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Perron-Frobenius, principal eigenvalue, Maximum Principle: a personal itinerary

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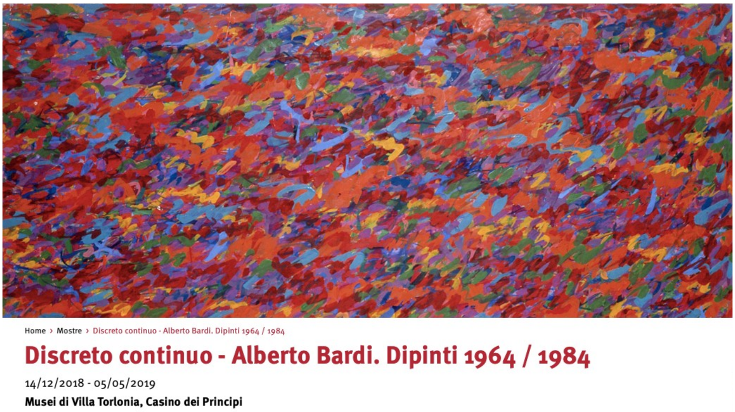
Dedicato ad Umberto con amicizia e gratitudine

Abstract. *This paper dedicated to Umberto Mosco revisits a line of my research starting in the early 70's in the aim of identifying a path connecting different concepts which play a role in elliptic pde's, spectral theory and optimal control.*



1. Optimal stopping of a finite state Markov chain

The starting point of my personal itinerary through the Perron-Frobenius theorem, the notion of principal eigenvalue and its relations with the Maximum Principle for elliptic equations goes back to the time of a collaboration with Massimo Lorenzani and Fabio Spizzichino [17, 18] on the optimal stopping time problem for a Markov chain with a finite number of possible states and an infinite horizon cost criterion.

Following the approaches of E.B. Dynkin [30] and of B.I. Grigelionis and A.N. Sirjaev [33] that optimal control problem can be formulated as the one to determine

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a vector $u \in \mathbb{R}^n$ satisfying the following system of inequalities

$$u \leq g \, , (P - I)u \leq c \, , \ (u - g) \cdot ((P - I)u - c) = 0 \tag{1.1}$$

where P is the $n \times n$ transition matrix of the Markov chain, g e c represent, respectively, stopping and transition costs while $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$. Observe that by the change of variable $x = u - g$, (1.1) can be rewritten as

$$x \leq 0 \, , Ax \leq f \, , \ x \cdot (Ax - f) = 0 \, , \tag{1.2}$$

with $A = P - I$ and $f = c - Ag$. In the early 70's problems of the type (1.2), known as linear complementarity problems, were the object of wide interest in the domain of optimisation theory and operations research. In this context, my tesi di laurea [15] advised by Umberto Mosco examined among other the relations of complementarity problems with obstacle problems in the theory of variational inequalities, see the seminal papers [38],[39] and also [20].
A comprehensive account on complementarity systems and their applications is the book [25].

A result in [26] guarantees that the complementarity problem (1.2) has a unique solution for any f if A is a Minkowski matrix, that is A is non singular and A^{-1} has nonnegative entries.

Coming back to (1.1), since $P = (p_{ij})$ is a stochastic matrix, that is

$$p_{ij} \geq 0 \ \ \forall i, j \in (1, \dots, n) \quad , \quad \sum_{j=1}^{n} p_{ij} = 1 \ \ \forall i \in (1, \dots, n) \, ,$$

the kernel of $A = P - I$ contains at least the vector $(1, \dots, 1)$. However, if $P = (p_{ij})$ is irreducible, i.e. there is no subset I of $(1, \dots, n)$ such that

$$I \neq \emptyset \quad , \quad I \neq (1, \dots, n) \ \text{ such that } \ p_{ij} = 0 \, \forall i \in I, \forall j \in (1, \dots, n) \setminus I \, ,$$

then as a consequence of the classical Perron-Frobenius theorem, see the next section, there exists a vector $m = (m_1, \dots, m_n)$ such that

$$m_i > 0 \ \ \forall i \in (1, \dots, n) \text{ and } \sum_{i}^{n} m_i = 1 \text{ such that } P^t m = m \, , \tag{1.3}$$

where P^t is the transpose of P.

Moreover, the vector m is an eigenvector of P^t associated to the simple eigenvalue $\lambda_{PF} = 1$, the kernel of $A = P - I$ is one dimensional and λ_{PF}, the Perron-Frobenius eigenvalue, is the largest eigenvalue of P^t. From the probabilistic point of view, m is the stationary probability distribution of the Markov chain.

In this framework, the main result in [18] is the following:

Theorem 1.1. *If problem (1.1) has a solution then necessarily*

$$c \cdot m \geq 0 \tag{1.4}$$

If (1.4) holds with a strict inequality then (1.1) has a unique solution for any g.

It is also possible to interpret the solution of problem (1.2) as the value of the ergodic problem associated to the regularized α-discounted problem

$$u \leq g \,, (P-I)u + \alpha u \leq c \,, \; (u-g)\cdot((P-I)u + \alpha u - c) = 0 \tag{1.5}$$

More precisely, if $c \cdot m > 0$ then the solutions u_α of (1.5) with $\alpha > 0$, converge as $\alpha \to 0^+$ to the unique solution of (1.1). This has also a dynamic interpretation since the asymptotic behaviour of the Markov chain is given by the relations

$$\lim_{t\to+\infty} \frac{1}{e\cdot x^t} x^t = m \quad , \quad \lim_{t\to+\infty} \frac{x_i^{t+1}}{x_i^t} = \lambda_{PF}$$

where $e = (1\ldots1)$, irregardless of the initial state x_0, and x^t is given recursively by

$$x^{t+1} = Px^t \;,\; t = 0,1,2,\ldots$$

2. The Perron-Frobenius theorem

For positive definite matrices all eigenvalues are positive while matrices $M > 0$, meaning that all entries $(m_{ij}) > 0$ are strictly positive, may have some negative eigenvalues but they admit nonetheless at least a positive eigenvalue, denoted by $\lambda_{PF}(M)$, which is real and larger than the modulus of the other eigenvalues. Moreover, $\lambda_{PF}(M)$ is associated to a 1-dimensional eigenspace generated by an eigenvector with strictly positive components.
This is the classical Perron-Frobenius theorem, see for example [32],[4].

A non algebraic approach to the proof of this theorem is based on an optimization principle described in the next result, providing a useful representation formula for $\lambda_{PF}(M)$ which, as we will see in the sequel, has a counterpart in the infinite dimensional setting related to differential or integral operators.

Theorem 2.1. *Let $M > 0$ and set*

$$S^-(M) = \{\lambda \geq 0 : \exists\, 0 \leq x \in R^n \text{ such that } Mx \geq \lambda x\}$$

Then the Perron-Frobenius eigenvalue $\lambda_{PF}(M)$ is given by the solution of the constrained optimization problem

$$\lambda_{PF}(M) = \sup_{\lambda\in S^-(M)} \lambda \tag{2.1}$$

and the max-min Collatz-Wielandt formula holds

$$\lambda_{PF}(M) = \max_{x\in\Sigma} \min_{i\in(1\ldots n)} \frac{\sum_{j=1}^n m_{ij}x_j}{x_i} \tag{2.2}$$

where Σ is the simplex $\{x \geq 0, \sum_{i=1}^n x_i = 1\}$.

For the proof of (2.2) one can see for example [46]. Let us sketch instead here that of (2.1). It is not restrictive to take $x \in \Sigma$; it is immediate to check that $S^-(M)$ is closed and bounded and that the sup in (2.1) is indeed a maximum.

Let λ^* be an optimal value and x^* an optimal point for (2.1), so that

$$\lambda^* > 0, x^* \geq 0, x^* \neq 0, Mx^* \geq \lambda^* x^* \tag{2.3}$$

Let us show first that

$$Mx^* = \lambda^* x^* \tag{2.4}$$

Assume by contradiction that this is not the case so that $(Mx^*)_k - \lambda^* x^*_k > 0$ for some k and look at $\tilde{x} = (x^*_1, \ldots, x^*_k + \varepsilon, \ldots, x^*_n)$ where $\epsilon > 0$. Observe that $0 \neq \tilde{x} \geq 0$ and that for $i \neq k$

$$(M\tilde{x})_i = (Mx^*)_i + m_{ik}\varepsilon > (Mx^*)_i \geq \lambda^* x^*_i = \lambda^* \tilde{x}_i$$

while

$$(M\tilde{x})_k - \lambda^* \tilde{x}_k = (Mx^*)_k - \lambda^* x^*_k - \varepsilon(\lambda^* - m_{kk}) > 0$$

for sufficiently small ε. This shows that for small $\varepsilon > 0$ we have $M\tilde{x} > \lambda^* \tilde{x}$, meaning that $M\tilde{x} \geq \lambda \tilde{x}$ for some $\lambda > \lambda^*$, contradicting the definition of λ^*.
The vector x^* is strictly positive: if $x^*_k = 0$ for some k from (2.4) it follows that $(Mx^*)_k = 0$ contradicting the fact that $Mx^* > 0$ which is a consequence of the assumption $M > 0$ and relation (2.3).

It remains to show that λ^* coincides indeed with $\lambda_{PF}(M)$: take any other eigenvalue- eigenvector pair (λ, z) for M and set $\overline{z} = (|z_1|, \ldots, |z_n|) \geq 0$.
Since $M > 0$ it follows that $M\overline{z} \geq |Mz| \geq |\lambda|\overline{z}$, implying $|\lambda| \leq \lambda^*$ and $\lambda^* = \lambda_{PF}(M)$.

Remark 2.2. Nonnegative matrices arise in many different situations including in second-order centered finite difference approximations of elliptic boundary value problems such as the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega \quad , \quad u = g \text{ on } \partial\Omega \, , \tag{2.5}$$

see for example [37].

The discretized problem with mesh h is a system of linear algebraic equations

$$\Delta_h U = G \tag{2.6}$$

governed by a square matrix Δ_h which is positive definite, and therefore invertible, having off-diagonal entries less or equal than 0.
Such matrices are known in linear algebra as Minkowski matrices and their inverse matrix have strictly positive entries, see [40]. Consequently, the Perron-Frobenius eigenvalue of Δ_h^{-1} is the reciprocal of the minimum eigenvalue of Δ_h and the following form of the Maximum Principle holds for solutions of (2.6)

$$G \geq 0 \text{ implies } U = \Delta_h^{-1} G \geq 0 \, ,$$

see [43] for several results concerning the validity of the Maximum Principle for linear algebraic systems. A toy example of such matrices are

$$\Delta_h = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \quad , \quad \Delta_h^{-1} = \frac{1}{56} \begin{pmatrix} 15 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 4 \end{pmatrix}$$

with Perron eigenvalue $\lambda_{PF}(\Delta_h^{-1}) = \dfrac{4+\sqrt{2}}{14}$.

3. Optimal stopping of reflected diffusions

An infinite dimensional version of the complementarity problem discussed in Section 2 is the oblique derivative problem with obstacle: determine a function $u\colon \Omega \to \mathbb{R}$, Ω bounded open subset of $\mathbb{R}^n$, satisfying the sistem

$$u \le 0\,, \quad Au \le f\,, \quad u(Au - f) = 0 \quad \text{in } \Omega \tag{3.1}$$

and the oblique derivative condition

$$Bu = \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} = 0 \quad \text{on } \partial\Omega\,. \tag{3.2}$$

The operator A in system (3.1) is a second-order uniformly elliptic operator of the form

$$A = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} \tag{3.3}$$

so that (3.1),(3.2) are the optimality conditions for the optimal stopping problem of the Markov process whose infinitesimal generator is A with reflection conditions on the boundary of Ω. In particular, if b is the normal vector to $\partial\Omega$, the boundary condition in (3.2) is the Neumann one.

A classical result, see for example [5], states that the boundary value problem

$$A^* m = 0 \quad (x \in \Omega) \quad , \quad B^* m = 0 \quad (x \in \partial\Omega) \tag{3.4}$$

where A^* and B^* are the adjoints of A and B, has a unique solution m with

$$m > 0 \ \text{ and } \ \frac{1}{|\Omega|} \int_\Omega m dx = 1\,.$$

The function m is the analogue of the Perron-Frobenius eigenvector associated to the eigenvalue 0 of A^* which, at least formally, plays the same role as the matrix $(P - I)^t$. The following result, formally very close to the one in Section 2, is taken from the work [16] in collaboration with Maria Giovanna Garroni:

Theorem 3.1. *Assume $f \in L^p$ and that the coefficients of A are Lipschitz continuous. If $\int_\Omega f m dx > 0$ then (3.1), (3.2) has a unique solution $u \in W^{2,p}(\Omega)$.*

In [16] it is also proved that, under assumption $\int_\Omega f m dx > 0$, then u is the weak limit as $\alpha \to 0^+$ in $W^{2,p}(\Omega)$ of the solutions u_α of the discounted problems

$$u_\alpha \leq 0\,, \quad Au_\alpha + \alpha u_\alpha \leq f\,, \quad u_\alpha(Au_\alpha + \alpha u_\alpha - f) = 0$$

In the case $\int_\Omega f m dx < 0$ or $\int_\Omega f m dx = 0$ different limiting behaviours occur, see [16]. A key tool in obtaining the a priori estimates needed in the proof is the general form of the Lewy-Stampacchia inequality, see [34].

The pde approach to ergodic problems in optimal control received considerable attention in recent years and different nonlinear models and boundary conditions have been treated by similar Fredholm alternative ideas. A relevant model is the Dirichlet problem

$$-\Delta u + |\nabla u|^2 = 1 \text{ in } \Omega \quad , \quad u = 0 \text{ on } \partial\Omega \tag{3.5}$$

occurring in optimal exit time problems.

The standard Hopf-Cole transformation linearizes the above problem and it can be proved that (3.5) has a solution if and only if $\lambda_1(-\Delta + 1, \Omega)$, the principal eigenvalue of the associated linear Dirichlet problem in Ω, is strictly positive.

The discounted problem

$$-\Delta u_\alpha + |\nabla u_\alpha|^2 + \alpha u_\alpha = 1 \text{ in } \Omega \quad , \quad u_\alpha = 0 \text{ on } \partial\Omega \tag{3.6}$$

has a solution for every $\alpha > 0$. When (3.5) has a solution then $u_\alpha \to u$ in H_0^1 as $\alpha \to 0^+$. On the other hand, when (3.5) has no solution then αu_α converges locally uniformly to a constant c_0 characterized as the unique number such that the differential problem with explosive boundary conditions

$$-\Delta v + |\nabla v|^2 + c_0 = 1 \text{ in } \Omega \quad , \quad v = +\infty \text{ on } \partial\Omega \tag{3.7}$$

has a solution. It turns out also that (3.5) has a unique solution if and only if $c_0 > 0$ while (3.5) has no solution if $c_0 \leq 0$. Moreover, it is relevant to point out that c_0 coincides with $\lambda_1(-\Delta + 1, \Omega)$ and also, especially in view of the next section, that the representation formula

$$c_0 = \sup\{c \in \mathbb{R} : \exists \psi \text{ such that } -\Delta\psi + |\nabla\psi|^2 + c \leq 1 \text{ in the viscosity sense}\}$$

holds. We refer to the nice paper [41] which clarifies the relations between principal eigenvalue, maximum principle and ergodic constant in Bellman type nonlinear pde's in the framework of more general, genuinely nonlinear, growth behaviour in the gradient term $|\nabla u|^q$ with $1 < q \leq 2$.

4. Elliptic operators: principal eigenvalue and Maximum Principle

This section reviews some results concerning the relations between the principal eigenvalue, or some generalized notion of it, and the validity of the weak Maximum Principle

$$(\mathbf{MP}) \qquad F(x, u, Du, D^2u) \geq 0 \text{ in } \Omega,\ u \leq 0 \text{ on } \partial\Omega \text{ implies } u \leq 0 \text{ in } \Omega\,.$$

Here $u \in USC(\overline{\Omega})$, the set of real-valued upper semicontinuous functions on $\overline{\Omega}$, F is a degenerate elliptic fully nonlinear mapping from $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ into $\mathbb{R}$ and $\mathcal{S}^n$ is the space of $n \times n$ symmetric matrices.

In what follows $\Omega \subset \mathbb{R}^n$ will be a general bounded domain with possibly irregular boundary and the differential inequality will be understood to hold in the viscosity sense, see [27].

I will review first the case when F is a linear uniformly elliptic operator in non-divergence form following the treatment in [7], in a second subsection I will report on a characterization result in [6] concerning degenerate fully nonlinear elliptic operators, a third one is dedicated to a very recent result in collaboration with Antonio Vitolo [23] for the case of cooperative systems.

4.1. Positivity of the principal eigenvalue and (MP): the uniformly elliptic case

Let us consider the linear operator

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u, \ \alpha\, I \leq A(x) \leq \beta\, I \tag{4.1}$$

with, say, continuous and bounded coefficients A, b, c, $0 < \alpha < \beta$. Note that in this section the sign convention adopted on the principal part of the operator is the opposite one with respect to Section 3, see (3.3).

Several sufficient conditions of different nature are known, see [42], for the validity of weak Maximum Principle in a bounded domain Ω, e.g.

- (i) $c(x) \leq 0$,
- (ii) exists $\psi > 0$ in $\overline{\Omega}$ such that $L[\psi] \leq 0$,
- (ii) Ω is narrow (i.e. contained in a suitably small strip).

Simple examples show however that none of these conditions is however necessary for the validity of the weak Maximum Principle.

What about sufficient and also necessary conditions for the validity of the Maximum Principle?

An important characterisation result due to Berestycki, Nirenberg and Varadhan [7] asserts that **(MP)** holds for L in a bounded domain Ω if and only if the number λ_1 defined by

$$\lambda_1 := \sup \{\lambda \in \mathbb{R} : \exists\, \psi > 0 \text{ in } \Omega \text{ such that } L[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\} \tag{4.2}$$

is strictly positive. In the definition of λ_1, $\psi \in W^{2,p}_{loc}(\Omega)$. Notably, this very nice numerical criterion was proved to hold under mild conditions on the coefficients and applies to a large class of domains with rough boundary $\partial\Omega$. In the above mentioned result the matrix $A(x)$ is required to be uniformly positive definite but not necessarily symmetric. Note that even for symmetric A the operator L is not in general self-adjoint due to the presence of the drift term b.

Nonetheless, in [7] it is proved that the number λ_1 shares some of the properties of the classical principal eigenvalue for $-L$ with Dirichlet conditions namely:

- there exists a principal eigenfunction $w_1 > 0$ in Ω such that $L[w_1] + \lambda_1 w_1 = 0$ in Ω, $w_1 = 0$ on $\partial\Omega$,
- w_1 is simple,
- $\mathrm{Re}\lambda \geq \lambda_1$ for any other eigenvalue λ of L.

The Berestycki-Nirenberg-Varadhan definition above can be expressed by the equivalent pointwise min-max formula

$$\lambda_1 = -\inf_{\psi(x)>0} \sup_{x\in\Omega} \frac{L\psi(x)}{\psi(x)} \tag{4.3}$$

where $\psi \in W^{2,p}_{loc}(\Omega)$. The same formula, under more restrictive conditions (smooth boundary, continuous coefficients), have been considered before in [29], see also [47]. A much older reference is [3] where the same min-max formula is proposed in the particular case $L = \Delta$.

Remark 4.1. Definition (4.2) is clearly reminiscent of that of Perron-Frobenius eigenvalue of a positive matrix, see Section 2, formulas (2.1) and (2.2).
This is formally justified by the observation that for the solution operator L^{-1} of the Dirichlet problem (2.5) the sign propagation property

$$L^{-1}g \geq 0 \text{ in } \Omega \quad \text{if} \quad g \geq 0 \text{ on } \partial\Omega$$

holds and the obvious fact that the maximum eigenvalue of a matrix is the minimum one for its inverse.

The existence of a positive eigenfunction associated to λ_1 in the Berestycki, Nirenberg and Varadhan setting follows from the Krein-Rutman theorem thanks to compactness estimates guaranteed by the uniform ellipticity of L and the boundedness of Ω. This existence argument, which is delicate in the infinite dimensional setting, is straightforward in finite dimensions, see the proof of Theorem 2.1.

Recent related research motivated by control and game theory is reported in [2]. In particular, some interesting infinite dimensional versions of the Collatz-Wielandt formulas have been proposed there.

Remark 4.2. For the self-adjoint operator $Lu(x) = \mathrm{div}(A(x)Du) + c(x)u$ the principal eigenvalue $\lambda_1 = \lambda_1(L, \Omega)$ is given by the classical Rayleigh-Ritz variational formula

$$\lambda_1 := \min_{\psi\in H^1_0(\Omega), ||\psi||_{L^2(\Omega)}=1} \int_\Omega \left(A(x)D\psi \cdot D\psi + c(x)\psi^2\right) dx$$

For linear operators in divergence form there is a vast literature on computational methods for the principal eigenvalue. On the other hand, for general non-divergence type elliptic operators such as in in (4.1) the Rayleigh-Ritz variational formula is not available anymore.

In [9] we developed a finite difference scheme for the computation of the principal eigenvalue and the principal eigenfunction of fully nonlinear uniformly elliptic operators based on the min-max formula (4.3) which can be seen as a pointwise alternative to the Rayleigh-Ritz integral formula. The approximation results in [9] seem to be new even in the linear case.

4.2. Positivity of a generalized principal eigenvalue and (MP): the degenerate elliptic case

A numerical criterion similar to the one in [7] may hold in the case L is degenerate elliptic, meaning by this that the matrix $A(x)$ is just positive semidefinite?

It is well-known that in this setting the Dirichlet problem is not well-posed in general. Let us mention in this respect the role played by the following condition: a boundary point $\xi \in \partial\Omega$ satisfies the Fichera condition, see [31], if either

$$A(\xi)Dd(\xi) \cdot Dd(\xi) > 0$$

or

$$A(\xi)Dd(\xi) \cdot Dd(\xi) = 0 \ \text{ and } \ \mathrm{Tr}(A(\xi)D^2d(\xi)) + b(\xi) \cdot Dd(\xi) < 0\,,$$

where d is the signed distance function from $\partial\Omega$, positive outside Ω.

Also, a principal eigenvalue and an associated eigenfunction may not exist in the degenerate elliptic case due to the lack of suitable a priori estimates, see however [44] for some positive results in this direction.

In this spirit, a similar numerical criterion for the validity of the weak Maximum Principle has been established in [6]. In order to deal with degeneracies a new index was introduced there by setting, for L as in (4.1),

$$\mu_1(L,\Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \Omega' \supset \overline{\Omega} \text{ and } \psi \in C(\Omega'), \psi > 0 : L[\psi] + \lambda\psi \le 0 \text{ in } \Omega'\}.$$

This rather implicit definition of the index $\mu_1(L,\Omega)$, requiring L to be defined on a larger domain, is motivated in particular by possible degeneracies of coefficients occurring on $\partial\Omega$. Observe, however, that using the monotonicity with respect to set inclusion of λ_1 as defined in (4.2) it is possible to show that

$$\mu_1(L,\Omega) = \sup_{\Omega' \supset \overline{\Omega}} \lambda_1(L,\Omega'). \tag{4.4}$$

The main result in [6] is as follows:

Theorem 4.3. *Let*

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u$$

with continuous coefficients $A(x), b(x), c(x)$ for x in a bounded domain Ω. Assume the monotonicity conditions

$$A(x)\xi \cdot \xi \ge 0 \ \textit{ for all } \ \xi \in \mathbb{R}^n \ \textit{(degenerate ellipticity)}$$

and

$$(b(x) - b(y)) \cdot (x - y) \geq -c|x - y|^2 \text{ for some } c > 0$$

and, moreover,

$$x \to \sqrt{A(x)} \text{ is Lipschitz continuous .}$$

Then,

$$(\mathbf{MP}) \quad L[u](x) \geq 0 \text{ in } \Omega,\ u \leq 0 \text{ on } \partial\Omega \text{ implies } u \leq 0 \text{ in } \Omega$$

holds for all upper semicontinuous functions u satisfying the above differential inequality in the viscosity sense if and only if

$$\mu_1(L, \Omega) > 0 \, . \tag{4.5}$$

Remark 4.4. Consider for simplicity the case where $L[u] = \mathrm{Tr}(AD^2u)$ and let $\Omega \subset B_R$ be contained in B_R, the ball of radius R centered at the origin. The function $\psi(x) = \frac{R^2}{2} - \frac{|x|^2}{2}$ is strictly positive in B_R and $D^2\psi = -I$.

If A is positive semidefinite with moreover $\mathrm{Tr}(A) > 0$, we deduce that

$$\mathrm{Tr}(AD^2\psi) + \lambda\psi = -\mathrm{Tr}(A) + \frac{\lambda}{2}\left(R^2 - |x|^2\right) \leq 0 \text{ in } B_R$$

provided that $\lambda \leq 2\mathrm{Tr}(A)/R^2$ for all $i = 1 \ldots N$. Therefore,

$$\overline{\mu}_1(L, \Omega) \geq \frac{2}{R^2}\,\mathrm{Tr}(A) > 0 \, .$$

Since A is positive semidefinite, the extra assumption $\mathrm{Tr}(A) > 0$ amounts to the requirement that the linear operator L is strictly elliptic at least in one coordinate direction. See [21, 22] for a completely different proof of the validity of the weak Maximum Principle in this setting.

Actually, the above result is proved in [6] for a quite general class of positively homogeneous fully nonlinear operators F which are just degenerate elliptic in the sense that the weak monotonicity condition

$$F(x, r, p, X + Y) \geq F(x, r, p, X)$$

holds true for any non-negative definite matrix $Y \in \mathcal{S}^n$, thus generalizing previous results for uniformly elliptic operators in [10]. Theorem 4.3 applies to some Hessian operators such as the uniformly elliptic Pucci maximal operator

$$P_{\gamma,\Gamma}(D^2u) := \Gamma\, \Sigma_{i \in I_+} \eta_i(D^2u) + \gamma\, \Sigma_{i \in I_-} \eta_i(D^2u)$$

(here, $0 < \gamma < \Gamma$ and I_+, I_- correspond, respectively, to positive and negative eigenvalues of D^2u) and the degenerate elliptic Harvey-Lawson Hessian operators

$$H_k(D^2u) = \eta_{n-k+1}(D^2u) + \cdots + \eta_n(D^2u),$$

k an integer between 1 and n. Here $\eta_1(D^2u) \leq \eta_2(D^2u) \leq \cdots \leq \eta_N(D^2u)$ are the ordered eigenvalues of the matrix D^2u, see also [12] for recent results in this direction.

4.3. The (MP) for cooperative systems

In the very recent paper [23] in collaboration with A.Vitolo we consider systems of elliptic partial differential inequalities of the form

$$L[u] + C(x)u \geq 0\,. \tag{4.6}$$

Here $u = (u_1 \dots u_N)$ is a vector-valued function $u\colon \mathbb{R}^n \to \mathbb{R}^N$, which is intended either as a row or as a column on the occasion. Furthermore, $C(x) = (c_{ij}(x))$ is a $N \times N$ matrix-valued function and $L = (L_1, \dots, L_N)$ is a matrix linear operator acting on u of the form

$$L_i[u] = \mathrm{Tr}(A_i(x)D^2u) + b_i(x) \cdot Du + c_i(x)u, \;\; i = 1 \dots N. \tag{4.7}$$

The vector differential inequality (4.6) is meant to hold componentwise

$$L_i[u] + \sum_{j=1}^{N} c_{ij}(x)u_j \geq 0, \;\; i = 1 \dots N \tag{4.8}$$

in the viscosity sense.

In a number of papers, see for example [45, 13, 1] for linear F_i and [14] for fully nonlinear operators, the validity of the weak Maximum Principle, namely the sign propagation property:

$$u_i \leq 0 \text{ on } \partial\Omega \text{ for all } i = 1 \dots N \;\text{ implies }\; u_i \leq 0 \text{ in } \Omega \text{ for all } i = 1 \dots N, \tag{4.9}$$

where Ω is a bounded domain of $\mathbb{R}^n$, has been established through different approaches. The results obtained in those papers require uniform ellipticity of the operators F_i and the crucial assumption that $C(x) = (c_{ij}(x))_{i,j=1\dots N}$ is a cooperative matrix. By this we mean that the following conditions hold:

$$c_{ij}(x) \geq 0 \;\; \forall\, i \neq j, \quad \sum_{j=1}^{N} c_{ij}(x) \leq 0, \quad i = 1 \dots N \tag{4.10}$$

Observe that cooperativity implies $c_{ii}(x) \leq -\sum_{j\neq i} c_{ij}(x) \leq 0$ for $i = 1 \dots N$, in agreement with what is well-known in relation with the weak Maximum Principle in the diagonal case $c_{ij} \equiv 0$ for $i \neq j$, see [8] for further properties of such matrices.

In [23] we obtained extensions of the results from the above mentioned papers to the case where the L_i are degenerate elliptic (actually, even fully nonlinear) through a reduction to the scalar case adapting an idea in [14] in combination with the main result in [6], reported here in Subsection 4.2, for degenerate elliptic operators.

Actually, we consider the the extremal scalar nonlinear operator

$$F(x,t,\xi,X) = L_1(x,t,\xi,X) \vee \dots \vee L_N(x,t,\xi,X) \equiv \max_{i=1\dots N} L_i(x,t,\xi,X) \tag{4.11}$$

Here and below we use the notation $s \vee t = \max{(s,t)}$. Note that F is a standard Bellman operator occurring in the Dynamic Programming approach to optimal control of possibly degenerate diffusions.

Our main result in this setting is as follows:

Theorem 4.5. *Suppose that the coefficients of the operator L and the entries of the cooperative matrix C are continuous functions of $x \in \Omega$, a bounded domain of $\mathbb{R}^n$. Then the validity of the sign propagation property*

$$w \leq 0 \text{ on } \partial\Omega \;\Rightarrow\; w \leq 0 \text{ in } \Omega$$

for all viscosity solutions $w \in C(\Omega; \mathbb{R})$ of the scalar nonlinear inequality $F[w] \geq 0$ implies that the same property (4.9) holds for all viscosity solutions $u \in C(\Omega; \mathbb{R}^N)$ of the linear system $L[u] + C(x)u \geq 0$ in Ω.

The proof relies on cooperativity of C and lattice properties of viscosity solutions: a first step is to observe fact that if $u = (u_1 \dots u_N)$ is a viscosity solution of (4.6) then

$$L_i(x, u_i^+, Du_i^+, D^2u_i^+) \geq -\sum_{j=1}^{N} c_{ij}(x) u_j^+(x) \tag{4.12}$$

where $u_i^+ = u_i \vee 0$. Next, some viscosity calculus allows to show that the vector function $u^* = u_1^+ \vee \cdots \vee u_N^+$ satisfies the inequality $F[u^*] \geq 0$ in the viscosity sense as well as the boundary condition $u^* \leq 0$ on $\partial\Omega$.

Now we let into the picture the numerical index associated to the nonlinear positively homogeneous of degree 1 degenerate elliptic operator F as defined in (4.11)

$$\mu_1(F, \Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \Omega' \supset \Omega \text{ and } \psi \in C(\Omega'), \psi > 0 : F[\psi] + \lambda\psi \leq 0 \text{ in } \Omega'\}$$

introduced in [6] which has been already considered in Subsection 4.2.

At this point is not difficult to conclude, taking into account the above observations and the main result Theorem 1.3 in [6] that if $\mu_1(F, \Omega) > 0$ then the weak Maximum Principle holds for solution of (4.6).

Example 4.6. Consider linear operators as in (4.7) with positive semidefinite matrices A^i with, just for simplicity, constant entries and zero lower order terms. Let Ω be contained in B_R, the ball of radius R centered at the origin. The function $\psi(x) = \frac{R^2}{2} - \frac{|x|^2}{2}$ is strictly positive in B_R and $D^2\psi = -I$. Then

$$\mathrm{Tr}(A^i D^2\psi) + \lambda\psi = -\mathrm{Tr}(A^i) + \frac{\lambda}{2}\left(R^2 - |x|^2\right) \leq 0 \text{ in } B_R,$$

provided that $\lambda \leq 2\mathrm{Tr}(A^i)/R^2$ for all $i = 1 \dots N$. From this is not difficult to derive the inequality

$$\mu_1(F, \Omega) \geq \frac{2}{R^2}\,\mathrm{Tr}(A^1) \wedge \cdots \wedge \mathrm{Tr}(A^n).$$

If $\mathrm{Tr}(A^i) > 0$ for each $i = 1 \dots N$, then $\mu_1(F, \Omega) > 0$ so that the weak Maximum Principle hold for the cooperative system (4.6). Since the A^i are positive semidefinite matrices, the extra assumption $\mathrm{Tr}(A^i) > 0$ amounts, as already noticed before, to the requirement that each linear operator is strictly elliptic at least in one coordinate direction, as for instance the directional elliptic operators considered in the scalar case in [21, 22].

I refer the reader to [23] for a general version of Theorem 4.5 including a large class of nonlinear degenerate elliptic systems as well for the discussion of more sophisticated examples.

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