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A study of some special rings by delta invariant

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Abstract. *This paper is devoted to study the regular, Gorenstein, generically Gorenstein and non-regular Gorenstein local rings by means of delta invariant.*

1. Introduction

Let R be a local ring. The delta invariant of a finite (i.e., finitely generated) module was defined by M. Auslander ([1]). For a finite R-module M, denote M^{cm} the sum of all submodules $\phi(L)$ of M, where L ranges over all maximal Cohen-Macaulay R-modules with no non-zero free direct summands and ϕ ranges over all R-linear homomorphisms from L to M. The δ invariant of M, denoted by $\delta_R(M)$, is defined to be $\mu_R(M/M^{\text{cm}})$, the minimal number of generators of the the quotient module M/M^{cm}.

A short exact sequence $0 \longrightarrow Y \longrightarrow X \xrightarrow{\varphi} M \longrightarrow 0$ of R-modules is called a Cohen-Macaulay approximation of M if X is a maximal Cohen-Macaulay R-module and Y has finite injective dimension over R ([12, Definition 11.8]). A Cohen-Macaulay approximation $0 \longrightarrow Y \longrightarrow X \xrightarrow{\varphi} M \longrightarrow 0$ of M is called minimal if each endomorphism ψ of X, with $\varphi \circ \psi = \varphi$, is an automorphism of X ([12, Definition 11.11]. If R is a Cohen-Macaulay ring with canonical module ω_R, then a minimal Cohen-Macaulay approximation of M exists and is unique up to isomorphism (see [12, Theorem 11.16], [1, Theorem 1.1]). If the sequence $0 \longrightarrow Y \longrightarrow X \xrightarrow{\varphi} M \longrightarrow 0$ is a minimal Cohen-Macaulay approximation of M, then $\delta_R(M)$ determines the maximal rank of a free direct summand of X (see [12, Exercise 11.47] and [12, Exercise 11.24]). For an integer $n \geq 0$ and an R–module M, $\delta^n_R(M) := \delta_R(\Omega^n_R(M))$ is denoted as the higher delta invariant, where $\Omega^n_R(M)$ is the n th syzygy module of M in its minimal free resolution (paragraph just after [2, Proposition 5.3]).

A commutative Noetherian ring R is called *generically Gorenstein* whenever $R_\mathfrak{p}$ is Gorenstein for every minimal prime ideal $\mathfrak{p}$ of R. It is well known that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with canonical module then R is generically Gorenstein if and only if the canonical module is isomorphic to an ideal of R (see [5, Proposition 3.3.18]). In section 2, we use the delta invariant in order to study rings to be generically Gorenstein, Gorenstein, or regular. Our first result is that a complete local ring $(R, \mathfrak{m}, k)$ is regular if and only if R is Gorenstein and a

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syzygy module of k has a cyclic direct summand R-module whose delta invariant is equal to 1 and satisfies an extra condition (see Theorem 2.3). Our second result, studies Cohen-Macaulay local rings with canonical modules which are Gorenstein (see Theorem 2.4). Also, we study Cohen-Macaulay local rings with canonical modules which are generically Gorenstein but not Gorenstein (see Theorem 2.5)

Section 3 is devoted to presenting a generalization of [16, Theorem 2.3] (see Corollary 3.2) and is devoted to study non-regular Gorenstein rings by means of higher delta invariant (see Corollary 3.4).

Throughout, $(R, \mathfrak{m})$ is a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and all modules are finite (i.e. finitely generated).

2. Generically Gorenstein, regular and Gorenstein rings

We recall the basic properties of the delta invariant.

Proposition 2.1 ([12, Corollary 11.26] and [4, Lemma 1.2])**.** *Let M and N be finite modules over a Gorenstein local ring $(R, \mathfrak{m}, k)$. Then the following statements hold true:*

(i) $\delta_R(M \oplus N) = \delta_R(M) + \delta_R(N)$*;*

(ii) If there is a an R-epimorphism $M \longrightarrow N$, then $\delta_R(M) \geq \delta_R(N)$;

(iii) $\delta_R(M) \leq \mu(M)$*;*

(iv) $\delta_R(k) = 1$ *if and only if R is regular;*

(v) $\delta_R(M) = \mu(M)$ *when* $\text{proj.dim}_R(M)$ *is finite.*

Assume that $(R, \mathfrak{m}, k)$ is a local ring with residue field k. In [7, Corollary 1.3], Dutta presents a characterization for R to be regular in terms of the admitting a syzygy of k with a free direct summand. Later on, Takahashi, in [14, Theorem 4.3], generalized the result in terms of the existence of a syzygy module of the residue field having a semidualizing module as its direct summand. Also Ghosh, Gupta and Puthenpurakal in [8, Theorem 3.7], have shown that the ring is regular if and only if a syzygy module of k has a non-zero direct summand of finite injective dimension.

Now I investigate these notions by means of delta invariant. Denote by $\Omega^i_R(k)$ the ith syzygy, in the minimal free resolution, of k.

Definition 2.2. *An R-module X is said to satisfy the condition $(*)$ whenever, for any X-regular element a, X/aX is indecomposable as R/aR-module.*

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be a complete local ring of dimension d. The following statements are equivalent:*

(i) R is a regular ring;

(ii) R is a Gorenstein ring and $\Omega_R^n(k)$ has a cyclic R-module as its direct summand whose delta invariant is 1 and satisfies the property $(*)$, for some $n \geq 0$.

Proof. (i)⇒(ii). $k = \Omega_R^0(k)$ fulfills our statement by Proposition 2.1.

(ii)⇒(i). Suppose that R is Gorenstein and, for an integer $n \geq 0$, $\Omega_R^n(k) \cong X \oplus Y$ for some R-modules X and Y such that $X \cong R/\mathrm{Ann}_R(X)$ with $\delta_R(X) = 1$. The case $n = 0$ implies that R is regular. So we may assume that $n \geq 1$.

We proceed by induction on d. For the case $d = 0$, if $\mathfrak{m} \neq 0$ then $\mathrm{Soc}\,(R) \neq 0$ and $R/\mathrm{Soc}\,(R)$ is maximal Cohen-Macaulay R-module with no free direct summand and so $\delta_R(R/\mathrm{Soc}\,(R)) = 0$. On the other hand, by [8, Lemma 2.1], $\mathrm{Soc}\,(R) \subseteq \mathrm{Ann}_R(\Omega_R^n(k)) = \mathrm{Ann}_R(X \oplus Y) \subseteq \mathrm{Ann}_R(X)$. Therefore the natural surjection $R/\mathrm{Soc}\,(R) \longrightarrow R/\mathrm{Ann}_R(X) \cong X$ implies that $1 = \delta_R(X) \leq \delta_R(R/\mathrm{Soc}\,(R)) = 0$ which is absurd. Hence $\mathfrak{m} = 0$ and $R = R/\mathfrak{m}$ is regular.

Now we suppose that $d \geq 1$ and the statement is settled for $d - 1$. As R is Cohen-Macaulay, we choose an R-regular element $y \in \mathfrak{m} \setminus \mathfrak{m}^2$. Hence y is $\Omega_R^n(k)$-regular and X-regular. We set $\overline{(-)} = (-) \otimes_R R/yR$. Note that $\overline{X}$ is a principal $\overline{R}$-module and that, by [15, Corollary 2.5] and Proposition 2.1, $1 = \delta_R(X) \leq \delta_{\overline{R}}(\overline{X}) \leq \mu(\overline{X}) = 1$. Note that, by [14, Proposition 5.2] , we have

$$\overline{\Omega_R^n(k)} \cong \Omega_{\overline{R}}^n(k) \oplus \Omega_{\overline{R}}^{n-1}(k).$$

Therefore we have $\overline{X} \oplus \overline{Y} \cong \overline{\Omega_R^n(k)} \cong \Omega_{\overline{R}}^n(k) \oplus \Omega_{\overline{R}}^{n-1}(k)$. But $\overline{X}$ is indecomposable $\overline{R}$-module so, by Krull-Schmit uniqueness theorem (see [11, Theorem 21.35]), $\overline{X}$ is direct summand of $\Omega_{\overline{R}}^{n-1}(k)$ or $\Omega_{\overline{R}}^n(k)$. Now our induction hypothesis implies that $\overline{R}$ is regular and so is R. □

Over a Gorenstein local ring R, Proposition 2.1 (iii) states that the inequality $\delta_R(M) \leq \mu(M)$.

In the following, we explore when equality holds true by means of Gorenstein dimensions. A finite R-module M is said to be *totally reflexive* if the natural map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism and

$$\mathrm{Ext}_R^i(M, R) = 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_R(M, R), R)$$

for all $i > 0$. An R-module M is said to have Gorenstein dimension $\leq n$, write $\text{G-dim}_R(M) \leq n$, if there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

of R modules such that each G_i is totally reflexive. Write $\text{G-dim}_R(M) = n$ if there is no such sequence with shorter length. If there is no such finite length exact sequence, we write $\text{G-dim}_R(M) = \infty$.

Our result indicates the existence of a finite length R- module M such that the equality $\delta_R(M) = \mu(M)$ holds true may put a strong condition on R. More precisely:

In [14, Theorem 6.5], it is shown that the local ring $(R, \mathfrak{m}, k)$ is Gorenstein if and only if $\Omega_R^n(k)$ has a G-projective summand for some n, $0 \leq n \leq \operatorname{depth} R + 2$.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a local ring. The following statements are equivalent:*

(i) *R is a Gorenstein ring;*

(ii) *There exists an R–module M such that $\delta_R(M) = \mu(M)$, $\mathfrak{m}^n M = 0$, and $\operatorname{G-dim}_R(\mathfrak{m}^{n-2} M^{\mathrm{cm}}) < \infty$ for some integer $n \geq 2$.*

Proof. Assume first that R is Gorenstein and that $\underline{x}$ is a maximal R–regular sequence. Thus there is a surjective homomorphism $R/\mathfrak{m}^t \longrightarrow R/\underline{x}R$ for some integer $t \geq 1$. As $\operatorname{proj.dim}(R/\underline{x}R) < \infty$, Proposition 2.1 implies that

$$1 = \mu(R/\underline{x}R) = \delta_R(R/\underline{x}R) \leq \delta_R(R/\mathfrak{m}^t) \leq \mu(R/\mathfrak{m}^t) = 1.$$

Therefore $\delta_R(R/\mathfrak{m}^t) = 1 = \mu(R/\mathfrak{m}^t)$. Now by setting $n = t+1 \geq 2$, the module $M := R/\mathfrak{m}^t$ trivially justifies claim (ii).

For the converse, consider the natural exact sequence

$$0 \longrightarrow (M^{\mathrm{cm}} + \mathfrak{m}M)/\mathfrak{m}M \longrightarrow M/\mathfrak{m}M \longrightarrow \frac{M/M^{\mathrm{cm}}}{\mathfrak{m}(M/M^{\mathrm{cm}})} \longrightarrow 0\,.$$

Now the equality $\delta_R(M) = \mu(M)$ implies that $M^{\mathrm{cm}} \subseteq \mathfrak{m}M$. As $\mathfrak{m}^n M = 0$, $\mathfrak{m}^{n-2}M^{\mathrm{cm}}$ is vector space. Our assumption $\operatorname{G-dim}_R(\mathfrak{m}^{n-2}M^{cm}) < \infty$ implies that $\operatorname{G-dim}_R(R/\mathfrak{m}) < \infty$. Hence R is Gorenstein by [6, Theorem 1.4.9]. □

Here is our observation which shows how one may characterize a Cohen-Macaulay local ring with canonical module to be generically Gorenstein by the δ-invariant.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ with canonical module ω_R. Then the following statements are equivalent:*

(a) *The ring R is a generically Gorenstein ring but not Gorenstein;*

(b) *There exists an ideal I of R such that:*

(i) $\delta_R(R/I) = 1$,

(ii) $\operatorname{ht}_R(I) = 1$,

(iii) *There exists a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & R/I & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \operatorname{Hom}_R(I, \omega_R) & \longrightarrow & \operatorname{Ext}^1_R(R/I, \omega_R) & \longrightarrow & 0 \end{array}$$

with isomorphism vertical maps.

Proof. (a)⇒(b). Assume that R is generically Gorenstein and that $\omega_R \ncong R$. As ω_R is an ideal of R, we consider the exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \stackrel{\pi}{\longrightarrow} M \longrightarrow 0\,,$$

where $M := R/\omega_R$. Let L be a maximal Cohen-Macaulay R-module with no free direct summands, $\phi : L \longrightarrow M$ an R-homomorphism. Applying the functor $\mathrm{Hom}_R(L, -)$ gives the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(L, \omega_R) \longrightarrow \mathrm{Hom}_R(L, R) \longrightarrow \mathrm{Hom}_R(L, M) \longrightarrow \mathrm{Ext}^1_R(L, \omega_R)\,.$$

As $\mathrm{Ext}^1_R(L, \omega_R) = 0$, there exists $\alpha \in \mathrm{Hom}_R(L, R)$ such that $\pi \circ \alpha = \phi$. If there exists $x \in L$ such that $\phi(x) \notin \mathfrak{m}M$ then we have $\alpha(x) \notin \mathfrak{m}$, i.e. $\alpha(x)$ is a unit and so α is an epimorphism which means L has a free direct summand which is not the case. Hence $\phi(L) \subseteq \mathfrak{m}M$. Therefore $M^{\mathrm{cm}} \subseteq \mathfrak{m}M$ and we have

$$\delta_R(M) = \mu(M/M^{\mathrm{cm}}) = \mathrm{vdim}\,_k(M/(M^{\mathrm{cm}} + \mathfrak{m}M)) = \mu(M/\mathfrak{m}M) = \mu(M) = 1.$$

Moreover, we have $\mathrm{Ext}^1_R\,(R/\omega_R, \omega_R) \cong R/\omega_R$ since R/ω_R is Gorenstein ring of dimension $d-1$, and $\mathrm{Hom}_R(\omega_R, \omega_R) \cong R$, $\mathrm{ht}\,_R\,(\omega_R) = 1$. Now that the statement (iii) follows naturally.

(b)⇒(a). As $\mathrm{ht}\,(I) = 1$, $I \nsubseteq \bigcup\limits_{\mathfrak{p} \in \mathrm{Ass}\,(R)} \mathfrak{p}$ and so $\mathrm{Hom}_R(R/I, \omega_R) = 0$. Hence, naturally, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R, \omega_R) \longrightarrow \mathrm{Hom}_R(I, \omega_R) \longrightarrow \mathrm{Ext}^1_R(R/I, \omega_R) \longrightarrow 0\,.$$

One has the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(R, \omega_R) & \longrightarrow & \mathrm{Hom}_R(I, \omega_R) & \longrightarrow & \mathrm{Ext}^1_R(R/I, \omega_R) & \longrightarrow & 0. \end{array}$$

Therefore we obtain, $I \cong \omega_R$ which means R is generically Gorenstein.

To see the final claim, assume contrarily that R is Gorenstein. Hence $\omega_R \cong R$ and $\mathrm{Hom}_R(R, \omega_R) \cong \mathrm{Hom}_R(I, \omega_R)$. Now, the commutative diagram (iii) implies that $R/I = 0$ so $\delta_R(R/I) = 0$ which is a contradiction. □

The notion of linkage of ideals in commutative algebra is invented by Peskine and Szpiro [13]. Two ideals I and J in a Cohen-Macaulay local ring R are said to be linked if there is a regular sequence $\underline{a}$ in their intersection such that $I = (\underline{a}) :_R J$ and $J = (\underline{a}) :_R I$. They have shown that the Cohen-Macaulay-ness property is preserved under linkage over Gorenstein local rings and provided a counterexample to show that the above result is no longer true if the base ring is Cohen-Macaulay but not Gorenstein. In the following, we investigate the situation over a Cohen-Macaulay local ring with canonical module and generalize the result of Peskine and Szpiro [13].

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Suppose that I and J are two ideals of R such that*

$$0 :_{\omega_R} I = J\omega_R, \quad 0 :_{\omega_R} J = I\omega_R, \quad \text{G-dim}_{R/I}(\omega_R/I\omega_R) < \infty$$

and also $\text{G-dim}_{R/J}(\omega_R/J\omega_R) < \infty$ (e.g. R is Gorenstein), then R/I is Cohen-Macaulay R-module if and only if R/J is Cohen-Macaulay R-module.

Proof. Assume that R/I is Cohen-Macaulay. Set $t := \text{grade}(I, R)$ so that $t = \text{ht}_R(I) = \dim R - \dim R/I$. If $t > 0$ then there exists an R-regular element x in I. As ω_R is maximal Cohen-Macaulay, x is also ω_R-regular which implies that $J\omega_R = (0 :_{\omega_R} I) = 0$. Hence $J = 0$ which is absurd. So assume that $t = 0$ which implies that R/I is maximal Cohen-Macaulay R–module so that $\text{Ext}^i_R(R/I, \omega_R) = 0$ for all $i \geq 1$. Apply the functor $\text{Hom}_R(-, \omega_R)$ on a minimal free resolution

$$\cdots \longrightarrow \overset{t_2}{\oplus} R \longrightarrow \overset{t_1}{\oplus} R \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

of R/I, to obtain the induced exact sequence

$$0 \longrightarrow \omega_R/J\omega_R \longrightarrow \overset{t_1}{\oplus}\omega_R \longrightarrow \overset{t_2}{\oplus}\omega_R \longrightarrow \cdots .$$

Splitting into the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_R/J\omega_R & \longrightarrow & \overset{t_1}{\oplus}\omega_R & \longrightarrow & C_1 & \longrightarrow & 0 \\ 0 & \longrightarrow & C_1 & \longrightarrow & \overset{t_2}{\oplus}\omega_R & \longrightarrow & C_2 & \longrightarrow & 0 \\ 0 & \longrightarrow & C_2 & \longrightarrow & \overset{t_3}{\oplus}\omega_R & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & & & \vdots & & & & \end{array}$$

where $C_i = \text{Im} f_{i+1}$ for $i \geq 1$, we obtain $\text{depth}_R(\omega_R/J\omega_R) = d$. Note that $\text{G-dim}_{R/J}(\omega_R/J\omega_R) < \infty$, implies that $d = \text{depth}_{R/J}(\omega_R/J\omega_R) \leq \text{depth}_{R/J}(R/J)$. Thus R/J is also a maximal Cohen-Macaulay R-module. □

To see some applications of Theorem 2.6, we refer to the nth δ-invariant of an R–module M as in the paragraph just after [2, Proposition 5.3].

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Let I and J be ideals of R.*

(a) If $0 :_{\omega_R} I = J\omega_R$ and R/I is a maximal Cohen-Macaulay R-module, then $\delta^i_R(J\omega_R) = 0$ for all $i \geq 1$.

(b) If $0 :_{\omega_R} I = J\omega_R$, $0 :_{\omega_R} J = I\omega_R$, R/I is a maximal Cohen-Macaulay R-module, and $\text{G-dim}_{R/J}(\omega_R/J\omega_R) < \infty$, then $\delta^i_R(I\omega_R) = 0$ for all $i \geq 1$.

Proof. (a). A similar argument as in the proof of Theorem 2.6, implies that $\operatorname{depth}_R(\omega_R/J\omega_R) = d$ and $\omega_R/J\omega_R$ is maximal Cohen-Macaulay. By the paragraph just after [2, Proposition 5.3], we get $\delta^i_R(J\omega_R) = 0$ for all $i \geq 1$.

(b). By Theorem 2.6, R/J is maximal Cohen-Macaulay R-module so, by part (a), $\delta^i_R(I\omega_R) = 0$ for all $i \geq 1$. □

3. Gorenstein non-regular rings

For an ideal I of a ring R, we set $G := gr_I(R)$ as the associated graded ring of R with respect to I.

Lemma 3.1. *Assume that $(R, \mathfrak{m})$ is a local ring and that I is an $\mathfrak{m}$-primary ideal of R such that I^i/I^{i+1} is a free R/I-module for all $i \geq 0$. Suppose that $x \in I \setminus I^2$ such that $x^* := x + I^2$ is a G-regular element in G. Set $\bar{R} = R/xR$. Then, for any $n \geq 0$, we have $\Omega^n_R(I^m) \otimes_R \bar{R} \cong \Omega^n_{\bar{R}}(I^{m-1}/I^m) \oplus \Omega^n_{\bar{R}}(I^m/xI^{m-1})$ for all $m \geq 1$.*

Proof. As x^* is a G-regular element in G, the map $I^{m-1}/I^m \xrightarrow{x\cdot} I^m/I^{m+1}$ is injective for all $m \geq 1$, to prove this claim, suppose that $t + I^m \in I^{m-1}/I^m$ such that $xt \in I^{m+1}$. Therefore $(x + I^2)(t + I^m) = xt + I^{m+1} = 0_G$. As x^* is a G regular element in G, then $t \in I^m$. Let $m \geq 1$. We prove the claim by induction on n. I claim that $I^m/xI^m \cong I^{m-1}/I^m \oplus I^m/xI^{m-1}$, to prove this claim, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{m-1}/I^m & \xrightarrow{x\cdot} & I^m/xI^m & \longrightarrow & I^m/xI^{m-1} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & I^{m-1}/I^m & \xrightarrow{x\cdot} & I^m/I^{m+1} & \longrightarrow & I^m/(xI^{m-1} + I^{m+1}) & \longrightarrow & 0 \end{array} .$$

As I is a $\mathfrak{m}$ primary ideal of R, we get $\dim(R/I) = \dim(R/\sqrt{I}) = \dim(R/\mathfrak{m}) = 0$. Therefore the injective map $I^{m-1}/I^m \xrightarrow{x.} I^m/I^{m+1}$ splits. Therefore the first row of the above diagram splits. Thus

$$\begin{aligned} \Omega^0_R(I^m) \otimes_R \bar{R} &= I^m \otimes_R \bar{R} \\ &\cong I^m/xI^m \\ &\cong I^{m-1}/I^m \oplus I^m/xI^{m-1} \\ &= \Omega^0_{\bar{R}}(I^{m-1}/I^m) \oplus \Omega^0_{\bar{R}}(I^m/xI^{m-1}), \end{aligned}$$

which proves the claim for $n = 0$.

Now we assume that $n > 0$ and the claim is settled for integers less than n. x is a regular element on both R and $\Omega^{n-1}_R(I^m)$ (since for all $m \geq 1$ the map $I^{m-1}/I^m \xrightarrow{x\cdot} I^m/I^{m+1}$ is injective).

Therefore a minimal free cover $0 \longrightarrow \Omega^n_R(I^m) \longrightarrow F \longrightarrow \Omega^{n-1}_R(I^m) \longrightarrow 0$ of $\Omega^{n-1}_R(I^m)$ gives a minimal cover

$$0 \longrightarrow \Omega^n_R(I^m) \otimes_R \bar{R} \longrightarrow F \otimes_R \bar{R} \longrightarrow \Omega^{n-1}_R(I^m) \otimes_R \bar{R} \longrightarrow 0$$

of $\Omega_R^{n-1}(I^m) \otimes_R \bar{R}$ over $\bar{R}$. Hence we get $\Omega_R^n(I^m) \otimes_R \bar{R} \cong \Omega_{\bar{R}}^1(\Omega_R^{n-1}(I^m) \otimes_R \bar{R})$. By the induction hypothesis we have

$$\begin{aligned}\Omega_R^n(I^m) \otimes_R \bar{R} &\cong \Omega_{\bar{R}}^1(\Omega_R^{n-1}(I^m) \otimes_R \bar{R}) \\ &\cong \Omega_{\bar{R}}^1(\Omega_{\bar{R}}^{n-1}(I^{m-1}/I^m) \oplus \Omega_{\bar{R}}^{n-1}(I^m/xI^{m-1})) \\ &\cong \Omega_{\bar{R}}^n(I^{m-1}/I^m) \oplus \Omega_{\bar{R}}^n(I^m/xI^{m-1}).\end{aligned}$$

□

It is shown by Yoshino [16, Theorem 2.3] that, in a complete non-regular Gorenstein local ring $(R, \mathfrak{m})$ with depth $(gr_{\mathfrak{m}}(R)) \geq d-1$, one has $\delta_R^n(R/\mathfrak{m}^m) = 0$ for all positive integers n and m, where $gr_{\mathfrak{m}}(R)$ denote the associated graded ring of R with respect to $\mathfrak{m}$. Now by following theorem which is a generalization of [16, Theorem 2.3], we approach our result (Corollary 3.4). Set depth $(G) =$ grade (G_+, G) where G_+ is the ideal which is generated by all elements with positive degree in G.

Theorem 3.2. *Suppose that $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension d with infinite residue field $R/\mathfrak{m}$. Assume that I is an $\mathfrak{m}$-primary ideal of R such that:*

(i) For any $i \geq 0$, I^i/I^{i+1} is free R/I-module, and

(ii) for any R-regular sequence $\mathbf{x} = x_1, \cdots, x_s$ in I with

$$x_i + (x_1, \cdots, x_{i-1}) \in (I/(x_1, \cdots, x_{i-1})) \setminus (I/(x_1, \cdots, x_{i-1}))^2, \quad 1 \leq i \leq s,$$

we have $\delta^n_{R/\mathbf{x}R}(R/I) = 0$ for all $n \geq 0$.

Then $\delta_R^n(R/I^m) = 0$ for all integers $n \geq d + 1 - depth G$ and all $m \geq 1$. In particular, if depth $G = d-1$, then $\delta_R^n(R/I^m) = 0$ for all $n \geq 2$ and all $m \geq 1$.

Proof. Let $m \geq 1$ and $t =$ depth (G). If $d = 0$ the result is trivial by [3, Corollary 1.2.5]. We assume that $d > 0$ and $n \geq d + 1 - t$.

If $t = 0$ then $n \geq d+1$ and the result is clear (since $\Omega_R^n(R/I^m)$ is a maximal Cohen-Macaulay module by [5, Exercises 2.1.26] and $\Omega_R^n(R/I^m)$ has a no free direct summand by [3, Corollary 1.2.5]). Now assume that $d > 0$ and $t > 0$. As $R/\mathfrak{m}$ is infinite implies that I has a superficial element $x \in I \setminus I^2$ ([10, Proposition 8.5.7]), and we get $x^* := x + I^2$ is a G-regular element on G by [9, Lemma 2.1]. Then the map $I^{m-1}/I^m \xrightarrow{x\cdot} I^m/I^{m+1}$ is injective. Set $\bar{R} = R/xR$ and $\bar{I} = I/xR$ and let $n \geq d - t + 1$. By Lemma 3.1 we have

$$\Omega_R^{n-1}(I^m) \otimes_R \bar{R} \cong \Omega_{\bar{R}}^{n-1}(I^{m-1}/I^m) \oplus \Omega_{\bar{R}}^{n-1}(I^m/xI^{m-1}).$$

On the other hand, x is $\Omega_R^{n-1}(I^m)$-regular; therefore by [15, Corollary 2.5] we have

$$\delta_R(\Omega_R^{n-1}(I^m) \leq \delta_{\bar{R}}(\Omega_R^{n-1}(I^m) \otimes_R \bar{R}).$$

Therefore

$$\begin{aligned}\delta^n_R(R/I^m) &= \delta_R(\Omega^{n-1}_R(I^m) \\ &\le \delta_{\bar{R}}(\Omega^{n-1}_R(I^m)\otimes_R \bar{R}) \\ &= \delta_{\bar{R}}(\Omega^{n-1}_{\bar{R}}(I^{m-1}/I^m)\oplus \Omega^{n-1}_{\bar{R}}(I^m/xI^{m-1})) \\ &= \delta_{\bar{R}}(\Omega^{n-1}_{\bar{R}}(I^{m-1}/I^m)) + \delta_{\bar{R}}(\Omega^{n-1}_{\bar{R}}(I^m/xI^{m-1})) \\ &= \delta^{n-1}_{\bar{R}}(I^{m-1}/I^m) + \delta^{n-1}_{\bar{R}}(I^m/xI^{m-1}).\end{aligned} \tag{3.1}$$

The injective map $I^{m-1}/I^m \xrightarrow{x\cdot} I^m/I^{m+1}$ implies, by induction on m, that $xI^{m-1} = xR\cap I^m$ so we get $(\bar{I})^m = I^m/(xR\cap I^m) = I^m/xI^{m-1}$; therefore

$$\begin{aligned}\delta^n_R(R/I^m) &\le \delta^{n-1}_{\bar{R}}(I^m/xI^{m-1}) + \delta^{n-1}_{\bar{R}}(I^{m-1}/I^m) \\ &= \delta^{n-1}_{\bar{R}}((\bar{I})^m) + \delta^{n-1}_{\bar{R}}(I^{m-1}/I^m)\,.\end{aligned}$$

Note that, by assumption, $I^{m-1}/I^m \cong \overset{a}{\oplus} R/I$ for some non-negative integer a, and hence, $\delta^{n-1}_{\bar{R}}(I^{m-1}/I^m) = a.\delta^{n-1}_{\bar{R}}(R/I) = a.0 = 0$.

If $d = 1$ then $\dim(\bar{R}) = 0$ so $\delta^{n-1}_{\bar{R}}((\bar{I})^m) = \delta_{\bar{R}}(\Omega^{n-1}_{\bar{R}}((\bar{I})^m)) = \delta^n_{\bar{R}}(\bar{R}/(\bar{I})^m) = 0$ hence the result is clear.

Suppose that $d \ge 2$. As $n \ge d-t+1 = (d-1)-(t-1)+1$, when $t = 1$, we have $n \ge (d-1)+1$ hence $\delta^n_{\bar{R}}(\bar{R}/(\bar{I})^m) = 0$, therefore $\delta^n_R(R/I^m) = 0$.

Set $\bar{G} = gr_{\bar{R}}(\bar{I})$. If $t \ge 2$ then $\operatorname{depth}(\bar{G}) = \operatorname{depth}(G/x^*G) = t-1 > 0$. As $\bar{R}/\bar{\mathfrak{m}} \cong R/\mathfrak{m}$ is infinite and $\dim(\bar{R}) > 0$ and $\operatorname{depth}(\bar{G}) > 0$, by [9, Lemma 2.1], there exists $\bar{y} = y + xR \in \bar{I}\setminus \bar{I}^2$ such that $\bar{y}^*$ is $\bar{G}$-regular. Therefore the map $\bar{I}^{m-1}/\bar{I}^m \xrightarrow{\bar{y}} \bar{I}^m/\bar{I}^{m+1}$ is injective. On the other hand, we have $(\bar{I})^m/(\bar{I})^{m+1} \cong I^m/(xI^{m-1}+I^{m+1})$ and $I^m/(xI^{m-1}+I^{m+1})$ is a direct summand of I^m/I^{m+1}. Therefore $(\bar{I})^m/(\bar{I})^{m+1}$ is a free $\bar{R}/\bar{I}$-module for any $i \ge 1$. Set $\bar{\bar{R}} = \bar{R}/\bar{y}\bar{R}$ and $\bar{\bar{I}} = \bar{I}/\bar{y}\bar{I}$. Then by the same argument as above we have $\delta^n_{\bar{R}}(\bar{R}/\bar{I}^m) \le \delta^n_{\bar{\bar{R}}}(\bar{\bar{R}}/(\bar{\bar{I}})^m) + \delta^{n-1}_{\bar{\bar{R}}}(\oplus R/I)$.

By our assumption $\delta^{n-1}_{\bar{\bar{R}}}(\oplus R/I)) = \delta^{n-1}_{R/(x,y)}(\oplus R/I)) = 0$. When $d = 2$, $\dim(\bar{\bar{R}}) = 0$ and so $\delta^n_{\bar{\bar{R}}}(\bar{\bar{R}}/(\bar{\bar{I}})^m) = 0$ and the result is clear. Suppose that $d \ge 3$. As $n \ge d-t+1 = (d-2)-(t-2)+1$, if $t = 2$ then $\delta^n_{\bar{\bar{R}}}(\bar{\bar{R}}/(\bar{\bar{I}})^m) = 0$, therefore (3.1) implies that

$$\begin{aligned}\delta^n_R(R/I^m) &\le \delta^n_{\bar{R}}(\bar{R}/(\bar{I})^m) + \delta^{n-1}_{\bar{R}}(\oplus R/I)) \\ &\le \delta^n_{\bar{\bar{R}}}(\bar{\bar{R}}/(\bar{\bar{I}})^m) + \delta^{n-1}_{R/(x,y)}(\oplus R/I) + \delta^{n-1}_{R/xR}(\oplus R/I) \\ &= 0.\end{aligned}$$

For the case $t \ge 3$, we proceed by the same argument as above to find $\delta^n_R(R/I^m) = 0$. □

Remark 3.3. Let $(R,\mathfrak{m})$ be a local ring. The ring R is regular if and only if R is Gorenstein and $\delta_R(M) > 0$ for all non-zero finitely generated R-module M.

Proof. Suppose that R is regular. Assume contrarily that there exists a non-zero R–module M such that $\delta_R(M) = 0$. By definition of delta, there exists a surjective homomorphism $X \longrightarrow M$ such that X is maximal Cohen-Macaulay R-module with no free direct summand. On the other hand, as R is regular, $\text{proj.dim}_R(X) = 0$ and so X is free a R-module which is not the case.

Conversely, by assumption and Proposition 2.1, $1 \leq \delta_R(R/\mathfrak{m}) \leq \mu(R/\mathfrak{m}) = 1$. Hence, by Proposition 2.1, R is regular. □

Corollary 3.4. *Suppose that $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension d such that $R/\mathfrak{m}$ is infinite. Consider the following statements:*

(a) R is not regular;

(b) There exists an $\mathfrak{m}$-primary ideal I of R such that

(i) I^i/I^{i+1} is free R/I-module for any $i \geq 0$, and

(ii) for any R-regular sequence $\mathbf{x} = x_1, \cdots, x_s$ in I such that

$$x_i + (x_1, \cdots, x_{i-1}) \in (I/(x_1, \cdots, x_{i-1})) \setminus (I/(x_1, \cdots, x_{i-1}))^2,\ 1 \leq i \leq s,$$

one has $\delta^n_{R/\mathbf{x}R}(R/I) = 0$ for all $n \geq 0$;

(c) There exists a non-zero ideal I of R such that $\delta^n_R(R/I^m) = 0$ for all integers $n \geq d - \text{depth}\,(G) + 1$ and $m \geq 1$.

Then the implications (a)⇒(b) and (b)⇒(c) hold true. If $\text{depth}\,(G) > \text{depth}_R(R/I)$, the statements (a), (b), and (c) are equivalent.

Proof. (a)⇒(b). We show that $I = \mathfrak{m}$ works. Assume that $\mathbf{x} = x_1, \cdots, x_s$ is R-regular sequence in $\mathfrak{m}$ such that $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $x_i + (x_1, \cdots, x_{i-1}) \in (\mathfrak{m}/(x_1, \cdots, x_{i-1})) \setminus (\mathfrak{m}/(x_1, \cdots, x_{i-1}))^2$. Set

$$\bar{R} = R/(x_1, \cdots, x_{s-1})R, \qquad \bar{\mathfrak{m}} = \mathfrak{m}/(x_1, \cdots, x_{s-1})R$$

and $\bar{x}_s = x_s + (x_1, \cdots, x_{s-1})$. As $\bar{R}/\bar{x}_s\bar{R}$ is not regular so by [2, Proposition 5.7] we have, for all $n \geq 0$,

$$\begin{aligned}
\delta^n_{R/(x_1,\cdots,x_s)R}(R/\mathfrak{m}) &= \delta^n_{(\bar{R}/\bar{x}_s\bar{R})}(R/\mathfrak{m}) \\
&= \delta^n_{(\bar{R}/\bar{x}_s\bar{R})}(\bar{R}/\bar{\mathfrak{m}}) \\
&= \delta^n_{(\bar{R}/\bar{x}_s\bar{R})}\big((\bar{R}/\bar{x}_s\bar{R})/(\bar{\mathfrak{m}}/\bar{x}_s\bar{R})\big) \\
&= 0.
\end{aligned}$$

(b)⇒(c). Apply Theorem 3.2.

(c)⇒(a). We assume that R is a regular ring. By our assumption we get $\delta^n_R(R/I) = 0$ for all integers $n \geq d - \text{depth}\,(G) + 1$. Therefore $\delta_R(\Omega^n_R(R/I)) = 0$ for all integers $n \geq d - \text{depth}\,(G) + 1$ and by Remark 3.3 we have $\Omega^n_R(R/I)) = 0$ for all integers $n \geq d - \text{depth}\,(G) + 1$. Then $\text{proj.dim}_R(R/I) \leq d - \text{depth}\,(G)$. On the other hand Auslander-Buchsbaum formula implies that $d - \text{depth}_R(R/I) = \text{proj.dim}_R(R/I) \leq d - \text{depth}\,(G)$ which contradicts that $\text{depth}\,(G) > \text{depth}_R(R/I)$. □

Lemma 3.5. *Assume that $(R, \mathfrak{m})$ is a 1-dimensional Gorenstein local ring and that I is an $\mathfrak{m}$-primary ideal of R such that $\mu_R(I) \geq 2$. Then $\delta_R(I^n/I^{n+1}) = 0$ and $\delta^m_R(R/I^n) = 0$ for all positive integers n and m.*

Proof. The assumption $\dim(R/I) = 0$ and the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

imply that I is maximal Cohen-Macaulay as an R–module. As $\mu_R(I) \geq 2$, I has no free direct summand and so $\delta_R(I) = 0$. For a finite R-module M, the natural epimorphism $\overset{\mu_R(M)}{\oplus} I \longrightarrow IM$, by Proposition 2.1, implies that $\delta_R(IM) \leq \delta_R(\overset{t}{\oplus} I) = 0$ which gives $\delta_R(IM) = 0$. Let n and m be positive integers. As $\Omega^m_R(R/I^n)$ is a maximal Cohen-Macaulay R-module for all $m \geq 1$, we have $\delta^m_R(R/I^n) = 0$ for all $m > 1$ (see the paragraph just after [2, Proposition 5.3]). For the case $m = 1$ we have $\delta^1_R(R/I^n) = \delta_R(\Omega^1_R(R/I^n)) = \delta_R(I^n) = \delta_R(II^{n-1}) = 0$. □

For an R-regular element x in $\mathfrak{m}$, we set $\overline{(-)} = (-) \otimes_R R/xR$. Recall from the first paragraph of Section 5 of [2] that an R-module M is called *weakly liftable* on $\overline{R}$ if M is a direct summand of $\overline{N}$ for some R-module N. The following result will be used in characterizing a ring to be non-regular Gorenstein of dimensions 2.

Proposition 3.6. *Assume that $(R, \mathfrak{m})$ is a 2-dimensional Gorenstein local ring. Then the following statements are equivalent:*

- *(a) R is not regular;*
- *(b) There exists an $\mathfrak{m}$-primary ideal I of R such that*
  - *(i) $\mu_R(I) \geq 3$,*
  - *(ii) I/I^2 is a free R/I-module,*
  - *(iii) There exists R-regular element $x \in I \setminus I^2$ such that the natural map $R/I \xrightarrow{x\cdot} I/I^2$ is injective and $\delta_{R/xR}(R/I) = 0$;*
- *(c) There exists a non-zero ideal I of R such that $\delta^n_R(R/I) = 0$ for all positive integers $n \geq 1$.*

Proof. (a)⇒(b). We set $I = \mathfrak{m}$. Choose an R-regular element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Hence the map $R/\mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2$, with $a + \mathfrak{m} \rightsquigarrow ax + \mathfrak{m}^2$, is injective. As R/xR is not regular, $\delta_{R/xR}(R/\mathfrak{m}) = \delta_{R/xR}((R/xR)/(\mathfrak{m}/xR)) = 0$ by [2, Proposition 5.7] and also it is clear $\mathfrak{m}/\mathfrak{m}^2$ is a free $R/\mathfrak{m}$ module.

(b)⇒(c). Set $\overline{R} = R/xR$ and assume that $n \geq 1$. As $\dim(\overline{R}) = 1$ and $\mu_{\overline{R}}(\overline{I}) \geq 2$, by Lemma 3.5 and assumption (iii), we have $\delta_{\overline{R}}(\Omega^n_{\overline{R}}(R/I)) = \delta_{\overline{R}}(\Omega^n_{\overline{R}}(\overline{R}/\overline{I})) = 0$ and $\delta_{\overline{R}}(\Omega^{n-1}_{\overline{R}}(R/I)) = \delta_{\overline{R}}(\Omega^{n-1}_{\overline{R}}(\overline{R}/\overline{I})) = 0$. On the other hand, $\dim R/I = 0$ and the exact sequence

$$0 \longrightarrow R/I \longrightarrow I/I^2 \longrightarrow I/(xR + I^2) \longrightarrow 0$$

imply that $\operatorname{proj.dim}_{R/I}(I/(xR + I^2)) = 0$ by Auslander-Buchsbaum formula. Therefore, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/I & \xrightarrow{x\cdot} & I/xI & \longrightarrow & I/xR & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R/I & \xrightarrow{x\cdot} & I/I^2 & \longrightarrow & I/(xR+I^2) & \longrightarrow & 0 \end{array}$$

with exact rows implies that the upper row splits. Hence R/I, as $\bar{R}$–module, is weakly liftable on $\bar{R}$.

Thus, by [14, Proposition 5.2], we obtain $\overline{\Omega^n_R(R/I)} \cong \Omega^n_{\overline{R}}(R/I) \oplus \Omega^{n-1}_{\overline{R}}(R/I)$. Therefore

$$\begin{aligned} \delta^n_R(R/I) = \delta_R(\Omega^n_R(R/I)) &\leq \delta_{\overline{R}}(\overline{\Omega^n_R(R/I)}) \\ &= \delta_{\overline{R}}(\Omega^n_{\overline{R}}(R/I)) + \delta_{\overline{R}}(\Omega^{n-1}_{\overline{R}}(R/I)) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

(c)⇒(a). By assumption $\delta_R(I) = 0$, therefore R is not regular. □

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