(\Omega)$). The equality $L_\varphi(\Omega) = E_\varphi(\Omega)$ hold if and only if φ satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note. The dual of $E_\varphi(\Omega)^*$ can be identified with $L_{\overline{\varphi}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u\in L_\varphi(\Omega)$ and $v\in L_{\overline{\varphi}}(\Omega)$.

We return now to the Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all function u such that u and its distibutional derivatives up to order 1 lie in $\in L_\varphi(\Omega)$ (resp. $\in E_\varphi(\Omega)$). For $u\in W^1L_\varphi(\Omega)$, we define

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u)$$

and,

$$\|u\|_{\varphi,1} = \|u\|_{\varphi,\Omega,1} = \inf\left\{\lambda\succ 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\right\}.$$

These functionals are a convex modular, and a norm on $W^1L_\varphi(\Omega)$ respectively, and $(W^1L_\varphi(\Omega), \|u\|_{\varphi,1})$ is a Banach space if φ satisfies the following condition: there exists a constant c such that

$$\inf_{x\in\Omega}\varphi(x,1) \geq c.$$

Thus $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of $\prod L_\varphi$, we have the weak topology $\sigma(\prod L_\varphi, \prod E_{\overline{\varphi}})$. The space $W_0^1E_\varphi(\Omega)$ (resp. $W_0^1L_\varphi(\Omega)$) is defined by the closure of $D(\Omega)$ in $W^1L_\varphi(\Omega)$ for the norm (resp. for the topology) $\sigma(\prod L_\varphi, \prod E_{\overline{\varphi}})$.

Definition 2.1. The sequence u_n converges to u in $L_\varphi(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_\varphi(\Omega)$) if $\varrho_{\varphi,\Omega}(\frac{u_n-u}{\lambda}) \to 0$ as $n \to \infty$ for some $\lambda \succ 0$.

Definition 2.2. The sequence u_n converges to u in $W^1L_\varphi(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1L_\varphi(\Omega)$) if $\overline{\varrho}_{\varphi,\Omega}(\frac{u_n-u}{\lambda}) \to 0$ as $n \to \infty$, for some $\lambda \succ 0$.

Lemma 2.3 (see [12, Lemma 4.1])**.** *Let φ be an Musielak-Orlicz function. If $u_n \in L_\varphi(\Omega)$ converges a.e. to u and u_n bounded in $L_\varphi(\Omega)$, then $u \in L_\varphi(\Omega)$ and $u_n \to u$ for the topology $\sigma(L_\varphi(\Omega), E_{\overline{\varphi}}(\Omega))$.*

3. Approximation result

The next lemma is preparatory for the proof of the Theorem 3.2.

Lemma 3.1. *For almost all $x_0 \in \Omega$, there exists a sequence $\alpha_k \succ 0$ with $\alpha_k \to 0$ as $k \to \infty$ such that $\displaystyle\int_{B(x_0,2\alpha)} \varphi\left(x, \frac{\lambda\,|u(x)-u(x_0)|}{\alpha_k}\right)\rho(x)dx \to 0$ as $k \to \infty$ for some $\lambda \succ 0$.*

Proof. Let $x_0 \in \Omega$. For each $t \succ 0$, we define the set $\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \succ t\}$. Let $\alpha_0 \succ 0$. For $\alpha \prec \alpha_0$, we consider the function $\phi_\alpha \colon \Omega_{2\alpha_0} \to \mathbb{R}$ defined by

$$\phi_\alpha(y) = \int_{B(y,2\alpha)} \varphi\left(x, \frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) dx. \tag{3.1}$$

Since

$$\phi_\alpha(y) = \int_\Omega \varphi\left(x, \frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) \chi_{B(y,2\alpha)}\,dx\,,$$

then the function $\phi_\alpha \colon \Omega_{2\alpha_0} \to \mathbb{R}$ is measurable; χ_E, as usual denotes the charateristic function of the set E. For all $\alpha_0 \succ 0$, we shall show that:

$$|\phi_\alpha(y)| \to 0 \quad \text{in} \quad L^1(\Omega_{2\alpha_0}) \quad \text{as} \quad \alpha \to 0, \quad \alpha \prec \alpha_0\,. \tag{3.2}$$

This obviously implies the statement of Lemma 3.1, (because if (3.2) is satisfied, then there is a subsequence α_k converges at 0 as $k \to \infty$ and such that $\phi_{\alpha_k}(y) \to 0$ a.e. y in $\Omega_{2\alpha_0}$). Since α_0 is arbitrary, then the previous convergence is true for a.e. x_0 in Ω.

To verify (3.2), we denotes by $u_\delta = u * \varphi_\delta$ the mollification of u, where $\varphi_\delta \in D(\mathbb{R}^N)$, $\varphi_\delta = 1$ for $|x| \geq \delta$, $\varphi_\delta \geq 0$ and $\int_{\mathbb{R}^N} \varphi_\delta(x)dx = 1$. Hence, φ_δ is well defined in $\Omega_{2\alpha_0}$ for $\delta \prec \alpha \prec \alpha_0$ and we have

$$\begin{aligned} \int_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy &= \int_{\Omega_{2\alpha_0}} \int_{B(y,2\alpha)} \varphi\left(x, \frac{\lambda\, |u(x) - u(x_0)|}{\alpha}\right) dx\, dy \\ &\leq \lim_{\delta\to 0} \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \varphi\left(x, \frac{\lambda\, |u_\delta(y-x) - u_\delta(y)|}{\alpha}\right) dx\, dy\,. \end{aligned}$$

Since u_δ is continuously differentiable, we may estimate

$$I_\alpha = \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \varphi\left(x, \frac{\lambda\, |u_\delta(y-x) - u_\delta(y)|}{\alpha}\right) dx\, dy.$$

Indeed, we have

$$\begin{aligned} I_\alpha &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \varphi\left(x, \frac{\lambda \int_0^1 |\nabla u_\delta(y-tx)|\, |x|\, dt}{\alpha}\right) dx\, dy \\ &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \varphi\left(x, 2\lambda \int_0^1 |\nabla u_\delta(y-tx)|\, dt\right) dx\, dy\,. \end{aligned}$$

Then, it follows by Jensen's inequality that

$$\begin{aligned} I_\alpha &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \int_0^1 \varphi(x, 2\lambda\, |\nabla u_\delta(y-tx)|) dt\; dx\; dy \\ &\overset{(*)}{=} \int_0^1 \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \varphi(x, 2\lambda \left| \int_{B(0,\delta)} \nabla u(y-tx-z)\varphi_\delta(z) dz \right|) dt\; dx\; dy \\ &\leq k_2 \int_0^1 \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \int_{B(0,\delta)} \varphi(x, 2k_1\lambda\, |\nabla u(y-tx-z)|) dt\; dx\; dy\; dz \\ &= k_2 \int_0^1 \int_{B(0,\delta)} \int_{\Omega_{2\alpha_0}} (\int_{B(0,2\alpha)} \varphi(x, 2k_1\lambda\, |\nabla u(y-tx-z)|) dx) dy\, dt\, dz \\ &\leq k_2 \int_0^1 \int_{B(0,\delta)} \int_{\Omega_{2\alpha_0}} (\frac{1}{N} \sum_{1\leq i\leq N} \int_{B(0,2\alpha)} \varphi(x, 2k_1\lambda \left| \frac{\partial u(y-tx-z)}{\partial x_i} \right|) dx) dy\, dt\, dz \\ &\leq \frac{k_3}{N} \int_0^1 \int_{B(0,\delta)} \int_{\Omega_{2\alpha_0}} dy\; dt\; dz \\ &\leq k_4 (\frac{\sigma_N}{N}) \delta^N \\ &\leq k_4 (\frac{\sigma_N}{N}) \alpha^N \quad (\text{because} \quad \alpha \succ \delta) \end{aligned}$$

for some positive constants k_1, k_2, k_3, and k_4, (σ_N denotes the measure of the unit sphere in $\mathbb{R}^N$). So we obtain $I_\alpha \to 0$ as $\alpha \to 0$. Then it follows for $\alpha_0 \succ 0$ that $\int_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy \to 0$ as $\alpha \to 0$ $\alpha \prec \alpha_0$, which allows us to conclude for almost every $x_0 \in \Omega$, that we have $\phi_{\alpha_k}(x_0) \to 0$as $k \to \infty$. To justify $(*)$, we recall that in $\Omega_{2\alpha_0}$ the differentiation and the mollification commutant for $\delta \prec \alpha \prec \alpha_0$, which proves the statement of Lemma 3.1. □

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, let φ be a Musielak-Orlicz function and $u \in W^1L_\varphi(\Omega)$, then for almost all $x_0 \in \Omega$, there is ball $B(x_0,\alpha)$ $(\alpha > 0)$ and a function $u_\alpha \in W^1L_\varphi(\Omega)$, such that:*
i) $u_\alpha \to u(mod)$ in $W^1L_\varphi(\Omega)$ as $\alpha \to 0$,
ii) $u_\alpha \equiv c(x_0,\alpha)$ in $B(x_0,\alpha)$, where $c(x_0,\alpha) = u(x_0)$.

Proof. Let Ψ_α be a C_0^∞ cut-off function with support in $B(0,2\alpha)$ such that $\Psi_\alpha \equiv 1$ in $B(0,\alpha)$ and $|\nabla\Psi_\alpha| \leq \frac{2}{\alpha}$. Let x_0 be a Lebesgue point of the function u in Ω , hence we can take $c(x_0,\alpha) = u(x_0)$. We define in Ω the function u_α by

$$u_\alpha(x) = u(x)(1 - \Psi_\alpha(x - x_0)) + u(x_0)\Psi_\alpha(x - x_0). \tag{3.3}$$

First we observe that $u_\alpha \in W^1L_\varphi(\Omega)$. In fact since $u \in W^1L_\varphi(\Omega)$, then there exist real numbers $\lambda_i \succ 0$, $0 \leq i \leq N$, such that

$$\int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda_0}\right) dx < \infty$$

and

$$\int_\Omega \varphi\left(x, \frac{1}{\lambda_i}\left|\frac{\partial u(x)}{\partial x_i}\right|\right) dx \prec \infty \text{ for } 1 \leq i \leq N$$

Let $\lambda \succ 0$, since $\varphi(x,\cdot)$ is a convex function, then

$$\begin{aligned}\int_\Omega \varphi(x, \frac{|u_\alpha(x)|}{\lambda})dx &\leq \frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}\,|u(x)(1-\Psi_\alpha(x-x_0))|)dx \\ &\quad + \frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}\,|u(x_0)\Psi_\alpha(x-x_0)|)dx \\ &\leq \frac{1}{2}\int_\Omega \varphi(x, 2k_1\frac{|u(x)|}{\lambda})dx + \frac{1}{2}\int_{B(0,2\alpha)} \varphi(x, \frac{2k'}{\lambda}\,|u(x_0|)dx \\ &\prec \infty\end{aligned}$$

where

$$k_1 = \sup_{B(0,2\alpha)} |1 - \Psi_\alpha(x - x_0)|, \quad \lambda = 2k_1\lambda_0 \quad \text{and} \quad k' = \sup_{B(0,2\alpha)} |\Psi_\alpha(x - x_0)|.$$

Remains to show that

$$\frac{\partial u_\alpha}{\partial x_i} \in L_\varphi(\Omega), \qquad 1 \leq i \leq N.$$

By a simple calculation we find that

$$\frac{\partial u_\alpha}{\partial x_i} = \frac{\partial u(x)}{\partial x_i}(1 - \Psi_\alpha(x - x_0)) + (u(x_0) - u(x))\frac{\partial \Psi_\alpha(x - x_0)}{\partial x_i}.$$

Then

$$\int_\Omega \varphi(x, \frac{1}{\lambda}\left|\frac{\partial u_\alpha}{\partial x_i}\right|)dx \;\le\; \frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}\left|(1-\Psi_\alpha(x-x_0)\frac{\partial u(x)}{\partial x_i}\right|)dx$$
$$+\frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}\right|)dx$$
$$\le\; \frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}k_1\left|\frac{\partial u(x)}{\partial x_i}\right|)dx$$
$$+\frac{1}{2}\int_\Omega \varphi(x, \frac{2}{\lambda}\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}\right|)dx.$$

Since $u \in W^1L_\varphi(\Omega)$, then the first term on the right side of the inequality is finite. In addition we will show in Lemma 3.1, that

$$I'_\alpha = \int_{\Omega_{2\alpha_0}}\int_{B(y,2\alpha)} \varphi\left(x, \frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) dx\, dy \prec \infty,$$

then

$$\int_{B(y,2\alpha)} \varphi\left(x, \frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) dx \prec \infty \quad \text{a.e,}$$

which implies that the second term is finite. Thus $u_\alpha \in W^1L_\varphi(\Omega)$. It is clear by using the Lebesgue theorem that

$$u_\alpha \to u\ (mod) \quad L_\varphi(\Omega) \quad \text{as} \quad \alpha \to 0. \tag{3.4}$$

therefore, it remains to show that

$$\frac{\partial u_{\alpha_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i}\ (mod) \quad L_\varphi(\Omega),\ 1 \le i \le N, \tag{3.5}$$

for the sequence α_k with $\alpha_k \to 0$ as $k \to \infty$. By a simple calculation we find that:

$$\frac{\partial(u-u_\alpha)(x)}{\partial x_i} = \frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x-x_0) + \frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}(u(x)-u(x_0))$$

and the convexity of $\varphi(x,.)$ we can write

$$\int_\Omega \varphi(x,\lambda\left|\frac{\partial(u-u_\alpha)(x)}{\partial x_i}\right|)dx \le \frac{1}{2}\int_\Omega \varphi(x, 2\lambda\left|\frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x-x_0)\right|)dx$$
$$+\frac{1}{2}\int_\Omega \varphi(x, 2\lambda\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}\right| dx.$$

By virtue of Lebesgue theorem, the first term in the expression right of the above inequality converges to zero as $\alpha \to 0$, so it remains to show that:

$$\int_\Omega \varphi(x, 2\lambda\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}\right|)dx \to 0 \quad \text{as} \quad \alpha \to 0 \tag{3.6}$$

Using the above Lemma 3.1 we conclude directly, which completes the proof of Theorem 3.2. □

Remark 3.3.

1. In the particular case when $\varphi(x,t) = M(t)\rho(x)$, M an N-function and ρ the weight function, we recover the statement of [9, Theorem 4.1].
2. In the particular case when $\varphi(x,t) = M(t)$, M an N-function , we recover the statement of [2, Lemma 2].
3. In the particular case when $\varphi(x,t) = \dfrac{|t|^p}{p}$, $1 \leq p \prec \infty$ we recover the statement of [10, Lemma 2-1].

4. Functional depending on x and ∇u

Let Ω be a bounded domain in $\mathbb{R}^N$, let φ be an Musielak-Orlicz function. We consider the functional $J\colon W^1L_\varphi(\Omega) \to \mathbb{R}$ defined as the following

$$J = \int_\Omega f(x, \nabla u)dx. \tag{4.1}$$

Where $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,\xi)| \leq T(x)G(|\xi|). \tag{4.2}$$

for some nondecreasing function $G\colon \mathbb{R} \to \mathbb{R}$ and some $T(x) \in L^1(\Omega)$. For each μ, we write H_μ for the level set of the functional J, i.e.,

$$H_\mu = \left\{ u \in W^1L_\varphi(\Omega) : J(u) = \mu \right\},$$

and $\overline{H}_\mu^w$ is the closure of H_μ in $W^1L_\varphi(\Omega)$ with respect to the weak topology $\sigma(\prod L_\varphi(\Omega), \prod E_{\overline{\varphi}}(\Omega))$.

Definition 4.1. A functional $J\colon W^1L_\varphi(\Omega) \to \mathbb{R}$ is called weakly lower semicontinuous at a level set H_μ. If $J(u) \leq \mu$ for all $u \in \overline{H}_\mu^w$.

Remark 4.2. Note that this definition does not imply that $J_{/\overline{H}_\mu^w}$ is weakly lower semicontinuous .

Definition 4.3. A function $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is called a Carathéodory-function if

- $f(\cdot, \xi)\colon \Omega \to \mathbb{R}$ is mesurable for all $\xi \in \mathbb{R}^N$,
- $f(x, \cdot)\colon \mathbb{R}^N \to \mathbb{R}$ is continuous for almost all $x \in \Omega$.

We have the following result:

Theorem 4.4 (see [2, Theorem 6])**.** *Let Ω be a bounded domain in $\mathbb{R}^N$.*

Let $J\colon W^1L_\varphi(\Omega) \to \mathbb{R}$ be a functional defined as in (4.1), with a Carathéodory function $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfying (4.2).

If J is weakly lower semicontinuous at nonvoid level set H_μ, then we have two alternatives: either μ is an extreme value of J or for almost all $x \in \Omega$ the function $f(x, \cdot)$ is convex.

Proposition 4.5. *The sequence of function $\hat{c}_n$ defined by*

$$\hat{c}_n(x) =< \xi^*, x > + \int_0^{<\xi-\xi^*,x>} g_\lambda(nt)dt$$

satisfying the following propreties:
i) $\hat{c}_n(x) \to \hat{c}_0(x)$ for almost all $x \in \Omega$ where $\hat{c}_0(x) =< \lambda\xi + (1-\lambda)\xi^*, x >$,
ii) $\hat{c}_n$ is bounded in $W^1L_\varphi(\Omega)$.

It is clear to check the two conditions of proposition (see [10] and [2]).

Remark 4.6. By Definition 4.3 the functional $J\colon W^1L_\varphi(\Omega) \to \mathbb{R}$ defined in (4.1) is continuous.

Proof of Theorem 4.4. Let assume that μ is not an extreme value of J, then we show that

$$f(x, \lambda\xi + (1-\lambda)\xi^*) \le \lambda f(x, \xi) + (1-\lambda) f(x, \xi^*)$$

for all $\lambda \in [0, 1]$, for all $\xi, \xi^* \in \mathbb{R}^N$ and for $a.e.$ $x \in \Omega$. We can assume that $\mu = 0$ and that there exist two functions $\hat{a}_1$ and $\hat{a}_2$ in $W^1L_\varphi(\Omega)$ such that $J(\hat{a}_1) \prec -\epsilon_0$ and $J(\hat{a}_2) \succ \epsilon_0$ for some $\epsilon_0 \succ 0$.

Let x_0 be a Lebesgue point for $f(x, \xi)$ for every $\xi \in \mathbb{Q}^N$. We can assume that $x_0 = 0$. By the continuity of the functional J and by Theorem 3.2, there exists a ball $B(0, R_0) \subset \Omega$ and there exist $\bar{b}, \bar{b_1}$ and $\bar{b_2}$ (see [10]) such that

$$\nabla\bar{b} = \nabla\bar{b_1} = \nabla\bar{b_2} = 0 \quad \text{on} \quad B(0, R_0). \tag{4.3}$$

$$J(\bar{b_1}) \prec \frac{7}{8}\epsilon_0, \quad J(\bar{b_2}) \succ \frac{7}{8}\epsilon_0 \quad \text{and} \quad \left|J(\bar{b})\right| \prec \frac{1}{8}\epsilon_0. \tag{4.4}$$

Moreover, for all function $\bar{a}$ satisfying $|J(\bar{a})| \prec \frac{7}{8}\epsilon_0$ there is $t_i \in [0, 1]$ with $i = i(\bar{a}) \in \{1, 2\}$ such that the function $\bar{c} = \bar{a} + t_i(\bar{b_i} - \bar{a})$ lies in the level set N_0 , i.e. $J(\bar{c}) = 0$.

Let us now fix $\lambda \in [0, 1] \bigcap \mathbb{Q}$ and $\xi, \xi^* \in \mathbb{Q}^N$. We define the sequence of functions

$$\hat{c}_n(x) =< \xi^*, x > + \int_0^{<\xi-\xi^*,x>} g_\lambda(nt)dt,$$

where $<,>$ denotes the usual inner product in $\mathbb{R}^N$ and

$$g_\lambda(x) = \begin{cases} 1 & \text{if} \quad 0 \prec t \prec \lambda \\ 0 & \text{if} \quad \lambda \prec t \prec 1 \end{cases}$$

We recall that

$$g_n(x) \rightharpoonup^* \lambda \quad \text{in} \quad L^\infty(\Omega)$$

and

$$(1 - g_n(x)) \rightharpoonup^* (1 - \lambda) \qquad \text{in} \qquad L^\infty(\Omega).$$

(see [10]). Moreover, $\hat{c}_n$ is bounded in $W^1L_\varphi(\Omega)$, and converges almost everywhere $\hat{c}_n(x) \to \hat{c}_0(x)$ where $\hat{c}_0(x) =< \lambda\xi + (1-\lambda)\xi^*, x >$ (see [2, 10]). Hence,

$$\hat{c}_n \to \hat{c}_0 \quad \text{in} \quad W^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\prod L_\varphi(\Omega), \prod E_{\bar{\varphi}}(\Omega)).$$

Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a C^∞-function with support in the interval $(-1, 1)$ and $\psi(t) = 1$ for all $|t| \prec \frac{1}{2}$. Defining $\bar{c}_R(x) = \psi\left(\frac{|x|}{R}\right)\hat{c}_0(x)$ for all $R \succ 0$, we have

$$\nabla\bar{c}_R(x) = \psi'\left(\frac{|x|}{R}\right)\frac{|x|}{R}\hat{c}_0(x) + \psi\left(\frac{|x|}{R}\right)\nabla\hat{c}_0(x).$$

Moreover, the function $\bar{c}_R(x) = \psi(\frac{|x|}{R})\hat{c}_0(x)$ satisfying the following properties (see [10, Proposition 3.1]):

$$|\nabla\bar{c}_R(x)| \le c \quad \text{in} \quad \Omega. \tag{4.5}$$

$$\int_{B(0,R)} f(x, \nabla\bar{c}_R(x))dx \to 0 \quad \text{as} \quad R \to 0. \tag{4.6}$$

Note that (4.2) is used for to prove (4.6). Next we consider the sequence $\hat{c}_n(x)$ in a ball $B(0, r)$, say. We will show that is possible alter each element of the sequence $\hat{c}_n(x)$ in such a manner that it coincides with limit $\hat{c}_0(x)$ in the boundary.

The following lemma is a generalization of [10, Proposition 3.2] to the Musielak-Orlicz-Sobolev spaces.

Lemma 4.7. *There exists a sequence $a_n(x) \in W^1L_\varphi(\Omega)$ such that:*
i) $a_n(x) = \hat{c}_0(x) =< \lambda\xi + (1-\lambda)\xi^*, x > 0$ in $\partial B(0, r)$,
ii) $a_n - \hat{c}_n \to 0$ (mod) in $W^1L_\varphi(\Omega)$ as $n \to \infty$,
iii) $a_n \to \hat{c}_0$ in $W^1L_\varphi(\Omega)$ for $\sigma(\prod L_\varphi(\Omega), \prod E_{\overline{\varphi}}(\Omega))$,
iv) $\|\nabla a_n\|_\infty + \|\nabla\hat{c}_n\|_\infty \le c$,
v) $\left|\int_{B(0,r)} f(x, \nabla\hat{c}_n)dx - \int_{B(0,r)} f(x, \nabla a_n)dx\right| \to 0$ as $n \to \infty$,
vi) $\int_{B(0,r)} f(x, \nabla a_n)dx \to 0$ as $r \to 0$ uniformly in n.

Now, we are in the position to complete the proof of Theorem 4.4. For $R \leq R_0$ and $r = \frac{R}{2}$, we define the sequence:

$$\widehat{b}_n(x) = \begin{cases} \bar{b}(x) & \text{if} \quad x \in \Omega \backslash B(0,R), \\ \bar{b}(x) + \bar{c}_R(x) & \text{if} \quad x \in B(0,R) \backslash B(0,r), \\ \bar{b}(x) + a_n(x) & \text{if} \quad x \in B(0,r); \end{cases}$$

which converges in $W^1L_\varphi(\Omega)$ for the weak topology $\sigma(\prod L_\varphi(\Omega), \prod E_{\overline{\varphi}}(\Omega))$ to

$$\widehat{b}_0(x) = \begin{cases} \bar{b}(x) & \text{for} \quad x \in \Omega \backslash B(0,R), \\ \bar{b}(x) + \bar{c}_{R}(x) & \text{for} \quad x \in B(0,R). \end{cases}$$

Combining (4.5), (4.6) and Lemma 4.7, (as in [10] and [2]), we have for $R > 0$ small enough $\left|J(\overline{b}_n)\right| < \frac{7}{8}\epsilon_0$ for all n. Hence for any n, we find numbers $t_n \in [0,1]$ and $i_n \in \{1,2\}$, such that for $b_n = \widehat{b}_n + t_n(\bar{b}_{i_n} - \widehat{b}_n)$ we have $J(b_n) = 0$. Now choosing a subsequence t_n such that $t_n \to t_0$ and $i_n = i$; $i \in \{1,2\}$, we have

$$b_n \to b_0 \quad \text{in} \quad W^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\textstyle\prod L_\varphi(\Omega), \prod E_{\overline{\varphi}}(\Omega)).$$

Because of the continuity of J with respect to the strong topology of $W^1L_\varphi(\Omega)$, we have

$$\lim\nolimits_{n\to\infty} J(\bar{b} + t_n(\bar{b}_{i_n} - \bar{b})) = J(\bar{b} + t_0(\bar{b}_i - \bar{b})),$$

and by construction

$$f(x, \nabla(\bar{b} + t_n(\bar{b}_i - \bar{b})) = f(x,0) \text{ in } B(0,R).$$

Since

$$\nabla\bar{b} = \nabla\bar{b}_1 = \nabla\bar{b}_2 = 0 \text{ in } B(0,R)$$

which yields

$$\lim_{n\to\infty} \int_{B(0,R)} f(x, \nabla b_n(x))dx \geq \int_{B(0,R)} f(x, \nabla b_0(x))dx.$$

Since $b_n = b_0$ in $B(0,R)\backslash B(0,r)$, $r = \frac{R}{2}$, we finally get

$$\begin{aligned} \int_{B(0,r)} f(x, \lambda\xi + (1-\lambda)\xi^*)dx &= \int_{B(0,r)} f(x, \nabla b_0(x))dx \\ &\leq \lim_{n\to\infty} \int_{B(0,r)} f(x, \nabla b_n)(x))dx \\ &= \lim_{n\to\infty} \int_{B(0,R)} f(x, \nabla a_n(x))dx \\ &= \lambda \int_{B(0,r)} f(x,\xi)dx + (1-\lambda) \int_{B(0,r)} f(x,\xi^*)dx. \end{aligned}$$

Since the above inequality can be obtained for all $B(0,r)$ with radius $r < \frac{R}{2}$, we conclude that $f(x_0, \lambda\xi + (1-\lambda)\xi^*) \leq \lambda f(x_0,\xi) + (1-\lambda) f(x_0,\xi^*)$ for all $\lambda \in [0,1] \cap \mathbb{Q}$

and all $\xi, \xi^* \in \mathbb{Q}^N$. It then follows by the continuity of $f(x,\xi)$ with respect to ξ, that the above inequality holds for all $\lambda \in [0,1]$ and all $\xi, \xi^* \in \mathbb{R}^N$. $\square$

We now prove Lemma 4.7.

Proof of Lemma 4.7. Let $\tilde{\omega}_\delta$ be a C^∞ -function with support in $[-1,1]$ such that $\tilde{\omega}_\delta = 1$ for all $|t| \prec 1-\delta$ and $\left|\tilde{\omega}_\delta'\right| \prec \frac{2}{\delta}$ for all t.

Defining the function $\omega_\delta(x) = \tilde{\omega}_\delta(\frac{|x|}{r})$ and $a_{n,\delta}(x) = \hat{c}_0(x) + \omega_\delta(x)(\hat{c}_n(x) - \hat{c}_0(x))$ by Proposition 3.2 in [10] we have that

$$|\nabla(\hat{c}_n(x) - \hat{c}_0(x))|\,(1-\omega_\delta(x)) \leq c' r(|\xi^*| + |\xi|)(1-\omega_\delta(x)), \tag{4.7}$$

$$|\nabla\omega_\delta(x)|\,|\hat{c}_n(x) - \hat{c}_0(x)| \leq O(n^{-1})\frac{1}{\delta}\chi_{\mathrm{supp}(\nabla\omega_\delta)}. \tag{4.8}$$

for some positive constants c and c'. Assume that the following formula is true

$$\begin{aligned} &\int_\Omega \varphi(x, |a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega \varphi(x, |\nabla(a_{n,\delta} - \hat{c}_n)|)dx \\ &\quad \leq O(\delta) + c\int_{B(0,r)} \varphi(x, |\nabla(\hat{c}_n(x) - \hat{c}_0(x))(1-\omega_\delta(x))|)dx \end{aligned} \tag{4.9}$$

hence we get

$$\omega_\delta(x) = \begin{cases} 0 & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 1 & \text{in} \quad B(0,(1-\delta)r) \\ \tilde{\omega_\delta}(\frac{|x|}{r}) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

which implies that

$$a_{n,\delta}(x) - \hat{c}_n(x) = \begin{cases} \hat{c_0}(x) - \hat{c}_n(x) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ (1-\tilde{\omega_\delta}(\frac{|x|}{r}))(\hat{c}_0(x) - \hat{c}_n(x)) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

and

$$\nabla(a_{n,\delta}(x) - \hat{c}_n(x)) = \begin{cases} \nabla(\hat{c_0}(x) - \hat{c}_n(x)) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ \nabla(\tilde{\omega_\delta}(\frac{|x|}{r}))(\hat{c}_n(x) - \hat{c}_0(x)) + & \\ (1-\tilde{\omega_\delta}(\frac{|x|}{r}))\nabla((\hat{c}_0(x) - \hat{c}_n(x))) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

Hence, we get the following estimate

$$\begin{aligned} &\int_\Omega \varphi(x, |a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega \varphi(x, |\nabla(a_{n,\delta} - \hat{c}_n)|)dx \\ &\quad \leq O(\delta) + c\int_{B(0,r)\backslash B(0,(1-\delta)r)} \varphi(x, |\nabla(\hat{c}_n(x) - \hat{c}_0(x))(1-\omega_\delta(x))|)dx \\ &\quad \leq O(\delta) + c\int_{B(0,r)\backslash B(0,(1-\delta)r)} \varphi(x, c_1 O(n^{-1})\frac{1}{\delta})dx. \end{aligned}$$

Selecting numbers δ_n such that $O(n^{-1})\frac{1}{\delta_n} = 1$, this implies that $O(\delta_n) = O(n^{-1})$ and $\delta_n \to 0$ as $n \to \infty$. Then, we conclude that

$$\int_\Omega \varphi(x, |a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega \varphi(x, |\nabla(a_{n,\delta} - \hat{c}_n)|)dx \\ \leq O(n^{-1}) + c\int_{B(0,r)\backslash B(0,(1-\delta)r)} \varphi(x, c_1 O(n^{-1})\frac{1}{\delta})dx$$

which converge to 0 as $n \to \infty$. We define the functions $a_n = a_{n,\delta}$ and we have $a_{n,\delta} - \hat{c}_n \to 0$ (mod)$W^1L_\varphi(\Omega)$ as $n \to 0$. Which gives (ii) in Lemma 4.7 and

$$a_n - \hat{c}_0 = (a_n - \hat{c}_n) + (\hat{c}_n - \hat{c}_0) \to 0 \qquad \text{in} \quad W^1L_\varphi(\Omega)$$

with respect to $\sigma(\prod L_\varphi(\Omega,), \prod E_{\overline{\varphi}}(\Omega))$ (because $(\hat{c}_n - \hat{c}_0) \to 0$ in $W^1L_\varphi(\Omega)$ for $\sigma(\prod L_\varphi(\Omega), \prod E_{\overline{\varphi}}(\Omega))$.

The properties i), iv) and vi) are satisfied by the definition of a_n. Now, it remains to prove the inequality (4.9). Indeed we can write

$$\int_\Omega \varphi(x, |a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega \varphi(x, |\nabla(a_{n,\delta} - \hat{c}_n)|)dx \\ = \int_{\overline{B}(0,r)} \varphi(x, |\hat{c}_n - \hat{c}_0|\,(1-\omega_\delta))dx + \int_{\Omega\backslash\overline{B}(0,r)} \varphi(x, |\hat{c}_n - \hat{c}_0|)dx \\ + \int_{\Omega\backslash\overline{B}(0,r)} \varphi(x, |\nabla(\hat{c}_n - \hat{c}_0)|)dx + \int_{\overline{B}(0,r)} \varphi(x, |\nabla(\hat{c}_n - \hat{c}_0)(1-\omega_\delta)|)dx.$$

Since

$$(1 - \omega_\delta(x)) \to 0 \text{ a.e. in } \overline{B}(0,r) \quad \text{as } \delta \to 0$$

and

$$\int_{\Omega\backslash\overline{B}(0,r)} \varphi(x, |\hat{c}_n - \hat{c}_0|)dx + \int_{\Omega\backslash\overline{B}(0,r)} \varphi(x, |\nabla(\hat{c}_n - \hat{c}_0)|)dx \to 0 \text{ as } n \to \infty,$$

then we conclude that

$$\int_\Omega \varphi(x, |a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega \varphi(x, |\nabla(a_{n,\delta} - \hat{c}_n)|)dx \\ \leq O(\delta) + c\int_{B(0,r)} \varphi(x, |\nabla(\hat{c}_n(x) - \hat{c}_0(x))(1 - \omega_\delta(x))|)dx.$$

Which implies the inequality (4.9). ☐

Corollary 4.8. *Under the same assumptions as in theorem suppose that there is a nonvoid weakly closed level set H_μ. If μ is not an extreme value of J, then the function $f(x, \nabla u(x))$ is affine in the gradient.*

Remark 4.9. 1) In the particular case when $\varphi(x,t) = M(t)$, M is an N-function, we recover the statement of [2, Theorem 6].
2)In the particular case when $\varphi(x,t) = \dfrac{|t|^p}{p}$,$1 \leq p \prec \infty$, we recover the statement of [10, Theorem 3-1].

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