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T-minima on convex sets and Mosco-convergence

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To Umberto, our teacher since 50+25 years.

Abstract. *Half century ago, Umberto Mosco was the "relatore di tesi (tesi about the Mosco-convergence) di laurea" of the first author; a quart of century ago, the first author was the "relatore di tesi di laurea" of the second author. The roots of this paper are the Mosco-convergence of convex sets and the minimization of integral functionals of the Calculus of Variations.*

We consider integral functionals of the type

$$J(v) = \int_\Omega j(x, Dv) - \int_\Omega f(x)\, v(x).$$

We study the existence of T-minima (infinite energy minima) on convex sets of the Sobolev space $W_0^{1,p}(\Omega)$ *and the stability of the T-minima under the Mosco-convergence of the convex sets.*

1. Introduction

Half century ago, Umberto Mosco was the "relatore di tesi (tesi about the Mosco-convergence) di laurea" of the first author; a quart of century ago, the first author was the "relatore di tesi di laurea" of the second author.

The roots of this paper are the Mosco-convergence of convex sets (see [15], [14]) and the minimization of integral functionals of the Calculus of Variations (see [12]) like

$$J(v) = \int_\Omega j(x, Dv) - \int_\Omega f(x)\, v(x), \tag{1.1}$$

even in the case of meagre summability of $f(x)$. Here Ω is a bounded subset of $\mathbb{R}^N$, $N > 2$, $f(x)$ belongs to some Lebesgue space and $j(x, \xi)$ is a real valued convex function on ξ, with growth of order p with respect to ξ.

If the datum f belongs to a "large Lebesgue space", as $L^1(\Omega)$, it is not possible to use the standard definition of minimum, because in the functional $J(v)$, $v \in W_0^{1,p}(\Omega)$, $p > 1$, the term $\int_\Omega f(x)\, v(x)$ is not well defined on the energy space $W_0^{1,p}(\Omega)$.

Nevertheless in [5] and [13] the authors give a suitable definition (T-minima), which coincides with the usual definition of minimum in the standard framework and with many useful properties (see [5] and [13] for the proofs). In some sense,

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we can say that T-minima are infinite energy minima, since they are outside of the energy space $W_0^{1,p}(\Omega)$. The main property is that the T-minimum is a solution of an appropriate Euler-Lagrange equation (see [4]).

We recall the following contributions about T-minima.

A The question of existence, uniqueness and continuous dependence with respect to the forcing terms has been successfully solved in [5] and [13].

B Summability of u and of Du (if the function f is not enough summable and we are beyond the duality pairing) are studied in [7] ("a tribute to Guido Stampacchia on the 30th anniversary of his death"), if $p = 2$, and in [9], if $p > 1$.

C The stability of the T-minima with respect to the $\Gamma(weak)$-convergence of the integral functionals has been studied in [13] and [9].

The aim of this paper is twofold.

- The study of the existence of T-minima on convex subsets of $W_0^{1,p}(\Omega)$.
- The study of the continuous dependence of T-minima if the convex subsets (of obstacle type) converge in the sense of Mosco.

The last subject is related to the results of [10], where the continuous dependence of the standard minima is studied if the convex subsets converge in the sense of Mosco.

In this paper we do not assume the differentiability of the functionals, since we do not use Euler-Lagrage equations.

1.1. Assumptions

Let $j(x,\xi)$ be a function defined in $\Omega \times \mathbb{R}^N$, satisfying the standard hypotheses of the integrands in the Calculus of Variations:

$$\begin{cases} \text{the function } j(x,\xi) \text{ is measurable with respect to } x \\ \text{and strictly convex with respect to } \xi, \end{cases} \tag{1.2}$$

there exist α, $\beta > 0$ such that

$$\alpha|\xi|^p \le j(x,\xi) \le \beta|\xi|^p,\ \forall\, \xi \in \mathbb{R}^N, \text{ a.e. in}\, \Omega, \quad 1 < p \le N. \tag{1.3}$$

We recall (see [18]) the definition of truncation $T_k\colon \mathbb{R} \mapsto \mathbb{R}$

$$T_k(t) = \begin{cases} t\ , & |t| \le k\ , \\ k\,\dfrac{t}{|t|}, & |t| > k\ , \end{cases}$$

and the definition of T-minima.

Definition 1.1 ([5]). Let $f \in L^1(\Omega)$. A measurable function u is a T-minimum for the functional

$$J(v) = \int_\Omega j(x, Dv) - \int_\Omega f(x)\, v(x) \tag{1.4}$$

if

$$\begin{cases} T_i(u) \in W_0^{1,p}(\Omega),\ \forall\ i \in \mathbb{R}^+ : \\ \displaystyle\int_{\{|u-\varphi|\le i\}} j(x, Du) - \int_\Omega f(x)\, T_i[u-\varphi] \le \int_{\{|u-\varphi|\le i\}} j(x, D\varphi), \\ \forall\ \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \forall\ i \in \mathbb{R}^+. \end{cases} \tag{1.5}$$

or equivalently (see [13])

$$\begin{cases} T_i(u) \in W_0^{1,p}(\Omega),\ \forall\ i \in \mathbb{R}^+ : \\ \displaystyle\int_\Omega j(x, D\{\varphi + T_i[u-\varphi]\}) - \int_\Omega f(x)\, T_i[u-\varphi] \le \int_\Omega j(x, D\varphi), \\ \forall\ \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \forall\ i \in \mathbb{R}^+. \end{cases} \tag{1.6}$$

Remark 1.2. Let us recall that for a measurable function u such that $T_i(u) \in W_0^{1,p}(\Omega)$ for every $i > 0$, there exists a measurable function $\Phi\colon \Omega \to \mathbb{R}^N$ such that $DT_i(u) = \Phi\chi_{\{|u|\le i\}}$ a.e. in Ω (see Lemma 2.1 in [4]). This function Φ, which is unique up to almost everywhere equivalence, will be denoted by Du. Note that Du coincides with the distributional gradient of u whenever $u \in L^1_{\mathrm{loc}}(\Omega)$, $T_i(u) \in W_0^{1,p}(\Omega)$ for every $i > 0$, and $Du \in L^1(\Omega, \mathbb{R}^N)$.

Minimization problems for integral functionals with nonregular data are also studied in many papers (see the References in [9]); here we recall [6, 8, 16, 17].

2. Existence of T-minima on convex sets of obstacle type

In this section, we assume (1.2), (1.3),

$$f \in L^1(\Omega), \tag{2.1}$$

and we consider the set

$$\mathcal{K}(\psi) = \{v \in W_0^{1,p}(\Omega) : v \ge \psi\}, \quad \psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

If $f \in L^m(\Omega)$, $m \ge 1$, as in [5], we define

$$J_n(v) = \int_\Omega j(x, Dv) - \int_\Omega f_n(x)\, v(x), \tag{2.2}$$

where $\{f_n\}$ is a sequence of bounded functions converging to f in $L^m(\Omega)$, and such that

$$|f_n(x)| \leq |f(x)|, \tag{2.3}$$

as $f_n(x) = T_n[f(x)]$ or $f_n(x) = \dfrac{f(x)}{1+\frac{1}{n}|f(x)|}$.

The existence of the minimum $w_n \in W_0^{1,p}(\Omega)$ of J_n on $\mathcal{K}(\psi)$ is classical. We will prove the following existence and uniqueness theorem.

Theorem 2.1. *Assume* (1.2), (1.3), *and* (2.1). *There exists a unique measurable function u such that*

$$\begin{cases} u \geq \psi \ \ a.e. \ \ in\ \Omega, \ \ T_i(u) \in W_0^{1,p}(\Omega) \quad \forall i > 0, \\[2ex] \displaystyle\int_\Omega j(x, D\{\varphi + T_i[u-\varphi]\}) - \int_\Omega f(x)\, T_i[u-\varphi] \leq \int_\Omega j(x, D\varphi), \\[2ex] \forall\ \ \varphi \in \mathcal{K}(\psi) \cap L^\infty(\Omega), \quad \forall\, i > 0. \end{cases} \tag{2.4}$$

Proof. In the definition of minimum

$$w_n \in \mathcal{K}(\psi): \ J_n(w_n) \leq J_n(v), \quad v \in \mathcal{K}(\psi),$$

we insert $v = w_n - T_i[w_n - \psi]$ and we have

$$\int_\Omega j(x, Dw_n) \leq \int_\Omega j(x, D\{w_n - T_i[w_n-\psi]\}) + \int_\Omega f_n(x) T_i[w_n - \psi],$$

that is

$$\int_{\{0 \leq w_n - \psi \leq i\}} j(x, Dw_n) \leq \int_{\{0 \leq w_n - \psi \leq i\}} j(x, D\psi) + i\, \|f\|_1 \tag{2.5}$$

and (since $\psi \in W_0^{1,p}(\Omega)$)

$$\alpha \int_{\{0 \leq w_n - \psi \leq i\}} |Dw_n|^p \leq \int_\Omega j(x, D\psi) + i\, \|f\|_1 \ \leq \beta \|\psi\|^p_{W_0^{1,p}(\Omega)} \ + i\, \|f\|_1\ , \tag{2.6}$$

which implies

$$\begin{aligned} \int_{\{0 \leq w_n - \psi \leq i\}} |D(w_n - \psi)|^p &\leq 2^{p-1} \left(\|\psi\|^p_{W_0^{1,p}(\Omega)} \ + \frac{\beta}{\alpha} \|\psi\|^p_{W_0^{1,p}(\Omega)} \ + i \frac{\|f\|_1}{\alpha} \right) \\ &\leq C\, i, \end{aligned}$$

for every $i \geq 1$. Together with the fact that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, this implies

$$\begin{aligned} \int_{\{|w_n| \leq j\}} |Dw_n|^p &\leq \int_{\{|w_n - \psi| \leq |\psi| + j\}} |Dw_n|^p \\ &\leq 2^{p-1} \int_{\{|w_n - \psi| \leq \tilde{M} + j\}} |D(w_n - \psi)|^p + 2^{p-1} \int_\Omega |D\psi|^p \\ &\leq 2^{p-1} C\, (j + \tilde{M}) + 2^{p-1} \int_\Omega |D\psi|^p \leq C\, j, \end{aligned} \tag{2.7}$$

where $\tilde{M} = \|\psi\|_{L^\infty(\Omega)}$ and $j \geq 1$. Following exactly the proofs of [4] (see Lemma 4.1 and Lemma 4.2), we can deduce the Marcinkiewicz estimates

$$\begin{cases} \text{measure}\{x : j < |w_n|\} \leq \dfrac{C}{j^{\frac{(p-1)N}{N-p}}}, \\ \text{measure}\{x : \lambda < |Dw_n|\} \leq \dfrac{C}{\lambda^{\frac{(p-1)N}{N-1}}}, \end{cases} \tag{2.8}$$

for every $j, \lambda \geq 1$. Moreover (see once again [4]), the following scheme of convergence holds true:

$$\begin{cases} w_{n_k} \text{ converges to } u \text{ a.e. in } \Omega, \\ \forall j > 0,\ T_j(w_{n_k}) \text{ weakly converges to } T_j(u) \text{ in } W_0^{1,p}(\Omega), \end{cases} \tag{2.9}$$

from which we also deduce that

$$\int_\Omega |DT_j(u)|^p \leq C(j+1),\ \forall j > 0,\ \text{and } u \geq \psi \text{ a.e. in } \Omega.$$

Let now $\varphi \in \mathcal{K}(\psi)$, $\varphi \in L^\infty(\Omega)$, and choose $v = w_n - T_i[w_n - \varphi]$ in the definition of minimum

$$J_n(w_n) \leq J_n(v), \quad v \in \mathcal{K}(\psi).$$

Then we have

$$\int_\Omega j(x, Dw_n) \leq \int_\Omega j(x, D\{w_n - T_i[w_n - \varphi]\}) + \int_\Omega f_n(x) T_i[w_n - \varphi].$$

If we rewrite the above inequality as

$$\int_\Omega j(x, D\{\varphi + T_i[w_n - \varphi]\}) \leq \int_\Omega j(x, D\varphi) + \int_\Omega f_n(x) T_i[w_n - \varphi],$$

using the weak lower semicontinuity in $W_0^{1,p}(\Omega)$ of the left hand side (see [12]) and the continuity of the right hand side, it is possible to pass to the limit, thanks to (2.9) and the boundedness of φ which imply the weak convergence in $W_0^{1,p}(\Omega)$ of $T_i[w_n - \varphi]$ to $T_i[u - \varphi]$ (up to a subsequence). Then we have

$$\int_\Omega j(x, D\{\varphi + T_i[u - \varphi]\}) \leq \int_\Omega j(x, D\varphi) + \int_\Omega f(x) T_i[u - \varphi],$$

that is u solves (2.4).

The uniqueness question can be treated exactly as in [13]. Assuming that u and v are both solutions to (2.4) we use the test function $\varphi = (T_h(u) + T_h(v))/2$, with $h > \max\{2i, \|\psi\|_{L^\infty}\}$, to be sure that $\varphi \geq \psi$ a.e. in Ω. The further restriction on h (in [13] $h > 2i$) has no effects in the proof since h is destined to go to $+\infty$. This ends the proof of the theorem. □

Remark 2.2. Note that the whole sequence $\{w_n\}$ is convergent thanks to the uniqueness of T-minima.

Remark 2.3. Let us note that if $p > 2 - \frac{1}{N}$ then $\frac{(p-1)N}{N-1} > 1$, thus the second Marcinkiewicz estimate in (2.8) implies that the sequence $\{Dw_n\}$ is bounded in the reflexive space $W_0^{1,q}(\Omega)$, $1 < q < \frac{(p-1)N}{N-1}$. Then the convergence

$$w_n \text{ weakly converges to } u \text{ in } W_0^{1,q}(\Omega),$$

can be included in (2.9). Moreover the T-minimum u is a Sobolev function belonging to $W_0^{1,q}(\Omega)$.

2.1. The Q-assumption

In this subsection, for the sake of simplicity, we assume $p = 2$. In [1], the following problem is studied. Assume that

$$f(x),\ a(x) \in L^1(\Omega), \tag{2.10}$$

and that there exists $Q > 0$ such that, for a.e. $x \in \Omega$,

$$|f(x)| \le Q\, a(x). \tag{2.11}$$

Then there exists u minimum on the space $W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),\ |u| \le Q : \\ \displaystyle\int_\Omega j(x, Du) + \frac{1}{2}\int_\Omega a(x)u^2 - \int_\Omega f(x)\, u(x) \\ \qquad \displaystyle\le \int_\Omega j(x, Dv) + \frac{1}{2}\int_\Omega a(x)v^2 - \int_\Omega f(x)\, v(x), \\ \forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}.$$

We point out the strong regularizing effect of the assumption (2.11): despite the L^1 summability of $f(x)$, the minimum u has finite energy and it is a bounded function. Thus, even if $f \in L^1(\Omega)$, the T-minimum framework is not needed.

Here we adapt the approach of [1] for the minimization on

$$\mathcal{K}(\psi) = \{v \in W_0^{1,2}(\Omega) : v \ge \psi\}$$

if

$$\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{2.12}$$

Theorem 2.4. *Assume (1.2), (1.3), (2.10), (2.11), (2.12). Then there exists a unique minimum u of*

$$\begin{cases} \psi \le u \in W_0^{1,2}(\Omega), : \\ \displaystyle\int_\Omega j(x, Du) + \frac{1}{2}\int_\Omega a(x)u^2 - \int_\Omega f(x)\, u(x) \\ \qquad \displaystyle\le \int_\Omega j(x, Dv) + \frac{1}{2}\int_\Omega a(x)v^2 - \int_\Omega f(x)\, v(x), \\ \forall\, v \in \mathcal{K}(\psi) \cap L^\infty(\Omega). \end{cases} \tag{2.13}$$

Moreover the following estimate holds

$$|u| \leq \max\{Q, \|\psi\|_{L^\infty(\Omega)}\}.$$

Proof. Define

$$I_n(v) = \int_\Omega j(x, Dv) + \frac{1}{2}\int_\Omega a_n(x)v^2 - \int_\Omega f_n(x)\, v(x),$$

where

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}, \qquad a_n(x) = \frac{a(x)}{1 + \frac{1}{n}a(x)},$$

in order that there exists the minimum u_n of I_n on the set

$$\{v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}.$$

Let $Q_0 \geq \max\{Q, \|\psi\|_{L^\infty(\Omega)}\}$. The minimality inequality $I_n(u_n) \leq I_n(T_{Q_0}(u_n))$, i.e.,

$$\int_\Omega j(x, Du_n) + \frac{1}{2}\int_\Omega a_n(x)|u_n|^2 - \int_\Omega f_n\, u_n$$
$$\leq \int_\Omega j(x, DT_{Q_0}(u_n)) + \frac{1}{2}\int_\Omega a_n(x)|T_{Q_0}(u_n)|^2 - \int_\Omega f_n\, T_{Q_0}(u_n)$$

implies, dropping a positive term, that

$$\frac{1}{2}\int_\Omega a_n(x)\Big[|u_n|^2 - |T_{Q_0}(u_n)|^2\Big] \leq \int_\Omega f_n\, G_{Q_0}(u_n) \leq \int_\Omega |f_n||G_{Q_0}(u_n)|,$$

and by (2.11)

$$\frac{1}{2}\int_\Omega a_n(x)\Big[|u_n|^2 - |T_{Q_0}(u_n)|^2\Big] \leq \int_\Omega Q_0\, a_n(x)\, |G_{Q_0}(u_n)|.$$

Manipulating the previous inequality we get

$$\frac{1}{2}\int_{\{|u_n|>Q_0\}} a_n(x)(|u_n| - Q_0)^2 \leq 0,$$

which implies $|u_n| \leq Q_0$.

Then the use of ψ as a test function gives, dropping a positive term,

$$\alpha \int_\Omega |Du_n|^2 \leq 2Q_0 \int_\Omega |f| + \beta\|\psi\|^2_{W_0^{1,2}(\Omega)} + Q_0 \int_\Omega a\,. \tag{2.14}$$

Thus the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and then there exist a function $u \in W_0^{1,2}(\Omega)$ and a subsequence, still denoted $\{u_n\}$, such that u_n converges to u weakly in $W_0^{1,2}(\Omega)$ and a.e. to u and $|u| \leq Q_0$. To conclude the proof, we use that u_n minimizes I_n, i.e.,

$$\int_\Omega j(x, Du_n) + \frac{1}{2}\int_\Omega a_n(x)|u_n|^2 - \int_\Omega f_n\, u_n \leq I_n(\varphi),$$

for every $\psi \leq \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Observe that we can pass to the limit in the first term (by weak lower semicontinuity in $W_0^{1,2}(\Omega)$), in the second term (by Fatou Lemma) and in the third term (by Lebesgue Theorem, since $|f_n\, u_n| \leq Q|f|$). Thus we proved the existence of a solution of the problem (2.13). The strict convexity of j leads to easily treat the uniqueness question using in (2.13), satisfied by u and $\tilde{u}$, both solutions, the test function $v = \dfrac{u+\tilde{u}}{2}$ and adding the two expressions. $\square$

3. T-minima on convex sets Mosco converging

The investigations of the properties of obstacle problems when the obstacle varies relies on a notion of convergence for sequences of convex sets introduced by U. Mosco in [15, 14].

There are many papers devoted to the Mosco-convergence; here we recall only [3, 2, 11, 10] and the references therein; in particular, in [10] there a large list of cases of Mosco-convergence of convex of obstacle type.

Definition 3.1. Let $\{K_n\}$ be a sequence of subsets of a Banach space X. The strong lower limit

$$s - \liminf_{n\to+\infty} K_n$$

of the sequence $\{K_n\}$ is the set of all $v \in X$ such that there exists a sequence $v_n \in K_n$, for n large, converging to v strongly in X. The weak upper limit

$$w - \limsup_{n\to+\infty} K_n$$

of the sequence $\{K_n\}$ is the set of all $v \in X$ such that there exists a sequence $\{v_k\}$ converging to v weakly in X and a sequence of integers n_k converging to $+\infty$, such that $v_k \in K_{n_k}$. The sequence $\{K_n\}$ converges to the set K in the sense of Mosco, shortly $K_n \xrightarrow{M} K$, if

$$s - \liminf_{n\to+\infty} K_n = w - \limsup_{n\to+\infty} K_n = K.$$

Let ψ_0, $\psi_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be such that

$$\|\psi_n\|_{L^\infty(\Omega)}, \|\psi_0\|_{L^\infty(\Omega)} \leq \tilde{M}, \tag{3.1}$$

and let us consider the two T-minimum problems for $f \in L^1(\Omega)$:

$$\begin{cases} T_i(u_n) \in W_0^{1,p}(\Omega),\ \forall i > 0,\ u_n \geq \psi_n \text{ a.e. in } \Omega, \\[2ex] \displaystyle\int_\Omega j(x, D\{\varphi + T_i[u_n - \varphi]\}) - \int_\Omega f(x)\, T_i[u_n - \varphi] \leq \int_\Omega j(x, D\varphi), \\[2ex] \forall\ \varphi \in \mathcal{K}(\psi_n) \cap L^\infty(\Omega), \quad \forall\ i > 0, \end{cases} \tag{3.2}$$

$$\begin{cases} T_i(u_0) \in W_0^{1,p}(\Omega), \ \forall i > 0, \ u_0 \geq \psi_0 \text{ a.e. in } \Omega, \\ \displaystyle\int_\Omega j(x, D\{\varphi + T_i[u_0 - \varphi]\}) - \int_\Omega f(x)\, T_i[u_0 - \varphi] \leq \int_\Omega j(x, D\varphi), \\ \forall\ \varphi \in \mathcal{K}(\psi_0) \cap L^\infty(\Omega), \quad \forall\ i > 0, \end{cases} \tag{3.3}$$

whose existence comes from the previous Section.

We will prove the following result.

Theorem 3.2. *Let us assume (1.2), (1.3), (2.1) and (3.1), with $\mathcal{K}(\psi_n)$ converging to $\mathcal{K}(\psi_0)$ in the sense of Mosco. Then, for every $j > 0$, $T_j(u_n)$ converges to $T_j(u_0)$ weakly in $W_0^{1,p}(\Omega)$. Moreover*

$$\lim_{n\to+\infty} \int_\Omega j(x, DT_j(u_n)) = \int_\Omega j(x, DT_j(u_0)), \quad \forall j \geq \tilde{M}. \tag{3.4}$$

Proof. To simplify the exposition, it is convenient to divide the proof into various steps.

Step 1 - We will prove that, for every $j > 0$, $T_j(u_n)$ converges weakly in $W_0^{1,p}(\Omega)$ (up to a subsequence not relabeled).

Let $v_0 \in \mathcal{K}(\psi_0) \cap L^\infty(\Omega)$; by the definition of Mosco-convergence, there exist $\tilde{v}_n \in \mathcal{K}(\psi_n)$ strongly convergent to v_0 in $W_0^{1,p}(\Omega)$. Let $M^* = \max\{\tilde{M}, \|v\|_{L^\infty(\Omega)}\}$ and $v_n = T_{M^*}(\tilde{v}_n)$. Then $v_n \in \mathcal{K}(\psi_n)$, the sequence $\{v_n\}$ is bounded in $L^\infty(\Omega)$ and it is strongly convergent to v_0 in $W_0^{1,p}(\Omega)$.

Let us insert the function $\varphi = v_n$ in (3.2); we get

$$\int_{\{|u_n - v_n| \leq i\}} j(x, Du_n) \leq \int_{\{|u_n - v_n| \leq i\}} j(x, Dv_n) + \int_\Omega f T_i[u_n - v_n] \tag{3.5}$$

Arguing as in the proof of Theorem 2.1 (see (2.5)), we deduce the analogous of (2.7) for u_n (recall that the sequence $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$):

$$\int_\Omega |DT_j(u_n)|^p \leq C(j+1), \tag{3.6}$$

for every $j > 0$. As before, this implies that, up to a subsequence not relabeled, u_n converges a.e. in Ω to a measurable function u^* and $T_j(u_n)$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_j(u^*)$ for every $j > 0$.

Step 2 - We will prove that $u^* \geq \psi_0$ a.e. in Ω.

We first recall that in [11] the author proved that if $\mathcal{K}(\psi_n)$ converges to $\mathcal{K}(\psi_0)$ in the sense of Mosco, then also $\mathcal{K}(T_j(\psi_n))$ converges to $\mathcal{K}(T_j(\psi_0))$ in the sense of Mosco, for every $j > 0$. By the weak convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_n)$ to $T_j(u^*)$ we deduce that $T_j(u^*) \geq T_j(\psi_0)$ a.e. in Ω, for every $j > 0$, so that $u^* \geq \psi_0$ a.e. in Ω.

Step 3 - We will prove that $u^* = u_0$.

We rewrite (3.5) as

$$\int_\Omega j(x, D\{v_n + T_i[u_n - v_n]\}) \le \int_\Omega j(x, Dv_n) + \int_\Omega f(x)\, T_i[u_n - v_n]. \tag{3.7}$$

Using the lower semicontinuity of the left hand side with respect to the weak convergence in $W_0^{1,p}(\Omega)$ (see [12]) and the continuity of the right hand side we can pass to the limit, proving that u^* is a solution of the T-minimum problem (3.3).

The uniqueness result about T-minima implies that $u^* = u_0$ and that it is the whole sequence $\{T_j(u_n)\}$ which converges to $T_j(u_0)$ weakly in $W_0^{1,p}(\Omega)$, for every $j > 0$.

Step 4 - We will prove (3.4).

Let $j \ge \tilde{M}$, then $T_j(u_0) \in \mathcal{K}(\psi_0) \cap L^\infty(\Omega)$. By the definition of Mosco-convergence, there exists a sequence $\Phi_n \in \mathcal{K}(\psi_n)$ strongly convergent to $T_j(u_0)$ in $W_0^{1,p}(\Omega)$ and, as in *Step 1*, we can assume that Φ_n is equi-bounded (by j in this case).

For $h > j$ let us insert in (3.2) the test function $v_n = T_h(u_n) - T_j(u_n) + \Phi_n \in \mathcal{K}(\psi_n) \cap L^\infty(\Omega)$:

$$\int_\Omega j(x, D\{v_n + T_i[u_n - v_n]\}) \le \int_\Omega j(x, Dv_n) + \int_\Omega f T_i[u_n - v_n].$$

We split the first integral in the right hand side where $|u_n| \le j$, $j < |u_n| \le h$ and $|u_n| > h$ obtaining

$$\int_\Omega j(x, Dv_n) = \int_{\{|u_n| \le j\}} j(x, D\Phi_n) + \int_{\{j < |u_n| \le h\}} j(x, D(u_n + \Phi_n)) + \int_{\{|u_n| > h\}} j(x, D\Phi_n).$$

Analogously, choosing $i \ge 2j$, the integral in the left hand side becomes

$$\int_\Omega j(x, D\{v_n + T_i[u_n - v_n]\}) = \int_{\{|u_n| \le j\}} j(x, Du_n) + \int_{\{j < |u_n| \le h\}} j(x, Du_n) + \int_{\{|u_n| > h\}} j(x, D\{\Phi_n + T_i[u_n - T_h(u_n) + T_j(u_n) - \Phi_n]\}).$$

To sum up

$$\begin{aligned}
&\int_\Omega j(x, DT_j(u_n)) + \int_{\{j<|u_n|\le h\}} j(x, Du_n)\\
&\quad + \int_{\{|u_n|>h\}} j(x, D\{\Phi_n + T_i[u_n - T_h(u_n) + T_j(u_n) - \Phi_n]\})\\
&\le \int_{\{|u_n|\le j\}} j(x, D\Phi_n) + \int_{\{j<|u_n|\le h\}} j(x, D(u_n + \Phi_n))\\
&\quad + \int_{\{|u_n|>h\}} j(x, D\Phi_n)\\
&\quad + \int_\Omega f T_i[u_n - T_h(u_n) + T_j(u_n) - \Phi_n].
\end{aligned}$$

Since the third integral in the left hand side is non-negative the previous estimate can be written as

$$\begin{aligned}
&\int_\Omega j(x, DT_j(u_n)) + \int_{\{j<|u_n|\le h\}} j(x, Du_n)\\
&\le \int_{\{|u_n|\le j\}} j(x, D\Phi_n) + \int_{\{j<|u_n|\le h\}} j(x, D(u_n + \Phi_n))\\
&\quad + \int_{\{|u_n|>h\}} j(x, D\Phi_n)\\
&\quad + \int_\Omega f T_i[u_n - T_h(u_n) + T_j(u_n) - \Phi_n].
\end{aligned} \tag{3.8}$$

Now we bring together the second terms of the left and of the right hand side, respectively, writing

$$\int_{\{j<|u_n|\le h\}} [j(x, D(u_n + \Phi_n)) - j(x, Du_n)];$$

we apply the following inequality

$$|j(x,\xi) - j(x,\eta)| \le C(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|,$$

which comes by the convexity of $j(x,\cdot)$ and by (1.3). Thus we have

$$\begin{aligned}
&\left| \int_{\{j<|u_n|\le h\}} [j(x, D(u_n + \Phi_n)) - j(x, Du_n)] \right|\\
&\le C \int_{\{j<|u_n|\le h\}} (1 + |Du_n|^{p-1} + |D\Phi_n|^{p-1})|D\Phi_n|\\
&\le C(1 + \|T_h(u_n)\|_{W_0^{1,p}(\Omega)}^{\frac{p}{p'}} + \|\Phi_n\|_{W_0^{1,p}(\Omega)}^{\frac{p}{p'}}) \left(\int_{\{|u_n|>j\}} |D\Phi_n|^p \right)^{\frac{1}{p}}\\
&\le C(h+1)^{\frac{1}{p'}} \left(\int_{\{|u_n|>j\}} |D\Phi_n|^p \right)^{\frac{1}{p}},
\end{aligned}$$

where we also used the Hölder inequality and (3.6). Let us note that when n tends to infinity, the last term goes to 0 since $D\Phi_n$ strongly converges to $DT_j(u_0)$ in $L^p(\Omega)$ and u_n converges to u_0 a.e. in Ω, from which $|D\Phi_n|^p\chi_{\{|u_n|>j\}}$ converges to 0 strongly in $L^1(\Omega)$. The convergences

$$\lim_{n\to\infty}\int_{\{|u_n|>h\}} j(x, D\Phi_n) = 0$$

and

$$\lim_{n\to\infty}\int_{\{|u_n|\le j\}} j(x, D\Phi_n) = \int_\Omega j(x, DT_j(u_0))$$

have an analogous justification, while for the first term in the left hand side of (3.8) we use the weak lower semicontinuity in $W_0^{1,p}(\Omega)$, so that

$$\begin{aligned}\int_\Omega j(x, DT_j(u_0)) &\le \liminf_{n\to+\infty}\int_\Omega j(x, DT_j(u_n)) \le \limsup_{n\to+\infty}\int_\Omega j(x, DT_j(u_n))\\ &\le \int_\Omega j(x, DT_j(u_0)) + \int_\Omega fT_i(u_0 - T_h(u_0)).\end{aligned}$$

Finally, passing to the limit as h goes to infinity we get the result. □

3.1. Convergence with the Q-condition

In this subsection, for the sake of simplicity, we assume $p = 2$. Under the assumptions (1.2), (2.10), (2.11), (2.12), (3.1), we consider minimization problems for the functional

$$I(v) = \int_\Omega j(x, Dv) + \frac{1}{2}\int_\Omega a(x)v^2 - \int_\Omega f(x)\,v(x),$$

on $\mathcal{K}(\psi_n)$ and on $\mathcal{K}(\psi_0)$: u_n is the minimum on $\mathcal{K}(\psi_n)$ and u_0 is the minimum on $\mathcal{K}(\psi_0)$ (their existence comes from Subsection 2.1). If $\mathcal{K}(\psi_n) \xrightarrow{M} \mathcal{K}(\psi_0)$, then, by (2.14), the sequence $\{u_n\} \in L^\infty(\Omega)$ is equi-bounded in $W_0^{1,2}(\Omega)$. Thus (up to a subsequence not relabeled) the sequence $\{u_n\}$ converges weakly in $W_0^{1,2}(\Omega)$ to some u^* and by the definition of Mosco-convergence we easily get that $u^* \in \mathcal{K}(\psi_0)$.

Let $v_0 \in \mathcal{K}(\psi_0)$; in the first step of the proof of the Theorem 3.2, we proved the existence of $v_n \in \mathcal{K}(\psi_n)$, with $\{v_n\}$ a bounded sequence in $L^\infty(\Omega)$ strongly convergent to v_0 in $W_0^{1,2}(\Omega)$. Then it is easy to pass to the limit (weak lower semicontinuity on the left hand side, continuity on the right hand side) in inequality

$$I(u_n) \le I(v_n)$$

and to deduce that u^* is a minimum of the integral functional I on $\mathcal{K}(\psi_0)$. Finally the uniqueness gives $u^* = u_0$.

Now we can prove the strong $W_0^{1,2}(\Omega)$-convergence of the sequence $\{u_n\}$. We consider a sequence $w_n \in \mathcal{K}(\psi_n)$, with $\{w_n\}$ equi-bounded in $L^\infty(\Omega)$ and strongly convergent to u_0 in $W_0^{1,2}(\Omega)$. The minimality of u_n

$$I(u_n) \le I(w_n)$$

and the property of I give the convergence

$$\int_\Omega j(x, Du_n) \to \int_\Omega j(x, Du_0).$$

This convergence is the convergence (12) of [10], which implies (the proof is not short) the convergence (20) of [10], that is the strong convergence of the sequence $\{u_n\}$ to u_0 in $W_0^{1,2}(\Omega)$. Thus we proved the following result.

Theorem 3.3. *Let us assume (1.2), (1.3), (2.10), (2.11), (3.1) and let u_n be the minimum of I on $\mathcal{K}(\psi_n)$ and let u_0 be the minimum of I on $\mathcal{K}(\psi_0)$, with $\mathcal{K}(\psi_n) \xrightarrow{M} \mathcal{K}(\psi_0)$. Then the sequence $\{u_n\}$ strongly converges to u_0 in $W_0^{1,2}(\Omega)$.*

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